Review

Ptolemy's Theorem in the Relativistic Model of Analytic Hyperbolic Geometry

Abraham A. Ungar [ID]

Department of Mathematics, North Dakota State University, Fargo, ND 58105, USA; abraham.ungar@ndsu.edu

Abstract: Ptolemy's Theorem in Euclidean geometry, named after the Greek astronomer and mathematician Ptolemy, is well-known. By means of the relativistic model of hyperbolic geometry, we translate Ptolemy's Theorem from Euclidean geometry into the hyperbolic geometry of Lobachevsky and Bolyai. The relativistic model of hyperbolic geometry is based on the Einstein addition of relativistically admissible velocities and, as such, it coincides with the well-known Beltrami–Klein ball model of hyperbolic geometry. The translation of Ptolemy's Theorem from Euclidean geometry into hyperbolic geometry is achieved by means of hyperbolic trigonometry, called *gyrotrigonometry*, to which the relativistic model of analytic hyperbolic geometry gives rise.

Keywords: Ptolemy's theorem; analytic hyperbolic geometry; einstein addition; gyrogroups; gyrovector spaces; gyrotrigonometry

MSC: 51M10; 83A05

1. Introduction

Analytic hyperbolic geometry is the hyperbolic geometry of Lobachevsky and Bolyai, studied analytically since 1988 [1]. In order to demonstrate the power and elegance of the novel discipline of analytic hyperbolic geometry, we review topics of analytic hyperbolic geometry, including the theory of gyrogroups and gyrovector spaces and gyrotrigonometry, which enable Ptolemy's Theorem to be translated in an elegant way from analytic Euclidean geometry into analytic hyperbolic geometry. Analytic Euclidean geometry involves vector addition and, in full analogy, analytic hyperbolic geometry involves Einstein addition.

Nature organizes itself using the language of symmetries. In particular, the symmetry group that regulates Einstein's special relativity theory is the Lorentz group. The Lorentz group, in turn, is parametrized by the velocity parameter which is governed by Einstein addition and multiplication. As such, the spaces of the velocity parameter give rise to gyrogroups, gyrovector spaces, and analytic hyperbolic geometry which are, respectively, fully analogous to groups, vector spaces, and analytic Euclidean geometry. These analogies are vividly demonstrated in several books [2–10] and many papers.

Unexpectedly, the resulting analogies that the Einstein addition captures enable Ptolemy's Theorem to be translated from Euclidean geometry into hyperbolic geometry. Ptolemy's Theorem concerns the situation depicted in Figure 5 and its hyperbolic counterpart is depicted in Figure 6.

Seemingly structureless, the Einstein addition of relativistically admissible velocities is neither commutative nor associative. However, it has been known since 1988 [1,2,11,12] that Einstein addition is both *gyrocommutative* and *gyroassociative*, thus forming a gyrocommutative gyrogroup operation in Einstein gyrogroups. Moreover, Einstein addition admits scalar multiplication, giving rise to Einstein gyrovector spaces. The latter form the algebraic setting for analytic hyperbolic geometry, just as vector spaces form the algebraic setting for analytic Euclidean geometry.

Analytic hyperbolic geometry, in turn, admits hyperbolic trigonometry, called *gyrotrigonometry*, which is analogous to trigonometry. Ptolemy's Theorem in Euclidean

geometry can be verified by means of trigonometry, as shown in Section 13. Accordingly, the *trigonometry – gyrotrigonometry duality* enables Ptolemy's Theorem to be translated from Euclidean geometry into hyperbolic geometry. Various different attempts to extend Ptolemy's Theorem to non-Euclidean geometries have been found in the literature as, for instance, in the Refs. [13,14], and references cited therein.

The gyrocommutative gyrogroup structure that the Einstein addition encodes gives rise to our *gyrolanguage* in which we prefix a gyro to any term that describes a concept in Euclidean geometry and in associative algebra to mean the analogous concept in hyperbolic geometry and in nonassociative algebra. The prefix "gyro" stems from "gyration", which is the mathematical abstraction of the special relativistic effect known as *"Thomas precession"* [2]. In gyrolanguage, thus, Einstein addition of vectors is a gyroaddition of gyrovectors.

In Sections 2–13 we review topics from the theory of gyrogroups, gyrovector spaces and analytic hyperbolic geometry and present a trigonometric proof of Ptolemy's Theorem. The reviewed topics are necessary for the introduction of the novel hyperbolic Ptolemy's Theorem and its application in Sections 14 and 15. As such, the present article is a review paper into which the novel hyperbolic Ptolemy's Theorem has been incorporated.

Accordingly, we start the unexpected journey to the hyperbolic Ptolemy's Theorem with a review of Einstein addition and the gyrogroup and gyrovector space structures that it encodes [15].

2. Einstein Addition

Let $c > 0$ be any positive constant and let $\mathbb{R}^n = (\mathbb{R}^n, +, \cdot)$ be the Euclidean n-space, $n \in \mathbb{N}$, endowed with the common vector addition, +, and inner product, $\cdot$. The space of all n-dimensional relativistically admissible velocities is the c-ball $\mathbb{R}^n_c$,

$$\mathbb{R}^n_c = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < c\}. \tag{1}$$

Einstein velocity addition is a binary operation, $\oplus$, in the c-ball $\mathbb{R}^n_c$ given by

$$\mathbf{u} \oplus \mathbf{v} = \frac{1}{1 + \frac{\mathbf{u} \cdot \mathbf{v}}{c^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}} \mathbf{v} + \frac{1}{c^2} \frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}} (\mathbf{u} \cdot \mathbf{v}) \mathbf{u} \right\}, \tag{2}$$

Refs. [2,4], ([16], Equation (2.9.2)), ([17], p. 55), [18], for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$. In physical applications $n = 3$, but in geometry $n \in \mathbb{N}$ is any natural number. Here, $\gamma_{\mathbf{v}}$ is the Lorentz gamma factor,

$$\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{v}\|^2}{c^2}}} \geq 1, \tag{3}$$

and $\mathbf{u} \cdot \mathbf{v}$ and $\|\mathbf{v}\|$ are the inner product and the norm in the ball, which the ball $\mathbb{R}^n_c$ inherits from its ambient space $\mathbb{R}^n$, and $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v} = \mathbf{v}^2$. A nonempty set with a binary operation is called a *groupoid*, so that the pair $(\mathbb{R}^n_c, \oplus)$ is an *Einstein groupoid*.

A useful identity that follows immediately from (3) is

$$\frac{\mathbf{v}^2}{c^2} = \frac{\|\mathbf{v}\|^2}{c^2} = \frac{\gamma_{\mathbf{v}}^2 - 1}{\gamma_{\mathbf{v}}^2}. \tag{4}$$

The constant $c > 0$ represents the vacuum speed of light. In the Euclidean–Newtonian limit of large c, $c \to \infty$, the ball $\mathbb{R}^n_c$ expands to the whole of its ambient space $\mathbb{R}^n$, as we see from (1), and the Einstein addition $\oplus$ in $\mathbb{R}^n_c$ descends to the common vector addition $+$ in $\mathbb{R}^n$, as we see from (2) and (3).

3. The Elegant Gyroformalism that Regulates Einstein Addition

Vector addition, +, in $\mathbb{R}^n$ is both commutative and associative. In contrast, Einstein addition, $\oplus$, in $\mathbb{R}^n_c$, given by (2), is seemingly structureless, being neither commutative nor associative. Strikingly, the deviation from both commutativity and associativity in Einstein addition is controlled by *gyrations*, as evidenced from (5)–(7).

Gyrations $\mathrm{gyr}[\mathbf{u}, \mathbf{v}] \in \mathrm{Aut}(\mathbb{R}^n_c, \oplus)$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$ are automorphisms given in terms of Einstein addition by the *gyrator equation*

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = \ominus(\mathbf{u} \oplus \mathbf{v}) \oplus \{\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w})\} \tag{5}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n_c$. Equation (5) presents the application to $\mathbf{w}$ of the gyration $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ generated by $\mathbf{u}$ and $\mathbf{v}$. Gyrations are automorphisms of the Einstein groupoid $(\mathbb{R}^n_c, \oplus)$, so that the *gyrator* gyr is the map

$$\mathrm{gyr} : \mathbb{R}^n_c \times \mathbb{R}^n_c \rightarrow \mathrm{Aut}(\mathbb{R}^n_c, \oplus). \tag{6}$$

An automorphism of a groupoid $(S, \oplus)$ is a bijective map f of S onto itself that respects the binary operation, that is, $f(a \oplus b) = f(a) \oplus f(b)$ for all $a, b \in S$. The set of all automorphisms of a groupoid $(S, \oplus)$ forms a group, denoted by $\mathrm{Aut}(S, \oplus)$, where the group operation is given by automorphism composition. To emphasize that the gyrations of an Einstein groupoid $(\mathbb{R}^n_c, \oplus)$ are automorphisms of the groupoid, gyrations are also called *gyroautomorphisms*.

Possessing their own rich structure, gyrations measure the extent to which Einstein addition deviates from commutativity and associativity, as we see from the gyrocommutative law and the gyroassociative law in the following list of identities [2–5]:

$$
\begin{array}{lll}
\mathbf{u} \oplus \mathbf{v} = \mathrm{gyr}[\mathbf{u}, \mathbf{v}](\mathbf{v} \oplus \mathbf{u}) & \text{Gyrocommutative Law} & \\
\mathbf{u} \oplus (\ominus \mathbf{u} \oplus \mathbf{w}) = \mathbf{w} & \text{Left Cancellation Law} & \\
\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} & \text{Left Gyroassociative Law} & \\
(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} = \mathbf{u} \oplus (\mathbf{v} \oplus \mathrm{gyr}[\mathbf{v}, \mathbf{u}]\mathbf{w}) & \text{Right Gyroassociative Law} & \\
\mathrm{gyr}[\mathbf{u} \oplus \mathbf{v}, \mathbf{v}] = \mathrm{gyr}[\mathbf{u}, \mathbf{v}] & \text{Left Reduction Property} & (7) \\
\mathrm{gyr}[\mathbf{u}, \mathbf{v} \oplus \mathbf{u}] = \mathrm{gyr}[\mathbf{u}, \mathbf{v}] & \text{Right Reduction Property} & \\
\mathrm{gyr}[\ominus \mathbf{u}, \ominus \mathbf{v}] = \mathrm{gyr}[\mathbf{u}, \mathbf{v}] & \text{Gyration Even Property} & \\
(\mathrm{gyr}[\mathbf{u}, \mathbf{v}])^{-1} = \mathrm{gyr}[\mathbf{v}, \mathbf{u}] & \text{Gyration Inversion Law} &
\end{array}
$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n_c$.

