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A Note on Certain General Transformation Formulas for the Appell and the Horn Functions

Insuk Kim [1,*] and Arjun K. Rathie [2]

1 Department of Mathematics Education, Wonkwang University, Iksan 54538, Republic of Korea
2 Department of Mathematics, Vedant College of Engineering and Technology, (Rajasthan Technical University), Bundi 323021, India
* Correspondence: iki@wku.ac.kr

Abstract: In a number of problems in applied mathematics, physics (theoretical and mathematical), statistics, and other fields the hypergeometric functions of one and several variables naturally appear. Hypergeometric functions in one and several variables have several known applications today. The Appell's four functions and the Horn's functions have shown to be particularly useful in providing solutions to a variety of problems in both pure and applied mathematics. The Hubbell rectangular source and its generalization, non-relativistic theory, and the hydrogen dipole matrix elements are only a few examples of the numerous scientific and chemical domains where Appell functions are used. The Appell series is also used in quantum field theory, especially in the evaluation of Feynman integrals. Additionally, since 1985, computational sciences such as artificial intelligence (AI) and information processing (IP) have used the well-known Horn functions as a key idea. In literature, there have been published a significant number of results of double series in particular of Appell and Horn functions in a series of interesting and useful research publications. We find three general transformation formulas between Appell functions F_2 and F_4 and two general transformation formulas between Appell function F_2 and Horn function H_4 in the present study, which are mostly inspired by their work and naturally exhibit symmetry. By using the generalizations of the Kummer second theorem in the integral representation of the Appell function F_2, this is accomplished. As special cases of our main findings, both previously known and new results have been found.

Keywords: hypergeometric function; generalized hypergeometric function; Kummer second theorem; Appell functions; Horn functions; integral representation

Citation: Kim, I.; Rathie, A.K. A Note on Certain General Transformation Formulas for the Appell and the Horn Functions. *Symmetry* **2023**, *15*, 696. https://doi.org/10.3390/sym15030696

Academic Editors: Djurdje Cvijović and Ioan Rașa

Received: 19 December 2022
Revised: 13 February 2023
Accepted: 17 February 2023
Published: 10 March 2023

1. Introduction

We begin by recalling the definition of the well-known and useful Pochhammer symbol (or the shifted factorial) $(a)_n$ defined for every complex number $a(\neq 0)$ by

$$(a)_n = \begin{cases} a(a+1)\cdots(a+n-1) & ; n \in \mathbb{N}, \\ 1 & ; n = 0. \end{cases} \tag{1}$$

In terms of the well-known gamma function, this can be written as

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

In the Gauss's hypergeometric function

$${}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix}; z\right],$$

there are two numerator parameters a and b and one denominator parameter c; z is called the variable of the function. A natural generalization of this function is accomplished by increasing any number of numerator and denominator parameters as follows:

$$ {}_pF_q\left[\begin{array}{ccc} a_1, & \cdots, & a_p \\ b_1, & \cdots, & b_q \end{array}; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \tag{2} $$

in which $(a)_n$ is the well-known Pochhammer symbol, already defined in (1). The function defined in (2) is called generalized hypergeometric function. Here p and q are nonnegative integers. It is also assumed that the variable z, the numerator parameters $a_1, \cdots, a_p$ and the denumerator parameters $b_1, \cdots, b_q$ can take on real or complex values with an exception that the denumerator parameters $b_1, \cdots, b_q$ should not be zero or a negative integer.

By ratio test, it can be easily see that the series ${}_pF_q$

(i) Converges for all $|z| < \infty$ if $p \leq q$;
(ii) Converges for all $|z| < 1$ if $p = q + 1$;
(iii) Diverges for all z, $z \neq 0$ if $p > q + 1$.

Moreover, if we set

$$ \delta = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i, $$

then the series ${}_pF_q$ with $p = q + 1$ is

(i) Absolutely convergent for $|z| = 1$, if $Re(\delta) > 0$;
(ii) Conditionally convergent for $|z| = 1$, $z \neq 1$ if $-1 < Re(\delta) \leq 0$;
(iii) Divergent for $|z| = 1$ if $Re(\delta) \leq -1$.

It is interesting to note that the generalized hypergeometric function (2) has symmetry in both the numerator parameters $a_1, a_2, \cdots, a_p$ and the denominator parameters $b_1, b_2, \cdots, b_q$. This means that every arrangement of the generalized hypergeometric function's numerator parameters $a_1, a_2, \cdots, a_p$ produces the same function, and every arrangement of the denominator parameters $b_1, b_2, \cdots, b_q$ generates the same function.

For a detailed account of hypergeometric function ${}_2F_1$ and generalized hypergeometric function ${}_pF_q$, the reader may consult the book by Andrews et al. [1], Bailey [2], Rainville [3], and Slater [4].

The vast popularity and immense usefulness of the hypergeometric function ${}_2F_1$ and the generalized hypergeometric function ${}_pF_q$ in one variable have inspired and stimulated a large number of mathematicians and researchers to study hypergeometric functions in two or more variables. In this regard, serious and very significant study of the functions of two variables initiated by Appell [5], who introduced the so-called four functions F_1, F_2, F_3, and F_4 named in the literature, the Appell functions in two variables which are the natural generalizations of the hypergeometric function ${}_2F_1$. It is interesting to mention here that Appell established the set of partial differential equations of which the above mentioned four functions are solutions. In addition to this, Appell found various very interesting reduction formulas together with expressions of these functions in terms of the hypergeometric function ${}_2F_1$. Moreover, in our present investigations, we are interested in the following two Appell functions [5] viz.

$$ F_2[\alpha, \mu_1, \mu_2; \nu_1, \nu_2; x_1, x_2] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\mu_1)_m(\mu_2)_n}{(\nu_1)_m(\nu_2)_n\, m!\, n!} x_1^m x_2^n, \tag{3} $$

provided $|x_1| + |x_2| < 1$,

$$F_4[\alpha, \beta; \gamma, \gamma^{'}; x, y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\gamma^{'})_n\, m!\, n!} x^m y^n, \tag{4}$$

provided $\sqrt{|x|} + \sqrt{|y|} < 1$.

