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On a Subfamily of q-Starlike Functions with Respect to m-Symmetric Points Associated with the q-Janowski Function

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Abstract: The main objective of this paper is to study a new family of analytic functions that are q-starlike with respect to m-symmetrical points and subordinate to the q-Janowski function. We investigate inclusion results, sufficient conditions, coefficients estimates, bounds for Fekete–Szego functional $|a_3 - \mu a_2^2|$ and convolution properties for the functions belonging to this new class. Several consequences of main results are also obtained.

Keywords: analytic functions; univalent functions; q-difference operator; q-starlike functions; m-symmetric points; Janowski functions; subordination; convolution

MSC: 30C45; 30C50

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1. Introduction

The concept of quantum calculus or q-calculus is ordinary calculus without the notion of limits. Recently, due to its wide applications in applied sciences [1–3], the area of q-calculus has attracted the serious attention of researchers [4–7]. Jackson [8,9] was the first who initiated the study of q-calculus by introducing q-analogue of ordinary derivative and integral. He defined and studied q-difference operator and q-integral operator in a systematic way. Ismail et al. [10] used the q-difference operator for the first time in geometric function theory by introducing the class of q-starlike functions. Later the study of q-calculus in geometric function theory was developed by many authors [11–14]. Several researchers studied the subclasses of analytic functions associated with the q-difference operator [13,15–19]. Raghavendar and Swaminathan [20] defined and studied some basic properties of q-close-to-convex functions. Agrawal and Sahoo [21] introduced the family of q-starlike functions of the order α. The aim of the present work is to explore some properties of a subclass of analytic functions associated with q-Janowski functions involving a q-difference operator.

Let $\mathcal{A}$ denote the family of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$. Any function f is said to be univalent in a domain $\mathbb{D}$ if it never takes the same value twice in $\mathbb{D}$. Let us denote by S the subclass of $\mathcal{A}$ consisting of univalent functions in E. For two functions f and $g \in \mathcal{A}$, we say that f is subordinate to g , written as $f \prec g$, if there exists a Schwartz function $w(z)$ which is analytic in E with $w(0) = 0$ and $|w(z)| < 1$, for all $z \in E$, such that $f(z) = g(w(z)), z \in E$. If $g(z)$ is univalent in E, then $f \prec g \Leftrightarrow f(0) = g(0)$ and $f(E) \subset g(E)$. For any function

f defined by Equation (1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the convolution or Hadamard product of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, z \in E. \tag{2}$$

For $0 < q < 1$, the q-difference operator of a function $f \in \mathcal{A}$ is defined as

$$\mathfrak{D}_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{z - qz}, & \text{if } z \neq 0, \\ f'(0), & \text{if } z = 0. \end{cases} \tag{3}$$

For $q \in \mathbb{C}, |q| \leq 1$, a natural generalization of Equation (3) is given by the convolution operator

$$\mathfrak{D}_q f(z) = \frac{1}{z}\left\{ f(z) * \frac{z}{(1-z)(1-qz)} \right\},$$

which for $q = 1$, becomes the derivative f' and for $q \in \mathbb{R}, 0 < q < 1$, is equivalent to Equation (3). For f given by Equation (1), we have

$$\mathfrak{D}_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \tag{4}$$

where

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + \sum_{k=1}^{n-1} q^k, n \in \mathbb{N}. \tag{5}$$

From Equations (4) and (5), it can be noted that $\lim_{q \to 1^-} \mathfrak{D}_q f(z) = f'(z)$ and $\lim_{q \to 1^-} [n]_q = n$.

We denote by P, the class of analytic functions p with

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{6}$$

such that $Re(p(z)) > 0$ in E. Any function of the form in Equation (6) is said to be in the class $P[A, B]$ if and only if

$$p(z) \prec \frac{1 + Az}{1 + Bz}, \; z \in E,$$

where $-1 \leq B < A \leq 1$. The class $P[A, B]$ was introduced by Janowski [22].

