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Geometry and Application in Economics of Fixed Point

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Abstract: Inspired by the reality that the collection of fixed/common fixed points can embrace any symmetrical geometric shape comparable to a disc, a circle, an elliptic disc, an ellipse, or a hyperbola, we investigate the subsistence of a fixed point and a common fixed point and study their geometry in a partial metric space by introducing some novel contractions and notions of a fixed ellipse-like curve and a common fixed ellipse-like curve which is symmetrical in shape but entirely different than that of an ellipse in a Euclidean space. We look at new hypotheses essential for the collection of nonunique fixed/common fixed points of some mathematical operators to incorporate an ellipse-like curve keeping in view the symmetry in fixed/common fixed points approaches. Appropriate nontrivial examples verify established conclusions. We conclude our work by applying our results to construct the mathematical model and solve the Production–Consumption Equilibrium problem of economics.

Keywords: cost-effectiveness; n-ellipse-like curve; 2-ellipse-like curve; mathematical operators; nonlinear systems; mathematical model; production–consumption equilibrium; partial metric space; symmetrical

Citation: Joshi, M.; Upadhyay, S.; Tomar, A.; Sajid, M. Geometry and Application in Economics of Fixed Point. *Symmetry* **2023**, *15*, 704. https://doi.org/10.3390/sym15030704

Academic Editors: Hasanen A. Hammad and Luis Manuel Sánchez Ruiz

Received: 14 February 2023
Revised: 27 February 2023
Accepted: 9 March 2023
Published: 11 March 2023

1. Introduction

A partial metric space is an endeavor to extend the metric space by substituting the zero self distance $d(\varrho_1, \varrho_1) = 0$ by the condition $\rho(\varrho_1, \varrho_1) \leq \rho(\varrho_1, \varrho_2)$. Nonzero self-distance appears to be reasonable in the case of finite sequences. In fact, partial metrics are more accommodating than the metrics because these induce partial orders and have more general topological properties as the self-distance of each of its points is not essentially zero. The inspiration at the back of the introduction of partial metric space was to improve and enhance the theorem of Banach [1], which could be applied to both the partially computed and totally computed sequences. S.G. Matthews [2], encouraged by an understanding of computer science, initiated it and revealed that Banach's conclusion could be epitomized by the partial metric conditions so that it can be applied in program verification. The uniqueness of the fixed points is applicable to relate denotational and operational semantic models. For this, the nonlinear systems or mathematical models are studied in a partial metric space by taking a contractive function and are demonstrated as the fixed points of the introduced function. Two models are concluded to be equivalent by the uniqueness of the fixed point. These are extremely beneficial in computer science, especially in the investigation of semantics and domains, and were introduced to study computer programs.

The presence of the fixed/common fixed point of a self map performs a key function in the theory of fixed points and has a lot of applications to numerous applications in day-to-day life (see [3–5], and so on). However, suppose a self map has nonunique fixed points. In that case, looking at the symmetrical geometrical figures embraced by the collection of fixed points or common fixed points is extremely appealing and natural. The exploration of novel contractions, which assures that a given closed figure is a fixed figure, may be considered as an imminent nonlinear problem having importance in the application as well as the theory of real-life problems through nonlinear systems. Numerous examples exist in the literature wherein the collection of fixed points incorporates a particular symmetrical geometric figure. In other words, a self map fixes a particular symmetrical geometrical figure. Nevertheless, it may not be true at all times. There may exist a self map that maps a particular symmetrical geometric diagram to itself but may not fix all of the points of that diagram. If interested in the symmetrical geometry of fixed points, we may refer [4–15] and so on.

In the present work, we examine novel hypotheses to find the symmetrical geometry of the fixed points by introducing some novel contractions and establishing the presence of fixed and common fixed points in complete partial metric spaces wherein the self distance is not necessarily zero. Furthermore, we announce a concept of a fixed n-ellipse-like curve and a common fixed n-ellipse-like curve, which are symmetrical in shape but entirely different than that of the ellipse in a Euclidean space. Next, we verify the established conclusions by nontrivial illustrative examples. These fixed n-ellipse-like curve conclusions encourage more applications and explorations in partial metric spaces. Furthermore, we utilize our conclusions to model and solve an initial value problem appearing in Production–Consumption Equilibrium. It is well-known that the consumption and production of material goods are related to each other, and not one or the other exists without the other. Consumption is the method of utilizing goods or services by deriving utility from them and subsequently fulfilling our needs through production which is an action embraced, where raw materials are converted into a finished good with the utilization of components of production such as land, labor and so on.

2. Preliminaries

We start with the discussion of partial metrics and the convergence which we will utilize in the subsequent sections.

Definition 1 ([2])**.** *A function* $\rho : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$ *is a partial metric if*
(ρ_1) $\rho(\varrho_1, \varrho_1) = \rho(\varrho_2, \varrho_2) = \rho(\varrho_1, \varrho_2)$ *iff* $\varrho_1 = \varrho_2$*;*
(ρ_2) $\rho(\varrho_1, \varrho_1) \leq \rho(\varrho_1, \varrho_2)$*;*
(ρ_3) $\rho(\varrho_1, \varrho_2) = \rho(\varrho_2, \varrho_1)$*;*
(ρ_4) $\rho(\varrho_1, \varrho_2) \leq \rho(\varrho_1, \varrho_3) + \rho(\varrho_3, \varrho_2) - \rho(\varrho_3, \varrho_3), \varrho_1, \varrho_2, \varrho_3 \in \mathcal{U}$*.*

A partial metric is a striving to generalize the metric by removing the hypothesis $d(\varrho_1, \varrho_1) = 0$ and adding the hypothesis $d(\varrho_1, \varrho_1) \leq d(\varrho_1, \varrho_2)$.

Example 1 ([2])**.** *A function* $\rho : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$ *is a partial metric if*

(i) $\mathcal{U} = \mathbb{R}^+$ *and* $\rho(\varrho_1, \varrho_2) = \max\{\varrho_1, \varrho_2\}$, $\varrho_1, \varrho_2 \in \mathcal{U}$*.*
(ii) $\mathcal{U} = \{[\varrho_1, \varrho_2] : \varrho_1, \varrho_2 \in \mathbb{R}, \varrho_1 \leq \varrho_2\}$ *and* $\rho([\varrho_1, \varrho_2], [\varrho_3, \varrho_4]) = \max\{\varrho_2, \varrho_4\} - \max\{\varrho_1, \varrho_3\}$*.*

Definition 2 ([2])**.** *Consider a sequence* $\{\varrho_n\}$ *in a partial metric space* $(\mathcal{U}, \rho)$*. Then,*

1. $\{\varrho_n\} \subseteq \mathcal{U}$ *is Cauchy if* $\lim_{n,m\to\infty} \rho(\varrho_n, \varrho_m)$ *exists and is finite;*
2. $\{\varrho_n\} \subseteq \mathcal{U}$ *converges to a point u iff* $\lim_{n\to\infty} \rho(\varrho_n, \varrho_1) = \rho(\varrho_1, \varrho_1)$*;*
3. $(\mathcal{U}, \rho)$ *is complete if each Cauchy sequence* $\{\varrho_n\}$ *converges to* $\varrho_1 \in \mathcal{U}$*, that is,* $\lim_{n\to\infty} \rho(\varrho_n, \varrho_1) = \lim_{n,m\to\infty} \rho(\varrho_n, \varrho_m) = \rho(\varrho_1, \varrho_1)$*.*

3. Main Results

We explore here the fixed point/common fixed point for some novel contractions and examine their symmetrical geometry by introducing notions of a fixed n-ellipse-like curve and a common fixed n-ellipse-like curve and framing novel hypotheses in a partial metric space, wherein a point's distance from itself is not necessarily zero.

We symbolize partial metric by ρ and partial metric space by $(\mathcal{U}, \rho)$.

Definition 3. *If κ_k are n collinear points in $\mathcal{U}$ and $a \in [0, \infty)$ such that $\sigma^n_{k=1}\rho(\kappa_k, \kappa_k) + \sigma^n_{k\neq j=1}$ $\rho(\kappa_k, \kappa_j) < a$, then n-ellipse-like curve $\mathcal{E}_\rho(\kappa_1, \kappa_2, \ldots, \kappa_n; a)$, which has foci at κ_k in $(\mathcal{U}, \rho)$, is a collection of points satisfying $\sigma^n_{k=1}\rho(\varrho_1, \kappa_k) = a$, $\varrho_1 \in \mathcal{U}$ and $k = 1, 2, \ldots, n$, that is, an n-ellipse-like curve is the locus of points, and the sum of whose distances to these n foci κ_k is a constant a.*

The midpoint C of the line joining $n-$foci is known as the center of the n-ellipse-like curve. The line that passes through $n-$foci is the principal axis and a line perpendicular to the principal axis passing through the center is the orthogonal principal axis.

Definition 4. *If κ_1 and κ_2 are any two points in $\mathcal{U}$ and $a \in [0, \infty)$ such that $\rho(\kappa_1, \kappa_1) + \rho(\kappa_2, \kappa_2) + \rho(\kappa_1, \kappa_2) < a$, then a 2-ellipse-like curve $\mathcal{E}_\rho(\kappa_1, \kappa_2; a)$, which has foci at κ_1 and κ_2 in $(\mathcal{U}, \rho)$, is a collection of points satisfying $\rho(\varrho_1, \kappa_1) + \rho(\varrho_1, \kappa_2) = a$, $\varrho_1 \in \mathcal{U}$, that is, a 2-ellipse is the locus of points, the sum of whose distances to the foci κ_1 and κ_2 is constant a.*

Definition 5. *Let $\mathcal{E}_\rho(\kappa_1, \kappa_2, \ldots, \kappa_n; a)$ be an n-ellipse-like curve having foci at $\kappa_i, i = 1, 2, \ldots, n$, in $(\mathcal{U},\rho)$. Then, $\mathcal{E}_\rho(\kappa_1, \kappa_2, \ldots, \kappa_n; a)$ is a fixed n-ellipse-like curve of $\mathcal{M} : \mathcal{U} \to \mathcal{U}$ if $\mathcal{M}\varrho_1 = \varrho_1, \varrho_1 \in \mathcal{E}_\rho(\kappa_1, \kappa_2, \ldots, \kappa_n; a), a \in [0, \infty)$.*

In particular, $\mathcal{E}_\rho(\kappa_1, \kappa_2; a)$ is a fixed 2-ellipse-like curve of $\mathcal{M} : \mathcal{U} \to \mathcal{U}$ if $\mathcal{M}\varrho_1 = \varrho_1, \varrho_1 \in \mathcal{E}_\rho(\kappa_1, \kappa_2; a), a \in [0, \infty)$.

Example 2. *Let $\mathcal{U} = \mathbb{R}^+$ be equipped with $\rho : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$, which is described as $\rho(\varrho_1, \varrho_2) = \max\{\varrho_1, \varrho_2\}, \varrho_1, \varrho_2 \in \mathcal{U}$; then, a 2-ellipse-like curve*

$$\begin{aligned}\mathcal{E}_\rho(3, 4; 13) &= \{\varrho_1 \in \mathcal{U} : \rho(3, \varrho_1) + \rho(4, \varrho_1) = 13\}\\ &= \{\varrho_1 \in \mathcal{U} : \max\{3, \varrho_1\} + \max\{4, \varrho_1\} = 13\}\\ &= \{6.5\},\end{aligned}$$

that is, a 2-ellipse-like curve centered at 3.5 having foci at 3 and 4, and $a = 13$ is $\{6.5\}$.

Example 3. *Let $\mathcal{A}$ = first quadrant of $\mathbb{R}^2$ and $\rho : \mathcal{U} \times \mathcal{U} \to \mathcal{R}^+$ be described as $\rho(\varrho_1, \varrho_2) = \max\{|\varrho_1|, |\varrho_2|\}$, $\varrho_1 = (\varrho_1, \varrho_2)$, $\varrho_2 = (\varrho_1^2, \varrho_2^2)$, $|\varrho_1| = \sqrt{\varrho_1^2 + \varrho_2^2}$, $|\varrho_2| = \sqrt{(\varrho_1^2)^2 + (\varrho_2^2)^2}$, and $\varrho_1, \varrho_2 \in \mathcal{U}$, then a 2-ellipse-like curve*

$$\begin{aligned}\mathcal{E}_\rho(\kappa_1, \kappa_2; a) &= \{\varrho_1 \in \mathcal{U} : \rho(\kappa_1, \varrho_1) + \rho(\kappa_2, \varrho_1) = a\}\\ &= \Big\{\varrho_1 \in \mathcal{U} : \max\{\sqrt{\frac{1}{4} + \frac{1}{4}}, \sqrt{\varrho_1^2 + \varrho_2^2}\} + \max\Big\{\sqrt{\frac{1}{4} + 4}, \sqrt{\varrho_1^2 + \varrho_2^2}\} = 6\Big\},\end{aligned} \tag{1}$$

that is, a 2-ellipse-like curve centered at $(\frac{1}{2}, \frac{5}{4})$ having foci at $\kappa_1 = (\frac{1}{2}, \frac{1}{2})$, $\kappa_2 = (\frac{1}{2}, 2)$, and $a = 6$ is given by Equation (1).