Furthermore, the following integral representation for the Appell function F_2 viz.

$$\begin{aligned} &F_2[\lambda, \mu_1, \mu_2; \nu_1, \nu_2; x_1, x_2] \\ &\quad = \frac{1}{\Gamma(\lambda)} \int_0^{\infty} e^{-t}\, t^{\lambda-1} {}_1F_1\left[\begin{matrix} \mu_1 \\ \nu_1 \end{matrix}\; ; x_1 t\right] {}_1F_1\left[\begin{matrix} \mu_2 \\ \nu_2 \end{matrix}\; ; x_2 t\right] dt, \end{aligned} \tag{5}$$

provided $Re(\lambda) > 0$ and $Re(x_1 + x_2) < 1$.

It should be noted that two most significant functions described in the paper, the generalized hypergeometric function ${}_pF_q$ and the two Appell's functions F_2 and F_4 involving two variables, naturally exhibit symmetry.

Moreover, in the theory of special functions in mathematics, there are the 34 distinct convergent hypergeometric functions in two independent variables enumerated by Horn [6] (including the above mentioned four Appell functions). The total 34 Horn functions can be further categorized into 14 complete hypergeometric functions and 20 confluent hypergeometric functions. For details, we refer Erdélyi [7]. However, here we would like to mention Horn H_4 function which is defined as follows [6]:

$$H_4[\alpha,\, \beta; \gamma,\, \delta; x, y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{2m+n}(\beta)_n}{(\gamma)_m(\delta)_n\, m!\, n!} x^m y^n, \tag{6}$$

provided $|x| < \gamma$ and $|y| < \delta$ with $4\gamma = (\delta - 1)^2$.

It is not out of place to mention here that the symmetry occurs in the numerator parameters μ_1 and μ_2 and also symmetry occurs in the denominator parameters ν_1 and ν_2 of the Appell functions F_2 while the symmetry occurs in the denominator parameters γ and $\gamma^{'}$ of the Appell function F_4 and the denominator parameters γ and δ of the Horn function H_4.

2. Preliminaries

We start with the following interesting transformation formula between Appell functions F_2 and F_4 was given by Bailey [8] viz.

$$\begin{aligned} &F_2[\lambda,\, \mu,\, \nu; 2\mu,\, 2\nu; 2x_1,\, x_2] \\ &= (1 - x_1 - x_2)^{-\lambda}\, F_4\left[\frac{1}{2}\lambda,\, \frac{1}{2}(\lambda+1); \mu + \frac{1}{2},\, \nu + \frac{1}{2}; \left(\frac{x_1}{1 - x_1 - x_2}\right)^2, \left(\frac{x_2}{1 - x_1 - x_2}\right)^2\right]. \end{aligned} \tag{7}$$

Similarly, the following interesting transformation formula between Appell function F_2 and Horn function H_4 is given by Erdélyi [7] viz.

$$\begin{aligned} &F_2[\lambda,\, \mu_1,\, \mu_2; 2\mu_1,\, \nu; 4x_1, x_2] \\ &\quad = (1 - 2x_1)^{-\lambda}\, H_4\left[\lambda,\, \mu_2; \mu_1 + \frac{1}{2},\, \nu; \left(\frac{x_1}{1 - 2x_1}\right)^2, \left(\frac{x_2}{1 - 2x_2}\right)\right]. \end{aligned} \tag{8}$$

For a detailed account of hypergeometric functions in two and more variables, the reader may consult the book by Srivastava and Karlsson [9].

We remark in passing that the results (7) and (8) can be established by employing the following Kummer's second theorem [3] viz.

$$e^{-\frac{1}{2}x}{}_1F_1\left[\begin{matrix}\lambda\\2\lambda\end{matrix};x\right]={}_0F_1\left[\begin{matrix}-\\\lambda+\frac{1}{2}\end{matrix};\frac{x^2}{16}\right]. \tag{9}$$

In 1995, Rathie and Nagar [10] established the following two results closely related to the Kummer's second theorem (9) viz.

$$e^{-\frac{1}{2}t}{}_1F_1\left[\begin{matrix}\lambda\\2\lambda+1\end{matrix};t\right]={}_0F_1\left[\begin{matrix}-\\\lambda+\frac{1}{2}\end{matrix};\frac{t^2}{16}\right]-\frac{t}{2(2\lambda+1)}\,{}_0F_1\left[\begin{matrix}-\\\lambda+\frac{3}{2}\end{matrix};\frac{t^2}{16}\right], \tag{10}$$

and

$$e^{-\frac{1}{2}t}{}_1F_1\left[\begin{matrix}\lambda\\2\lambda-1\end{matrix};t\right]={}_0F_1\left[\begin{matrix}-\\\lambda-\frac{1}{2}\end{matrix};\frac{t^2}{16}\right]+\frac{t}{2(2\lambda-1)}\,{}_0F_1\left[\begin{matrix}-\\\lambda+\frac{1}{2}\end{matrix};\frac{t^2}{16}\right], \tag{11}$$

by employing two results closely related to Gauss's second summation theorem. These results are also recorded in [11].