A function $f \in S$ is said to belong to the class S^* of starlike functions if, and only if,

$$Re\left(\frac{zf'(z)}{f(z)} \right) > 0, \; z \in E,$$

which is equivalent to

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \; z \in E. \tag{7}$$

Let $S^*[A, B]$ denote the subclass of S, defined by

$$S^*[A, B] = \left\{ f \in S : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}, \; z \in E \right\}.$$

It is known that $S^*[1 - 2\alpha, -1] = S^*(\alpha)$ is the class of starlike functions of the order α $(0 \leq \alpha < 1)$. This class consists of the functions $f \in S$ with the property that

$$Re\left(\frac{zf'(z)}{f(z)} \right) > \alpha, \; z \in E. \tag{8}$$

Chand and Singh [23] introduced and studied the class $S^*[m]$ of starlike functions with respect to m-symmetric points, consisting of the functions $f \in S$ such that

$$\frac{zf'(z)}{f_m(z)} \prec \frac{1+z}{1-z}, \; z \in E, \tag{9}$$

where $f_m(z) = \frac{1}{m}\sum_{j=0}^{m-1} \epsilon_m^{-j} f(\epsilon_m^j z)$, $\epsilon_m = \exp(\frac{2\pi\iota}{m})$ and $m \in \mathbb{N}$. It can be noted that

$$f_m(z) = z + \sum_{n=2}^{\infty} \delta_{m,n} a_n z^n, \tag{10}$$

where

$$\delta_{m,n} = \begin{cases} 0 \;\text{ if } & \dfrac{n-1}{m} \notin \mathbb{N}, \\ 1 \;\text{ if } & \dfrac{n-1}{m} \in \mathbb{N}. \end{cases}$$

Kwon and Sim [24] studied the class $S^*[m, A, B]$ which consists of the functions $f \in S$ such that

$$\frac{zf'(z)}{f_m(z)} \prec \frac{1+Az}{1+Bz}, \; z \in E. \tag{11}$$

In [10], Ismail et al. introduced the class S_q^* as:

Definition 1. *A function $f \in \mathcal{A}$ is said to belong to the class S_q^* if*

$$\left| \frac{z\mathfrak{D}_q f(z)}{f(z)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q},$$

or equivalently

$$\frac{z\mathfrak{D}_q f(z)}{f(z)} \prec \frac{1+z}{1-qz}, \; z \in E. \tag{12}$$

It can be noted that if $q = 1$ then Equation (12) coincides with Equation (7), that is, for $q = 1$, $S_q^* = S^*$.

By taking motivation from above-mentioned work and using a q-difference operator $\mathfrak{D}_q$, for $0 < q \leq 1$, we introduce the following new subclasses of analytic functions.

Definition 2. *A function $f \in \mathcal{A}$ is said to be in class $S_q^*[m]$ if*

$$\left| \frac{z\mathfrak{D}_q f(z)}{f_m(z)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q},$$

or equivalently

$$\frac{z\mathfrak{D}_q f(z)}{f_m(z)} \prec \frac{1+z}{1-qz}, z \in E. \tag{13}$$

Definition 3. *A function $f \in \mathcal{A}$ is in class $S_q^*[m, A, B]$ if*

$$\frac{z\mathfrak{D}_q f(z)}{f_m(z)} \prec \frac{1+A_q z}{1+B_q z}, \; z \in E, \tag{14}$$

where,

$$A_q = \frac{(A+1)+(A-1)q}{2}, \quad B_q = \frac{(B+1)+(B-1)q}{2},$$

and $-1 \leq B < A \leq 1$.

It is important to note that $-q \leq B_q < A_q \leq 1$.

Remark 1. *It is worth mentioning that the functions in class $S_q^*[m, A, B]$ are not necessarily to be univalent in E. For example, $f(z) = z + \frac{3}{5}z^2$ is not univalent in E but one can easily verify that it belongs to classes $S_{0.5}^*[2, 1, -0.4]$ and $S_{0.5}^*[1, 0.95, 0.5]$.*