Remark 1. *A Scottish mathematician and scientist, James Clerk Maxwell [16], was the first one to study multifocal ellipse. Following Maxwell [16] and, Erdös and Vincze [7], we conclude that an n-ellipse-like curve is the generalization of the 2-ellipse-like curve, which allows more than two foci and is also known as a multifocal ellipse. The n-ellipse-like curve with one focus is the circle (1- ellipse-like curve), and with two foci is the 2-ellipse-like curve. Noticeably, all of these shapes are symmetrical.*

We denote the family of monotone increasing as well as continuous functions $\psi : [0, \infty) \to [0, \infty)$ satisfying $\psi(0) = 0$ by ψ and the set of lower semi-continuous functions $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(0) = 0$ by ϕ.

Definition 6. *Let $\psi \in \psi$, $\phi \in \phi$. A self map $\mathcal{M} : \mathcal{U} \to \mathcal{U}$ of a partial metric space $(\mathcal{U}, \rho)$ is known as a partial $(\psi, \phi)-$contraction if*

$$\psi(\rho(\mathcal{M}\varrho_1, \mathcal{M}\varrho_2)) \leq \psi(M(\varrho_1, \varrho_2)) - \phi(N(\varrho_1, \varrho_2)),\ \varrho_1 \neq \varrho_2,\ \forall \varrho_1, \varrho_2 \in \mathcal{U}, \tag{2}$$

where

$$M(\varrho_1, \varrho_2) = \max\Big\{\rho(\varrho_1, \varrho_2), \rho(\varrho_1, \mathcal{M}\varrho_1), \rho(\varrho_2, \mathcal{M}\varrho_2), \frac{1}{2}(\rho(\varrho_1, \mathcal{M}\varrho_1) + \rho(\varrho_2, \mathcal{M}\varrho_2)), \frac{1}{2}(\rho(\varrho_1, \mathcal{M}\varrho_2) + \rho(\varrho_2, \mathcal{M}\varrho_1))\Big\};$$

$$N(\varrho_1, \varrho_2) = \max\Big\{\rho(\varrho_1, \varrho_2), \rho(\varrho_1, \mathcal{M}\varrho_1), \rho(\varrho_2, \mathcal{M}\varrho_2), \frac{\rho(\varrho_1, \mathcal{M}\varrho_1)(\rho(\varrho_1, \mathcal{M}\varrho_2) + \rho(\varrho_2, \mathcal{M}\varrho_1) - \rho(\varrho_2, \mathcal{M}\varrho_2))}{1 + \rho(\varrho_1, \varrho_2)}\Big\}.$$

Definition 7. *Let $\psi \in \psi$, $\phi \in \phi$. A self map $\mathcal{M} : \mathcal{U} \to \mathcal{U}$ of $(\mathcal{U}, \rho)$ is known as a partial $(\psi, \phi)_{\mathcal{E}_\rho}-$contraction if*

$$\psi(\rho(\varrho_1, \mathcal{M}\varrho_1)) \leq \psi(M'(\varrho_1, \varrho_2, \varrho_3)) - \phi(N'(\varrho_1, \varrho_2, \varrho_3)),\ \varrho_1 \neq \varrho_2 \neq \varrho_3,\ \varrho_1 \neq \mathcal{M}\varrho_1,\ \forall \varrho_1, \varrho_2, \varrho_3 \in \mathcal{U}, \tag{3}$$

where

$$M'(\varrho_1, \varrho_2, \varrho_3) = \max\{\rho(\varrho_1, \varrho_2) + \rho(\varrho_1, \varrho_3), \rho(\varrho_1, \mathcal{M}\varrho_1), \rho(\varrho_2, \mathcal{M}\varrho_2), \frac{1}{3}(\rho(\varrho_1, \mathcal{M}\varrho_1) + \rho(\varrho_1, \mathcal{M}\varrho_2) + \rho(\varrho_1, \mathcal{M}\varrho_3) + \rho(\varrho_2, \mathcal{M}\varrho_2) + \rho(\varrho_3, \mathcal{M}\varrho_3) + \rho(\varrho_2, \varrho_3)), \rho(\varrho_1, \mathcal{M}\varrho_2) + \rho(\varrho_1, \mathcal{M}\varrho_3)\};$$

$$N'(\varrho_1, \varrho_2, \varrho_3) = \max\Big\{\rho(\varrho_1, \varrho_2) + \rho(\varrho_1, \varrho_3), \rho(\varrho_1, \mathcal{M}\varrho_1), \rho(\varrho_2, \mathcal{M}\varrho_2), \rho(\varrho_3, \mathcal{M}\varrho_3), \frac{\rho(\varrho_1, \mathcal{M}\varrho_1)(\rho(\varrho_1, \mathcal{M}\varrho_2) - \rho(\varrho_2, \mathcal{M}\varrho_2))}{1 + \rho(\varrho_1, \mathcal{M}\varrho_2)}, \frac{\rho(\varrho_1, \mathcal{M}\varrho_1)(\rho(\varrho_1, \mathcal{M}\varrho_2) - \rho(\varrho_3, \mathcal{M}\varrho_3))}{1 + \rho(\varrho_1, \mathcal{M}\varrho_3)}\Big\}.$$

Definition 8. *Let $\psi \in \psi$, $\phi \in \phi$. The self maps $\mathcal{M}, \mathcal{N} : \mathcal{U} \to \mathcal{U}$ of $(\mathcal{U}, \rho)$ are a partial $(\psi, \phi)-$contraction for a pair of maps if*

$$\psi(\rho(\mathcal{M}\varrho_1, \mathcal{N}\varrho_2)) \leq \psi(M''(\varrho_1, \varrho_2)) - \phi(N''(\varrho_1, \varrho_2)),\ \varrho_1 \neq \varrho_2,\ \forall \varrho_1, \varrho_2 \in \mathcal{U}, \tag{4}$$

where

$$M''(\varrho_1, \varrho_2) = \max\{\rho(\varrho_1, \varrho_2), \rho(\varrho_1, \mathcal{M}\varrho_1), \rho(\varrho_2, \mathcal{N}\varrho_2), \frac{1}{2}(\rho(\varrho_1, \mathcal{M}\varrho_1) + \rho(\varrho_2, \mathcal{N}\varrho_2))\};$$

$$N''(\varrho_1, \varrho_2) = \max\Big\{\rho(\varrho_1, \varrho_2), \rho(\varrho_1, \mathcal{M}\varrho_1), \rho(\varrho_2, \mathcal{N}\varrho_2), \frac{\rho(\varrho_1, \mathcal{M}\varrho_1)(\rho(\varrho_1, \mathcal{N}\varrho_2) + \rho(\varrho_2, \mathcal{M}\varrho_1) - \rho(\varrho_2, \mathcal{N}\varrho_2))}{1 + \rho(\varrho_1, \varrho_2)}\Big\}.$$

Definition 9. *Let $\psi \in \psi$, $\phi \in \phi$. The self maps $\mathcal{M}$, $\mathcal{N} : \mathcal{U} \to \mathcal{U}$ of $(\mathcal{U}, \rho)$ are a partial $(\psi, \phi)_{\mathcal{E}_\subset}-$contraction for a pair of maps if*

$$\psi(\rho(\mathcal{M}\varrho_1, \mathcal{N}\varrho_1)) \leq \psi(M'''(\varrho_1, \varrho_2, \varrho_3)) - \phi(N'''(\varrho_1, \varrho_2, \varrho_3)),\ \varrho_1 \neq v \neq \varrho_3,\ \varrho_1 \neq \mathcal{M}\varrho_1 \neq \mathcal{N}\varrho_1,\ \forall \varrho_1, \varrho_2, \varrho_3 \in \mathcal{U}, \tag{5}$$

where

$$M'''(\varrho_1, \varrho_2, \varrho_3) = \max\{\rho(\varrho_1, \varrho_2) + \rho(\varrho_1, \varrho_3), \rho(\varrho_1, \mathcal{M}\varrho_1), \rho(\varrho_2, \mathcal{N}\varrho_2), \frac{1}{3}(\rho(\varrho_1, \mathcal{M}\varrho_1) + \rho(\varrho_1, \mathcal{N}\varrho_2) + \rho(\varrho_1, \mathcal{N}\varrho_3) + \rho(\varrho_2, \mathcal{N}\varrho_2) + \rho(\varrho_3, \mathcal{N}\varrho_3) + \rho(\varrho_2, \varrho_3)), \rho(\varrho_1, \mathcal{M}\varrho_2) + \rho(\varrho_1, \mathcal{M}\varrho_3)\};$$

$$N'''(\varrho_1, \varrho_2, \varrho_3) = \max\Big\{\rho(\varrho_1, \varrho_2) + \rho(\varrho_1, \varrho_3), \rho(\varrho_1, \mathcal{M}\varrho_1), \rho(\varrho_2, \mathcal{N}\varrho_2), \rho(\varrho_3, \mathcal{N}\varrho_3), \frac{\rho(\varrho_1, \mathcal{M}\varrho_1)(\rho(\varrho_1, \mathcal{N}\varrho_2) - \rho(\varrho_2, \mathcal{M}\varrho_2))}{1 + \rho(\varrho_1, \mathcal{N}\varrho_2)}, \frac{\rho(\varrho_1, \mathcal{M}\varrho_1)(\rho(\varrho_1, \mathcal{N}\varrho_3) - \rho(\varrho_3, \mathcal{M}\varrho_3))}{1 + \rho(\varrho_1, \mathcal{N}\varrho_3)}\Big\}.$$

Theorem 1. *Let $\mathcal{M} : \mathcal{U} \to \mathcal{U}$ be a partial $(\psi, \phi)-$contraction of a complete partial metric space $(\mathcal{U}, \rho)$. Then, $\mathcal{M}$ has exactly one fixed point, and the sequence of iterates $\{\varrho_n\}$ converges to a unique fixed point in $\mathcal{U}$.*

Proof. Let the initial point $\varrho_0 \in \mathcal{U}$ and the Picard sequence $\{\varrho_n\} \subseteq \mathcal{U}$ be $\varrho_{n+1} = \mathcal{M}\varrho_n$, $n \in \mathbb{N}_0$. If $\varrho_{n+1} = \varrho_n, n \in \mathbb{N}_0$, then ϱ_n is a fixed point of $\mathcal{M}$. Hence the conclusion. Let $\varrho_n \neq \varrho_{n+1}$, $n \in \mathbb{N}_0$. Substituting $\varrho_1 = \varrho_n$ and $\varrho_2 = \varrho_{n+1}$ into inequality (2),

$$\psi(\rho(\mathcal{M}\varrho_n, \mathcal{M}\varrho_{n+1})) \leq \psi(M(\varrho_n, \varrho_{n+1})) - \phi(N(\varrho_n, \varrho_{n+1})),$$

where

$$\begin{aligned} M(\varrho_n, \varrho_{n+1}) = \max\{&\rho(\varrho_n, \varrho_{n+1}), \rho(\varrho_n, \mathcal{M}\varrho_n), \rho(\varrho_{n+1}, \mathcal{M}\varrho_{n+1}), \\ &\frac{1}{2}(\rho(\varrho_n, \mathcal{M}\varrho_n) + \rho(\varrho_{n+1}, \mathcal{M}\varrho_{n+1})), \\ &\frac{1}{2}(\rho(\varrho_n, \mathcal{M}\varrho_{n+1}) + \rho(\varrho_{n+1}, \mathcal{M}\varrho_n))\} \\ &= \max\{\rho(\varrho_n, \varrho_{n+1}), \rho(\varrho_n, \varrho_{n+1}), \rho(\varrho_{n+1}, \varrho_{n+2}), \\ &\frac{1}{2}(\rho(\varrho_n, \varrho_{n+1}) + \rho(\varrho_{n+1}, \varrho_{n+2})), \\ &\frac{1}{2}(\rho(\varrho_n, \varrho_{n+2}) + \rho(\varrho_{n+1}, \varrho_{n+1}))\}. \end{aligned}$$

Since,

$$\begin{aligned} &\rho(\varrho_n, \varrho_{n+2}) \leq \rho(\varrho_n, \varrho_{n+1}) + \rho(\varrho_{n+1}, \varrho_{n+2}) - \rho(\varrho_{n+1}, \varrho_{n+1}), \\ &\text{that is, } \rho(\varrho_n, \varrho_{n+2}) + \rho(\varrho_{n+1}, \varrho_{n+1}) \leq \rho(\varrho_n, \varrho_{n+1}) + \rho(\varrho_{n+1}, \varrho_{n+2}), \\ &\text{that is, } \frac{1}{2}(\rho(\varrho_n, \varrho_{n+2}) + \rho(\varrho_{n+1}, \varrho_{n+1})) \leq \frac{1}{2}(\rho(\varrho_n, \varrho_{n+1}) + \rho(\varrho_{n+1}, \varrho_{n+2})) \\ &\qquad\qquad \leq \max\{\rho(\varrho_n, \varrho_{n+1}), \rho(\varrho_{n+1}, \varrho_{n+2})\}. \end{aligned} \tag{6}$$