In 2019, Mathur and Solanki [12,13], by employing the results (9), (10), and (11) in the integral representation (5) of F_2, established certain transformation formulas between F_2, F_4 and H_4 out of which some were corrected very recently by Mohammed et al. [14] in the following form:

$$\begin{aligned}
&F_2[\lambda,\,\mu,\,\nu;2\mu+1,\,2\nu+1;\,2x_1,\,x_2]\\
&=(1-x_1-x_2)^{-\lambda}\left\{F_4\left[\frac{1}{2}\lambda,\,\frac{1}{2}(\lambda+1);\mu+\frac{1}{2},\,\nu+\frac{1}{2};\left(\frac{x_1}{1-x_1-x_2}\right)^2,\left(\frac{x_2}{1-x_1-x_2}\right)^2\right]\right.\\
&-\frac{\lambda x_1}{(2\mu+1)(1-x_1-x_2)}\,F_4\left[\frac{1}{2}(\lambda+1),\,\frac{1}{2}\lambda+1;\mu+\frac{3}{2},\,\nu+\frac{1}{2};\left(\frac{x_1}{1-x_1-x_2}\right)^2,\left(\frac{x_2}{1-x_1-x_2}\right)^2\right]\\
&-\frac{\lambda x_2}{(2\nu+1)(1-x_1-x_2)}\,F_4\left[\frac{1}{2}(\lambda+1),\frac{1}{2}\lambda+1;\mu+\frac{1}{2},\nu+\frac{3}{2};\left(\frac{x_1}{1-x_1-x_2}\right)^2,\left(\frac{x_2}{1-x_1-x_2}\right)^2\right]\\
&+\frac{\lambda(\lambda+1)x_1x_2}{(2\mu+1)(2\nu+1)(1-x_1-x_2)^2}\\
&\left.\times F_4\left[\frac{1}{2}\lambda+1,\frac{1}{2}\lambda+\frac{3}{2};\mu+\frac{3}{2},\nu+\frac{3}{2};\left(\frac{x_1}{1-x_1-x_2}\right)^2,\left(\frac{x_2}{1-x_1-x_2}\right)^2\right]\right\},
\end{aligned} \tag{12}$$

provided $Re(\lambda)>0$ and $Re(x_1+x_2)<\frac{1}{2}$.

$$\begin{aligned}
&F_2[\lambda,\,\mu,\,\nu;2\mu-1,\,2\nu-1;\,2x_1,\,x_2]=(1-x_1-x_2)^{-\lambda}\\
&\times\left\{F_4\left[\frac{1}{2}\lambda,\,\frac{1}{2}(\lambda+1);\mu-\frac{1}{2},\,\nu-\frac{1}{2};\left(\frac{x_1}{1-x_1-x_2}\right)^2,\left(\frac{x_2}{1-x_1-x_2}\right)^2\right]\right.\\
&+\frac{\lambda x_1}{(2\mu-1)(1-x_1-x_2)}\\
&\times F_4\left[\frac{1}{2}(\lambda+1),\,\frac{1}{2}\lambda+1;\mu+\frac{1}{2},\,\nu-\frac{1}{2};\left(\frac{x_1}{1-x_1-x_2}\right)^2,\left(\frac{x_2}{1-x_1-x_2}\right)^2\right]\\
&+\frac{\lambda x_2}{(2\nu-1)(1-x_1-x_2)}\\
&\times F_4\left[\frac{1}{2}(\lambda+1),\frac{1}{2}\lambda+1;\mu-\frac{1}{2},\nu+\frac{1}{2};\left(\frac{x_1}{1-x_1-x_2}\right)^2,\left(\frac{x_2}{1-x_1-x_2}\right)^2\right]\\
&+\frac{\lambda(\lambda+1)x_1x_2}{(2\mu-1)(2\nu-1)(1-x_1-x_2)^2}\\
&\left.\times F_4\left[\frac{1}{2}\lambda+1,\frac{1}{2}\lambda+\frac{3}{2};\mu+\frac{1}{2},\nu+\frac{1}{2};\left(\frac{x_1}{1-x_1-x_2}\right)^2,\left(\frac{x_2}{1-x_1-x_2}\right)^2\right]\right\},
\end{aligned} \tag{13}$$

provided $Re(\lambda) > 0$ and $Re(x_1 + x_2) < \frac{1}{2}$.

$$F_2[\lambda, \mu_1, \mu_2; 2\mu_1 + 1, \nu; 4x_1, x_2]$$
$$= (1-2x_1)^{-\lambda}\left\{H_4\left[\lambda, \mu_2; \mu_1 + \frac{1}{2}, \nu; \left(\frac{x_1}{1-2x_1}\right)^2, \left(\frac{x_2}{1-2x_1}\right)\right]\right.$$
$$\left. - \frac{2\lambda x_1}{(2\mu_1+1)(1-2x_1)} H_4\left[\lambda+1, \mu_2; \mu_1 + \frac{3}{2}, \nu; \left(\frac{x_1}{1-2x_1}\right)^2, \left(\frac{x_2}{1-2x_1}\right)\right]\right\}, \tag{14}$$

provided $Re(\lambda) > 0$ and $Re(4x_1 + x_2) < 1$, and

$$F_2[\lambda, \mu_1, \mu_2; 2\mu_1 - 1, \nu; 4x_1, x_2]$$
$$= (1-2x_1)^{-\lambda}\left\{H_4\left[\lambda, \mu_2, \mu_1 - \frac{1}{2}, \nu; \left(\frac{x_1}{1-2x_1}\right)^2, \left(\frac{x_2}{1-2x_1}\right)\right]\right.$$
$$\left. + \frac{2\lambda x_1}{(2\mu_1-1)(1-2x_1)} H_4\left[\lambda+1, \mu_2; \mu_1 + \frac{1}{2}, \nu; \left(\frac{x_1}{1-2x_1}\right)^2, \left(\frac{x_2}{1-2x_1}\right)\right]\right\}, \tag{15}$$

provided $Re(\lambda) > 0$ and $Re(4x_1 + x_2) < 1$.

We remark in passing that the results (12) and (13) are closely related to the result (7) from Bailey while the results (14) and (15) are closely related to the result (8) from Erdélyi.