4. From Einstein Velocity Addition to Gyrogroups

Guided by analogies with groups, the key features of Einstein groupoids $(\mathbb{R}^n_c, \oplus)$, $n \in \mathbb{N}$, suggest the formal gyrogroup Definition in which gyrogroups form a most natural generalization of groups.

Definition 1 (Binary Operations). *A binary operation $+$ in a set S is a function $+ : S \times S \rightarrow S$. We use the notation $a + b$ to denote $+(a, b)$ for any $a, b \in S$.*

Definition 2 (Groupoids, Automorphisms). *A groupoid $(S, +)$ is a nonempty set, S, with a binary operation, $+$. An automorphism ϕ of a groupoid $(S, +)$ is a bijective self-map of S which respects its groupoid operation, that is, $\phi(a + b) = \phi(a) + \phi(b)$ for all $a, b \in S$. The automorphisms of a groupoid $(S, +)$ form a group denoted by $\mathrm{Aut}(S, +)$.*

Definition 3 (Gyrogroups ([3], Definition 2.5)). *A groupoid* $(G, \oplus)$ *is a gyrogroup if its binary operation satisfies the following axioms. In G there is at least one element,* 0, *called a left identity, satisfying*

(G1) $\qquad 0 \oplus a = a$

for all $a \in G$. *There is an element* $0 \in G$ *satisfying axiom (G1) such that for each* $a \in G$ *there is an element* $\ominus a \in G$, *called a left inverse of* a, *satisfying*

(G2) $\qquad \ominus a \oplus a = 0 \, .$

Moreover, for any $a, b, c \in G$ *there exists an automorphism* $\mathrm{gyr}[a, b] \in \mathrm{Aut}(G, \oplus)$ *such that the binary operation obeys the left gyroassociative law*

(G3) $\qquad a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c \, .$

The automorphism $\mathrm{gyr}[a, b]$ *of G is called the gyroautomorphism, or the gyration, of G generated by* $a, b \in G$. *The operator* $\mathrm{gyr} : G \times G \to \mathrm{Aut}(G, \oplus)$ *is called the gyrator of G. Finally, the gyroautomorphism* $\mathrm{gyr}[a, b]$ *generated by any* $a, b \in G$ *obeys the left reduction axiom*

(G4) $\qquad \mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b] \, .$

As in group theory, we use the notation $a \ominus b = a \oplus (\ominus b)$ in gyrogroup theory as well.

In full analogy with groups, gyrogroups split up into gyrocommutative and non-gyrocommutative ones.

Definition 4 (Gyrocommutative Gyrogroup ([3], Definition 2.6)). *A gyrogroup* $(G, \oplus)$ *is gyrocommutative if its binary operation obeys the gyrocommutative law*

(G5) $\qquad a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$

for all $a, b \in G$.

The theory of gyrogroups and gyrovector spaces was studied in [2–10]. An attractive review of gyrogroup theory can be found in ([19], Sections 2–12), and an attractive review of gyrogroup and gyrovector space theory can be found in [15].

The abstract gyrocommutative gyrogroup is an algebraic structure derived from Einstein addition $\oplus$ in $\mathbb{R}^n_c$. Indeed, Einstein groupoids $(\mathbb{R}^n_c, \oplus)$, $n \in \mathbb{N}$, are gyrocommutative gyrogroups. Gyrogroups, both gyrocommutative and nongyrocommutative, abound in group theory as demonstrated, for instance, in the Refs. [19–24]. Gyrogroups share Remarkable analogies with groups studied, for instance, in the Refs. [25–31].

5. Gyrovector Spaces

Einstein addition admits scalar multiplication between real numbers and relativistically admissible velocity vectors, giving rise to Einstein gyrovector spaces. As an example, Einstein scalar multiplication enables hyperbolic lines to be determined analytically (see Figure 2), just as Euclidean lines are commonly determined analytically (see Figure 1). Along with Remarkable analogies that Einstein scalar multiplication shares with the common scalar multiplication in vector spaces there is a striking disanalogy. Einstein scalar multiplication does not distribute over Einstein addition. However, a weaker law, called the *monodistributive law*, remains valid. Einstein gyrovector spaces form the algebraic setting for the Cartesian–Beltrami–Klein ball model of hyperbolic geometry, just as how vector spaces form the algebraic setting for the standard Cartesian model of Euclidean geometry.

Guided by properties of Einstein scalar multiplication, the formal Definition of real inner product gyrovector spaces (gyrovector spaces in short) follows in Definition 6.

Definition 5 (Real Inner Product Vector Spaces). *A real inner product vector space* $(\mathbb{V}, +, \cdot)$ *(vector space, in short) is a real vector space together with a map*

$$\mathbb{V} \times \mathbb{V} \to \mathbb{R}, \qquad (\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} \cdot \mathbf{v} \, , \qquad (8)$$

called a real inner product, satisfying the following properties for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{V}$ *and* $r \in \mathbb{R}$:

(1) $\quad \mathbf{v} \cdot \mathbf{v} \geq 0$, *with equality if, and only if,* $\mathbf{v} = 0$.

(2) $\quad \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(3) $(\mathbf{u} + \mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$

(4) $(r\mathbf{u})\cdot\mathbf{v} = r(\mathbf{u}\cdot\mathbf{v})$.

The norm $\|\mathbf{v}\|$ of $\mathbf{v} \in \mathbb{V}$ is given by the equation $\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}$.

Note that the properties of vector spaces imply (i) the Cauchy–Schwarz inequality

$$|\mathbf{u}\cdot\mathbf{v}| \leq \|\mathbf{u}\|\,\|\mathbf{v}\| \tag{9}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{V}$; and (ii) the *positive definiteness* of the inner product, according to which $\mathbf{u}\cdot\mathbf{v} = 0$ for all $\mathbf{u} \in \mathbb{V}$ implies $\mathbf{v} = 0$ [32].

Definition 6 (Real Inner Product Gyrovector Spaces ([9], Definition 3.2)). *A real inner product gyrovector space $(G, \oplus, \otimes)$ (gyrovector space, in short) is a gyrocommutative gyrogroup $(G, \oplus)$ that obeys the following axioms:*

(1)   G is a subset of a real inner product vector space $\mathbb{V}$ called the ambient space of G, $G \subset \mathbb{V}$, from which it inherits its inner product, $\cdot$, and norm, $\|\cdot\|$, which are invariant under gyroautomorphisms, that is,

(V1) $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{a}\cdot\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a}\cdot\mathbf{b}$ *Inner Product Gyroinvariance*
for all points $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G$.

(2)   G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

(V2) $1\otimes\mathbf{a} = \mathbf{a}$ *Identity Scalar Multiplication*

(V3) $(r_1 + r_2)\otimes\mathbf{a} = r_1\otimes\mathbf{a}\oplus r_2\otimes\mathbf{a}$ *Scalar Distributive Law*

(V4) $(r_1 r_2)\otimes\mathbf{a} = r_1\otimes(r_2\otimes\mathbf{a})$ *Scalar Associative Law*

(V5) $\dfrac{|r|\otimes\mathbf{a}}{\|r\otimes\mathbf{a}\|} = \dfrac{\mathbf{a}}{\|\mathbf{a}\|},$ $\mathbf{a} \neq \mathbf{0}, r \neq 0$ *Scaling Property*

(V6) $\mathrm{gyr}[\mathbf{u}, \mathbf{v}](r\otimes\mathbf{a}) = r\otimes\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{a}$ *Gyroautomorphism Property*

(V7) $\mathrm{gyr}[r_1\otimes\mathbf{v}, r_2\otimes\mathbf{v}] = I$ *Identity Gyroautomorphism.*

(3)   Real, one-dimensional vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of one-dimensional "vectors" (see, for instance, [33])

(V8) $\|G\| = \{\pm\|\mathbf{a}\| : \mathbf{a} \in G\} \subset \mathbb{R}$ *Vector Space*
with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in G$,

(V9) $\|r\otimes\mathbf{a}\| = |r|\otimes\|\mathbf{a}\|$ *Homogeneity Property*

(V10) $\|\mathbf{a}\oplus\mathbf{b}\| \leq \|\mathbf{a}\|\oplus\|\mathbf{b}\|$ *Gyrotriangle Inequality.*

Remark 1. *We use the notation $(r_1\otimes\mathbf{a})\oplus(r_2\otimes\mathbf{b}) = r_1\otimes\mathbf{a}\oplus r_2\otimes\mathbf{b}$, and $\mathbf{a}\otimes r = r\otimes\mathbf{a}$. Our ambiguous use of $\oplus$ and $\otimes$ in Definition 6 as interrelated operations in the gyrovector space $(G, \oplus, \otimes)$ and in its associated vector space $(\|G\|, \oplus, \otimes)$ should raise no confusion since the sets in which these operations operate are always clear from the context. These operations in the former (gyrovector space $(G, \oplus, \otimes)$) are nonassociative–nondistributive gyrovector space operations, and in the latter (vector space $(\|G\|, \oplus, \otimes)$) are associative–distributive vector space operations. Additionally, the gyro-addition $\oplus$ is gyrocommutative in the former and commutative in the latter.*

While each of the operations $\oplus$ and $\otimes$ has distinct interpretations in the gyrovector space G and in the vector space $\|G\|$, they are related to one another by the gyrovector space axioms $(V9)$ and $(V10)$. The analogies that conventions about the ambiguous use of $\oplus$ and $\otimes$ in G and $\|G\|$ share with similar vector space conventions are obvious. Indeed, in vector spaces we use (i) the same notation, $+$, for the addition operation between vectors and between their magnitudes, and (ii) the same notation for the scalar multiplication between two scalars and between a scalar and a vector. In full analogy, in gyrovector spaces we use (i) the same notation, $\oplus$, for the gyroaddition operation

between gyrovectors and between their magnitudes, in (V10), and (ii) the same notation, $\otimes$, for the scalar gyromultiplication between two scalars and between a scalar and a gyrovector, in (V9).