Now, $M(\varrho_n, \varrho_{n+1}) \leq \max\{\rho(\varrho_n, \varrho_{n+1}), \rho(\varrho_{n+1}, \varrho_{n+2})\}$

$$\begin{aligned} N(\varrho_n, \varrho_{n+1}) &= \max\Big\{\rho(\varrho_n, \varrho_{n+1}), \rho(\varrho_n, \mathcal{M}\varrho_n), \rho(\varrho_{n+1}, \mathcal{M}\varrho_{n+1}), \\ &\frac{\rho(\varrho_n, \mathcal{M}\varrho_n)(\rho(\varrho_n, \mathcal{M}\varrho_{n+1}) - \rho(\varrho_{n+1}, \mathcal{M}\varrho_{n+1}) - \rho(\varrho_{n+1}, \mathcal{M}\varrho_n))}{1 + \rho(\varrho_n, \varrho_{n+1})}\Big\} \\ &= \max\Big\{\rho(\varrho_n, \varrho_{n+1}), \rho(\varrho_n, \varrho_{n+1}), \rho(\varrho_{n+1}, \varrho_{n+2}), \\ &\frac{\rho(\varrho_n, \varrho_{n+1})(\rho(\varrho_n, \varrho_{n+2}) - \rho(\varrho_{n+1}, \varrho_{n+2}) - \rho(\varrho_{n+1}, \varrho_{n+1}))}{1 + \rho(\varrho_n, \varrho_{n+1})}\Big\} \\ N(\varrho_n, \varrho_{n+1}) &\leq \max\{\rho(\varrho_n, \varrho_{n+1}), \rho(\varrho_{n+1}, \varrho_{n+2})\}. \end{aligned} \tag{7}$$

Hence,

$$\begin{aligned} \psi(\rho(\varrho_{n+1}, \varrho_{n+2})) \leq\ &\psi(\max\{\rho(\varrho_n, \varrho_{n+1}), \rho(\varrho_{n+1}, \varrho_{n+2})\})- \\ &\phi(\max\{\rho(\varrho_n, \varrho_{n+1}), \rho(\varrho_{n+1}, \varrho_{n+2})\}). \end{aligned} \tag{8}$$

When $\rho(\varrho_n, \varrho_{n+1}) \leq \rho(\varrho_{n+1}, \varrho_{n+2})$, then

$$\begin{aligned} \psi(\rho(\varrho_{n+1}, \varrho_{n+2})) &\leq \psi(\rho(\varrho_{n+1}, \varrho_{n+2})) - \phi(\rho(\varrho_{n+1}, \varrho_{n+2})) \\ &< \psi(\rho(\varrho_{n+1}, \varrho_{n+2})), \end{aligned} \tag{9}$$

a contradiction. Hence, $\rho(\varrho_{n+1}, \varrho_{n+2}) \leq \rho(\varrho_n, \varrho_{n+1})$, that is, $\{\rho(\varrho_n, \varrho_{n+1})\}$ is the sequence of positive real numbers, which is decreasing.

Thus, $\lim_{n\to\infty} \rho(\varrho_n, \varrho_{n+1}) = l \ \ (l \geq 0)$.

Now, taking $n \to \infty$ in inequality (8),

$$\begin{aligned} \psi(l) &\leq \psi(l) - \lim_{n\to\infty} \inf \phi(\rho(\varrho_n, \varrho_{n+1})) \\ &\leq \psi(l) - \phi(l) \\ &< \psi(l), \end{aligned} \tag{10}$$

a contradiction, i.e., $\lim_{n\to\infty} \rho(\rho_n, \varrho_{n+1}) = 0$.

Since

$$d_\rho(\varrho_n, \varrho_{n+1}) = 2\rho(\varrho_n, \varrho_{n+1}) - \rho(\varrho_n, \varrho_n) - \rho(\varrho_{n+1}, \varrho_{n+1})$$
$$d_\rho(\varrho_n, \varrho_{n+1}) \leq 2\rho(\varrho_n, \varrho_{n+1}) \to 0.$$

Now, we demonstrate that $\lim_{n,m\to\infty} \rho(\varrho_n, \varrho_m) = 0$, using a contradiction method.

Suppose $\lim_{n,m\to\infty} \rho(\varrho_n, \varrho_m) \neq 0$. Then, there exist two sub-sequences $\{\varrho_{n_k}\}$ and $\{\varrho_{m_k}\}$ and an $\epsilon > 0$ for which

$$k < m_k < n_k, \;\; \rho(\varrho_{m_k}, \varrho_{n_k}) > \epsilon, \tag{11}$$

that is,

$$\rho(\varrho_{m_k}, \varrho_{n_k-1}) < \epsilon. \tag{12}$$

Now, using inequalities (11) and (12)

$$\begin{aligned} \epsilon < \rho(\varrho_{m_k}, \varrho_{n_k}) &\leq \rho(\varrho_{m_k}, \varrho_{n_k-1}) + \rho(\varrho_{n_k-1}, \varrho_{n_k}) - \rho(\varrho_{n_k-1}, \varrho_{n_k-1}) \\ &< \epsilon + \rho(\varrho_{n_k-1}, \varrho_{n_k}) \\ &< \epsilon, \text{ as } k \to \infty, \end{aligned}$$

i.e., $\lim_{k\to\infty} \rho(\varrho_{m_k}, \varrho_{n_k}) = \epsilon$.

Now,

$$\begin{aligned} \rho(\varrho_{m_k-1}, \varrho_{n_k-1}) &\leq \rho(\varrho_{m_k-1}, \varrho_{n_k}) + \rho(\varrho_{n_k}, \varrho_{n_k-1}) - \rho(\varrho_{n_k}, \varrho_{n_k}) \\ &\leq \rho(\varrho_{m_k-1}, \varrho_{n_k}) + \rho(\varrho_{n_k}, \varrho_{n_k-1}) \\ &\leq \rho(\varrho_{m_k-1}, \varrho_{m_k}) + \rho(\varrho_{m_k}, \varrho_{n_k}) - \rho(\varrho_{m_k}, \varrho_{m_k}) + \rho(\varrho_{n_k}, \varrho_{n_k-1}) \\ &\leq \rho(\varrho_{m_k-1}, \varrho_{m_k}) + \rho(\varrho_{m_k}, \varrho_{n_k}) + \rho(\varrho_{n_k}, \varrho_{n_k-1}) \\ &\to \epsilon, \text{ as } k \to \infty, \end{aligned}$$

i.e., $\lim_{k\to\infty} \rho(\varrho_{m_k-1}, \varrho_{n_k-1}) = \epsilon$.

Now, from inequality (2)

$$\begin{aligned} \psi(\rho(\varrho_{m_k}, \varrho_{n_k})) &= \psi(\rho(\mathcal{M}\varrho_{m_k-1}, \mathcal{M}\varrho_{n_k-1})) \\ &\leq \psi(M(\varrho_{m_k-1}, \varrho_{n_k-1})) - \phi(N(\varrho_{m_k-1}, \varrho_{n_k-1})), \end{aligned} \tag{13}$$

where

$$\begin{aligned} M(\varrho_{m_k-1}, \varrho_{n_k-1}) = &\max\{\rho(\varrho_{m_k-1}, \varrho_{n_k-1}), \rho(\varrho_{m_k-1}, \mathcal{M}\varrho_{m_k-1}), \rho(\varrho_{n_k-1}, \mathcal{M}\varrho_{n_k-1}), \\ &\frac{1}{2}(\rho(\varrho_{m_k-1}, \mathcal{M}\varrho_{m_k-1}) + \rho(\varrho_{n_k-1}, \mathcal{M}\varrho_{n_k-1})), \\ &\frac{1}{2}(\rho(\varrho_{m_k-1}, \mathcal{M}\varrho_{n_k-1}) + \rho(\varrho_{n_k-1}, \mathcal{M}\varrho_{m_k-1}))\} \\ = &\max\{\rho(\varrho_{m_k-1}, \varrho_{n_k-1}), \rho(\varrho_{m_k-1}, \varrho_{m_k}), \rho(\varrho_{n_k-1}, \varrho_{n_k}), \\ &\frac{1}{2}(\rho(\varrho_{m_k-1}, \varrho_{m_k}) + \rho(\varrho_{n_k-1}, \varrho_{n_k})), \\ &\frac{1}{2}(\rho(\varrho_{m_k-1}, \varrho_{n_k}) + \rho(\varrho_{n_k-1}, \varrho_{m_k}))\}. \end{aligned} \tag{14}$$

$$\begin{aligned}\frac{1}{2}(\rho(\varrho_{m_k-1},\varrho_{n_k})+\rho(\varrho_{n_k-1},\varrho_{m_k})) \leq& \frac{1}{2}(\rho(\varrho_{m_k-1},\varrho_{m_k})+\rho(\varrho_{m_k},\varrho_{n_k})-\rho(\varrho_{m_k},\varrho_{m_k})\\ &+\rho(\varrho_{n_k-1},\varrho_{n_k})+\rho(\varrho_{n_k},\varrho_{m_k})-\rho(\varrho_{n_k},\varrho_{n_k}))\\ \leq& \frac{1}{2}(\rho(\varrho_{m_k-1},\varrho_{m_k})+\rho(\varrho_{m_k},\varrho_{n_k})\\ &+\rho(\varrho_{n_k-1},\varrho_{n_k})+\rho(\varrho_{n_k},\varrho_{m_k})).\end{aligned} \tag{15}$$

Using inequality (15) and letting $k \to \infty$ in inequality (14),

$$\lim_{k\to\infty} M(\varrho_{m_k-1},\varrho_{n_k-1}) = \epsilon. \tag{16}$$

$$\begin{aligned}N(\varrho_{m_k-1},\varrho_{n_k-1}) =& \max\Big\{\rho(\varrho_{m_k-1},\varrho_{n_k-1}),\rho(\varrho_{m_k-1},\mathcal{M}\varrho_{m_k-1}),\rho(\varrho_{n_k-1},\mathcal{M}\varrho_{n_k-1}),\\ &\frac{\rho(\varrho_{m_k-1},\mathcal{M}\varrho_{m_k-1})\rho(\varrho_{m_k-1},\mathcal{M}\varrho_{n_k-1})}{1+\rho(\varrho_{m_k-1},\mathcal{M}\varrho_{n_k-1})},\\ &\frac{\rho(\varrho_{n_k-1},\mathcal{M}\varrho_{n_k-1})\rho(\varrho_{n_k-1},\mathcal{M}\varrho_{m_k-1})}{1+\rho(\varrho_{n_k-1},\mathcal{M}\varrho_{m_k-1})}\Big\}\\ =& \max\Big\{\rho(\varrho_{m_k-1},\varrho_{n_k-1}),\rho(\varrho_{m_k-1},\varrho_{m_k}),\rho(\varrho_{n_k-1},\varrho_{n_k}),\\ &\frac{\rho(\varrho_{m_k-1},\varrho_{m_k})\rho(\varrho_{m_k-1},\varrho_{n_k})}{1+\rho(\varrho_{m_k-1},\varrho_{n_k})},\frac{\rho(\varrho_{n_k-1},\varrho_{n_k})\rho(\varrho_{n_k-1},\varrho_{m_k})}{1+\rho(\varrho_{n_k-1},\varrho_{m_k})}\Big\}\\ &\to \epsilon, \text{ as } k\to\infty,\end{aligned} \tag{17}$$

that is,

$$\lim_{k\to\infty} N(\varrho_{m_k-1},\varrho_{n_k-1}) = \epsilon. \tag{18}$$

Using inequalities (16) and (18) in inequality (13) and taking $k \to \infty$

$$\lim_{k\to\infty} \psi(\rho(\varrho_{m_k},\varrho_{n_k})) \leq \psi(\epsilon)-\phi(\epsilon)$$
$$\psi(\epsilon) < \psi(\epsilon),$$

a contradiction. Thus, $\lim_{m,n\to\infty}\rho(\varrho_m,\varrho_n)=0$, and sequence $\{\varrho_n\}$ is a Cauchy sequence in a complete partial metric space $(\mathcal{U},\rho)$. As a result, a point $\varrho_1 \in \mathcal{U}$ which satisfies

$$\lim_{n,m\to\infty}\rho(\varrho_m,\varrho_n) = 0 = \lim_{n\to\infty}\rho(\varrho_n,\varrho_1) = \rho(\varrho_1,\varrho_1).$$