In order to establish our general results, we shall need the following two results recorded in [15] written here in a slightly different form, valid for $i \in \mathbb{N}_0$ viz.

$${}_1F_1\left[\begin{matrix} \alpha \\ 2\alpha + i \end{matrix}; 2x\right]$$
$$= e^x \sum_{k=0}^{i} \frac{(-i)_k (2\alpha-1)_k\, x^k}{(2\alpha+i)_k (\alpha - \frac{1}{2})_k\, 2^k\, k!}\, {}_0F_1\left[\begin{matrix} - \\ \alpha + \frac{1}{2} + k \end{matrix}; \frac{x^2}{4}\right], \tag{16}$$

and

$${}_1F_1\left[\begin{matrix} \alpha \\ 2\alpha - i \end{matrix}; 2x\right]$$
$$= e^x \sum_{k=0}^{i} \frac{(-1)^k (-i)_k (2\alpha - 2i - 1)_k\, x^k}{(2\alpha-i)_k (\alpha - \frac{1}{2} - i)_k\, 2^k\, k!}\, {}_0F_1\left[\begin{matrix} - \\ \alpha + k - i + \frac{1}{2} \end{matrix}; \frac{x^2}{4}\right]. \tag{17}$$

Appell functions have numerous uses in several physical and chemical domains. In this follow-up, we refer to [16] for application in radiation field problems, [17] for application in Hubbell rectangular sources and its generalization, [18] for application in non-relativistic theory and [19] for application in the hydrogen dipole matrix element. Additionally, it can be shown in the interesting works [20,21], the Appell series is used in quantum field theory, specifically in the evaluation of Feynman integrals. In the computational sciences, including artificial intelligence (AI) and information processing (IP), the well-known Horn functions have also been used since 1985 as a fundamental idea. The papers [22–24] provide more information.

In recent years, in a series of research papers, Brychkov and Saad [25–27], Brychkov [28,29], Brychkov et al. [30], and Brychkov and Savischenko [31–37] have established a large number of results on Appell and Horn functions.

Inspired mainly by their work, in this paper, our main objective is to generalize Bailey's result (7) and Erdélyi's result (8) in the most general form. For this, the rest of the paper is organized as follows: In Section 3, we shall establish three general transformation formulas between Appell functions F_2 and F_4 while in Section 3, two general transformation formulas between Appell function F_2 and Horn function H_4. The results obtained earlier by Mohammed et al. [14] (which are corrected forms of the results from Mathur and Solanki) follow special cases of our main findings. A few new interesting special cases have also been mentioned.

3. Transformation Formulas between Appell Functions F_2 and F_4

In this section, we shall establish three general transformation formulas between Appell functions F_2 and F_4 asserted in the following theorem.

Theorem 1. *For $Re(\lambda) > 0$, $Re(x_1 + x_2) < \frac{1}{2}$ and $i, j \in \mathbb{N}_0$, the following transformation formulas hold true.*

$$\begin{aligned} &F_2[\lambda,\, \mu,\, \nu; 2\mu + i,\, 2\nu + j; 2x_1,\, 2x_2] = (1 - x_1 - x_2)^{-\lambda} \\ &\times \sum_{k_1=0}^{i}\sum_{k_2=0}^{j} \frac{(-i)_{k_1}\,(-j)_{k_2}\,(2\mu-1)_{k_1}\,(2\nu-1)_{k_2}\,(\lambda)_{k_1+k_2}\,x_1^{k_1}\,x_2^{k_2}}{(2\mu+i)_{k_1}\,(2\nu+j)_{k_2}\,2^{k_1+k_2}\,(k_1)!\,(k_2)!\,(1-x_1-x_2)^{k_1+k_2}} \\ &\times F_4\left[\frac{1}{2}(\lambda + k_1 + k_2),\, \frac{1}{2}(\lambda + k_1 + k_2 + 1); \mu + \frac{1}{2} + k_1,\, \nu + \frac{1}{2} + k_2;\right. \\ &\qquad\qquad \left.\left(\frac{x_1}{1 - x_1 - x_2}\right)^2, \left(\frac{x_2}{1 - x_1 - x_2}\right)^2\right], \end{aligned} \tag{18}$$

$$\begin{aligned} &F_2[\lambda,\, \mu,\, \nu; 2\mu + i,\, 2\nu - j; 2x_1,\, 2x_2] = (1 - x_1 - x_2)^{-\lambda} \\ &\times \sum_{k_1=0}^{i}\sum_{k_2=0}^{j} \left\{ \frac{(-1)^{k_2}\,(-i)_{k_1}\,(-j)_{k_2}\,(\lambda)_{k_1+k_2}}{2^{k_1+k_2}\,(k_1)!\,(k_2)!} \right. \\ &\times \frac{(2\mu-1)_{k_1}\,(2\nu-2j-1)_{k_2}\,x_1^{k_1}\,x_2^{k_2}}{(2\mu+i)_{k_1}\,(2\nu-j)_{k_2}\,(\mu-\frac{1}{2})_{k_1}\,(\nu-\frac{1}{2}-j)_{k_2}\,(1-x_1-x_2)^{k_1+k_2}} \\ &\times F_4\left[\frac{1}{2}(\lambda + k_1 + k_2),\, \frac{1}{2}(\lambda + k_1 + k_2 + 1); \mu + \frac{1}{2} + k_1,\, \nu + k_2 - j + \frac{1}{2};\right. \\ &\qquad\qquad \left.\left.\left(\frac{x_1}{1 - x_1 - x_2}\right)^2, \left(\frac{x_2}{1 - x_1 - x_2}\right)^2\right]\right\}, \end{aligned} \tag{19}$$