1.1. Special Cases

1. For $m = 1, S_q^*[m] = S_q^*$ is the class studied by Ismail et al. [10].
2. For $q = 1$, $S_q^*[m] = S^*[m]$ is the class studied by Chand and Singh [23].
3. For $m = 1$ and $q = 1$, $S_q^*[m] = S^*$ is the familiar class of starlike functions.
4. $S_q^*[m, 1, -1] = S_q^*[m]$ is the class of q-starlike functions with respect to m-symmetric points defined by Equation (13).
5. For $q = 1$, $S_q^*[m, A, B] = S^*[m, A, B]$ is the class studied by Kwon and Sim [24].
6. For $q = 1$, $S_q^*[1, A, B] = S^*[A, B]$ is the class studied by Janowski [22].
7. For $q = 1$, $S_q^*[1, 1-2\alpha, -1] = S^*(\alpha)$ is the class of starlike functions of the order α, defined and studied by Roberston [25].
8. For $q = 1$, $S_q^*[1, 1, -1] = S^*$ is the class of starlike functions defined by Alexander [26].
9. For $q = 1$, $S_q^*[2, 1, -1] = S^*[2]$ is the class of odd starlike functions studied by Sakaguchi [27].

1.2. Geometrical Interpretation

A function $f \in \mathcal{A}$ is in the class $S_q^*[m, A, B]$ if, and only if, $\frac{z\mathfrak{D}_q f(z)}{f_m(z)}$ takes all values in the circular domain centred at $\frac{1 - A_q B_q}{1 - B_q^2}$ and radius $\frac{A_q - B_q}{1 - B_q^2}$.

2. A Set of Lemmas

The following lemmas are needed to prove our main results in the subsequent section.

Lemma 1 ([28]). *If $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is an analytic function with a positive real part in E and μ is a complex number, then $|c_2 - \mu c_1^2| \leq 2\max\{1, |2\mu - 1|\}$. This result is sharp for the functions given by*

$$p(z) = \frac{1+z}{1-z}, \qquad p(z) = \frac{1+z^2}{1-z^2}, \; z \in E.$$

Lemma 2 ([29]). *If $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is an analytic function with a positive real part in E, then $|c_n| \leq 2$. This result is sharp for the function given by*

$$p(z) = \frac{1+z}{1-z}, \; z \in E.$$

Lemma 3 ([30]). *Let $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$. Then*

$$\frac{1 + A_1 z}{1 + B_1 z} \prec \frac{1 + A_2 z}{1 + B_2 z}, \; z \in E.$$

Lemma 4 ([31]). *Let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, $q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n$ and $p \prec q$ in E. If $q(z)$ is convex univalent in E, then*

$$|p_n| \leq q_1, \text{ for all } n \geq 1.$$

3. Main Results

Theorem 1. *Let $f \in S_q^*[m, A, B]$, then $f_m \in S_q^*[1, A, B]$.*

Proof. Since $f \in S_q^*[m, A, B]$, therefore

$$\frac{z\mathfrak{D}_q f(z)}{f_m(z)} \prec \frac{1 + A_q z}{1 + B_q z},\ z \in E.$$

Replacing z by $z\epsilon_m^j$ where $j = 0, 1, 2, \ldots, m-1$, we have

$$\frac{\epsilon_m^j z(\mathfrak{D}_q f)(\epsilon_m^j z)}{f_m(\epsilon_m^j z)} \prec \frac{1 + A_q \epsilon_m^j z}{1 + B_q \epsilon_m^j z} \prec \frac{1 + A_q z}{1 + B_q z},\ z \in E.$$

Using the properties $f_m(\epsilon_m^j z) = \epsilon_m^j f_m(z)$ and $(\mathfrak{D}_q f)(\epsilon_m^j z) = \epsilon_m^{-j}(\mathfrak{D}_q(f(\epsilon_m^j z)))$, we obtain

$$\frac{\epsilon_m^{-j} z\mathfrak{D}_q(f(\epsilon_m^j z))}{f_m(z)} \prec \frac{1 + A_q z}{1 + B_q z},\ z \in E.$$

Since $\frac{1 + A_q z}{1 + B_q z}$ is a convex function, therefore applying summation $\sum_{j=0}^{m-1}$ and dividing by m, we obtain

$$\frac{z\mathfrak{D}_q(f_m(z))}{f_m(z)} \prec \frac{1 + A_q z}{1 + B_q z},\ z \in E.$$

That is $f_m \in S_q^*[1, A, B]$. □

Theorem 2. *If* $-1 \le D \le B < A \le C \le 1$, *then* $S_q^*[m, A, B] \subset S_q^*[m, C, D]$.