Suppose $\varrho_1 \neq \mathcal{M}\varrho_1$; then,

$$\begin{aligned}\psi(\rho(\mathcal{M}\varrho_1,\varrho_n)) &= \psi(\rho(\mathcal{M}\varrho_1,\mathcal{M}\varrho_{n-1}))\\ &\leq \psi(M(\varrho_1,\varrho_{n-1}))-\phi(N(\varrho_1,\varrho_{n-1})),\end{aligned} \tag{19}$$

where

$$\begin{aligned}M(\varrho_1,\varrho_{n-1}) =& \max\{\rho(\varrho_1,\varrho_{n-1}),\rho(\varrho_1,\mathcal{M}\varrho_1),\rho(\varrho_{n-1},\mathcal{M}\varrho_{n-1}),\\ &\frac{1}{2}(\rho(\varrho_1,\mathcal{M}\varrho_1)+\rho(\varrho_{n-1},\mathcal{M}\varrho_{n-1})),\frac{1}{2}(\rho(\varrho_1,\mathcal{M}\varrho_{n-1})+\rho(\varrho_{n-1},\mathcal{M}\varrho_1))\}\\ =& \max\{\rho(\varrho_1,\varrho_{n-1}),\rho(\varrho_1,\mathcal{M}\varrho_1),\rho(\varrho_{n-1},\varrho_n),\\ &\frac{1}{2}(\rho(\varrho_1,\mathcal{M}\varrho_1)+\rho(\varrho_{n-1},\varrho_n)),\frac{1}{2}(\rho(\varrho_1,\varrho_n)+\rho(\varrho_{n-1},\mathcal{M}\varrho_1))\}\\ &\to \max\{\rho(\varrho_1,\varrho_1),\rho(\varrho_1,\mathcal{M}\varrho_1)\}, \text{ as } n\to\infty,\\ &= \rho(\varrho_1,\mathcal{M}\varrho_1).\end{aligned} \tag{20}$$

$$\begin{aligned}
N(\varrho_1, \varrho_{n-1}) = \max\Big\{&\rho(\varrho_1, \varrho_{n-1}), \rho(\varrho_1, \mathcal{M}\varrho_1), \rho(\varrho_{n-1}, \mathcal{M}\varrho_{n-1}),\\
&\frac{\rho(\varrho_1, \mathcal{M}\varrho_1)\rho(\varrho_1, \mathcal{M}\varrho_{n-1})}{1+\rho(\varrho_1, \mathcal{M}\varrho_{n-1})}, \frac{\rho(\varrho_{n-1}, \mathcal{M}\varrho_{n-1})\rho(\varrho_{n-1}, \mathcal{M}\varrho_1)}{1+\rho(\varrho_{n-1}, \mathcal{M}\varrho_1)}\Big\}\\
&\to \max\{\rho(\varrho_1, \varrho_1), \rho(\varrho_1, \mathcal{M}\varrho_1)\}, \text{ as } n \to \infty,\\
&= \rho(\varrho_1, \mathcal{M}\varrho_1).
\end{aligned} \tag{21}$$

Now, using properties of functions ψ and ϕ and inequalities (20) and (21) in inequality (19)

$$\begin{aligned}
\psi(\rho(\mathcal{M}\varrho_1, \varrho_1)) &\leq \psi(\rho(\varrho_1, \mathcal{M}\varrho_1) - \phi(\rho(\varrho_1, \mathcal{M}\varrho_1))\\
&< \psi(\rho(\varrho_1, \mathcal{M}\varrho_1)),
\end{aligned}$$

a contradiction, that is, $\mathcal{M}\varrho_1 = \varrho_1$.

Next, suppose $\mathcal{M}$ has more than one fixed point, and ϱ_2 is one more fixed point of $\mathcal{M}$, which is different from ϱ_1, that is, $\varrho_1 \neq \varrho_2$. Now, using inequality (2),

$$\begin{aligned}
\psi(\rho(\mathcal{M}\varrho_1, \mathcal{M}\varrho_2)) \leq \psi\Big(&\max\Big\{\rho(\varrho_1, \varrho_2), \rho(\varrho_1, \mathcal{M}\varrho_1), \rho(\varrho_2, \mathcal{M}\varrho_2), \frac{1}{2}(\rho(\varrho_1, \mathcal{M}\varrho_1) + \rho(\varrho_2, \mathcal{M}\varrho_2)),\\
&\frac{1}{2}(\rho(\varrho_1, \mathcal{M}\varrho_2) + \rho(\varrho_2, \mathcal{M}\varrho_1))\Big\}\Big) - \phi\Big(\max\Big\{\rho(\varrho_1, \varrho_2), \rho(\varrho_1, \mathcal{M}\varrho_1), \rho(\varrho_2, \mathcal{M}\varrho_2),\\
&\frac{\rho(\varrho_1, \mathcal{M}\varrho_1)\rho(\varrho_1, \mathcal{M}\varrho_2)}{1+\rho(\varrho_1, \mathcal{M}\varrho_2)}, \frac{\rho(\varrho_2, \mathcal{M}\varrho_2)\rho(\varrho_2, \mathcal{M}\varrho_1)}{1+\rho(\varrho_2, \mathcal{M}\varrho_1)}\Big\}\Big)\\
\psi(\rho(\varrho_1, \varrho_2)) \leq \psi\Big(&\max\Big\{\rho(\varrho_1, \varrho_2), \rho(\varrho_1, \varrho_1), \rho(\varrho_2, \varrho_2), \frac{1}{2}(\rho(\varrho_1, \varrho_1) + \rho(\varrho_2, \varrho_2)),\\
&\frac{1}{2}(\rho(\varrho_1, \varrho_2) + \rho(\varrho_2, \varrho_1))\Big\}\Big) - \phi\Big(\max\Big\{\rho(\varrho_1, \varrho_2), \rho(\varrho_1, \varrho_1), \rho(\varrho_2, \varrho_2),\\
&\frac{\rho(\varrho_1, \varrho_1)\rho(\varrho_1, \varrho_2)}{1+\rho(\varrho_1, \varrho_2)}, \frac{\rho(\varrho_2, \varrho_2)\rho(\varrho_2, \varrho_1)}{1+\rho(\varrho_2, \varrho_1)}\Big\}\Big)\\
\psi(\rho(\varrho_1, \varrho_2)) \leq{}& \psi(\rho(\varrho_1, \varrho_2)) - \phi(\rho(\varrho_1, \varrho_2))\\
\psi(\rho(\varrho_1, \varrho_2)) <{}& \psi(\rho(\varrho_1, \varrho_2)),
\end{aligned} \tag{22}$$

a contradiction, that is, $\varrho_1 = \varrho_2$. □

Example 4. *Let $\mathcal{U} = \mathbb{R}^+$ be equipped with partial metric $\rho : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$, which is described as $\rho(\varrho_1, \varrho_2) = \max\{\varrho_1, \varrho_2\}$. Clearly, $(\mathcal{U}, \rho)$ is complete.*

Let $\psi(\varrho_1) = \varrho_1^2$ and $\phi(\varrho_1) = \begin{cases} \frac{\varrho_1^2}{2} + 1, & \varrho_1 \neq 0 \\ 0, & \varrho_1 = 0 \end{cases}$.

Let a self-map $\mathcal{M} : \mathcal{U} \to \mathcal{U}$ be described as $\mathcal{M}\varrho_1 = \begin{cases} 0, & \varrho_1 \in [0, 10] \\ \frac{\varrho_1}{50}, & \text{otherwise} \end{cases}$, $\varrho_1 \in \mathcal{U}$.

Then, $\mathcal{M}$ validates the partial $(\psi, \phi)-$contraction 1. Consequently, $\mathcal{M}$ has exactly one fixed point 0, and the sequence $\{\frac{n}{n^2+1}\}$ converges to 0.

Example 5. *Let $\mathcal{U} = \{\varrho_n = 2n : n \in \mathbb{N} \cup \{0\}\}$ and a partial metric $\rho : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$ be described as $\rho(\varrho_1, \varrho_2) = \max\{\varrho_1, \varrho_2\}$. Clearly, $(\mathcal{U}, \rho)$ is complete.*

Let $\psi(\varrho_1) = \varrho_1^2 + \varrho_1$ and $\phi(\varrho_1) = \begin{cases} 0, & \varrho_1 = 0 \\ \frac{\varrho_1}{10} + 1, & \varrho_1 \neq 0 \end{cases}$, $\varrho_1 \in \mathcal{U}$.

Let a self-map $\mathcal{M} : \mathcal{U} \to \mathcal{U}$ be described as $\mathcal{M}\varrho_1 = \begin{cases} \varrho_0, & \varrho_1 = \varrho_0 \\ \frac{\varrho_n}{2}, & \varrho_1 = \varrho_n,\ n \geq 1 \end{cases}$.

Now, we assert that $\mathcal{M}$ validates partial $(\psi, \phi)-$contraction, that is,

$$\frac{\psi(\rho(\mathcal{M}\varrho_n, \mathcal{M}\varrho_m))}{\psi(M(\varrho_n, \varrho_m)) - \phi(N(\varrho_1, \varrho_2))} \leq 1.$$

Case (i) *When $n = 0$ and $m \geq 1$,*

$$\frac{\psi(\rho(\mathcal{M}\varrho_n, \mathcal{M}\varrho_m))}{\psi(M(\varrho_n, \varrho_m)) - \phi(N(\varrho_1, \varrho_2))} = \frac{\psi(m)}{\psi(2m) - \phi(2m)}$$
$$= \frac{5(m^2 + m)}{20m^2 + 9m - 5} \leq \frac{1}{4} < 1.$$

Case (ii) *When $n > m \geq 1$*

$$\frac{\psi(\rho(\mathcal{M}\varrho_n, \mathcal{M}\varrho_m))}{\psi(M(\varrho_n, \varrho_m)) - \phi(N(\varrho_1, \varrho_2))} = \frac{\psi(n)}{\psi(2n) - \phi(2n)}$$
$$= \frac{5(n^2 + n)}{20n^2 + 9n - 5} \leq \frac{1}{4} < 1.$$

Thus, $\mathcal{M}$ is a partial $(\psi, \phi)-$contraction and has exactly one fixed point 0, and the constant sequence $\{0\}$ converges to 0.

Theorem 2. *Let $(\mathcal{U}, \rho)$ be a complete partial metric space, and maps $\mathcal{M}$, $\mathcal{N} : \mathcal{U} \to \mathcal{U}$ are partial $(\psi, \phi)-$contractions. Then, $\mathcal{M}$ has a common fixed point satisfying $\mathcal{M}\varrho_1 = \mathcal{N}\varrho_1 = \varrho_1$.*

Proof. Let $\varrho_0 \in \mathcal{U}$. Let the sequence $\{\varrho_n\}$, $n \in \{0\} \cup \mathbb{N}$ be described as

$$\varrho_{2n+1} = \mathcal{M}\varrho_{2n}, \;\; \varrho_{2n+2} = \mathcal{N}\varrho_{2n+1}. \tag{23}$$

If $\varrho_{2n} = \varrho_{2n+1}$, then ϱ_{2n} is a fixed point of $\mathcal{M}$. Thus,

$$\begin{aligned}\rho(\varrho_{2n+1}, \varrho_{2n+2}) &= \rho(\mathcal{M}\varrho_{2n}, \mathcal{N}\varrho_{2n+1}) \\ &\leq \psi(M''(\varrho_{2n}, \varrho_{2n+1})) - \phi(N''(\varrho_{2n}, \varrho_{2n+1})),\end{aligned}$$

where

$$\begin{aligned}M''(\varrho_{2n}, \varrho_{2n+1}) &= \max\{\rho(\varrho_{2n}, \varrho_{2n+1}), \rho(\varrho_{2n}, \mathcal{M}\varrho_{2n}), \rho(\varrho_{2n+1}, \mathcal{N}\varrho_{2n+1}), \\ &\frac{1}{2}(\rho(\varrho_{2n}, \mathcal{M}\varrho_{2n}) + \rho(\varrho_{2n+1}, \mathcal{N}\varrho_{2n+1}))\} \\ &= \max\{\rho(\varrho_{2n}, \varrho_{2n+1}), \rho(\varrho_{2n}, \varrho_{2n+1}), \rho(\varrho_{2n+1}, \varrho_{2n+2}), \\ &\frac{1}{2}(\rho(\varrho_{2n}, \varrho_{2n+1}) + \rho(\varrho_{2n+1}, \varrho_{2n+2})), \frac{1}{2}(\rho(\varrho_{2n}, \varrho_{2n+2}) + \rho(\varrho_{2n+1}, \varrho_{2n+1}))\} \\ &\leq \max\{\rho(\varrho_{2n}, \varrho_{2n+1}), \rho(\varrho_{2n+1}, \varrho_{2n+2}), \\ &\frac{1}{2}(\rho(\varrho_{2n}, \varrho_{2n+1}) + \rho(\varrho_{2n+1}, \varrho_{2n+2}) - \rho(\varrho_{2n+1}, \varrho_{2n+1}) + \rho(\varrho_{2n+1}, \varrho_{2n+1}))\} \\ &\leq \max\{\rho(\varrho_{2n}, \varrho_{2n+1}), \rho(\varrho_{2n+1}, \varrho_{2n+2})\}\end{aligned}$$