Immediate consequences of the gyrovector space axioms are presented in the following Theorem.

Theorem 1. *Let $(G, \oplus, \otimes)$ be a gyrovector space whose ambient vector space is $\mathbb{V}$, and let 0, $\mathbf{0}$ and $\mathbf{0}_{\mathbb{V}}$ be the neutral elements of $(\mathbb{R}, +)$, $(G, \oplus)$ and $(\mathbb{V}, +)$, respectively. Then, for all $n \in \mathbb{N}$, $r \in \mathbb{R}$, and $\mathbf{a} \in G$,*

(1) $0 \otimes \mathbf{a} = \mathbf{0}$
(2) $n \otimes \mathbf{a} = \mathbf{a} \oplus \; \ldots \; \oplus \mathbf{a}$ *(n terms).*
(3) $(-r) \otimes \mathbf{a} = \ominus(r \otimes \mathbf{a}) =: \ominus r \otimes \mathbf{a}$
(4) $r \otimes (\ominus \mathbf{a}) = \ominus r \otimes \mathbf{a}$
(5) $r \otimes \mathbf{0} = \mathbf{0}$
(6) $\|\ominus \mathbf{a}\| = \|\mathbf{a}\|$
(7) $\mathbf{0} = \mathbf{0}_{\mathbb{V}}$
(8) $r \otimes \mathbf{a} = \mathbf{0} \iff (r = 0 \text{ or } \mathbf{a} = \mathbf{0})$.

Proof. (1)   follows from the scalar distributive law $(V3)$,

$$r \otimes \mathbf{a} = (r + 0) \otimes \mathbf{a} = r \otimes \mathbf{a} \oplus 0 \otimes \mathbf{a}, \tag{10}$$

so that, by a left cancellation, $0 \otimes \mathbf{a} = \ominus(r \otimes \mathbf{a}) \oplus (r \otimes \mathbf{a}) = \mathbf{0}$.
(2)   follows from $(V2)$, and the scalar distributive law $(V3)$. Indeed, with "$\ldots$" signifying "n terms", we have

$$\mathbf{a} \oplus \ldots \oplus \mathbf{a} = 1 \otimes \mathbf{a} \oplus \ldots \oplus 1 \otimes \mathbf{a} = (1 + \ldots + 1) \otimes \mathbf{a} = n \otimes \mathbf{a}. \tag{11}$$

(3)   results from (1) and the scalar distributive law $(V3)$,

$$\mathbf{0} = 0 \otimes \mathbf{a} = (r - r) \otimes \mathbf{a} = r \otimes \mathbf{a} \oplus (-r) \otimes \mathbf{a}, \tag{12}$$

implying $\ominus(r \otimes \mathbf{a}) = (-r) \otimes \mathbf{a}$.
(4)   results from (3) and the scalar associative law,

$$r \otimes (\ominus \mathbf{a}) = r \otimes ((-1) \otimes \mathbf{a}) = (-r) \otimes \mathbf{a} = \ominus r \otimes \mathbf{a}. \tag{13}$$

(5)   follows from (1), $(V4)$, $(V3)$, (3),

$$
\begin{aligned}
r \otimes \mathbf{0} &= r \otimes (0 \otimes \mathbf{a}) \\
&= r \otimes ((1 - 1) \otimes \mathbf{a}) \\
&= (r(1 - 1)) \otimes \mathbf{a} \\
&= (r - r) \otimes \mathbf{a} \\
&= r \otimes \mathbf{a} \oplus (-r) \otimes \mathbf{a} \\
&= r \otimes \mathbf{a} \oplus (\ominus(r \otimes \mathbf{a})) \\
&= r \otimes \mathbf{a} \ominus r \otimes \mathbf{a} \\
&= \mathbf{0}.
\end{aligned}
\tag{14}
$$

(6)   follows from (3), the homogeneity property $(V9)$, and $(V2)$,

$$\|\ominus \mathbf{a}\| = \|(-1) \otimes \mathbf{a}\| = |-1| \otimes \|\mathbf{a}\| = 1 \otimes \|\mathbf{a}\| = \|1 \otimes \mathbf{a}\| = \|\mathbf{a}\|. \tag{15}$$

(7)   results from (5), $(V9)$, and $(V8)$ as follows.

$$\|\mathbf{0}\| = \|2 \otimes \mathbf{0}\| = 2 \otimes \|\mathbf{0}\| = \|\mathbf{0}\| \oplus \|\mathbf{0}\|, \tag{16}$$

implying $\|\mathbf{0}\| = \|\mathbf{0}\| \ominus \|\mathbf{0}\| = 0$ in the vector space $(\|G\|, \oplus, \otimes)$. This equation, $\|\mathbf{0}\| = 0$, is valid in the vector space $\mathbb{V}$ as well, where it implies $\mathbf{0} = \mathbf{0}_{\mathbb{V}}$.

(8)   results from the following considerations. Suppose $r \otimes \mathbf{a} = \mathbf{0}$, but $r \neq 0$. Then, by $(V1)$, $(V4)$ and (5) we have

$$\mathbf{a} = 1 \otimes \mathbf{a} = (1/r) \otimes (r \otimes \mathbf{a}) = (1/r) \otimes \mathbf{0} = \mathbf{0}. \tag{17}$$

The proof is thus complete. $\square$

Clearly, in the special case when all the gyrations of a gyrovector space are trivial, the gyrovector space descends to a vector space.

In general, gyroaddition does not distribute with scalar multiplication,

$$r \otimes (\mathbf{a} \oplus \mathbf{b}) \neq (r \otimes \mathbf{a}) \oplus (r \otimes \mathbf{b}) = r \otimes \mathbf{a} \oplus r \otimes \mathbf{b}. \tag{18}$$

However, gyrovector spaces possess a weak distributive law, called the monodistributive law, presented in the following Theorem.

Theorem 2 (The Monodistributive Law). *A gyrovector space $(G, \oplus, \otimes)$ possesses the monodistributive law*

$$r \otimes (r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}) = r \otimes (r_1 \otimes \mathbf{a}) \oplus r \otimes (r_2 \otimes \mathbf{a}) \tag{19}$$

for all $r, r_1, r_2 \in \mathbb{R}$ and $\mathbf{a} \in G$.

Proof. The proof follows from the scalar distributive law $(V3)$ and the scalar associative law $(V4)$,

$$
\begin{aligned}
r \otimes (r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}) &= r \otimes \{(r_1 + r_2) \otimes \mathbf{a}\} \\
&= (r(r_1 + r_2)) \otimes \mathbf{a} \\
&= (rr_1 + rr_2) \otimes \mathbf{a} \\
&= (rr_1) \otimes \mathbf{a} \oplus (rr_2) \otimes \mathbf{a} \\
&= r \otimes (r_1 \otimes \mathbf{a}) \oplus r \otimes (r_2 \otimes \mathbf{a}).
\end{aligned}
\tag{20}
$$

$\square$

6. Einstein Gyrovector Spaces

The rich structure of Einstein addition is not limited to its gyrocommutative gyrogroup structure. Indeed, Einstein addition admits scalar multiplication, giving rise to the Einstein gyrovector space. Remarkably, the resulting Einstein gyrovector spaces form the algebraic setting for the Cartesian–Beltrami–Klein ball model of hyperbolic geometry, just as vector spaces form the algebraic setting for the standard Cartesian model of Euclidean geometry.

Let $k \otimes \mathbf{v} = \mathbf{v} \oplus \mathbf{v} \dots \oplus \mathbf{v}$ (k terms) be the Einstein addition, (2), of k copies of $\mathbf{v} \in \mathbb{R}_c^n$, defined inductively as

$$(k+1) \otimes \mathbf{v} = \mathbf{v} \oplus (k \otimes \mathbf{v}), \qquad 1 \otimes \mathbf{v} = \mathbf{v}, \tag{21}$$

for any $\mathbf{v} \in \mathbb{R}_c^n$. Then,

$$k \otimes \mathbf{v} = c \frac{\left(1 + \dfrac{\|\mathbf{v}\|}{c}\right)^k - \left(1 - \dfrac{\|\mathbf{v}\|}{c}\right)^k}{\left(1 + \dfrac{\|\mathbf{v}\|}{c}\right)^k + \left(1 - \dfrac{\|\mathbf{v}\|}{c}\right)^k} \frac{\mathbf{v}}{\|\mathbf{v}\|}. \tag{22}$$

The Definition of scalar multiplication in an Einstein gyrovector space requires analytically continuing k off the positive integers. Accordingly, the integer multiplication (22) suggests the following Definition of scalar multiplication.

Definition 7 (Einstein Scalar Multiplication; Einstein Gyrovector Spaces). *An Einstein gyrovector space* $(\mathbb{R}^n_s, \oplus, \otimes)$ *is an Einstein gyrogroup* $(\mathbb{R}^n_s, \oplus)$ *with scalar multiplication* $\otimes$ *given by*

$$
r \otimes \mathbf{v} = s \frac{\left(1 + \frac{\|\mathbf{v}\|}{s}\right)^r - \left(1 - \frac{\|\mathbf{v}\|}{s}\right)^r}{\left(1 + \frac{\|\mathbf{v}\|}{s}\right)^r + \left(1 - \frac{\|\mathbf{v}\|}{s}\right)^r} \frac{\mathbf{v}}{\|\mathbf{v}\|} = s \tanh(r \tanh^{-1} \frac{\|\mathbf{v}\|}{s}) \frac{\mathbf{v}}{\|\mathbf{v}\|}, \tag{23}
$$

where r is any real number, $r \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}^n_s$, $\mathbf{v} \neq \mathbf{0}$, and $r \otimes \mathbf{0} = \mathbf{0}$, and with which we use the notation $\mathbf{v} \otimes r = r \otimes \mathbf{v}$.

Einstein gyrovector spaces turn out to be concrete realizations of the abstract gyrovector space in Definition 6. In fact, Definition 6 is motivated by considering key features of Einstein addition and scalar multiplication as axioms.