Proof. The proof follows directly by using $-q \le D_q \le B_q < A_q \le C_q \le 1$, and applying Lemma 3. □

Theorem 3. *Let* $f \in \mathcal{A}$ *be given in Equation (1) and satisfies*

$$\sum_{n=2}^{\infty}\big(([n]_q - \delta_{m,n}) + |A_q\delta_{m,n} - B_q[n]_q|\big)|a_n| \le A_q - B_q, \tag{15}$$

then $f \in S_q^*[m, A, B]$.

Proof. Let the Inequality (15) hold. Then from Equations (4) and (10)

$$\begin{aligned}
\left|\frac{\dfrac{z\mathfrak{D}_q f(z)}{f_m(z)} - 1}{A_q - B_q\dfrac{z\mathfrak{D}_q f(z)}{f_m(z)}}\right| &= \left|\frac{z\mathfrak{D}_q f(z) - f_m(z)}{A_q f_m(z) - B_q z\mathfrak{D}_q f(z)}\right| \\
&= \left|\frac{(z + \sum_{n=2}^{\infty}[n]_q a_n z^n) - (z + \sum_{n=2}^{\infty}\delta_{m,n}a_n z^n)}{A_q(z + \sum_{n=2}^{\infty}\delta_{m,n}a_n z^n) - B_q(z + \sum_{n=2}^{\infty}[n]_q a_n z^n)}\right| \\
&= \left|\frac{\sum_{n=2}^{\infty}([n]_q - \delta_{m,n})a_n z^{n-1}}{(A_q - B_q) + \sum_{n=2}^{\infty}(A_q\delta_{m,n} - B_q[n]_q)a_n z^{n-1}}\right| \\
&\le \frac{\sum_{n=2}^{\infty}|[n]_q - \delta_{m,n}||a_n||z^{n-1}|}{(A_q - B_q) - \sum_{n=2}^{\infty}|A_q\delta_{m,n} - B_q[n]_q||a_n||z^{n-1}|} \\
&\le \frac{\sum_{n=2}^{\infty}([n]_q - \delta_{m,n})|a_n|}{(A_q - B_q) - \sum_{n=2}^{\infty}|A_q\delta_{m,n} - B_q[n]_q||a_n|} \le 1,
\end{aligned}$$

then by maximum modulus theorem

$$\frac{z\mathfrak{D}_q f(z)}{f_m(z)} \prec \frac{1+A_q z}{1+B_q z},\ z \in E.$$

□

Using $q = 1$ and $m = 1$ in Theorem 3, we obtain the results of Ahuja [32].

Corollary 1. *If the function f defined by Equation (1) satisfies the inequality*

$$\sum_{n=2}^{\infty}((n-1)+|A-nB|)|a_n| \leq A-B,$$

then $f \in S^*[A,B]$.

Using $q = 1$ and $m = 1, A = 1-2\alpha, B = -1$ in Theorem 3, we obtain the following result of Silverman [33].

Corollary 2. *If the function f defined by Equation (1) satisfies the inequality*

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \leq 1-\alpha,$$

then $f \in S^*(\alpha)$.

Theorem 4. *For any $l,k \in \mathbb{N}$ with $l \leq k$, the polynomial functions*

$$p_l(z) = z + \frac{A_q - B_q}{k(1+|B_q|)} \sum_{n=2}^{l+1} \frac{z^n}{[n]_q}, \tag{16}$$

belong to the class $S_q^*[m,A,B]$ for all $m \geq k$.

Proof. By choosing $a_n = \dfrac{A_q - B_q}{k(1+|B_q|)[n]_q}$ for $n = 2,3\ldots,l+1$ and $a_n = 0$ for $n \geq l+2$ in Equation (15) and then applying Theorem 3 we obtain the required result. □

Theorem 5. *For any $l,k \in \mathbb{N}$ with $l \leq k$, the polynomial functions*

$$p_l(z) = z + \sum_{n=2}^{l+1} \frac{z^n}{k[n]_q}, \tag{17}$$

belong to the class $S_q^*[m]$ for all $m \in \mathbb{N}$.