and

$$\begin{aligned}N''(\varrho_{2n}, \varrho_{2n+1}) &= \max\Big\{\rho(\varrho_{2n}, \varrho_{2n+1}), \rho(\varrho_{2n}, \mathcal{M}\varrho_{2n}), \rho(\varrho_{2n+1}, \mathcal{M}\varrho_{2n+1}), \\ &\frac{\rho(\varrho_{2n}, \mathcal{M}\varrho_{2n})(\rho(\varrho_{2n}, \mathcal{N}\varrho_{2n+1}) + \rho(\varrho_{2n+1}, \mathcal{M}\varrho_{2n}) - \rho(\varrho_{2n+1}, \mathcal{N}\varrho_{2n+1}))}{1 + \rho(\varrho_{2n}, \varrho_{2n+1})}\Big\} \\ &= \max\Big\{\rho(\varrho_{2n}, \varrho_{2n+1}), \rho(\varrho_{2n}, \varrho_{2n+1}), \rho(\varrho_{2n+1}, \varrho_{2n+2}), \\ &\frac{\rho(\varrho_{2n}, \varrho_{2n+1})(\rho(\varrho_{2n}, \varrho_{2n+2}) + \rho(\varrho_{2n+1}, \varrho_{2n+1}) - \rho(\varrho_{2n+1}, \varrho_{2n+2}))}{1 + \rho(\varrho_{2n}, \varrho_{2n+1})}\Big\}\end{aligned}$$

$$\begin{aligned}
&\leq \max\Big\{\rho(\varrho_{2n},\varrho_{2n+1}),\rho(\varrho_{2n},\varrho_{2n+1}),\rho(\varrho_{2n+1},\varrho_{2n+2}),\\
&\frac{\rho(\varrho_{2n},\varrho_{2n+1})\rho(\varrho_{2n},\varrho_{2n+1})}{1+\rho(\varrho_{2n},\varrho_{2n+1})}\Big\}\\
&\leq \max\{\rho(\varrho_{2n},\varrho_{2n+1}),\rho(\varrho_{2n+1},\varrho_{2n+2})\}
\end{aligned}$$

$$\rho(\varrho_{2n+1},\varrho_{2n+2}) \leq \psi(\max\{\rho(\varrho_{2n},\varrho_{2n+1}),\rho(\varrho_{2n+1},\varrho_{2n+2})\}) - \phi(\max\{\rho(\varrho_{2n},\varrho_{2n+1}),\rho(\varrho_{2n+1},\varrho_{2n+2})\}).$$

Proceeding as in Theorem 1, sequence $\{\varrho_n\}$ is a Cauchy in a complete partial metric space $(\mathcal{U},\rho)$. Hence, $\varrho_1 \in \mathcal{U}$, which satisfies

$$\lim_{n,m\to\infty}\rho(\varrho_m,\varrho_n) = 0 = \lim_{n\to\infty}\rho(\varrho_n,\varrho_1) = \rho(\varrho_1,\varrho_1).$$

Suppose that $\varrho_1 \neq \mathcal{N}\varrho_1$ and $\{\varrho_{2n_k}\}$ are a subsequence of $\{\varrho_{2n}\}$ and hence of $\{\varrho_n\}$

$$\begin{aligned}
\psi(\rho(\varrho_{2n_k+1},\mathcal{N}\varrho_1)) &= \psi(\rho(\mathcal{M}''\varrho_{2n_k},\mathcal{N}''\varrho_1))\\
&\leq \psi(M(\varrho_{2n_k},\varrho_1)) - \phi(N(\varrho_{2n_k},\varrho_1)),
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
M''(\varrho_{2n_k},\varrho_1) = \max\{&\rho(\varrho_{2n_k},\varrho_1),\rho(\varrho_{2n_k},\mathcal{M}\varrho_{2n_k}),\rho(\varrho_1,\mathcal{N}\varrho_1),\\
&\frac{1}{2}(\rho(\varrho_{2n_k},\mathcal{M}\varrho_{2n_k})+\rho(\varrho_1,\mathcal{N}\varrho_1))\}\\
&= \max\{\rho(\varrho_{2n_k},\varrho_1),\rho(\varrho_{2n_k},\varrho_{2n_k+1}),\rho(\varrho_1,\mathcal{N}\varrho_1),\\
&\frac{1}{2}(\rho(\varrho_{2n_k},\varrho_{2n_k+1}))+\rho(\varrho_1,\mathcal{N}\varrho_1)\}\\
&\to \max\{\rho(\varrho_1,\varrho_1),\rho(\varrho_1,\mathcal{N}\varrho_1)\},\text{ as } n\to\infty,\\
&= \rho(\varrho_1,\mathcal{N}\varrho_1)
\end{aligned} \tag{25}$$

$$\begin{aligned}
N''(\varrho_{2n},\varrho_1) &= \max\Big\{\rho(\varrho_{2n},\varrho_1),\rho(\varrho_{2n},\mathcal{M}\varrho_{2n}),\rho(\varrho_1,\mathcal{N}\varrho_1),\\
&\frac{\rho(\varrho_{2n},\mathcal{M}\varrho_{2n})(\rho(\varrho_{2n},\mathcal{N}\varrho_1)+\rho(\varrho_1,\mathcal{M}\varrho_{2n})-\rho(\varrho_1,\mathcal{N}\varrho_1))}{1+\rho(\varrho_{2n},\varrho_1)}\Big\}\\
&= \max\Big\{\rho(\varrho_{2n},\varrho_1),\rho(\varrho_{2n},\varrho_{2n+1}),\rho(\varrho_1,\mathcal{N}\varrho_1),\\
&\frac{\rho(\varrho_{2n},\varrho_{2n+1})(\rho(\varrho_{2n},\mathcal{N}\varrho_1)+\rho(\varrho_1,\varrho_{2n+1})-\rho(\varrho_1,\mathcal{N}\varrho_1))}{1+\rho(\varrho_{2n},\varrho_1)}\Big\}\\
&\to \max\{\rho(\varrho_1,\varrho_1),\rho(\varrho_1,\mathcal{N}\varrho_1)\},\text{ as } n\to\infty,\\
&= \rho(\varrho_1,\mathcal{N}\varrho_1).
\end{aligned} \tag{26}$$

Now, using properties of functions ψ,ϕ and substituting inequalities (25) and (26) in inequality (24),

$$\begin{aligned}
\psi(\rho(\varrho_1,\mathcal{N}\varrho_1)) &\leq \psi(\rho(\varrho_1,\mathcal{N}\varrho_1) - \phi(\rho(\varrho_1,\mathcal{N}\varrho_1))\\
&< \psi(\rho(\varrho_1,\mathcal{N}\varrho_1)),
\end{aligned}$$

a contradiction, that is, $\mathcal{N}\varrho_1 = \varrho_1$.

Similarly, if we choose $\{\varrho_{2n_k+1}\}$ to be a subsequence of $\{\varrho_{2n+1}\}$ and hence of $\{\varrho_n\}$, we obtain $\mathcal{M}\varrho_1 = \varrho_1$ and hence $\mathcal{M}\varrho_1 = \mathcal{N}\varrho_1 = \varrho_1$.

Next, suppose $\mathcal{M}$ and $\mathcal{N}$ have more than one common fixed point, and ϱ_2 is one more common fixed point of $\mathcal{M}$ different from ϱ_1, that is, $\varrho_1 \neq \varrho_2$. Now, using inequality (4)

$$\begin{aligned}
\psi(\rho(\mathcal{M}\varrho_1,\mathcal{N}\varrho_2)) &\leq \psi\Big(\max\{\rho(\varrho_1,\varrho_2),\rho(\varrho_1,\mathcal{M}\varrho_1),\rho(\varrho_2,\mathcal{N}\varrho_2),\frac{1}{2}(\rho(\varrho_1,\mathcal{M}\varrho_1)+\rho(\varrho_2,\mathcal{N}\varrho_2))\}\Big),\\
&-\phi\Big(\max\{\rho(\varrho_1,\varrho_2),\rho(\varrho_1,\mathcal{M}\varrho_1),\rho(\varrho_2,\mathcal{M}\varrho_2),\\
&\frac{\rho(\varrho_1,\mathcal{M}\varrho_1)(\rho(\varrho_1,\mathcal{N}\varrho_2)+\rho(\varrho_2,\mathcal{M}\varrho_1)-\rho(\varrho_2,\mathcal{N}\varrho_2))}{1+\rho(\varrho_1,\varrho_2)}\}\Big)\\
\psi(\rho(\varrho_1,\varrho_2)) &\leq \psi\Big(\max\{\rho(\varrho_1,\varrho_2),\rho(\varrho_1,\varrho_1),\rho(\varrho_2,\varrho_2),\frac{1}{2}(\rho(\varrho_1,\varrho_1)+\rho(\varrho_2,\varrho_2))\}\Big),\\
&-\phi\Big(\max\{\rho(\varrho_1,\varrho_2),\rho(\varrho_1,\varrho_1),\rho(\varrho_2,\varrho_2),\frac{\rho(\varrho_1,\varrho_1)(\rho(\varrho_1,\varrho_2)+\rho(\varrho_2,\varrho_1)-\rho(\varrho_2,\varrho_2))}{1+\rho(\varrho_1,\varrho_2)}\}\Big)\\
\psi(\rho(\varrho_1,\varrho_2)) &\leq \psi(\rho(\varrho_1,\varrho_2))-\phi(\rho(\varrho_1,\varrho_2))\\
\psi(\rho(\varrho_1,\varrho_2)) &< \psi(\rho(\varrho_1,\varrho_2)),
\end{aligned} \tag{27}$$

a contradiction. Hence, $\varrho_1=\varrho_2$. □

Next, we give the subsequent example to justify Theorem 2 and to indicate the significant fact that the continuity or compatibility (or their weaker variants (see Tomar and Karapinar [17] and Singh and Tomar [18]) are not essential for the presence of a unique common fixed point satisfying a partial $(\psi,\phi)-$contraction.

Example 6. *Let $\mathcal{U}=\{\varrho_n=n:n\in\mathbb{N}\cup\{0\}\}$ and partial metric $\rho:\mathcal{U}\times\mathcal{U}\to\mathbb{R}^+$ be described as $\rho(\varrho_1,\varrho_2)=\max\{\varrho_1,\varrho_2\}$. Then, $(\mathcal{U},\rho)$ is a complete partial metric space. Let $\psi(\varrho_1)=\varrho_1^2+\varrho_1$ and $\phi(\varrho_1)=\begin{cases}0, & \varrho_1=0\\ \frac{\varrho_1}{10}+1, & \varrho_1\neq 0\end{cases}$, $\varrho_1\in\mathcal{U}$.*

Let self-maps $\mathcal{M},\mathcal{N}:\mathcal{U}\to\mathcal{U}$ be described as

$$\mathcal{M}\varrho_1=\begin{cases}\varrho_0, & \varrho_1=\varrho_0\\ \varrho_n-1 & \varrho_1=\varrho_n,\ n\geq 1\end{cases} \text{ and } \mathcal{N}\varrho_1=\begin{cases}\varrho_0, & \varrho_1=\varrho_0\\ \varrho_n+1 & \varrho_1=\varrho_n,\ n\geq 1\end{cases}.$$

Now, we assert that $\mathcal{M}$ and $\mathcal{N}$ validate partial $(\psi,\phi)-$contraction for a pair of maps, that is,

$$\frac{\psi(\rho(\mathcal{M}\varrho_n,\mathcal{N}\varrho_m))}{\psi(M''(\varrho_n,\varrho_m))-\phi(N''(\varrho_1,\varrho_2))}\leq 1.$$

Case (i) When $n=0$ and $m\geq 1$,

$$\begin{aligned}
\frac{\psi(\rho(\mathcal{M}\varrho_n,\mathcal{N}\varrho_m))}{\psi(M''(\varrho_n,\varrho_m))-\phi(N''(\varrho_1,\varrho_2))} &= \frac{\psi(m+1)}{\psi(m+1)-\phi(m+1)}\\
&= \frac{10(m^2+3m+2)}{10m^2+29m+9}<1.
\end{aligned}$$

Case (ii) When $m>n\geq 1$,

$$\begin{aligned}
\frac{\psi(\mathcal{M}\varrho_n,\mathcal{M}\varrho_m))}{\psi(M(\varrho_n,\varrho_m))-\phi(N(\varrho_1,\varrho_2))} &= \frac{\psi(m+1)}{\psi(m+1)-\phi(n+1)}\\
&= \frac{10(m^2+3m+2)}{10m^2+29m+9}<1.
\end{aligned}$$

Thus, $\mathcal{M}$ and $\mathcal{N}$ satisfy partial $(\psi,\phi)-$contraction for a pair of maps and have exactly one common fixed point 0.