and

$$\begin{aligned} &F_2[\lambda,\, \mu,\, \nu; 2\mu - i,\, 2\nu - j; 2x_1,\, 2x_2] = (1 - x_1 - x_2)^{-\lambda} \\ &\times \sum_{k_1=0}^{i}\sum_{k_2=0}^{j} \left\{ \frac{(-1)^{k_1+k_2}\,(-i)_{k_1}\,(-j)_{k_2}\,(\lambda)_{k_1+k_2}}{2^{k_1+k_2}\,(k_1)!\,(k_2)!} \right. \\ &\times \frac{(2\mu-2i-1)_{k_1}\,(2\nu-2j-1)_{k_2}\,x_1^{k_1}\,x_2^{k_2}}{(2\mu-i)_{k_1}\,(2\nu-j)_{k_2}\,(\mu-\frac{1}{2}-i)_{k_1}\,(\nu-\frac{1}{2}-j)_{k_2}\,(1-x_1-x_2)^{k_1+k_2}} \\ &\times F_4\left[\frac{1}{2}(\lambda + k_1 + k_2),\, \frac{1}{2}(\lambda + k_1 + k_2 + 1); \mu + k_1 - i + \frac{1}{2},\, \nu + k_2 - j + \frac{1}{2};\right. \\ &\qquad\qquad \left.\left.\left(\frac{x_1}{1 - x_1 - x_2}\right)^2, \left(\frac{x_2}{1 - x_1 - x_2}\right)^2\right]\right\}. \end{aligned} \tag{20}$$

Proof. In order to establish our first result (18) asserted in the Theorem 1, we proceed as follows. For this in the integral representation (5) of F_2, if we set $\mu_1 = \mu$, $\nu_1 = 2\mu + i$, $\mu_2 = \nu$, $\nu_2 = 2\nu + j$ and replacing x_1 and x_2 by $2x_1$ and $2x_2$, respectively, then for $i, j \in \mathbb{N}_0$, we have

$$\begin{aligned} &F_2[\lambda,\, \mu,\, \nu; 2\mu + i,\, 2\nu + j; 2x_1,\, 2x_2] \\ &= \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-t}\, t^{\lambda-1}\, {}_1F_1\left[\begin{matrix} \mu \\ 2\mu + i \end{matrix}; 2x_1 t\right] {}_1F_1\left[\begin{matrix} \nu \\ 2\nu + j \end{matrix}; 2x_2 t\right] dt. \end{aligned}$$

Using the result (16) in both ${}_1F_1$ functions, we have after some simplification

$$F_2[\lambda,\,\mu,\,\nu;2\mu+i,2\nu+j;2x_1,2x_2]=\frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-t}\,t^{\lambda-1}$$
$$\times\left\{e^{x_1t}\sum_{k_1=0}^{i}C_1(k_1)\,x_1^{k_1}\,t^{k_1}\,{}_0F_1\left[\begin{matrix}-\\ \mu+\frac{1}{2}+k_1\end{matrix};\frac{x_1^2\,t^2}{4}\right]\right.$$
$$\left.\cdot\,e^{x_2t}\sum_{k_2=0}^{j}C_2(k_2)\,x_2^{k_2}\,t^{k_2}\,{}_0F_1\left[\begin{matrix}-\\ \nu+\frac{1}{2}+k_2\end{matrix};\frac{x_2^2\,t^2}{4}\right]\right\}dt,$$

where

$$C_1(k_1)=\frac{(-i)_{k_1}(2\mu-1)_{k_1}}{(2\mu+i)_{k_1}(\mu-\frac{1}{2})_{k_1}\,2^{k_1}(k_1)!},$$

and

$$C_2(k_2)=\frac{(-j)_{k_2}(2\nu-1)_{k_2}}{(2\nu+j)_{k_2}(\nu-\frac{1}{2})_{k_2}\,2^{k_2}(k_2)!}.$$

Now, changing the order of integration and summation, we have

$$F_2[\lambda,\,\mu,\,\nu;2\mu+i,2\nu+j;2x_1,x_2]$$
$$=\sum_{k_1=0}^{i}\sum_{k_2=0}^{j}C_1(k_1)\,C_2(k_2)\,x_1^{k_1}\,x_2^{k_2}\frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-t(1-x_1-x_2)}\,t^{\lambda+k_1+k_2-1}$$
$$\times\,{}_0F_1\left[\begin{matrix}-\\ \mu+\frac{1}{2}+k_1\end{matrix};\frac{x_1^2\,t^2}{4}\right]{}_0F_1\left[\begin{matrix}-\\ \nu+\frac{1}{2}+k_2\end{matrix};\frac{x_2^2\,t^2}{4}\right]dt.$$

Next, expressing both ${}_0F_1$ functions as series, change the order of integration and summation, we have

$$F_2[\lambda,\,\mu,\,\nu;2\mu+i,2\nu+j;2x_1,x_2]$$
$$=\sum_{k_1=0}^{i}\sum_{k_2=0}^{j}C_1(k_1)\,C_2(k_2)\,x_1^{k_1}\,x_2^{k_2}$$
$$\times\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{x_1^{2m}\,x_2^{2n}}{(\mu+\frac{1}{2}+k_1)_m(\nu+\frac{1}{2}+k_2)_n\,2^{2m+2n}\,m!\,n!}\right.$$
$$\left.\times\,\frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-(1-x_1-x_2)t}\,t^{\lambda+k_1+k_2+2m+2n-1}dt\right\}.$$

Evaluating the Gamma integral and noting that

$$\frac{\Gamma(\lambda+k_1+k_2+2m+2n)}{\Gamma(\lambda)}$$
$$=2^{2m+2n}\left(\frac{1}{2}(\lambda+k_1+k_2)\right)_{m+n}\left(\frac{1}{2}(\lambda+k_1+k_2+1)\right)_{m+n}(\lambda)_{k_1+k_2},$$

and summing up the series, we after some simplification, easily arrive at the right-hand side of (18). This completes the proof of the first result (18) asserted in the Theorem 1. In exactly the same manner, the results (19) and (20) can be established. □

Corollaries

In this subsection, we shall mention special cases of the results (18), (19) and (20) asserted in the Theorem 1.