Proof. By choosing $A = 1,\ B = -1,\ a_n = \dfrac{1}{k[n]_q}$ for $n = 2,3\ldots,l+1$ and $a_n = 0$ for $n \geq l+2$ in Equation (15) and then applying Theorem 3 we obtain the required result. □

Theorem 6. *Let $f \in S_q^*[m,A,B]$ is given by Equation (1) then*

$$|a_2| \leq \frac{A_q - B_q}{[2]_q - \delta_{m,2}}, \tag{18}$$

and

$$|a_3| \leq \begin{cases} \dfrac{A_q - B_q}{[3]_q - \delta_{m,3}}, & \text{for all } m \geq 2 \text{ or } m = 1 \text{ with } A < b, \\ \dfrac{(A_q - B_q[2]_q)(A_q - B_q)}{([2]_q - 1)([3]_q - 1)}, & \text{if } m = 1 \text{ and } A \geq b, \end{cases} \tag{19}$$

where $b = (1+q)B + \dfrac{3q-q^2}{1+q}$.

Proof. Let $f \in S_q^*[m, A, B]$ is given by Equation (1) then

$$\frac{z\mathfrak{D}_q f(z)}{f_m(z)} = \frac{1 + A_q w(z)}{1 + B_q w(z)}. \tag{20}$$

From Equations (4) and (10) we have

$$\begin{aligned}\frac{z\mathfrak{D}_q f(z)}{f_m(z)} &= 1 + ([2]_q - \delta_{m,2})a_2 z + (([3]_q - \delta_{m,3})a_3 \\ &\quad + (\delta_{m,2}^2 - [2]_q\delta_{m,2})a_2^2)z^2 + \cdots\end{aligned} \tag{21}$$

Similarly for $w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots$, we have

$$\frac{1 + A_q w(z)}{1 + B_q w(z)} = 1 + (A_q - B_q)w_1 z + ((A_q - B_q)w_2 - B_q(A_q - B_q)w_1^2)z^2 + \cdots$$

Next we calculate the values of w_1 and w_2. Taking

$$\frac{1 + w(z)}{1 - w(z)} = p(z) = 1 + c_1 z + c_2 z^2 + \cdots, \tag{22}$$

we obtain

$$1 + 2w_1 z + (2w_2 + 2w_1^2)z^2 + \cdots = 1 + c_1 z + c_2 z^2 + \cdots. \tag{23}$$

Comparing Equations (22) and (23), we have

$$w_1 = \frac{c_1}{2} \; and \; w_2 = \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right). \tag{24}$$

Therefore,

$$\begin{aligned}\frac{1 + A_q w(z)}{1 + B_q w(z)} &= 1 + \left(\frac{A_q - B_q}{2}\right)c_1 z \\ &\quad + \left(\frac{A_q - B_q}{2}\right)\left(c_2 - (1 + B_q)\frac{c_1^2}{2}\right)z^2 + \cdots.\end{aligned} \tag{25}$$

From Equations (21) and (25), we see that

$$([2]_q - \delta_{m,2})a_2 = \frac{(A_q - B_q)}{2}c_1, \tag{26}$$

and

$$([3]_q - \delta_{m,3})a_3 + (\delta_{m,2}^2 - [2]_q\delta_{m,2})a_2^2 = \frac{(A_q - B_q)}{2}\left(c_2 - (1 + B_q)\frac{c_1^2}{2}\right). \tag{27}$$

From Equation (26), we obtain

$$a_2 = \frac{(A_q - B_q)}{2([2]_q - \delta_{m,2})}c_1. \tag{28}$$

Using Equation (28) in Equation (27) we obtain

$$a_3 = \frac{(A_q - B_q)}{2([3]_q - \delta_{m,3})}c_2 - \frac{[2]_q(1+B_q) - \delta_{m,2}(1+A_q)}{4([2]_q - \delta_{m,2})([3]_q - \delta_{m,3})}(A_q - B_q)c_1^2$$
$$= \frac{(A_q - B_q)}{2([3]_q - \delta_{m,3})}\left(c_2 - \left(\frac{[2]_q(1+B_q) - \delta_{m,2}(1+A_q)}{2([2]_q - \delta_{m,2})}\right)c_1^2\right). \quad (29)$$