Remark 2. *Theorems 1 and 2 evidenced a common fixed point for two self maps in a partial metric space via partial $(\psi,\phi)-$contraction, wherein we have neither utilized compatibility nor any of its* *variants (see Singh and Tomar [18]). In addition, we have neither utilized continuity nor any of its variants (see Tomar and Karapinar [17]). Theorems 1 and 2 are improvement and generalization of contractions used in [1–3,6,19], and so on to partial metric spaces for discontinuous self maps. Illustrative examples have demonstrated that these extensions, improvements and generalizations are genuine. Furthermore, by taking distinct values of ψ and ϕ, we achieve some novel conclusions and generalizations of celebrated and recent conclusions in the literature.*

Now, motivated by Joshi et al. [8], we investigate the subsistence of a fixed 2-ellipse-like curve to explore the symmetrical geometry of fixed points in a partial metric space.

Theorem 3. *Let $\mathcal{E}(\kappa_1, \kappa_2, a)$ be an ellipse-like curve in a partial metric space $(\mathcal{U}, \rho)$. If map $\mathcal{M} : \mathcal{U} \to \mathcal{U}$ is a partial $(\psi, \phi)_{\mathcal{E}_{\subset}}-$contraction and $\varrho_1 \neq \kappa_1 \neq \kappa_2$, $\varrho_1, \kappa_1, \kappa_2 \in \mathcal{U}$, $a = \frac{1}{2}$ $\inf\{\rho(\varrho_1, \mathcal{M}\varrho_1) : \varrho_1 \neq \mathcal{M}\varrho_1, \varrho_1 \in \mathcal{U}\}$ and $\kappa_1 = \mathcal{M}\kappa_1$, $\kappa_2 = \mathcal{M}\kappa_2$, then $\mathcal{E}_{\subset}(\kappa_1, \kappa_2; a)$ is a fixed 2-ellipse-like curve of $\mathcal{M}$.*

Proof. Let $\mathcal{M}\varrho_1 \neq \varrho_1$, $\varrho_1 \in \mathcal{E}_\rho(\kappa_1, \kappa_2; a)$ be any arbitrary point and, by definition of a,

$$\rho(\kappa_1, \kappa_1) + \rho(\kappa_2, \kappa_2) + \rho(\kappa_1, \kappa_2) < a < \rho(\varrho_1, \mathcal{M}\varrho_1). \tag{28}$$

Now,

$$\psi(\rho(\varrho_1, \mathcal{M}\varrho_1)) \leq \psi(M(\varrho_1, \kappa_1, \kappa_2)) - \phi(N(\varrho_1, \kappa_1, \kappa_2)), \tag{29}$$

where

$$\begin{aligned}
M(\varrho_1, \kappa_1, \kappa_2) = \max\{&\rho(\varrho_1, \kappa_1) + \rho(\varrho_1, \kappa_2), \rho(\varrho_1, \mathcal{M}\varrho_1), \rho(\kappa_1, \mathcal{M}\kappa_1),\\
&\frac{1}{3}(\rho(\varrho_1, \mathcal{M}\varrho_1) + \rho(\varrho_1, \mathcal{M}\kappa_1) + \rho(\varrho_1, \mathcal{M}\kappa_2) + \rho(\kappa_1, \mathcal{M}\kappa_1) + \rho(\kappa_2, \mathcal{M}\kappa_2) + \rho(\kappa_1, \kappa_2)),\\
&\rho(\varrho_1, \mathcal{M}\kappa_1) + \rho(\varrho_1, \mathcal{M}\kappa_2)\}\\
= \max\{&\rho(\varrho_1, \kappa_1) + \rho(\varrho_1, \kappa_2), \rho(\varrho_1, \mathcal{M}\varrho_1), \rho(\kappa_1, \kappa_1),\\
&\frac{1}{3}(\rho(\varrho_1, \mathcal{M}\varrho_1) + \rho(\varrho_1, \kappa_1) + \rho(\varrho_1, \kappa_2) + \rho(\kappa_1, \kappa_1) + \rho(\kappa_2, \kappa_2) + \rho(\kappa_1, \kappa_2)),\\
&\rho(\varrho_1, \kappa_1) + \rho(\varrho_1, \kappa_2)\}\\
< \max\{&a, \rho(\varrho_1, \mathcal{M}\varrho_1), a, \frac{1}{3}(a + a + a), a\}, \text{ (using Equation (28))}\\
= \rho(&\varrho_1, \mathcal{M}\varrho_1),
\end{aligned}$$

and

$$\begin{aligned}
N(\varrho_1, \kappa_1, \kappa_2) = \max\Big\{&\rho(\varrho_1, \kappa_1) + \rho(\varrho_1, \kappa_2), \rho(\varrho_1, \mathcal{M}\varrho_1), \rho(\kappa_1, \mathcal{M}\kappa_1), \rho(\kappa_2, \mathcal{M}\kappa_2),\\
&\frac{\rho(\varrho_1, \mathcal{M}\varrho_1)(\rho(\varrho_1, \mathcal{M}\kappa_1) - \rho(\kappa_1, \mathcal{M}\kappa_1))}{1 + \rho(\varrho_1, \mathcal{M}\kappa_1)}, \frac{\rho(\varrho_1, \mathcal{M}\varrho_1)(\rho(\varrho_1, \mathcal{M}\kappa_2) - \rho(\kappa_2, \mathcal{M}\kappa_2))}{1 + \rho(\varrho_1, \mathcal{M}\kappa_2)}\Big\}\\
= \max\Big\{&\rho(\varrho_1, \kappa_1) + \rho(\varrho_1, \kappa_2), \rho(\varrho_1, \mathcal{M}\varrho_1), \rho(\kappa_1, \kappa_1), \rho(\kappa_2, \kappa_2),\\
&\frac{\rho(\varrho_1, \mathcal{M}\varrho_1)(\rho(\varrho_1, \kappa_1) - \rho(\kappa_1, \kappa_1))}{1 + \rho(\varrho_1, \kappa_1)}, \frac{\rho(\varrho_1, \mathcal{M}\varrho_1)(\rho(\varrho_1, \kappa_2) - \rho(\kappa_2, \kappa_2))}{1 + \rho(\varrho_1, \kappa_2)}\Big\}\\
< \max\{&a, \rho(\varrho_1, \mathcal{M}\varrho_1), \rho(\kappa_1, \kappa_1), \rho(\kappa_2, \kappa_2), \rho(\varrho_1, \mathcal{M}\varrho_1), \rho(\varrho_1, \mathcal{M}\varrho_1)\}\\
= \rho(&\varrho_1, \mathcal{M}\varrho_1).
\end{aligned}$$

Now,

$$\begin{aligned}
\psi(\rho(\varrho_1, \mathcal{M}\varrho_1) &\leq \psi(\rho(\varrho_1, \mathcal{M}\varrho_1)) - \phi(\rho(\varrho_1, \mathcal{M}\varrho_1))\\
&< \psi(\rho(\varrho_1, \mathcal{M}\varrho_1)),
\end{aligned} \tag{30}$$

a contradiction. Hence, $\mathcal{M}\varrho_1 = \varrho_1, \varrho_1 \in \mathcal{E}_\rho(\kappa_1, \kappa_2, \varrho_1)$, that is, $\mathcal{E}_\rho(\kappa_1, \kappa_2; a)$ is a fixed 2-ellipse-like curve of $\mathcal{M}$. □

The subsequent example validates Theorem 3.

Example 7. *Let $\mathcal{M} = \mathbb{R}^+$ and $\rho : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ be described as* $\rho(\varrho_1, \varrho_2) = \max\{\varrho_1, \varrho_2\} + |\varrho_1 - \varrho_2|$. *The* 2*-ellipse-like curve*

$$\begin{aligned}\mathcal{E}_\rho(5,6,21) &= \{\varrho_1 \in \mathcal{U} : \rho(5,\varrho_1) + \rho(6,\varrho_1) = 21\} \\ &= \{\varrho_1 \in \mathcal{U} : \max\{5,\varrho_1\} + |5-\varrho_1| + \max\{6,\varrho_1\} + |6-\varrho_1| = 21\} \\ &= \{\frac{1}{2}, 8\}.\end{aligned} \tag{31}$$

Let $\psi(\varrho_1) = \varrho_1$ *and* $\phi(\varrho_1) = \begin{cases} 0, & \varrho_1 = 0 \\ \frac{\varrho_1}{10} + 1, & otherwise \end{cases}$. *Define a self map* $\mathcal{M} : \mathcal{U} \to \mathcal{U}$ *as*

$$\mathcal{M}\varrho_1 = \begin{cases} \varrho_1, & \varrho_1 \in \mathcal{E}_\rho(5,6,21) \\ 5, & \varrho_1 = 5 \\ 6, & \varrho_1 = 6 \\ \varrho_1 + 21, & otherwise \end{cases}.$$

Now, $a = \frac{1}{2}\min\{\rho(\varrho_1, \mathcal{M}\varrho_1) : \varrho_1 \neq \mathcal{M}\varrho_1\} = 21$.

Then, a self map $\mathcal{M}$ validates the postulates of Theorem 3. Noticeably, $\mathcal{M}$ fixes the 2*-ellipse-like curve* $\mathcal{E}_\rho(5,6,21)$.

Corollary 1. *Let $\mathcal{E}_\rho(\kappa_1, \kappa_2; a)$ be a* 2*-ellipse-like curve in a partial metric space $(\mathcal{U}, \rho)$. If map $\mathcal{M} : \mathcal{U} \to \mathcal{U}$ satisfies*

$$\begin{aligned}\rho(\varrho_1, \mathcal{M}\varrho_1) \leq \mu \max\{&\rho(\varrho_1,\varrho_2), \rho(\varrho_1,\mathcal{M}\varrho_1), \rho(\varrho_2,\mathcal{M}\varrho_2), \frac{1}{2}(\rho(\varrho_1,\mathcal{M}\varrho_1) + \rho(\varrho_2,\mathcal{M}\varrho_2)), \\ &\frac{1}{2}(\rho(\varrho_1,\mathcal{M}\varrho_2) + \rho(\varrho_2,\mathcal{M}\varrho_1))\},\ \mu \in [0,1) \text{ and } \varrho_1, \varrho_2 \in \mathcal{U},\end{aligned} \tag{32}$$

with $\varrho_1 \neq \kappa_1$, $\varrho_1 \neq \kappa_2 \in \mathcal{U}$ and $\kappa_1 = \mathcal{M}\kappa_1$, $\kappa_2 = \mathcal{M}\kappa_2$, $\varrho_1 \in \mathcal{E}_\rho(\kappa_1, \kappa_2, a)$, then $\mathcal{E}_\rho(\kappa_1, \kappa_2; a)$ being a fixed ellipse-like curve of $\mathcal{M}$.

Corollary 2. *Let $\mathcal{E}_\rho(\kappa_1, \kappa_2; a)$ be a* 2*-ellipse-like curve in a partial metric space $(\mathcal{U}, \rho)$. If map $\mathcal{M} : \mathcal{U} \to \mathcal{U}$ satisfies*

$$\begin{aligned}\rho(\varrho_1, \mathcal{M}\varrho_1) \leq \mu \max\Big\{&\rho(\varrho_1,\varrho_2), \rho(\varrho_1,\mathcal{M}\varrho_1), \rho(\varrho_2,\mathcal{M}\varrho_2), \frac{1}{2}(\rho(\varrho_1,\mathcal{M}\varrho_1) + \rho(\varrho_2,\mathcal{M}\varrho_2)), \\ &\frac{1}{2}(\rho(\varrho_1,\mathcal{M}\varrho_2) + \rho(\varrho_2,\mathcal{M}\varrho_1)), \frac{\rho(\varrho_1,\mathcal{M}\varrho_1)(\rho(\varrho_1,\mathcal{M}\varrho_2) - \rho(\varrho_2,\mathcal{M}\varrho_2))}{1 + \rho(\varrho_1,\mathcal{M}\varrho_2)}, \\ &\frac{\rho(\varrho_1,\mathcal{M}\varrho_1)(\rho(\varrho_1,\mathcal{M}\varrho_2) - \rho(\varrho_3,\mathcal{M}\vartheta))}{1 + \rho(\varrho_1,\mathcal{M}\vartheta)}\Big\},\ \mu \in [0,1) \text{ and } \varrho_1, \varrho_2 \in \mathcal{U},\end{aligned} \tag{33}$$

with $\varrho_1 \neq \kappa_1$, $\varrho_1 \neq \kappa_2 \in \mathcal{U}$ and $\kappa_1 = \mathcal{M}\kappa_1$, $\kappa_2 = \mathcal{M}\kappa_2$, $\varrho_1 \in \mathcal{E}_\rho(\kappa_1, \kappa_2, a)$, then $\mathcal{E}_\rho(\kappa_1, \kappa_2; a)$ is a fixed 2*-ellipse-like curve of $\mathcal{M}$.*