7. Linking Einstein Addition to Hyperbolic Geometry

The Einstein distance function, $d(\mathbf{u}, \mathbf{v})$ in an Einstein gyrovector space $(\mathbb{R}^n_c, \oplus, \otimes)$ is given by the equation

$$
d(\mathbf{u}, \mathbf{v}) = \|\ominus \mathbf{u} \oplus \mathbf{v}\| = \|\mathbf{u} \ominus \mathbf{v}\|, \tag{24}
$$

$\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$. We call it a *gyrodistance function* in order to emphasize the analogies it shares with its Euclidean counterpart, the distance function $\|\mathbf{u} - \mathbf{v}\|$ in $\mathbb{R}^n$. Among these analogies is the gyrotriangle inequality according to which ([4], Theorem 3.46)

$$
\|\mathbf{u} \oplus \mathbf{v}\| \leq \|\mathbf{u}\| \oplus \|\mathbf{v}\| \tag{25}
$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$.

In a two-dimensional Einstein gyrovector space $(\mathbb{R}^2_c, \oplus, \otimes)$ the squared gyrodistance between a point $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_c$ and an infinitesimally nearby point $\mathbf{x} + d\mathbf{x} \in \mathbb{R}^2_c$, $d\mathbf{x} = (dx_1, dx_2)$, is defined by the equation ([4], Section 7.5) ([3], Section 7.5) ([5], Section 7.5)

$$
\begin{aligned}
ds^2 &= \|(\mathbf{x} + d\mathbf{x}) \ominus \mathbf{x}\|^2 \\
&= E dx_1^2 + 2F dx_1 dx_2 + G dx_2^2 + \dots
\end{aligned} \tag{26}
$$

where, if we use the notation $r^2 = x_1^2 + x_2^2$, we have

$$
E = c^2 \frac{c^2 - x_2^2}{(c^2 - r^2)^2}
$$

$$
F = c^2 \frac{x_1 x_2}{(c^2 - r^2)^2} \tag{27}
$$

$$
G = c^2 \frac{c^2 - x_1^2}{(c^2 - r^2)^2}.
$$

The triple $(g_{11}, g_{12}, g_{22}) = (E, F, G)$ along with $g_{21} = g_{12}$ is known in differential geometry as the metric tensor g_{ij} [34]. It turns out to be the metric tensor of the Beltrami–Klein disc model of hyperbolic geometry ([35], p. 220). Hence, ds^2 in (26)–(27) is the Riemannian line element of the Beltrami–Klein disc model of hyperbolic geometry, linked to Einstein velocity addition (2), and to Einstein gyrodistance function (24) [36].

The link between Einstein gyrovector spaces and the Beltrami–Klein ball model of hyperbolic geometry, already anticipated by Fock ([18] p. 39), has thus been established in (24)–(27) in two dimensions. The extension of the link to higher dimensions is presented in ([2], Section 9, Chaper 3), ([3], Section 7.5), ([4], Section 7.5) and [36]. For a brief account of the history of linking Einstein's velocity addition law with hyperbolic geometry, see ([37],

p. 943). A study of Einstein addition within the framework of differential geometry is presented in [38–40].

8. Euclidean Lines

In order to set the road to lines in analytic hyperbolic geometry, in this section we present analytically the well-known Euclidean lines. We introduce Cartesian coordinates into $\mathbb{R}^n$ in the usual way in order to specify uniquely each point P of the Euclidean n-space $\mathbb{R}^n$ by an n-tuple of real numbers, called the coordinates, or components, of P. Cartesian coordinates provide a method of indicating the position of points and rendering graphs on a two-dimensional Euclidean plane $\mathbb{R}^2$ and in a three-dimensional Euclidean space $\mathbb{R}^3$.

As an example, Figure 1 presents a Euclidean plane $\mathbb{R}^2$ equipped with an unseen Cartesian coordinate system Σ. The position of points A and B and their midpoint m_{AB} with respect to Σ are shown. The missing Cartesian coordinates in Figure 1 are shown in ([9], Figure 3.3).

The set of all points

$$A + (-A + B)t \,, \tag{28}$$

$t \in \mathbb{R}$, forms a Euclidean line. The segment AB on this line, corresponding to $0 \leq t \leq 1$, and a generic point P on the segment, are shown in Figure 1. Being collinear, the points A, P and B obey the triangle equality
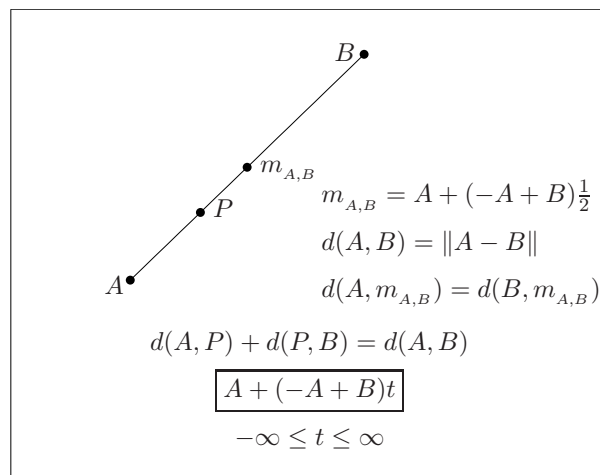
$$d(A, P) + d(P, B) = d(A, B) \,, \tag{29}$$

where

$$d(A, B) = \| -A + B \| \tag{30}$$

is the Euclidean distance function in $\mathbb{R}^n$.

Figure 1 demonstrates the use of the standard Cartesian model of Euclidean geometry for graphical presentations. In a fully analogous way, Figure 2 demonstrates the use of the Cartesian–Beltrami–Klein model of hyperbolic geometry, as we will see in Section 9.
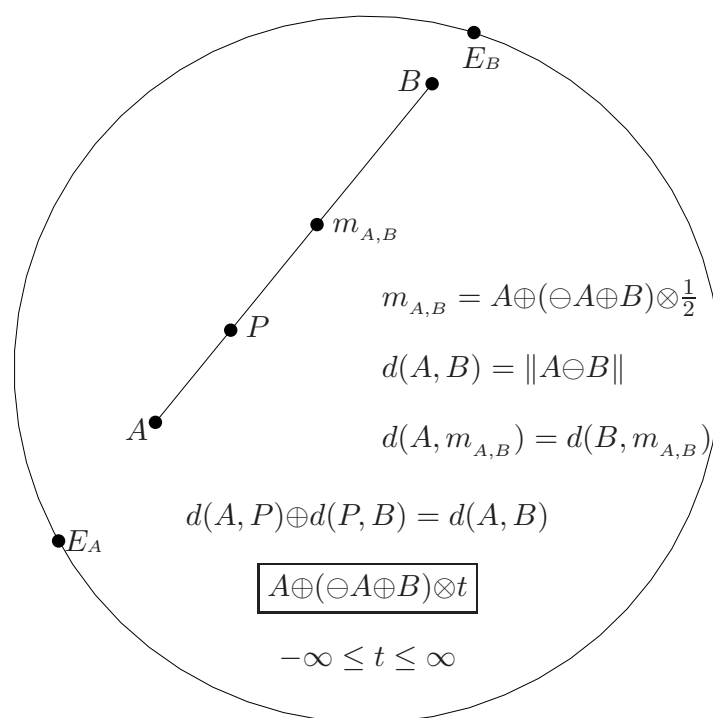

Figure 1. The Euclidean line. The line $A + (-A + B)t$, $t \in \mathbb{R}$, in a Euclidean plane is shown. The points A and B correspond to $t = 0$ and $t = 1$, respectively. The point P is a generic point on the line through the points A and B lying between these points. The sum, $+$, of the distance from A to P and from P to B equals the distance from A to B. The point $m_{A,B}$ is the midpoint of the points A and B, corresponding to $t = 1/2$. This figure sets the stage for its hyperbolic counterpart in Figure 2.

9. Gyrolines—The Hyperbolic Lines

Let $A, B \in \mathbb{R}^n_s$ be two distinct points of the Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, and let $t \in \mathbb{R}$ be a real parameter. Then, in full analogy with the Euclidean line (28), shown in Figure 1, the graph of the set of all points

$$A \oplus (\ominus A \oplus B) \otimes t \,, \tag{31}$$

$t \in \mathbb{R}$, in the Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ is a chord of the ball $\mathbb{R}^n_s$. This chord is a geodesic line of the Cartesian–Beltrami–Klein ball model of hyperbolic geometry, shown in Figure 2 for $n = 2$. The geodesic line (31) is the unique geodesic passing through the points A and B. It passes through the point A when $t = 0$ and, owing to the left cancellation law in (7), it passes through the point B when $t = 1$. Furthermore, it passes through the midpoint $m_{A,B}$ of A and B when $t = 1/2$. Accordingly, the *gyrosegment* that joins the points A and B in Figure 2 is obtained from the gyroline (31) with $0 \le t \le 1$.

Figure 2. Gyroline, the hyperbolic line. The gyroline $L_{AB} = A \oplus (\ominus A \oplus B) \otimes t$, $t \in \mathbb{R}$, that passes through the points A and B in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ is a geodesic line in the Beltrami-Klein ball model of hyperbolic geometry, fully analogous to the straight line $A + (-A + B)t$, $t \in \mathbb{R}$, in the Euclidean geometry of $\mathbb{R}^n$. The points A and B correspond to $t = 0$ and $t = 1$, respectively. The point P is a generic point on the gyroline through the points A and B lying between these points. The Einstein sum, $\oplus$, of the gyrodistance from A to P and from P to B equals the gyrodistance from A to B. The point $m_{A,B}$ is the gyromidpoint of the points A and B, corresponding to $t = 1/2$. The analogies between lines and gyrolines, as illustrated in Figures 1 and 2, are obvious. The gyroline L_{AB} approaches the boundary of $\mathbb{R}^n_s$ at its boundary points E_A and E_B. The boundary points of gyroline L_{AB} are determined by A and B in ([9], Secttion 5.9).