Applying Lemma 2, we obtain the first part of the result. For the second part, using Lemma 1 we obtain

$$|a_3| \le \frac{(A_q - B_q)}{[3]_q - \delta_{m,3}} \max\left\{1, \left|\frac{[2]_q B_q - \delta_{m,2} A_q}{[2]_q - \delta_{m,2}}\right|\right\}.$$

Note that

$$\left|\frac{[2]_q B_q - \delta_{m,2} A_q}{[2]_q - \delta_{m,2}}\right| \le 1 \text{ for all } m \ge 2 \text{ or } m = 1 \text{ with } A < b,$$
$$\left|\frac{[2]_q B_q - \delta_{m,2} A_q}{[2]_q - \delta_{m,2}}\right| \ge 1 \text{ if } m = 1 \text{ and } A \ge b.$$

Hence, the result follows. □

Theorem 7. *If $f \in S_q^*[m, A, B]$ is given by (1), then*

$$|a_n| \le \frac{A_q - B_q}{[n]_q - \delta_{m,n}} \prod_{j=2}^{n-1}\left(1 + \frac{\delta_{m,j}(A_q - B_q)}{[j]_q - \delta_{m,j}}\right), \text{ for all } n \ge 3. \quad (30)$$

Proof. Let

$$\frac{z\mathfrak{D}_q f(z)}{f_m(z)} = p(z). \quad (31)$$

Using Equations (4), (10) and $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ in Equation (31), we have

$$z + \sum_{n=2}^{\infty}[n]_q a_n z^n = \left(1 + \sum_{n=1}^{\infty} p_n z^n\right)\left(z + \sum_{n=2}^{\infty} \delta_{m,n} a_n z^n\right). \quad (32)$$

Comparing the coefficients of z^n, we obtain

$$([n]_q - \delta_{m,n})a_n = p_{n-1} + \sum_{k=1}^{n-2} \delta_{m,k+1} a_{k+1} p_{n-k-1}. \quad (33)$$

Since

$$p(z) \prec \frac{1 + A_q z}{1 + B_q z} = 1 + (A_q - B_q)z - B_q(A_q - B_q)z^2 + \cdots, \quad (34)$$

therefore, by Lemma 4

$$|p_n| \le A_q - B_q \text{ for all } n \ge 1. \quad (35)$$

Taking the absolute value of Equation (33) and using Equation (35), we obtain the following inequality

$$|a_n| \le \frac{A_q - B_q}{[n]_q - \delta_{m,n}}\left(1 + \sum_{k=1}^{n-2} \delta_{m,k+1}|a_{k+1}|\right). \quad (36)$$

We shall use principle of mathematical induction to prove Equation (30). By taking $n = 3$ in Equation (36) and using Equation (18), we have

$$|a_3| \le \frac{A_q - B_q}{[3]_q - \delta_{m,3}}\left(1 + \frac{\delta_{m,2}(A_q - B_q)}{[2]_q - \delta_{m,2}}\right), \quad (37)$$

which shows that Equation (30) is true for $n = 3$. Let us suppose Equation (30) holds for all $n \leq s$. For $n = s + 1$ from Equation (36), we see that

$$
\begin{aligned}
|a_{s+1}| &\leq \frac{A_q - B_q}{[s+1]_q - \delta_{m,s+1}}(1 + \delta_{m,2}|a_2| + \delta_{m,3}|a_3| + \cdots + \delta_{m,s}|a_s|) \\
&\leq \frac{A_q - B_q}{[s+1]_q - \delta_{m,s+1}}\left\{1 + \frac{\delta_{m,2}(A_q - B_q)}{[2]_q - \delta_{m,2}} + \frac{\delta_{m,3}(A_q - B_q)}{[3]_q - \delta_{m,3}}\left(1 + \frac{\delta_{m,2}(A_q - B_q)}{[2]_q - \delta_{m,2}}\right)\right. \\
&\quad \left. + \cdots + \delta_{m,s}\frac{A_q - B_q}{[s]_q - \delta_{m,s}}\prod_{j=2}^{s-1}\left(1 + \frac{\delta_{m,j}(A_q - B_q)}{[j]_q - \delta_{m,j}}\right)\right\} \\
&\leq \frac{A_q - B_q}{[s+1]_q - \delta_{m,s+1}}\prod_{j=2}^{s}\left(1 + \frac{\delta_{m,j}(A_q - B_q)}{[j]_q - \delta_{m,j}}\right),
\end{aligned}
$$

which shows the result is true for $n = s + 1$. Hence Equation (30) holds for all $n \geq 3$. □