Theorem 4. *Let $\mathcal{E}_\rho(\kappa_1, \kappa_2, \ldots, n; a)$ be an n-ellipse-like curve in a partial metric space $(\mathcal{U}, \rho)$. If map $\mathcal{M} : \mathcal{U} \to \mathcal{U}$ is a partial $(\psi, \phi)_{\mathcal{E}_\mathbb{C}}-$contraction and $\varrho_1 \neq \kappa_i$, $i = 1, 2, \ldots, v$ and $\varrho_1, \kappa_i \in \mathcal{U}$, $a = \frac{1}{2}\inf\{\rho(\varrho_1, \mathcal{M}\varrho_1) : \varrho_1 \neq \mathcal{M}\varrho_1, \varrho_1 \in \mathcal{U}\}$ and $\kappa_i = \mathcal{M}\kappa_i$, $i = 1, 2, \ldots, n$, then $\mathcal{E}_\rho(\kappa_1, \kappa_2, \ldots, \kappa_n; a)$ is a fixed n-ellipse-like curve of $\mathcal{M}$.*

Proof. It may be concluded on similar lines as Theorem 3. □

To obtain a common fixed ellipse-like curve, first define

$$a_1 = \frac{1}{2}\inf\{\rho(\varrho_1, \mathcal{M}\varrho_1) : \varrho_1 \neq \mathcal{M}\varrho_1, \varrho_1 \in \mathcal{U}\}$$
$$a_2 = \frac{1}{2}\inf\{\rho(\varrho_1, \mathcal{N}\varrho_1) : \varrho_1 \neq \mathcal{N}\varrho_1, \varrho_1 \in \mathcal{U}\}$$
$$a_3 = \frac{1}{2}\inf\{\rho(\mathcal{M}\varrho_1, \mathcal{N}\varrho_1) : \mathcal{M}\varrho_1 \neq \mathcal{N}\varrho_1, \varrho_1 \in \mathcal{U}\}$$

and

$$a^* = \min\{a_1, a_2, a_3\}. \tag{34}$$

Theorem 5. *Let $\mathcal{E}_\rho(\kappa_1, \kappa_2; a^*)$ be a 2-ellipse-like curve in a partial metric space. If maps $\mathcal{M}, \mathcal{N} : \mathcal{U} \to \mathcal{U}$ are satisfying partial $(\psi, \phi)_{\mathcal{E}_\subset}-$contraction for a pair of maps and map $\mathcal{M}$ satisfies partial $(\psi, \phi)_{\mathcal{E}_\subset}-$contraction with $\varrho_1 \neq \kappa_1 \neq \kappa_2$, $\varrho_1, \kappa_1, \kappa_2 \in \mathcal{U}$ and $\kappa_1 = \mathcal{M}\kappa_1$, $\kappa_2 = \mathcal{M}\kappa_2$, then $\mathcal{E}_\rho(\kappa_1, \kappa_2; a^*)$ is a common fixed 2-ellipse-like curve of maps $\mathcal{M}$ and $\mathcal{N}$.*

Proof. Let $u \in \mathcal{E}_\rho(\kappa_1, \kappa_2; a)$ be any arbitrary point and $\varrho_1 \neq \mathcal{M}\varrho_1 \neq \mathcal{N}\varrho_1$. By definition of a^*,

$$\rho(\kappa_1, \kappa_1) + \rho(\kappa_2, \kappa_2) + \rho(\kappa_1, \kappa_2) < a^* \leq \min\{\rho(\varrho_1, \mathcal{M}\varrho_1), \rho(\varrho_1, \mathcal{N}\varrho_1), \rho(\mathcal{M}\varrho_1, \mathcal{N}\varrho_1)\}. \tag{35}$$

Let $\mathcal{M}$ satisfy partial $(\psi, \phi)_{\mathcal{E}_\subset}$ contraction. Thus, using Theorem 1,

$$\mathcal{M}\varrho_1 = \varrho_1. \tag{36}$$

Now,

$$\psi(\rho(\mathcal{M}\varrho_1, \mathcal{N}\varrho_1)) \leq \psi(M(\varrho_1, \kappa_1, \kappa_2)) - \phi(N(\varrho_1, \kappa_1, \kappa_2)), \tag{37}$$

where

$$\begin{aligned}
M(\varrho_1, \kappa_1, \kappa_2) = \max\Big\{&\rho(\varrho_1, \kappa_1) + \rho(\varrho_1, \kappa_2), \rho(\varrho_1, \mathcal{M}\varrho_1), \rho(\kappa_1, \mathcal{N}\kappa_1), \\
&\frac{1}{3}(\rho(\varrho_1, \mathcal{M}\varrho_1) + \rho(\varrho_1, \mathcal{N}\kappa_1) + \rho(\varrho_1, \mathcal{N}\kappa_2) + \rho(\kappa_1, \mathcal{N}\kappa_1) + \rho(\kappa_2, \mathcal{N}\kappa_2) + \rho(\kappa_1, \kappa_2)), \\
&\rho(\varrho_1, \mathcal{M}\kappa_1) + \rho(\varrho_1, \mathcal{M}\kappa_2)\Big\} \\
= \max\Big\{&\rho(\varrho_1, \kappa_1) + \rho(\varrho_1, \kappa_2), \rho(\varrho_1, \mathcal{M}\varrho_1), \rho(\kappa_1, \kappa_1), \\
&\frac{1}{3}(\rho(\varrho_1, \mathcal{M}\varrho_1) + \rho(\varrho_1, \kappa_1) + \rho(\varrho_1, \kappa_2) + \rho(\kappa_1, \kappa_1) + \rho(\kappa_2, \kappa_2) + \rho(\kappa_1, \kappa_2)), \\
&\rho(\varrho_1, \kappa_1) + \rho(\varrho_1, \kappa_2)\Big\} \\
\leq \max\Big\{&a, \rho(\varrho_1, \mathcal{M}\varrho_1), a, \frac{1}{3}(a + a + a), a\Big\}, \text{ (using Equation (28))} \\
= \rho(&\varrho_1, \mathcal{M}\varrho_1),
\end{aligned}$$

and

$$\begin{aligned}
N(\varrho_1, \kappa_1, \kappa_2) = \max\Big\{&\rho(\varrho_1, \kappa_1) + \rho(\varrho_1, \kappa_2), \rho(\varrho_1, \mathcal{M}\varrho_1), \rho(\kappa_1, \mathcal{N}\kappa_1), \rho(\kappa_2, \mathcal{N}\kappa_2), \\
&\frac{\rho(\varrho_1, \mathcal{M}\varrho_1)(\rho(\varrho_1, \mathcal{N}\kappa_1) - \rho(\kappa_1, \mathcal{M}\kappa_1))}{1 + \rho(\varrho_1, \mathcal{N}\kappa_1)}, \frac{\rho(\varrho_1, \mathcal{M}\varrho_1)(\rho(\varrho_1, \mathcal{N}\kappa_2) - \rho(\kappa_2, \mathcal{M}\kappa_2))}{1 + \rho(\varrho_1, \mathcal{N}\kappa_2)}\Big\} \\
= \max\Big\{&\rho(\varrho_1, \kappa_1) + \rho(\varrho_1, \kappa_2), \rho(\varrho_1, \mathcal{M}\varrho_1), \rho(\kappa_1, \kappa_1), \rho(\kappa_2, \kappa_2), \\
&\frac{\rho(\varrho_1, \mathcal{M}\varrho_1)(\rho(\varrho_1, \kappa_1) - \rho(\kappa_1, \kappa_1))}{1 + \rho(\varrho_1, \kappa_1)}, \frac{\rho(\varrho_1, \mathcal{M}\varrho_1)(\rho(\varrho_1, \kappa_2) - \rho(\kappa_2, \kappa_2))}{1 + \rho(\varrho_1, \kappa_2)}\Big\}
\end{aligned}$$

$$\leq \max\{a, \rho(\varrho_1, \mathcal{M}\varrho_1), \rho(\kappa_1, \kappa_1), \rho(\kappa_2, \kappa_2), \rho(\varrho_1, \mathcal{M}\varrho_1), \rho(\varrho_1, \mathcal{M}\varrho_1)\}$$
$$= \rho(\varrho_1, \mathcal{M}\varrho_1).$$

Now,

$$\begin{aligned}\psi(\rho(\mathcal{M}\varrho_1, \mathcal{N}\varrho_1) &\leq \psi(\rho(\varrho_1, \mathcal{M}\varrho_1)) - \phi(\rho(\varrho_1, \mathcal{M}\varrho_1)) \\ &< \psi(\rho(\varrho_1, \mathcal{M}\varrho_1)),\end{aligned}$$

i.e.,

$$\rho(\mathcal{M}\varrho_1, \mathcal{N}\varrho_1) < \rho(\varrho_1, \mathcal{M}\varrho_1) < \rho(\mathcal{M}\varrho_1, \mathcal{M}\varrho_1),$$

a contradiction. Hence, $\mathcal{M}\varrho_1 = \mathcal{N}\varrho_1 = \varrho_1, \varrho_1 \in \mathcal{E}_\rho(\kappa_1, \kappa_2; a^*)$, that is, a fixed 2-ellipse-like curve of $\mathcal{M}$. □

The following example validates Theorems 5.

Example 8. *Let $\mathcal{M} = \mathbb{R}^+$ and a partial metric $\rho : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ be described as: $\rho(\varrho_1, \varrho_2) = \max\{\varrho_1, \varrho_2\} + |\varrho_1 - \varrho_2|$. The 2-ellipse-like curve*

$$\begin{aligned}\mathcal{E}_\rho(2, 6, 14) &= \{\varrho_1 \in \mathcal{U} : \rho(2, \varrho_1) + \rho(6, \varrho_1) = 21\} \\ &= \{\varrho_1 \in \mathcal{U} : \max\{2, \varrho_1\} + |2 - \varrho_1| + \max\{6, \varrho_1\} + |6 - \varrho_1| = 14\} \\ &= \{1, 4\}.\end{aligned} \tag{38}$$

Let $\psi(\varrho_1) = \varrho_1$ and $\phi(\varrho_1) = \begin{cases} 0, & \varrho_1 = 0 \\ \frac{\varrho_1}{20} + 1, & otherwise \end{cases}$.
Let self maps $\mathcal{M}, \mathcal{N} : \mathcal{U} \to \mathcal{U}$ be described as

$$\mathcal{M}\varrho_1 = \begin{cases} \varrho_1, & \varrho_1 < 8 \\ \varrho_1 + 10, & otherwise \end{cases} \quad and \quad \mathcal{N}\varrho_1 = \begin{cases} \varrho_1, & \varrho_1 < 8 \\ \varrho_1 + 20, & otherwise \end{cases}.$$

Since $a_1 = \frac{1}{2}\min\{\rho(\varrho_1, \mathcal{M}\varrho_1) : \varrho_1 \neq \mathcal{M}\varrho_1\} = 14$, $a_2 = \frac{1}{2}\min\{\rho(\varrho_1, \mathcal{N}\varrho_1) : \varrho_1 \neq \mathcal{N}\varrho_1\} = 24$, and $a_3 = \frac{1}{2}\min\{\rho(\mathcal{M}\varrho_1, \mathcal{N}\varrho_1) : \mathcal{M}\varrho_1 \neq \mathcal{N}\varrho_1\} = 19$. Now, $a^* = \min\{a_1, a_2, a_3\}$ $= 14$.

Then, a self map $\mathcal{M}$ validates all the assumptions of Theorem 5. Noticeably, $\mathcal{M}$ fixes the 2-ellipse-like curve $\mathcal{E}_\rho(2, 6, 14)$.