Corollary 1. *In (18), if we take $i = 0$, we obtain the following result which is also of general character:*

$$
\begin{aligned}
&F_2[\lambda,\, \mu,\, \nu; 2\mu,\, 2\nu + j; 2x_1,\, 2x_2] \\
&= (1 - x_1 - x_2)^{-\lambda} \sum_{k_2=0}^{j} \frac{(-j)_{k_2}\,(2\mu - 1)_{k_2}\,(\lambda)_{k_2}}{(2\nu + j)_{k_2}\, 2^{k_2}\,(k_2)!} \left(\frac{x_2}{1 - x_1 - x_2}\right)^{k_2} \\
&\times F_4\left[\frac{1}{2}(\lambda + k_2),\, \frac{1}{2}(\lambda + k_2 + 1); \mu + \frac{1}{2},\, \nu + \frac{1}{2} + k_2; \left(\frac{x_1}{1 - x_1 - x_2}\right)^2, \left(\frac{x_2}{1 - x_1 - x_2}\right)^2\right].
\end{aligned}
$$

In particular, for $j = 1$, we obtain

$$
\begin{aligned}
&F_2[\lambda,\, \mu,\, \nu; 2\mu,\, 2\nu + 1; 2x_1,\, 2x_2] = (1 - x_1 - x_2)^{-\lambda} \\
&\times \left\{F_4\left[\frac{1}{2}\lambda,\, \frac{1}{2}(\lambda + 1); \mu + \frac{1}{2},\, \nu + \frac{1}{2}; \left(\frac{x_1}{1 - x_1 - x_2}\right)^2, \left(\frac{x_2}{1 - x_1 - x_2}\right)^2\right]\right. \\
&- \frac{\lambda x_2}{(2\nu + 1)\,(1 - x_1 - x_2)} \\
&\left.\times F_4\left[\frac{1}{2}(\lambda + 1),\, \frac{1}{2}\lambda + 1; \mu + \frac{1}{2},\, \nu + \frac{3}{2}; \left(\frac{x_1}{1 - x_1 - x_2}\right)^2, \left(\frac{x_2}{1 - x_1 - x_2}\right)^2\right]\right\}.
\end{aligned}
$$

Corollary 2. *In (19), if we take $i = 0$, we obtain the following result which is also of general character:*

$$
\begin{aligned}
&F_2[\lambda,\, \mu,\, \nu; 2\mu,\, 2\nu - j; 2x_1,\, 2x_2] \\
&= (1 - x_1 - x_2)^{-\lambda} \sum_{k_2=0}^{j} \frac{(-1)^{k_2}\,(-j)_{k_2}\,(2\nu - 2j - 1)_{k_2}\,(\lambda)_{k_2}}{(2\nu - j)_{k_2}\,(\nu - \frac{1}{2} - j)_{k_2}\, 2^{k_2}\,(k_2)!} \left(\frac{x_2}{1 - x_1 - x_2}\right)^{k_2} \\
&\times F_4\left[\frac{1}{2}(\lambda + k_2),\, \frac{1}{2}(\lambda + k_2 + 1); \mu + \frac{1}{2},\, \nu + k_2 - j + \frac{1}{2}; \left(\frac{x_1}{1 - x_1 - x_2}\right)^2, \left(\frac{x_2}{1 - x_1 - x_2}\right)^2\right].
\end{aligned}
$$

In particular, for $j = 1$, we obtain

$$
\begin{aligned}
&F_2[\lambda,\, \mu,\, \nu; 2\mu, 2\nu - 1; 2x_1,\, 2x_2] = (1 - x_1 - x_2)^{-\lambda} \\
&\times \left\{F_4\left[\frac{1}{2}\lambda,\, \frac{1}{2}(\lambda + 1); \mu + \frac{1}{2},\, \nu - \frac{1}{2}; \left(\frac{x_1}{1 - x_1 - x_2}\right)^2, \left(\frac{x_2}{1 - x_1 - x_2}\right)^2\right]\right. \\
&+ \frac{\lambda x_2}{(2\nu - 1)\,(1 - x_1 - x_2)} \\
&\left.\times F_4\left[\frac{1}{2}(\lambda + 1),\, \frac{1}{2}\lambda + 1; \mu + \frac{1}{2},\, \nu + \frac{1}{2}; \left(\frac{x_1}{1 - x_1 - x_2}\right)^2, \left(\frac{x_2}{1 - x_1 - x_2}\right)^2\right]\right\}.
\end{aligned}
$$

Remark 1. *In (18) or (19) or (20), if we set $i = j = 0$, we at once obtain a known result (7) according to Bailey.*

Remark 2. *In (18), if we take $i = j = 1$, we obtain a known result from Mathur and Solanki in the corrected form (12) given very recently by Mohammed et al.*