Theorem 8. *Let $f \in S_q^*[m, A, B]$, then for any $\mu \in \mathbb{C}$, we have*

$$
|a_3 - \mu a_2^2| \leq \frac{(A_q - B_q)}{[3]_q - \delta_{m,3}} \max\{1, |2\lambda - 1|\},
$$

where

$$
\lambda = \frac{[2]_q(1 + B_q) - \delta_{m,2}(1 + A_q)}{2([2]_q - \delta_{m,2})} + \mu\frac{(A_q - B_q)([3]_q - \delta_{m,3})}{2([2]_q - \delta_{m,2})^2}. \tag{38}
$$

Proof. From Equations (28) and (29) we have

$$
\begin{aligned}
a_3 - \mu a_2^2 &= \frac{(A_q - B_q)}{2([3]_q - \delta_{m,3})}c_2 - \frac{[2]_q(1 + B_q) - \delta_{m,2}(1 + A_q)}{4([2]_q - \delta_{m,2})([3]_q - \delta_{m,3})}(A_q - B_q)c_1^2 \\
&\quad - \mu\left(\frac{(A_q - B_q)c_1}{2([2]_q - \delta_{m,2})}\right)^2 \\
&= \frac{(A_q - B_q)}{2([3]_q - \delta_{m,3})}\left(c_2 - \lambda c_1^2\right),
\end{aligned}
$$

where λ is given by Equation (38). Applying Lemma 1, we obtain the required result. □

For $m = 1$ and $q = 1$, we obtain the following result for the class defined by Janowski [22].

Corollary 3. *Let $f \in S^*[A, B]$ then*

$$
|a_3 - \mu a_2^2| \leq \frac{A - B}{2} \max\{1, |2B - A + 2\mu(A - B)|\}.
$$

For $m = 1, A = 1 - 2\alpha, B = -1$ and $q = 1$, we obtain the following result which is a special case of the result proved in [34] and can be found in [35].

Corollary 4. *Let $f \in S^*(\alpha)$, then*

$$
|a_3 - \mu a_2^2| \leq (1 - \alpha) \max\{1, |3 - 2\alpha - 4\mu(1 - \alpha)|\}.
$$

For $m = 1, A = 1, B = -1$ and $q = 1$, we obtain the following familiar Fekete–Szego inequality for starlike functions.

Corollary 5. *Let $f \in S^*$, then $|a_3 - \mu a_2^2| \leq \max\{1, |3 - 4\mu|\}$.*

Theorem 9. *If $f \in S_q^*[m, A, B]$ given by Equation (1) is univalent, then $f(E)$ contains an open disc of radius*

$$r_0 = \frac{[2]_q - \delta_{m,2}}{2([2]_q - \delta_{m,2}) + (A_q - B_q)}.$$

Proof. Let $\omega_0 \neq 0$ be a complex number such that $f(z) \neq \omega_0$ for $z \in E$. Then

$$g(z) = \frac{f(z)}{1 - \frac{f(z)}{\omega_0}} = \frac{\omega_0 f(z)}{\omega_0 - f(z)} = z + \left(a_2 + \frac{1}{\omega_0}\right)z^2 + \ldots.. \tag{39}$$

is analytic and univalent, therefore,

$$\left|a_2 + \frac{1}{\omega_0}\right| \leq 2. \tag{40}$$

The triangle inequality yields

$$\left|\frac{1}{\omega_0}\right| - |a_2| \leq 2.$$

Using Equation (18) we obtain

$$\left|\frac{1}{\omega_0}\right| \leq \frac{2([2]_q - \delta_{m,2}) + (A_q - B_q)}{[2]_q - \delta_{m,2}},$$

which implies

$$|\omega_0| \geq \frac{[2]_q - \delta_{m,2}}{2([2]_q - \delta_{m,2}) + (A_q - B_q)}, \tag{41}$$

which shows the image of E under $f(z)$ must cover an open disk with centre at the origin and radius r_0. □

Theorem 10. *If $f \in S_q^*[m, A, B]$, then*

$$\frac{1}{z}\left\{f(z) * \left(\frac{z}{(1-z)(1-qz)}(1 + B_q e^{i\theta}) - (1 + A_q e^{i\theta})h(z)\right)\right\} \neq 0, \tag{42}$$

where $h(z) = z + \sum_{n=2}^{\infty} \delta_{m,n} z^n$ and $0 \leq \theta < 2\pi$. The converse holds if $\frac{f_m(z)}{z} \neq 0$ for all $z \in E$.