Each point of (31) with $0 < t < 1$ is said to lie *between* A and B. Thus, for instance, the point P in Figure 2 lies between the points A and B. As such, the points A, P and B obey the *gyrotriangle equality* according to which

$$d(A, P) \oplus d(P.B) = d(A, B) \,, \tag{32}$$

where

$$d(A, B) = \|\ominus A \oplus B\| \tag{33}$$

is the hyperbolic distance function in $\mathbb{R}_c^n$, called the *gyrodistance function*, in full analogy with the triangle equality (29) in Euclidean geometry shown in Figure 1. The points in Figure 2 are drawn with respect to an unseen Cartesian coordinate system. The missing Cartesian coordinates for the hyperbolic disc in Figure 2 are shown in ([9], Figure 3.4).
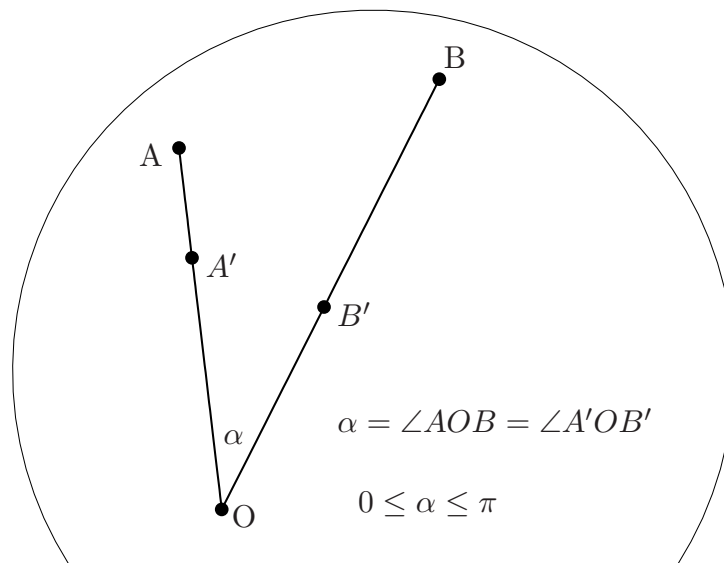
10. Gyroangles—The Hyperbolic Angles

Viewed in the Euclidean plane, the angle α in Figure 3 is given by the equation

$$\cos \alpha = \frac{-O + A}{\| - O + A\|} \cdot \frac{-O + B}{\| - O + B\|} . \tag{34}$$

We wish to find the analogous counterpart of (34) in the hyperbolic plane, in order to translate the common trigonometry in the Euclidean plane into gyrotrigonometry in the hyperbolic plane.

The analogies between lines and gyrolines boil down to the translation of $+$ and $-$ into $\oplus$ and $\ominus$. These, in turn, suggest corresponding analogies between angles and gyroangles. Indeed, in full analogy with the notions of distance and angle, the notion of the gyroangle is deduced from the notion of the gyrodistance. Let O, A and B be any three distinct points in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$. The resulting gyrosegments OA and OB that emanate from the point O include a gyroangle $\alpha = \angle AOB$ with vertex O, as shown in Figure 3 for $n = 2$.

$$\alpha = \angle AOB = \angle A'OB'$$

$$0 \le \alpha \le \pi$$

$$\cos \alpha := \frac{\ominus O \oplus A}{\|\ominus O \oplus A\|} \cdot \frac{\ominus O \oplus B}{\|\ominus O \oplus B\|} = \frac{\ominus O \oplus A'}{\|\ominus O \oplus A'\|} \cdot \frac{\ominus O \oplus B'}{\|\ominus O \oplus B'\|}$$

Figure 3. Gyroangles share remarkable analogies with angles, allowing the use of the elementary trigonometric functions cos, sin, etc., in gyrotrigonometry as well. Let A' and B' be points different from O, lying arbitrarily on the gyrosegments OA and OB, respectively, that emanate from a common point O in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ as shown here for $n = 2$. The measure of the gyroangle α formed by the two gyrosegments OA and OB or, equivalently, formed by the two gyrosegments OA' and OB', is given by $cos\alpha$, as shown here. In full analogy with angles, the measure of gyroangle α is independent of the choice of A' and B'.

Following the analogies between gyrolines and lines, the radian measure of gyroangle α in Figure 3 is, suggestively, given by the equation

$$\cos\alpha = \frac{\ominus O \oplus A}{\|\ominus O \oplus A\|} \cdot \frac{\ominus O \oplus B}{\|\ominus O \oplus B\|} . \tag{35}$$

Here, $(\ominus O \oplus A)/\|\ominus O \oplus A\|$ and $(\ominus O \oplus B)/\|\ominus O \oplus B\|$ are unit *gyrovectors*, and cos is the common cosine function of trigonometry, which we apply to the inner product between unit gyrovectors rather than unit vectors. Accordingly, in the context of gyrovector spaces rather than vector spaces, we refer the function "cosine" of trigonometry to as the function "gyrocosine" of gyrotrigonometry. Similarly, all the other elementary trigonometric functions and their interrelationships survive unimpaired in their transition from the common trigonometry in Euclidean spaces $\mathbb{R}^n$ to a corresponding gyrotrigonometry in Einstein gyrovector spaces $\mathbb{R}_s^n$, as demonstrated in ([9], Chaper 7).

The center $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}_s^n$ of the ball $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$ is conformal (to Euclidean geometry) in the sense that the measure of any gyroangle with vertex $\mathbf{0}$ is equal to the measure of its Euclidean counterpart. Indeed, if $O = \mathbf{0}$ then (35) descends to

$$\cos\alpha = \frac{A}{\|A\|} \cdot \frac{B}{\|B\|} , \tag{36}$$

which is indistinguishable from its Euclidean counterpart.

We thus encounter the *cosine – gyrocosine duality*, according to which $\cos\alpha$ can be viewed simultaneously as (i) the cosine of an angle α; and as (ii) the gyrocosine of a gyroangle α. More about the trigonometric – gyrotrigonometric duality is presented in Section 11. It is this duality that enables Ptolemy's Theorem to be translated from Euclidean geometry into hyperbolic geometry, as we will see in Secttions 13 and 14.

Unlike Euclidean geometry, where the differences $-O + A$ and $A - O$ are equal, in general, the gyrodifferences $\ominus O \oplus A$ and $A \ominus O$ are distinct. The presence of the gyrodifferences $\ominus O \oplus A$ and $\ominus O \oplus B$ in the gyroangle (35), rather then $A \ominus O$ and $B \ominus O$, is dictated by the demand that gyroangles must be *gyroinvariant*, that is, invariant under gyromotions. Being invariant under the gyromotions of $\mathbb{R}_s^n$, which are left gyrotranslations and rotations about the origin, gyroangles gain geometric significance, so that they are geometric objects of the hyperbolic geometry of the Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$. In more detail, this study is found in ([9], Chaper 3).

11. Trigonometry—Gyrotrigonometry Duality

The special positions of a gyroangle are positions where the gyroangle vertex coincides with the origin O of its gyrovector space. The measure of a gyroangle in a special position in $\mathbb{R}_s^n$ equals its measure when viewed as a corresponding angle in the Euclidean geometry of $\mathbb{R}^n$, as demonstrated in (36). Hence, the origin of an Einstein gyrovector space $\mathbb{R}_s^n$ is said to be *conformal*.

The result that every gyroangle can be left gyrotranslated without distortion to special positions where it can be viewed, without distorting its measure, as a Euclidean angle is *crucially important* in the gyrotrigonometry of Einstein gyrovector spaces. It implies that every trigonometric identity of trigonometric functions remains valid in gyrotrigonometry, giving rise to a corresponding gyrotrigonometric identity of gyrotrigonometric functions. Accordingly, we refer to these identities as trigonometric/gyrotrigonometric identities.

Thus, for instance, the familiar trigonometric identities

$$\cos^2 \alpha + \sin^2 \alpha = 1$$

$$\cot \frac{\alpha}{2} = \frac{1 + \cos \alpha}{\sin \alpha} \tag{37}$$

$$\sin^2 \frac{\alpha}{2} = \frac{1 - \cos \alpha}{2},$$

$\alpha \in \mathbb{R}$, are valid trigonometrically, where α is given by (34). Remarkably, they remain valid gyrotrigonometrically as well, where α is given by (35).

To see that trigonometric identities such as (37) remain valid in gyrotrigonometry, we left gyrotranslate the gyroangle α to a special position, where it can be viewed as an angle satisfying (37). This angle α can be left gyrotranslated back to its original position, where it can no longer be viewed as an angle. However, since gyroangles are invariant under left gyrotranslations, gyroangle α still obeys (37) regardless of whether it is located in a special position. Accordingly, one can use a computer algebra system for symbolic manipulation, like Mathematica or Maple, to manipulate gyrotrigonometric expressions in gyrotrigonometry. Accordingly, while a computer algebra system such as Mathematica or Maple is designed to deal with symbolic manipulation in trigonometry, it can be used to deal with symbolic manipulation in gyrotrigonometry as well.

As a Remarkable consequence of this intimate relationship between trigonometric and gyrotrigonometric functions, we find in the Ref. [8] the following result: Gyrobarycentric coordinates of gyrotriangle gyrocenters that are determined in terms of gyrotriangle gyroangles survive unimpaired, in form, in the transition from hyperbolic to Euclidean geometry.

A vivid example of the use of the trigonometry–gyrotrigonometry duality, according to which every trigonometric identity can simultaneously be viewed both trigonometrically and gyrotrigonometrically, is provided by the trigonometric identity

$$\sin \frac{\alpha}{2} \sin \frac{\gamma}{2} + \sin \frac{\beta}{2} \sin \frac{\delta}{2} = \sin \frac{\alpha + \beta}{2} \sin \frac{\beta + \gamma}{2} \tag{38}$$

for any $\alpha, \beta, \gamma \in \mathbb{R}$, where $\delta = 2\pi - (\alpha + \beta + \gamma)$. Identity (38) can be realized both trigonometrically and gyrotrigonometrically.

Realizing identity (38) trigonometrically yields in Section 13 the famous Ptolemy's Theorem in the Euclidean plane.

In full analogy, realizing identity (38) gyrotrigonometrically yields in Section 14 the novel Ptolemy's Theorem in the hyperbolic plane.

12. The Law of Gyrocosines

Let ABC be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}^n_s, \otimes, \oplus)$ along with its standard notation shown in Figure 4. According to ([9], Section 7.3), the gyrotriangle ABC obeys the following three identities, each of which represents its *law of gyrocosines*,

$$\begin{aligned}
\gamma_a &= \gamma_b \gamma_c (1 - b_s c_s \cos \alpha) \\
\gamma_b &= \gamma_a \gamma_c (1 - a_s c_s \cos \beta) \\
\gamma_c &= \gamma_a \gamma_b (1 - a_s b_s \cos \gamma).
\end{aligned} \tag{39}$$

The elements a, b, c and a_s, b_s, c_s and the gyroangles α, β, γ in the law of gyrocosines (39) are defined in Figure 4. The gamma factors in (39) are defined in (3).