Corollary 3. *In (19), if we take $i = j = 1$, we obtain the following interesting result in compact form.*

$$
\begin{aligned}
&F_2[\lambda,\, \mu,\, \nu; 2\mu+1,\, 2\nu-1; 2x_1,\, 2x_2] = (1-x_1-x_2)^{-\lambda} \\
&\times \left\{ F_4\left[\frac{1}{2}\lambda,\, \frac{1}{2}(\lambda+1); \mu+\frac{1}{2},\, \nu-\frac{1}{2}; \left(\frac{x_1}{1-x_1-x_2}\right)^2, \left(\frac{x_2}{1-x_1-x_2}\right)^2\right]\right. \\
&- \frac{\lambda x_1}{(2\mu+1)\,(1-x_1-x_2)} \\
&\times F_4\left[\frac{1}{2}(\lambda+1),\, \frac{1}{2}\lambda+1; \mu+\frac{3}{2},\, \nu-\frac{1}{2}; \left(\frac{x_1}{1-x_1-x_2}\right)^2, \left(\frac{x_2}{1-x_1-x_2}\right)^2\right] \\
&+ \frac{\lambda x_2}{(2\nu-1)\,(1-x_1-x_2)} \\
&\times F_4\left[\frac{1}{2}(\lambda+1),\, \frac{1}{2}\lambda+1; \mu+\frac{1}{2},\, \nu+\frac{1}{2}; \left(\frac{x_1}{1-x_1-x_2}\right)^2, \left(\frac{x_2}{1-x_1-x_2}\right)^2\right] \\
&- \frac{\lambda(\lambda+1)\,x_1\,x_2}{(2\mu+1)\,(2\nu-1)\,(1-x_1-x_2)^2} \\
&\times \left. F_4\left[\frac{1}{2}\lambda+1,\, \frac{1}{2}\lambda+\frac{3}{2}; \mu+\frac{3}{2},\, \nu+\frac{1}{2}; \left(\frac{x_1}{1-x_1-x_2}\right)^2, \left(\frac{x_2}{1-x_1-x_2}\right)^2\right]\right\}.
\end{aligned}
$$

Remark 3. *In* (20)*, if we take* $i = j = 1$*, we obtain another known result from Mathur and Solanki in the corrected form* (13) *given very recently by Mohammed et al.*

Similarly, other results can be obtained.

4. Transformation Formulas between Appell Functions F_2 and Horn Function H_4

In this section, we shall establish two general transformation formulas between Appell function F_2 and Horn function H_4 asserted in the following theorem.

Theorem 2. *For* $Re(\lambda) > 0$*,* $Re(4x_1 + x_2) < 1$ *and* $i \in \mathbb{N}_0$*, the following transformation formulas hold true.*

$$
\begin{aligned}
&F_2[\lambda, \mu_1,\, \mu_2; 2\mu_1+i,\, \nu; 4x_1,\, x_2] \\
&= (1-2x_1)^{-\lambda} \sum_{k=0}^{i} \frac{(-i)_k\,(2\mu_1-1)_k\,(\lambda)_k\,x_1^k}{(2\mu_1+i)_k(\mu_1-\frac{1}{2})_k\,(1-2x_1)^k\,k!} \\
&\times H_4\left[\lambda+k,\, \mu_2;\, \mu_1+k+\frac{1}{2},\, \nu; \left(\frac{x_1}{1-2x_1}\right)^2, \left(\frac{x_2}{1-2x_1}\right)\right],
\end{aligned} \tag{21}
$$

and

$$
\begin{aligned}
&F_2[\lambda, \mu_1,\, \mu_2; 2\mu_1-i,\, \nu; 4x_1,\, x_2] \\
&= (1-2x_1)^{-\lambda} \sum_{k=0}^{i} \frac{(-1)^k(-i)_k\,(2\mu_1-2i-1)_k\,(\lambda)_k\,x_1^k}{(2\mu_1-i)_k\,(\mu_1-i-\frac{1}{2})_k\,(1-2x_1)^k\,k!} \\
&\times H_4\left[\lambda+k,\, \mu_2;\, \mu_1-i+\frac{1}{2}+k,\, \nu; \left(\frac{x_1}{1-2x_1}\right)^2, \left(\frac{x_2}{1-2x_1}\right)\right].
\end{aligned} \tag{22}
$$

Proof. In order to establish first result (21) asserted in the Theorem 2, we proceed as follows. For this, in the integral representation (5) of F_2, if we set $\nu_1 = 2\mu_1 + i$, $\nu_2 = \nu$ and replacing x_1 by $4x_1$, then for $i \in \mathbb{N}_0$, we have

$$F_2[\lambda,\, \mu_1,\, \mu_2; 2\mu_1 + i,\, \nu; 4x_1,\, x_2]$$
$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-t}\, t^{\lambda-1}\, {}_1F_1\left[\begin{array}{c} \mu_1 \\ 2\mu_1 + i \end{array}; x_1\, t\right] {}_1F_1\left[\begin{array}{c} \mu_2 \\ \nu \end{array}; x_2\, t\right] dt.$$

Using the result (16) in the first ${}_1F_1$ function, we have after some simplification

$$F_2[\lambda,\, \mu_1,\, \mu_2; 2\mu_1 + i,\, \nu; 4x_1,\, x_2]$$
$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-t}\, t^{\lambda-1} \left\{ e^{2x_1 t} \sum_{k=0}^{i} C(k)\, x_1^k\, t^k\, {}_0F_1\left[\begin{array}{c} - \\ \mu_1 + k + \frac{1}{2} \end{array}; x_1^2\, t^2\right] \right\}$$
$$\times\, {}_1F_1\left[\begin{array}{c} \mu_2 \\ \nu \end{array}; x_2\, t\right] dt,$$

where

$$C(k) = \frac{(-i)_k\, (2\mu_1 - 1)_k}{(2\mu_1 + i)_k\, (\mu_1 - \frac{1}{2})_k\, (k)!}.$$

Now, changing the order of integration and summation, we have

$$F_2[\lambda,\, \mu_1,\, \mu_2; 2\mu_1 + i,\, \nu; 4x_1,\, x_2] = \sum_{k=0}^{i} C(k)\, x_1^k$$
$$\times \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-(1-2x_1)t}\, t^{\lambda+k-1}\, {}_0F_1\left[\begin{array}{c} - \\ \mu_1 + k + \frac{1}{2} \end{array}; x_1^2\, t^2\right] {}_1F_1\left[\begin{array}{c} \mu_2 \\ \nu \end{array}; x_2\, t\right] dt.$$