Proof. Assume that $f \in S_q^*[m, A, B]$,, then we have $\frac{z\mathfrak{D}_q f(z)}{f_m(z)} \prec \frac{1 + A_q z}{1 + B_q z}$ if, and only if, $\frac{z\mathfrak{D}_q f(z)}{f_m(z)} \neq \frac{1 + A_q e^{i\theta}}{1 + B_q e^{i\theta}}$ for all $z \in E$ and $0 \leq \theta < 2\pi$. The last condition can be written as

$$\frac{1}{z}\left\{z\mathfrak{D}_q f(z)(1 + B_q e^{i\theta}) - f_m(z)(1 + A_q e^{i\theta})\right\} \neq 0. \tag{43}$$

On the other hand,

$$z\mathfrak{D}_q f(z) = f(z) * \frac{z}{(1-z)(1-qz)}$$

and

$$f_m(z) = z + \sum_{n=2}^{\infty} \delta_{m,n} a_n z^n = (z + \sum_{n=2}^{\infty} a_n z^n) * (z + \sum_{n=2}^{\infty} \delta_{m,n} z^n) = f(z) * h(z).$$

Substituting values in Equation (43), we have

$$\frac{1}{z}\left\{f(z)*\frac{z}{(1-z)(1-qz)}(1+B_q e^{i\theta})-f(z)*h(z)(1+A_q e^{i\theta})\right\}\neq 0, \tag{44}$$

which implies Equation (42).

Conversely, if the assumption in Equation (42) holds for $0\leq\theta<2\pi$ and $\frac{f_m(z)}{z}\neq 0$ for all $z\in E$, then the function $g(z)=\frac{z\mathfrak{D}_q f(z)}{f_m(z)}$ is analytic in E and $g(0)=1$. Since we have shown that Equations (42) and (43) are equivalent, therefore

$$\frac{z\mathfrak{D}_q f(z)}{f_m(z)}\neq\frac{1+A_q e^{i\theta}}{1+B_q e^{i\theta}}. \tag{45}$$

For $\psi(z)=\frac{1+A_q z}{1+B_q z}$ and $z\in E$, Relation (45) shows that $g(E)\cap\psi(\partial E)=\phi$. Therefore, the simply connected domain $g(E)$ is contained in a connected component of $\mathbb{C}-\psi(\partial E)$. Using the fact that $g(0)=\psi(0)$ together with the univalence of function ψ, it follows that $g\prec\psi$ which shows that $f\in S_q^*[m,A,B]$. □

4. Conclusions

The q-calculus is an important area of study in the field of mathematics. It usually deals with the generalization of differential and integral operators. In recent years, it has attracted many researchers due to its wide range of applications in different fields of sciences such as quantum mechanics, physics, special functions, orthogonal polynomials, combinatorics and the related areas. This article concerns a generalization of the class of starlike functions using the q-difference operator and the concepts of m-symmetrical points. This work includes sufficiency criteria, coefficient estimates, bounds for Fekete–Szego functional and convolution results for a newly defined class. During this study, it is noted that the classes defined by the q-difference operator are larger than that defined by ordinary derivatives because they also contains non-univalent functions. For example, the function $f(z)=z+\frac{3}{5}z^2$ is not univalent in E but this belongs to classes $S_{0.5}^*[2,1,-0.4]$ and $S_{0.5}^*[1,0.95,0.5]$. By using the technique presented in this article, an infinite sequence of functions can be generated for a wide range of subclasses of analytic functions which are special cases of the newly defined class. Hopefully, the results proved in this article will be beneficial to researchers in the field of geometric function theory.

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