In Section 13 we will apply the common law of cosines in trigonometry and, in full analogy, In Section 14 we will apply the law of gyrocosines (39) in gyrotrigonometry.
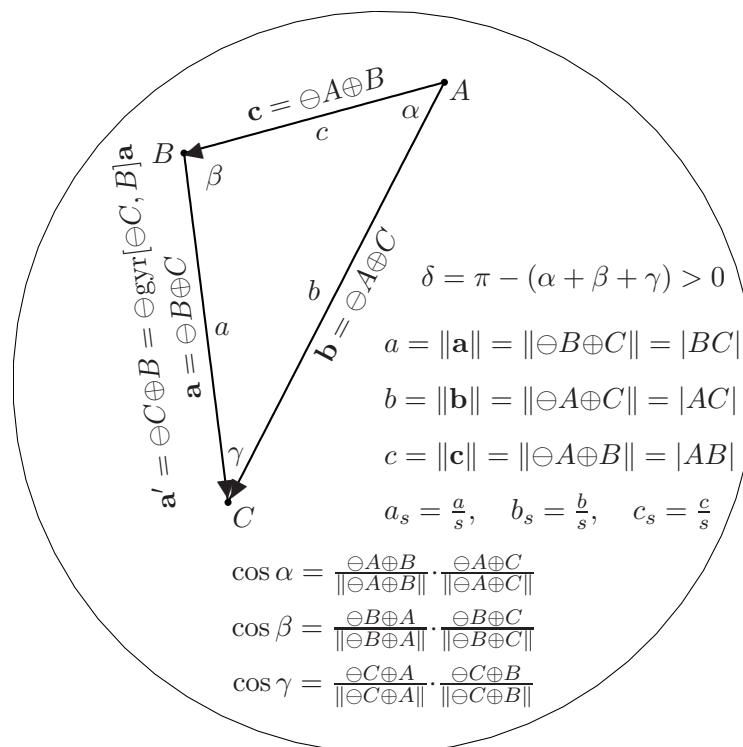
Figure 4. Gyrotriangle *ABC*, along with its standard notation, in an Einstein gyrovector space. The notation that we use with a gyrotriangle *ABC*, its gyrovector sides, and its gyroangles in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ is shown here for the Einstein gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$.

13. Ptolemy's Theorem in the Euclidean Plane

The proof of the hyperbolic Ptolemy's Theorem is a vivid example of the use of the trigonometry−gyrotrigonometry duality.

Let us consider the trigonometric identity

$$\sin\frac{\alpha}{2}\,\sin\frac{\gamma}{2} + \sin\frac{\beta}{2}\,\sin\frac{\delta}{2} = \sin\frac{\alpha+\beta}{2}\,\sin\frac{\beta+\gamma}{2} \tag{40}$$

for all angles $\alpha, \beta, \gamma \in \mathbb{R}$, where $\delta = 2\pi - (\alpha + \beta + \gamma)$. Note that owing to the condition $\alpha + \beta + \gamma + \delta = 2\pi$ the angle $(\alpha + \beta)/2$ can be replaced with the angle $(\gamma + \delta)/2$ in (40).

In this section we show that the trigonometric identity (40) is equivalent to Ptolemy's Theorem in the Euclidean plane, described in Figure 5.

Being a trigonometric identity, (40) can be viewed as a gyrotrigonometric identity as well, as explained in Section 11.

In Section 14 we will show that Identity (40), viewed gyrotrigonometrically, gives rise to the hyperbolic Ptolemy's Theorem in the hyperbolic plane.

In the context of Euclidean geometry, we use the notation

$$|AB| = \| - A + B\| \tag{41}$$

for any $A, B \in \mathbb{R}^2$. Let $A, B, C, D \in \mathbb{R}^2$ be four points such that *ABCD* is a cyclic quadrilateral inscribed in a circle centered at *O* with radius $r = |OA| = |OB| = |OC| = |OD|$ in a Euclidean plane, as shown in Figure 5.

As indicated in Figures 5 and 6, *A*, *B*, *C*, *D* are points lying on a circle and so arranged that the order of the points as one traverses the circle either anticlockwise or clockwise, as shown. Accordingly, the sides *AB* and *CD* do not intersect inside the circle.
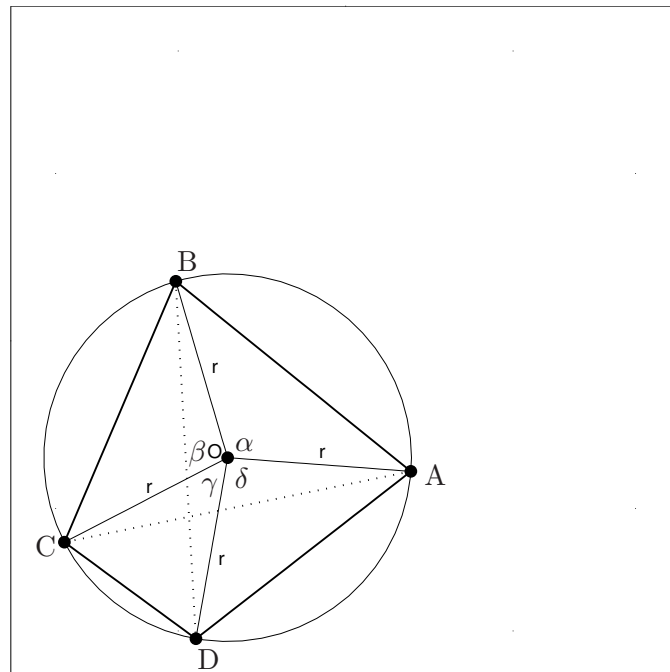
Figure 5. Illustrating Ptolemy's Theorem in the Euclidean plane. *ABCD* is a cyclic quadrilateral. It is inscribed in its circumcircle centered at its circumcenter *O*, with radius $r = \|-O+A\| = \|-O+B\| = \|-O+C\| = \|-O+D\|$. The *O*-vertex angles α, β, γ and δ satisfy the equation $\alpha + \beta + \gamma + \delta = 2\pi$. Ptolemy's Theorem asserts that $|AB||CD| + |AD||BC| = |AC||BD|$, where $|AB| = \|-A+B\|$, *etc.* The hyperbolic counterpart of this figure is shown in Figure 6.
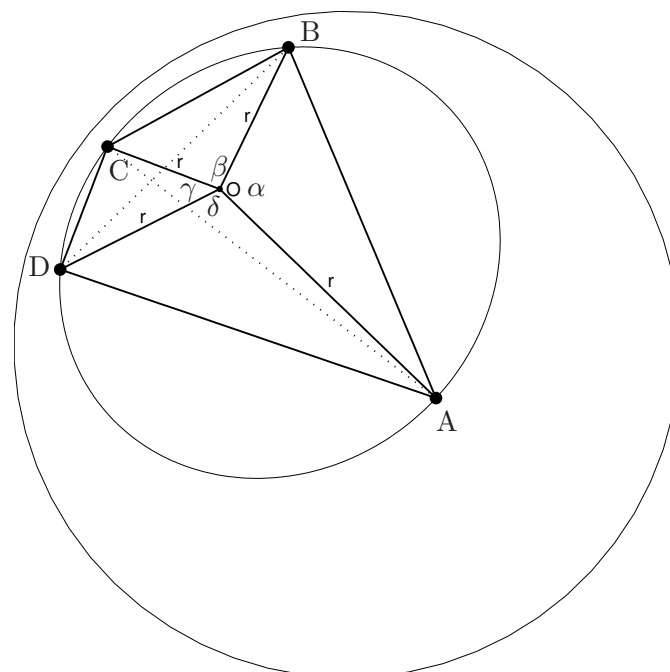
Figure 6. Illustrating Ptolemy's Theorem in the hyperbolic plane regulated by Einstein gyrovector plane $(\mathbb{R}_c^2, \oplus, \otimes)$. *ABCD* is a gyrocyclic gyroquadrilateral. It is inscribed in its circumgyrocircle gyrocentered at its circumgyrocenter *O*, with gyroradius $r = \|\ominus O \oplus A\| = \|\ominus O \oplus B\| = \|\ominus O \oplus C\| = \|\ominus O \oplus D\|$. The *O*-gyrovertex gyroangles α, β, γ and δ satisfy the equation $\alpha + \beta + \gamma + \delta = 2\pi$. The Hyperbolic Ptolemy's Theorem is fully analogous to its Euclidean counterpart, asserting that $|AB|_h|CD|_h + |AD|_h|BC|_h = |AC|_h|BD|_h$, where $|AB|_h$, etc., is defined by (59) along with (48).

Then, by the law of cosines, applied to triangle ABO in Figure 5, we have

$$
\begin{aligned}
|AB|^2 &= 2r^2 - 2r^2 \cos \alpha \\
&= 2r^2(1 - cos\alpha) \\
&= 4r^2 \sin^2 \frac{\alpha}{2} \, .
\end{aligned}
\tag{42}
$$

Accordingly, in the notation of Figure 5, we have

$$
\begin{aligned}
|AB| &= 2r \sin \frac{\alpha}{2} \\
|BC| &= 2r \sin \frac{\beta}{2} \\
|CD| &= 2r \sin \frac{\gamma}{2} \\
|AD| &= 2r \sin \frac{\delta}{2} \\
|AC| &= 2r \sin \frac{\alpha + \beta}{2} \\
|BD| &= 2r \sin \frac{\beta + \gamma}{2} \, .
\end{aligned}
\tag{43}
$$

Hence,

$$
\begin{aligned}
|AB||CD| &= 4r^2 \sin \frac{\alpha}{2} \sin \frac{\gamma}{2} \\
|BC||AD| &= 4r^2 \sin \frac{\beta}{2} \sin \frac{\delta}{2} \\
|AC||BD| &= 4r^2 \sin \frac{\alpha + \beta}{2} \sin \frac{\beta + \gamma}{2} \, .
\end{aligned}
\tag{44}
$$

By means of (44) and (40) we obtain Identity (45) of the following Ptolemy's Theorem.