Next, expressing both ${}_0F_1$ and ${}_1F_1$ generalized hypergeometric function as series, changing the order of integration and summation, we have

$$F_2[\lambda,\, \mu_1,\, \mu_2; 2\mu_1 + i,\, \nu; 4x_1,\, x_2] = \sum_{k=0}^{i} C(k)\, x_1^k$$

$$\times \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{x_1^{2m}\, x_2^{2n}\, (\mu_2)_n}{(\mu_1 + k + \frac{1}{2})_m\, (\nu)_n\, m!\, n!} \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-(1-2x_1)t}\, t^{\lambda+k+2m+n-1} dt \right\}.$$

Evaluating the Gamma integral and noting that

$$\frac{\Gamma(\lambda + k + 2m + n)}{\Gamma(\lambda)} = (\lambda + k)_{2m+n}\, (\lambda)_k,$$

and summing up the series, we after some simplification, easily arrive at the right-hand side of (21). This completes the proof of the first result (21) asserted in the Theorem 2. In exactly the same manner the result (22) can be established. □

Corollaries

In this subsection, we shall mention some known as well as new special cases of the results (21) and (22) asserted in the Theorem 2.

Remark 4. *In (21) or (22), if we take $i = 0$, the result (8) follows in view of the work conducted by Erdélyi [7], (Eq. (4.7), p. 239).*

Remark 5. *In (21), if we take $i = 1$, we obtain a known result from Mathur and Solanki in the corrected form (14) given very recently by Mohammed et al.*

Corollary 4. *In* (21)*, if we take* $i = 2$*, we obtain the following interesting contiguous result.*

$$
\begin{aligned}
&F_2[\lambda, \mu_1, \mu_2; 2\mu_1 + 2, \nu; 4x_1, x_2] \\
&= (1 - 2x_1)^{-\lambda} \left\{ H_4\left[\lambda, \mu_2; \mu_1 + \frac{1}{2}, \nu; \left(\frac{x_1}{1 - 2x_1}\right)^2, \left(\frac{x_2}{1 - 2x_1}\right)^2\right] \right. \\
&- \frac{2\lambda\, x_1}{(\mu_1 + 1)\,(1 - 2x_1)} H_4\left[\lambda + 1, \mu_2; \mu_1 + \frac{3}{2}, \nu; \left(\frac{x_1}{1 - 2x_1}\right)^2, \left(\frac{x_2}{1 - 2x_1}\right)^2\right] \\
&+ \frac{4\lambda\,(\lambda + 1)\, x_1^2}{(\mu_1 + 1)\,(2\mu_1 + 1)\,(2\mu_1 + 3)\,(1 - 2x_1)^2} \\
&\left. \times H_4\left[\lambda + 1, \mu_2; \mu_1 + \frac{5}{2}, \nu; \left(\frac{x_1}{1 - 2x_1}\right)^2, \left(\frac{x_2}{1 - 2x_1}\right)^2\right] \right\}.
\end{aligned}
$$

Remark 6. *In* (22)*, if we take* $i = 1$*, we obtain another known result from Mathur and Solanki in the corrected form* (15) *given recently by Mohammed et al.*

Corollary 5. *In* (22)*, if we take* $i = 2$*, we obtain the following interesting contiguous result.*

$$
\begin{aligned}
&F_2[\lambda, \mu_1, \mu_2; 2\mu_1 - 2, \nu; 4x_1, x_2] \\
&= (1 - 2x_1)^{-\lambda} \left\{ H_4\left[\lambda, \mu_2; \mu_1 - \frac{3}{2}, \nu; \left(\frac{x_1}{1 - 2x_1}\right)^2, \left(\frac{x_2}{1 - 2x_1}\right)^2\right] \right. \\
&- \frac{2\lambda\, x_1}{(\mu_1 - 1)\,(1 - 2x_1)} H_4\left[\lambda + 1, \mu_2; \mu_1 - \frac{1}{2}, \nu; \left(\frac{x_1}{1 - 2x_1}\right)^2, \left(\frac{x_2}{1 - 2x_1}\right)^2\right] \\
&+ \frac{4\lambda\,(\lambda + 1)\,(\mu_1 - 2)}{(\mu_1 - 1)\,(2\mu_1 - 1)\,(2\mu_1 - 3)^2} \\
&\left. \times H_4\left[\lambda + 2, \mu_2; \mu_1 + \frac{1}{2}, \nu; \left(\frac{x_1}{1 - 2x_1}\right)^2, \left(\frac{x_2}{1 - 2x_1}\right)^2\right] \right\}.
\end{aligned}
$$

Similarly, other results can be obtained.

5. Conclusions

In this study, three general transformation formulas between Appell functions F_2 and F_4 and two general transformation formulas between Appell function F_2 and Horn function H_4 have been established. To achieve this, in the integral formulation of the Appell function F_2, generalizations of the Kummer second theorem are used. As special cases of our major discoveries, both previously known and new results have been found. The findings in this work are thought to be novel to the literature and provide a significant advance to the understanding of generalized hypergeometric functions of one and two variables. It is believed that the findings in this work may have possible applications in a variety of physical and chemical domains, such as Hubbell's radiation field problems, Hubbell rectangular source and its generalization, non-relativistic theory, and hydrogen dipole matrix element. The Appell series is also used in quantum field theory, namely for the evaluation of Feynman integrals. Furthermore, since 1985, computational sciences such as artificial intelligence (AI) and information processing (IP) have used the well-known Horn functions as a key notion. A forthcoming paper in this area will explore the applications of the findings from this work as well as the geometrical interpretation of the general transformation formulas for the Appell and Horn functions.

Author Contributions: Writing—original draft, I.K. and A.K.R.; writing—review and editing, I.K. and A.K.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: The research work of Insuk Kim was supported by Wonkwang University in 2022.

Conflicts of Interest: The authors have no conflicts of interest.

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