Theorem 6. *Let $\mathcal{E}_\rho(\kappa_1, \kappa_2, \ldots, \kappa_v; a^*)$ be a $v-$ellipse in a partial metric space. If maps $\mathcal{M}, \mathcal{N} : \mathcal{U} \to \mathcal{U}$ are satisfying partial $(\psi, \phi)_{\mathcal{E}_\subset}-$ contraction for a pair of maps and $\mathcal{M}$ satisfies $(\psi, \phi)_{\mathcal{E}_\subset}-$ contraction with $\varrho_1 \neq \kappa_i, i = 1, 2, \ldots, v$ and $\varrho_1, \kappa_i \in \mathcal{U}$ and $\kappa_i = \mathcal{M}\kappa_i, i = 1, 2, \ldots, n$, then $\mathcal{E}_\rho(\kappa_1, \kappa_2, \ldots, \kappa_n; a^*)$ is a common fixed n-ellipse-like curve of maps $\mathcal{M}$ and $\mathcal{N}$.*

Proof. It may be concluded on a similar pattern as that of Theorem 5. □

Remark 3. *An ellipse-like curve in partial metric space is an enhancement of an ellipse in metric space (see [4,8]).*

Remark 4. *A fixed ellipse-like curve of the self map is not always unique (see Example 7). If the foci κ_1 and κ_2 of an ellipse-like curve are concurrent, then fixed ellipse-like curve conclusions diminish to corresponding fixed circle conclusions. Furthermore, Examples 2, 3, 7 and 8 demonstrate the significant fact that the shape of the ellipse-like curve, which is symmetrical, may alter by altering the center, principal axis, foci, or the partial metric under consideration.*

4. Application in Production–Consumption Equilibrium

We utilize our conclusions to construct a mathematical model and solve an initial value problem emerging in the dynamic market equilibrium problem, which is a significant problem in economics. For production $\alpha_{\mathcal{P}}$ and consumption $\alpha_{\mathcal{C}}$, whether the prices are rising or falling, day-to-day pricing trends, as well as prices, demonstrate an important impact on markets. Consequently, the economist is interested in knowing the current price $P(t)$. Now, assume

$$\alpha_{\mathcal{P}} = \beta_1 + \gamma_1 P(t) + \delta_1 \frac{dP(t)}{dt} + \varrho_1 \frac{d^2P(t)}{dt^2}$$

$$\alpha_{\mathcal{C}} = \beta_2 + \gamma_2 P(t) + \delta_2 \frac{dP(t)}{dt} + \varrho_2 \frac{d^2P(t)}{dt^2},$$

initially $P(0) = 0$, $\frac{dP}{dt}(0) = 0$, where β_1, β_2, γ_1, γ_2, δ_1, δ_2, ϱ_1, and ϱ_2 are constants. Dynamic economic equilibrium is the condition wherein there is a balance among market forces, that is, the current prices become stable between production and consumption, that is, $\alpha_{\mathcal{P}} = \alpha_{\mathcal{C}}$. Thus,

$$\begin{aligned}
&\beta_1 + \gamma_1 P(t) + \delta_1 \frac{dP(t)}{dt} + \varrho_1 \frac{d^2P(t)}{dt^2} = \beta_2 + \gamma_1 P(t) + \delta_2 \frac{dP(t)}{dt} + \varrho_2 \frac{d^2P(t)}{dt^2}, \\
&(\beta_1 - \beta_2) + (\gamma_1 - \gamma_2)P(t) + (\delta_1 - \delta_2)\frac{dP(t)}{dt} + (\varrho_1 - \varrho_2)\frac{d^2P(t)}{dt^2} = 0, \\
&\varrho \frac{d^2P(t)}{dt^2} + \delta \frac{dP(t)}{dt} + \gamma P(t) = -\beta, \\
&\frac{d^2P(t)}{dt^2} + \frac{\delta}{\varrho}\frac{dP(t)}{dt} + \frac{\gamma}{\varrho}P(t) = -\frac{\beta}{\varrho},
\end{aligned}$$

where $\beta = \beta_1 - \beta_2, \gamma = \gamma_1 - \gamma_2, \delta = \delta_1 - \delta_2$, and $\varrho = \varrho_1 - \varrho_2$.

Now, our initial value problem is modeled as:

$$P''(t) + \frac{\delta}{\varrho}P'(t) + \frac{\gamma}{\varrho}P(t) = -\frac{\beta}{\varrho}, \text{ with } P(0) = 0 \text{ and } P'(0) = 0. \tag{39}$$

If we study production and consumption duration time T, problem (39) is equivalent to

$$P(t) = \int_0^T \mathcal{G}(t, t^*) K(t^*, t, P(t)) dt, \tag{40}$$

where Green function $\mathcal{G}(t, t^*)$ is

$$\mathcal{G}(t, t^*) = \begin{cases} te^{\frac{\gamma}{2\delta}(t^* - t)}, & 0 \le t \le s \le T \\ se^{\frac{\gamma}{2\delta}(t - t^*)}, & 0 \le s \le t \le T \end{cases}$$

and $K : [0, T] \times \mathcal{U}^2 \to \mathbb{R}$ is a continuous function.

Let an operator $\mathcal{M} : \mathcal{U} \to \mathcal{U}$ be described as

$$\mathcal{M}P(t) = \int_0^T \mathcal{G}(t, t^*) K(t^*, t, P(t)) dt \tag{41}$$

Now, the solution to the dynamic market equilibrium problem, which is expressed as (39), is a fixed point of $\mathcal{M}$ (41). Actually, the current price $P(t)$ is regulated by (39). Let $\mathcal{C}[0, T]$ symbolize the family of real continuous functions on $[0, T]$, and we write $\mathcal{U} = \mathcal{C}[0, T]$. Define a distance function $\rho : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$ as $\rho(\varrho_1, \varrho_2) = \max\{\|\varrho_1\|, \|\varrho_2\|\}$, $\varrho_1, \varrho_2 \in \mathcal{U}$,where $\|\varrho_1\| = \sup_{t \in [0,T]} |\varrho_1(t)|$. Clearly, $(\mathcal{U}, \rho)$ is a complete partial metric space.

Theorem 7. *Consider the operator $\mathcal{M} : \mathcal{U} \to \mathcal{U}$ (41) in a complete partial metric space $(\mathcal{U}, \rho)$, satisfying*

1. *a continuous function $\mathcal{G} : \mathcal{U}^2 \to \mathbb{R}$ that satisfies*

$$\sup_{s\in[0,T]} \int_0^T \mathcal{G}(t,t^*)dt \leq \frac{2\delta}{\gamma} Te^{\frac{\gamma T}{2\delta}};$$

2. $\max\{\|K(t^*,t,P_1(t))\|, \|K(t^*,t,P_2(t))\|\} \leq \frac{\gamma}{4\delta T} e^{-\frac{\gamma T}{2\delta}} \max\{\|P_1(t)\|, \|P_2(t)\|\};$
3. $\|K(t^*,t,P_i(t))\| \leq \frac{\gamma}{2\delta T} e^{-\frac{\gamma T}{2\delta}} \|P_i(t)\|,\ i = 1,2.$

Then, the dynamic market equilibrium problem (39) has exactly one solution.

Proof. Let $\psi(\varrho_1) = \varrho_1$, $\phi(\varrho_1) = \frac{\varrho_1}{2}$, $P_1(t), P_2(t) \in \mathcal{U}$. Using assumptions (1) and (2), we obtain

$$\begin{aligned}
\psi(\rho(\mathcal{M}P_1(t),\mathcal{M}P_2(t))) &= \max\{\|\mathcal{M}P_1(t)\|, \|\mathcal{M}P_2(t)\|\} \\
&= \max\left\{\left\| \int_0^T \mathcal{G}(t,t^*)K(t^*,t,P_1(t))dt \right\|, \left\| \int_0^T \mathcal{G}(t,t^*)K(t^*,t,P_2(t))dt \right\| \right\} \\
&= \max\left\{ \sup_{t\in[0,T]} \left| \int_0^T \mathcal{G}(t,t^*)K(t^*,t,P_1(t))dt \right|, \sup_{t\in[0,T]} \left| \int_0^T \mathcal{G}(t,t^*)K(t^*,t,P_2(t))dt \right| \right\} \\
&\leq \max\left\{ \sup_{t\in[0,T]} \int_0^T |\mathcal{G}(t,t^*)| \{ \sup_{t\in[0,T]} |K(t^*,t,P_1(t))|, \sup_{t\in[0,T]} |K(t^*,t,P_2(t))| \} dt \right\} \\
&\leq \max\{ \sup_{t\in[0,T]} |K(t^*,t,P_1(t))|, \sup_{t\in[0,T]} |K(t^*,t,P_2(t))| \} \sup_{s\in[0,T]} \int_0^T \mathcal{G}(t,t^*)dt \\
&\leq \max\{ \frac{\gamma}{8\delta T} e^{-\frac{\gamma T}{2\delta}} [\sup_{t\in[0,T]} |P_1(t)|, \sup_{t\in[0,T]} |P_2(t)|] \} \sup_{s\in[0,T]} \left[\int_0^s se^{\frac{\gamma}{2\delta}(t-t^*)}dt + \int_s^T te^{\frac{\gamma}{2\delta}(t-t^*)}dt \right] \\
&< \max\{\|P_1(t)\|, \|P_2(t)\|\} \frac{\gamma}{8\delta T} e^{-\frac{\gamma T}{2\delta}} \frac{2\delta}{\gamma} Te^{\frac{\gamma T}{2\delta}} \\
&= \frac{1}{4} \rho(P_1(t), P_2(t)).
\end{aligned} \tag{42}$$

$$\begin{aligned}
\rho(P_i(t),\mathcal{M}P_i(t)) &= \max\{\|P_i(t)\|, \left\| \int_0^T \mathcal{G}(t,t^*)K(t^*,t,P_i(t))dt \right\| \} \\
&\leq \max\{\|P_i(t)\|, \frac{2\delta}{\gamma} Te^{\frac{\gamma T}{2\delta}} \frac{\gamma}{2\delta T} e^{-\frac{\gamma T}{2\delta}} \|P_i(t)\| \} \\
&= \rho(P_i(t), P_i(t)),\ i = 1,2.
\end{aligned}$$

$$\begin{aligned}
\rho(P_i(t),\mathcal{M}P_j(t)) &= \max\{\|P_i(t)\|, \left\| \int_0^T \mathcal{G}(t,t^*)K(t^*,t,P_j(t))dt \right\| \} \\
&\leq \max\{\|P_i(t)\|, \frac{2\delta}{\gamma} Te^{\frac{\gamma T}{2\delta}} \frac{\gamma}{2\delta T} e^{-\frac{\gamma T}{2\delta}} \|P_j(t)\| \} \\
&= \rho(P_i(t), P_j(t)),\ i \neq j.
\end{aligned}$$

$$\begin{aligned}
\psi(M(P_1(t),P_2(t))) - \phi(N(P_1(t),P_2(t))) &\leq \rho(P_1(t),P_2(t)) - \frac{1}{2}\rho(P_1(t),P_2(t)) \\
&= \frac{1}{2}\rho(P_1(t),P_2(t)).
\end{aligned} \tag{43}$$

Now, combining inequalities (4) and (43)

$$\psi(\rho(\mathcal{M}P_1(t),\mathcal{M}P_2(t))) \leq \psi(M(P_1(t),P_2(t))) - \phi(N(P_1(t),P_2(t))),\ \forall P_1(t),P_2(t) \in \mathcal{U},$$

where,

$$M(P_1(t), P_2(t)) = \max\{\rho(P_1(t), P_2(t)), \rho(P_1(t), \mathcal{M}P_1(t)), \rho(P_2(t), \mathcal{M}P_2(t)), \frac{1}{2}(\rho(P_1(t), \mathcal{M}P_1(t)) + \rho(P_2(t), \mathcal{M}P_2(t))), \frac{1}{2}(\rho(P_1(t), \mathcal{M}P_2(t)) + \rho(P_2(t), \mathcal{M}P_1(t)))\}$$

and

$$N(P_1(t), P_2(t)) = \max\Big\{\rho(P_1(t), P_2(t)), \rho(P_1(t), \mathcal{M}P_1(t)), \rho(P_2(t), \mathcal{M}P_2(t)), \frac{\rho(P_1(t), \mathcal{M}P_1(t))(\rho(P_1(t), \mathcal{M}P_2(t)) + \rho(P_2(t), \mathcal{M}P_1(t)) - \rho(P_2(t), \mathcal{M}P_2(t)))}{1 + \rho(P_1(t), P_2(t))}\Big\}.$$

Thus, all the postulates of Theorem 1 are validated. Hence, an initial value problem (39) has exactly one solution in $\mathcal{U}$. □

5. Conclusions

This current work is motivated by the symmetrical geometry of fixed points performing a remarkable role in nonlinear real-world problems or nonlinear systems and is fascinating and innovative. We have demonstrated the subsistence of a fixed point, common fixed point, ellipse-like curve and common fixed ellipse-like curve for some mathematical operators in a partial metric space by initiating some novel contractions and notions which are completely different from that of the ellipse in a Euclidean space. Appropriate nontrivial examples have validated all the conclusions to compare with the existing ones. As a result, we have explored the symmetrical geometry of the fixed points as well as common fixed points in a partial metric space. Established theorems and corollaries are improved and enhanced variants of renowned conclusions wherein compatibility and continuity (neither their variants) have not been utilized. It is relevant to examine suitable postulates which exclude the possibility of an identity map in Theorems 1, 3 and 4 and Corollaries 1 and 2 in some future work. Toward the end, we investigated the initial value problem appearing in Production–Consumption Equilibrium, which determines the significance of our conclusions. Our conclusions would provide a specific procedure and directions for further investigating/modeling nonlinear systems involving some suitable mathematical operators in a partial metric space which is fascinating in view of the reality that partial metric allows a self distance that is not zero.

Author Contributions: Conceptualization, M.J., S.U., A.T. and M.S.; methodology, M.J. and S.U.; formal analysis, M.J.; investigation, M.J. and S.U.; writing—original draft preparation, M.J. and S.U.; writing—review and editing, A.T. and M.S.; supervision, A.T.; funding acquisition, M.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Deanship of Scientific Research, Qassim University, Saudi Arabia.

Data Availability Statement: Not applicable.

Acknowledgments: Researchers would like to thank the Deanship of Scientific Research, Qassim University for funding publication of this project.

Conflicts of Interest: The authors declare no conflict of interest.

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