Theorem 3 (Ptolemy's Theorem in the Eucledean Plane). *Let $ABCD$ be a cyclic quadrilateral, shown in Figure 5. Then, the product of the diagonals equals the sum of the products of the opposite sides, that is,*

$$
|AB||CD| + |AD||BC| = |AC||BD| \, .
\tag{45}
$$

In the special case when the two diagonals AC and BD of the cyclic quadrilateral $ABCD$ in Figure 5 intersect at the circumcenter O, we have $\alpha = \gamma$, $\beta = \delta$ and $\alpha + \beta = \beta + \gamma$, implying $|AB| = |CD|$, $|BC| = |AD|$ and $|AC| = |BD|$. Consequently, in this special case triangle ABD is right-angled, and Ptolemy's Identity (45) descends to the Pythagorean identity

$$
|AB|^2 + |AD|^2 = |BD|^2 \, .
\tag{46}
$$

14. Ptolemy's Theorem in the Hyperbolic Plane

Identity (40) is viewed in Section 13 trigonometrically. Contrastingly, in this section we view it gyrotrigonometrically.

Accordingly, let us consider the gyrotrigonometric identity

$$
\sin \frac{\alpha}{2} \sin \frac{\gamma}{2} + \sin \frac{\beta}{2} \sin \frac{\delta}{2} = \sin \frac{\alpha + \beta}{2} \sin \frac{\beta + \gamma}{2}
\tag{47}
$$

for all gyroangles $\alpha, \beta, \gamma \in \mathbb{R}$, where $\delta = 2\pi - (\alpha + \beta + \gamma)$. Note that owing to the condition $\alpha + \beta + \gamma + \delta = 2\pi$ the gyroangle $(\alpha + \beta)/2$ can be replaced with the gyroangle $(\gamma + \delta)/2$ in (47).

In this section we show that the gyrotrigonometric identity (47) gives rise to Ptolemy's Theorem in the hyperbolic plane.

In full analogy with (41), in the context of hyperbolic geometry we use the notation

$$|AB| = \|\ominus A \oplus B\| \tag{48}$$

for any $A, B \in \mathbb{R}_c^2$, where $\oplus$ denotes Einstein addition in $\mathbb{R}_c^2$.

It should be noted that in the Euclidean limit, $s \to \infty$, the hyperbolic $|AB|$, given by (48), descends to the Euclidean $|AB|$, given by (41), since

$$\lim_{s \to \infty} \|\ominus A \oplus B\| = \| - A + B\|. \tag{49}$$

Let $A, B, C, D \in \mathbb{R}_c^2$ be four points such that $ABCD$ is a gyrocyclic gyroquadrilateral inscribed in a gyrocircle gyrocentered at O, with gyroradius $r = |OA| = |OB| = |OC| = |OD|$ in the hyperbolic plane regulated by the Einstein gyrovector plane $(\mathbb{R}_c^2, \oplus, \otimes)$, as shown in Figure 6.

Then, by the law of gyrocosines (39) applied to gyrotriangle ABO in Figure 6, we have

$$\gamma_{|AB|} = \gamma_r^2 (1 - r_s^2 \cos \alpha) \tag{50}$$

where we use the usual notation $\gamma_{|AB|} = (1 - |AB|^2/s^2)^{-1/2}$ and $r_s = r/s$, $\gamma_r = (1 - r_s^2)^{-1/2}$.

Solving (50) for $\cos \alpha$, noting (4), yields

$$\cos \alpha = \frac{\gamma_r^2 - \gamma_{|AB|}}{\gamma_r^2 - 1} \tag{51}$$

so that

$$\sin^2 \frac{\alpha}{2} = \frac{1 - \cos \alpha}{2} = \frac{1}{2} \frac{\gamma_{|AB|} - 1}{\gamma_r^2 r_s^2} \tag{52}$$

and, hence,

$$\sin \frac{\alpha}{2} = \frac{1}{\sqrt{2}} \frac{\sqrt{\gamma_{|AB|} - 1}}{\gamma_r \, r_s}. \tag{53}$$

Repeating the result in (53) to the O-gyrovertex gyroangles $\alpha, \beta, \gamma, \delta, \alpha + \beta$ and $\beta + \gamma$ in Figure 6, where $\alpha + \beta + \gamma + \delta = 2\pi$, yields

$$
\begin{aligned}
\sin \frac{\alpha}{2} &= \frac{1}{\sqrt{2}\gamma_r \, r_s} \sqrt{\gamma_{|AB|} - 1} \\
\sin \frac{\beta}{2} &= \frac{1}{\sqrt{2}\gamma_r \, r_s} \sqrt{\gamma_{|BC|} - 1} \\
\sin \frac{\gamma}{2} &= \frac{1}{\sqrt{2}\gamma_r \, r_s} \sqrt{\gamma_{|CD|} - 1} \\
\sin \frac{\delta}{2} &= \frac{1}{\sqrt{2}\gamma_r \, r_s} \sqrt{\gamma_{|AD|} - 1} \\
\sin \frac{\alpha + \beta}{2} &= \frac{1}{\sqrt{2}\gamma_r \, r_s} \sqrt{\gamma_{|AC|} - 1} \\
\sin \frac{\beta + \gamma}{2} &= \frac{1}{\sqrt{2}\gamma_r \, r_s} \sqrt{\gamma_{|BD|} - 1}.
\end{aligned}
\tag{54}
$$

By means of (4) we have

$$\gamma_{|AB|}^2 - 1 = \frac{1}{s^2} \gamma_{|AB|}^2 |AB|^2 \tag{55}$$

so that

$$\gamma_{|AB|} - 1 = \frac{1}{s^2} \frac{\gamma_{|AB|}^2}{1 + \gamma_{|AB|}} |AB|^2 \tag{56}$$

and, hence,

$$\sqrt{\gamma_{|AB|} - 1} = \frac{1}{s} \frac{\gamma_{|AB|}}{\sqrt{1 + \gamma_{|AB|}}} |AB|. \tag{57}$$

Repeating the result in (57) to the gyrocircle gyrochords AB, BC, CD, AD, AC and BD in Figure 6 yields

$$
\begin{aligned}
\sqrt{\gamma_{|AB|} - 1} &= \frac{1}{s} \frac{\gamma_{|AB|}}{\sqrt{(1 + \gamma_{|AB|})}} |AB| =: \frac{1}{s} |AB|_h \\[2mm]
\sqrt{\gamma_{|BC|} - 1} &= \frac{1}{s} \frac{\gamma_{|BC|}}{\sqrt{(1 + \gamma_{|BC|})}} |BC| =: \frac{1}{s} |BC|_h \\[2mm]
\sqrt{\gamma_{|CD|} - 1} &= \frac{1}{s} \frac{\gamma_{|CD|}}{\sqrt{(1 + \gamma_{|CD|})}} |CD| =: \frac{1}{s} |CD|_h \\[2mm]
\sqrt{\gamma_{|AD|} - 1} &= \frac{1}{s} \frac{\gamma_{|AD|}}{\sqrt{(1 + \gamma_{|AD|})}} |AD| =: \frac{1}{s} |AD|_h \\[2mm]
\sqrt{\gamma_{|AC|} - 1} &= \frac{1}{s} \frac{\gamma_{|AC|}}{\sqrt{(1 + \gamma_{|AC|})}} |AC| =: \frac{1}{s} |AC|_h \\[2mm]
\sqrt{\gamma_{|BD|} - 1} &= \frac{1}{s} \frac{\gamma_{|BD|}}{\sqrt{(1 + \gamma_{|BD|})}} |BD| =: \frac{1}{s} |BD|_h
\end{aligned}
\tag{58}
$$

where we define

$$|AB|_h := \frac{\gamma_{|AB|}}{\sqrt{1 + \gamma_{|AB|}}} |AB| \tag{59}$$

and so forth. We call $|AB|_h$ the *h*-modified $|AB|$.

The equations in (54) and (58) yield

$$
\begin{aligned}
\sin \frac{\alpha}{2} &= \frac{1}{\sqrt{2}\gamma_r \, r} |AB|_h \\[2mm]
\sin \frac{\beta}{2} &= \frac{1}{\sqrt{2}\gamma_r \, r} |BC|_h \\[2mm]
\sin \frac{\gamma}{2} &= \frac{1}{\sqrt{2}\gamma_r \, r} |CD|_h \\[2mm]
\sin \frac{\delta}{2} &= \frac{1}{\sqrt{2}\gamma_r \, r} |AD|_h \\[2mm]
\sin \frac{\alpha + \beta}{2} &= \frac{1}{\sqrt{2}\gamma_r \, r} |AC|_h \\[2mm]
\sin \frac{\beta + \gamma}{2} &= \frac{1}{\sqrt{2}\gamma_r \, r} |BD|_h.
\end{aligned}
\tag{60}
$$

Finally, the gyrotrigonometric identity (47) along with (60) yields Identity (61) of the following Hyperbolic Ptolemy's Theorem.

Theorem 4 (Ptolemy's Theorem in the Hyperbolic Plane). *Let ABCD be a gyrocyclic gyro-quadrilateral, shown in Figure 6. Then, the product of the h-modified gyrodiagonals equals the sum of the products of the h-modified opposite gyrosides, that is,*

$$|AB|_h |CD|_h + |AD|_h |BC|_h = |AC|_h |BD|_h. \tag{61}$$

Clearly, in the Euclidean limit, $s \to \infty$, the hyperbolic Ptolemy's Identity (61) descends to its Euclidean counterpart (45).

15. Gyrodiametric Gyrotriangles

Definition 8. (Gyrodiametric Gyrotriangles). *A triangle is diametric if one of its sides coincides with a diameter of its circumcircle. In full analogy, a gyrotriangle is gyrodiametric if one of its gyrosides coincides with a gyrodiameter of its circumgyrocircle.*

A diametric triangle is right-angled, the angle opposite to the diametric side being $\pi/2$. In contrast, non-Euclidean gyrodiametric gyrotriangles are not right gyroangled. However, they obey the three equations that are shown in Figure 7.

$$\theta = \alpha_1 + \alpha_2$$

$$\theta = \frac{\pi}{2} - \frac{1}{2}\delta_{ABC}$$

$$|AB|_h^2 + |BC|_h^2 = |AC|_h^2$$
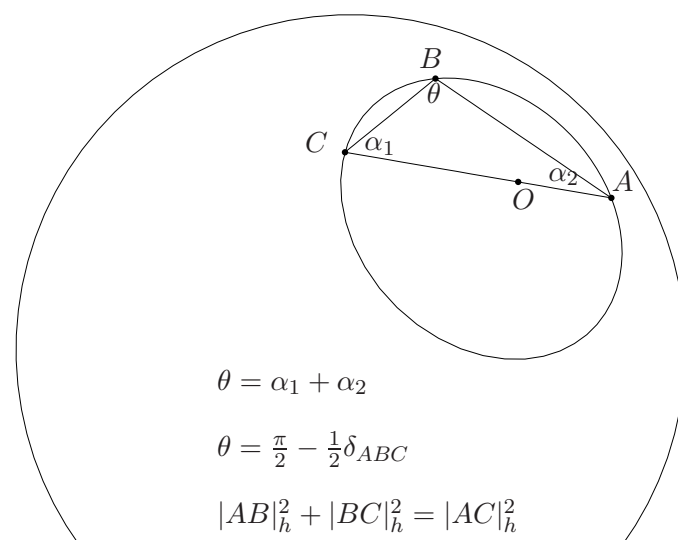
Figure 7. A Gyrodiametric Gyrotriangle. The circumgyrocenter, O, of gyrotriangle ABC is contained in gyroside AC of the gyrotriangle. Hence, the gyrotriangle is gyrodiametric, obeying the three equations that are shown in the Figure, where δ_{ABC} is the defect of gyrotriangle ABC.

The first two properties of gyrodiametric gyrotriangles in Figure 7 are

$$\theta = \alpha_1 + \alpha_2$$
$$\theta = \frac{\pi}{2} - \frac{1}{2}\delta_{ABC} \tag{62}$$

where δ_{ABC} is the defect of gyrotriangle ABC. These properties are established in ([9], Section 8.11). The third property of gyrodiametric gyrotriangles in Figure 7 is a Pythagorean-like identity. It is established in (63) as a special case of Ptolemy's Theorem in the hyperbolic plane.

In the special case when the two gyrodiagonals AC and BD of the gyrocyclic gyro-quadrilateral $ABCD$ in Figure 6 intersect at the circumgyrocenter O, we have the gyroangle equalities $\alpha = \gamma$, $\beta = \delta$ and $\alpha + \beta = \beta + \gamma$, implying by (60) the equations $|AB|_h = |CD|_h$, $|BC|_h = |AD|_h$ and $|AC|_h = |BD|_h$. Consequently, In this special case the hyperbolic Ptolemy Identity (61) descends to the Pythagorean-like identity

$$|AB|_h^2 + |BC|_h^2 = |AC|_h^2 \tag{63}$$

for the gyrodiametric gyrotriangle *ABC*, shown in Figure 7.

Clearly, in the Euclidean limit, $s \rightarrow \infty$, the Pythagorean-like identity (63) descends to the Pythagorean identity (46).

By means of (58), the Pythagorean-like identity (63) can be written as

$$(\gamma_{|AB|} - 1) + (\gamma_{|BC|} - 1) = \gamma_{|AC|} - 1. \tag{64}$$

However, in the Euclidean limit, $s \rightarrow \infty$, (64) tends to the trivial identity $0 = 0$.

References

1. Ungar, A.A. Thomas rotation and the parametrization of the Lorentz transformation group. *Found. Phys. Lett.* **1998**, *1*, 57–89. [CrossRef]
2. Ungar, A.A. *Beyond the Einstein Addition Law and Its Gyroscopic Thomas Precession: The Theory of Gyrogroups and Gyrovector Spaces, Volume 117 of Fundamental Theories of Physics*; Kluwer Academic Publishers Group: Dordrecht, The Netherlands, 2001.
3. Ungar, A.A. *Analytic Hyperbolic Geometry: Mathematical Foundations and Applications*; World Scientific Publishing Co. Pte. Ltd.: Hackensack, NJ, USA, 2005.
4. Ungar, A.A. *Analytic Hyperbolic Geometry and Albert Einstein's Special Theory of Relativity*; World Scientific Publishing Co. Pte. Ltd.: Hackensack, NJ, USA, 2008.
5. Ungar, A.A. *Analytic Hyperbolic Geometry and Albert Einstein's Special Theory of Relativity*, 2nd ed.; World Scientific Publishing Co. Pte. Ltd.: Hackensack, NJ, USA, 2022.
6. Ungar, A.A. *A Gyrovector Space Approach to Hyperbolic Geometry*; Morgan & Claypool Pub.: San Rafael, CA, USA, 2009.
7. Ungar, A.A. *Hyperbolic Triangle Centers: The Special Relativistic Approach*; Springer: New York, NY, USA, 2010.
8. Ungar, A.A. *Barycentric Calculus in Euclidean and Hyperbolic Geometry: A Comparative Introduction*; World Scientific Publishing Co. Pte. Ltd.: Hackensack, NJ, USA, 2010.
9. Ungar, A.A. *Analytic Hyperbolic Geometry in n Dimensions: An Introduction*; CRC Press: Boca Raton, FL, USA, 2015.
10. Ungar, A.A. *Beyond Pseudo-Rotations in Pseudo-Euclidean Spaces: An Introduction to the Theory of Bi-Gyrogroups and Bi-Gyrovector Spaces*; Mathematical Analysis and its Applications; Elsevier/Academic Press: Amsterdam, The Netherlands, 2018.
11. Ungar, A.A. The Thomas rotation formalism underlying a nonassociative group structure for relativistic velocities. *Appl. Math. Lett.* **1988**, *1*, 403–405. [CrossRef]
12. Ungar, A.A. Thomas precession and its associated grouplike structure. *Amer. J. Phys.* **1991**, *59*, 824–834. [CrossRef]
13. Platis, I.D.; Sönmez, N. The ptolemaean inequality in the closure of complex hyperbolic planes. *Turk. J. Math.* **2017**, *41*, 1108–1120. [CrossRef]
14. Valentine, J. An analogue of ptolemy's Theorem and its converse in hyperbolic geometry. *Pac. J. Math.* **1970**, *34*, 817–825. [CrossRef]
15. Ungar, A.A. The intrinsic beauty, harmony and interdisciplinarity in Einstein velocity addition law: Gyrogroups and gyrovector spaces. *Math. Interdisc. Res.* **2016**, *1*, 5–51.
16. Sexl, R.U.; Urbantke, H.K. *Relativity, Groups, Particles*; Springer Physics; Springer: Vienna, Austria, 2001.
17. Møller, C. *The Theory of Relativity*; Clarendon Press: Oxford, UK, 1952.
18. Fock, V. *The Theory of Space, Time and Gravitation*; 2nd revised ed.; Kemmer, N., Translator; The Macmillan Co.: New York, NY, USA, 1964.
19. Ungar, A.A. A spacetime symmetry approach to relativistic quantum multi-particle entanglement. *Symmetry* **2020**, *12*, 1259. [CrossRef]
20. Foguel, T.; Ungar, A.A. Involutory decomposition of groups into twisted subgroups and subgroups. *J. Group Theory* **2000**, *3*, 27–46. [CrossRef]
21. Foguel, T.; Ungar, A.A. Gyrogroups and the decomposition of groups into twisted subgroups and subgroups. *Pac. J. Math* **2001**, *197*, 1–11. [CrossRef]
22. Mahdavi, S.; Ashrafi, A.R.; Salahshour, M.A.; Ungar, A.A. Construction of 2-gyrogroups in which every proper subgyrogroup is either a cyclic or a dihedral group. *Symmetry* **2021**, *13*, 316. [CrossRef]
23. Suksumran, T.; Ungar, A.A. Bi-gyrogroup: The group-like structure induced by bi-decomposition of groups. *Math. Interdisc. Res.* **2016**, *1*, 111–142.
24. Ashrafi, A.R.; Nezhaad, K.M.; Salahshour, M.A. Construction of all gyrogroups of orders at most 31. *arXiv*, **2022**, arXiv:2209.04948.
25. Ferreira, M.; Suksumran, T. Orthogonal gyrodecompositions of real inner product gyrogroups. *Symmetry* **2020**, *12*, 941. [CrossRef]
26. Suksumran, T. The algebra of gyrogroups: Cayley's Theorem, Lagrange's Theorem and isomorphism Theorems. In *Essays in Mathematics and its Applications: In Honor of Vladimir Arnold*; Rassias, T.M., Pardalos, P.M., Eds.; Springer: New York, NY, USA, 2016; pp. 369–437.

27. Suksumran, T. Gyrogroup actions: A generalization of group actions. *J. Algebra* **2016**, *454*, 70–91. [CrossRef]
28. Suksumran, T. Involutive groups, unique 2-divisibility, and related gyrogroup structures. *J. Algebra Appl.* **2017**, *16*, 175. [CrossRef]
29. Suksumran, T.; Ungar, A.A. Gyrogroups and the Cauchy property. *Quasigroups Relat. Syst.* **2016**, *24*, 277–286.
30. Suksumran, T.; Wiboonton, K. Isomorphism Theorems for gyrogroups and L-subgyrogroups. *J. Geom. Symmetry Phys.* **2015**, *37*, 67–83.
31. Suksumran, T.; Wiboonton, K. Lagrange's Theorem for gyrogroups and the cauchy property. *Quasigroups Relat. Syst.* **2015**, *22*, 283–294.
32. Marsden, J.E. *Elementary Classical Analysis*; W. H. Freeman and Co.: San Francisco, CA, USA, 1974.
33. Carchidi, M.A. Generating exotic-looking vector spaces. *College Math. J.* **1998**, *29*, 304–308. [CrossRef]
34. Kreyszig, E. *Differential Geometry*; Dover Publications Inc.: New York, NY, USA, 1991.
35. McCleary, J. *Geometry from a Differentiable Viewpoint*; Cambridge University Press: Cambridge, UK, 1994.
36. Ungar, A.A. Gyrovector spaces and their differential geometry. *Nonlinear Funct. Anal. Appl.* **2005**, *10*, 791–834.
37. Rhodes, J.A.; Semon, M.D. Relativistic velocity space, Wigner rotation, and thomas precession. *Amer. J. Phys.* **2004**, *72*, 943–960. [CrossRef]
38. Barabanov, N.E.; Ungar, A.A. Binary operations in the unit ball—A differential geometry approach. *Symmetry* **2020**, *12*, 1178. [CrossRef]
39. Barabanov, N.E.; Ungar, A.A. Differential geometry and binary operations. *Symmetry* **2020**, *12*, 1525. [CrossRef]
40. Barabanov, N.E. Isomorphism of binary operations in differential geometry. *Symmetry* **2020**, *12*, 1634. [CrossRef]