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# On Non-Zero Vertex Signed Domination

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**Abstract:** For a graph $G = (V, E)$ and a function $f : V \to \{-1, +1\}$, if $S \subseteq V$ then we write $f(S) = \sum_{v \in S} f(v)$. A function $f$ is said to be a non-zero vertex signed dominating function (for short, NVSDF) of $G$ if $f(N[v]) = 0$ holds for every vertex $v$ in $G$, and the non-zero vertex signed domination number of $G$ is defined as $\gamma_{sb}(G) = \max\{f(V)|f$ is an NVSDF of $G\}$. In this paper, the novel concept of the non-zero vertex signed domination for graphs is introduced. There is also a special symmetry concept in graphs. Some upper bounds of the non-zero vertex signed domination number of a graph are given. The exact value of $\gamma_{sb}(G)$ for several special classes of graphs is determined. Finally, we pose some open problems.

**Keywords:** graph; non-zero vertex signed domination; non-zero vertex signed domination number

**MSC:** 05C05; 05C69

## 1. Introduction

We use Bondy and Murty [1] for terminology and notation not defined here and consider finite, undirected, and simple graph only.

Let $G = (V, E)$ be a graph with the vertex set $V$ and edge set $E$, the order $n = |V(G)|$ of $G$ is the number of its vertices, and its size is $m = |E(G)|$. For $v \in V(G)$, the open neighborhood of $v$ in $G$ is denoted as $N_G(v) = \{u \in V|uv \in E\}$ and $N_G[v] = N_G(v) \cup \{v\}$ for the closed one, $d_G(v) = |N_G(v)|$ is the degree of $v$ in $G$, where $d_G(v)$, $N_G(v)$, and $N_G[v]$ can be abbreviated to $d(v)$, $N(v)$, and $N[v]$, respectively. Then $\delta = \delta(G)$ and $\Delta = \Delta(G)$ denote the minimum degree and maximum degree of $G$. If $S \in V(G)$, then $G[S]$ denotes the subgraph of $G$ induced by $S$. If $A \subseteq V$, $B \subseteq V$ and $A \cap B = \varnothing$, then we write $E(A, B) = \{uv \in E|u \in A, v \in B\}$. For simplicity, if $u, v \in V$, then the symbol $u \sim v$ denotes $uv \in E(G)$.

For any two disjoint graphs $G_1$ and $G_2$, then $G_1 \vee G_2$ denotes the join graph of $G_1$ and $G_2$, where $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv|u \in V(G_1), v \in V(G_2)\}$.

For any two disjoint graphs $G$ and $H$, then $G \Box H$ denotes the product graph of $G$ and $H$, where $V(G \Box H) = \{(u, v)|u \in V(G), v \in V(H)\}$ and $E(G \Box H) = \{(u_1, v_1)(u_2, v_2)|\, u_1 = u_2 and\ v_1 \sim v_2\ or\ v_1 = v_2\ and\ u_1 \sim u_2\}$.

The domination of graphs is an important area in graph theory, and the domination parameters and their variations have been widely studied.

In 1995, J.E. Dunbar et al. [2] first proposed the concept of signed domination of a graph, and E.J. Cockayne et al. [3] generalized it and defined many variations on dominating functions of graphs. Thus, many variations on domination concepts were introduced, such as the signed total domination [4], the clique signed domination [5], the signed Roman domination [6], the signed clique domination [7], the signed and minus domination [8], and so on.

For convenience, given a graph $G = (V, E)$ and a function $f : V \to R$, if $S \subseteq V$, then we write $f(S) = \sum_{v \in S} f(v)$.

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Let $G = (V, E)$ be a graph, a function $f : V \to \{-1, +1\}$ is said to be a signed dominating function (SDF) of $G$, if $f(N[v]) \geq 1$ for every vertex $v \in V(G)$, and the signed domination number of $G$ is defined as:

$$\gamma_s(G) = \min\{f(V)|f \text{ is an SDF of } G\}.$$

In the recent past, B. Xu et al. [9] introduced the following concept of the balanced domination of graphs:

**Definition 1.** *For a graph $G = (V, E)$, a function $f : V \to \{-1, 0, +1\}$ is said to be a balanced dominating function (BDF) of $G$ if $f(N[v]) = 0$ holds for every vertex$v \in V(G)$, and the balanced domination number of $G$ is defined as:*

$$\gamma_b(G) = \max\{f(V)|f \text{ is a BDF of } G\}.$$

Combining the signed domination and balanced domination of graphs, we introduce non-zero vertex signed domination of graphs as following:

**Definition 2.** *For a graph $G = (V, E)$, if there exists a function $f : V \to \{-1, +1\}$ such that $f(N[v]) = 0$ holds for every vertex $v \in V(G)$, then $f$ is said to be a non-zero vertex signed dominating function (NVSDF) of $G$, the graph is called a non-zero vertex signed graph, and the non-zero vertex signed domination number of $G$ is defined as*

$$\gamma_{sb}(G) = \max\{f(V)|f \text{ is an NVSDF of } G\}.$$

A non-zero vertex signed dominating function of $G$ is said to be maximum if $\gamma_{sb}(G) = f(V)$. Obviously, if $f$ is a non-zero vertex signed dominating function of $G$, then $-f$ is also a vertex signed dominating function of $G$, then $\gamma_{sb}(G) \geq 0$ for all non-zero vertex signed graph $G$.

We are interested in the non-zero vertex signed domination in graphs. In this paper, we give some upper bounds of the non-zero vertex signed domination number of a graph and determined the exact value of $\gamma_{sb}(G)$ for several special classes of graphs $G$. Finally, we pose some open problems.

We begin with some basic properties that will be helpful to obtain our results.

**Lemma 1.** *For any non-zero vertex signed graph $G$, if $f$ is a non-zero vertex signed dominating function of $G$, then $f$ is also a non-zero vertex dominating function of $G$. Thus, $\gamma_{sb}(G) \leq \gamma_b(G)$ holds for any non-zero vertex signed graph $G$.*

**Lemma 2.** *For any non-zero vertex signed graph $G$ of order $n$, then*

(1) *The degree of each vertex is odd;*
(2) *$n$ is even;*
(3) *$\gamma_{sb}(G)$ is even.*

**Proof.**

(1) For any non-zero vertex signed dominating function $f$ of $G$, since $f(N[v]) = 0$ holds for every vertex $v$ in $G$, $|N[v]| = d(v) + 1$ is even. Thus, $d(v)$ is odd.
(2) Note that $\sum_{v \in V(G)} d(v) = 2m$ is even and $d(v)$ is odd, which implies that $|V(G)| = n$ is even.
(3) Let $A = \{v \in V(G)|f(v) = 1\}$, $B = \{v \in V(G)|f(v) = -1\}$, $|A| = t$, we have $|B| = n - t$ and $\gamma_{sb}(G) = |A| - |B| = 2t - n$. Note that $n$ is even, then $\gamma_{sb}(G)$ is even.

This completes the proof. □

**Lemma 3.** *For any non-zero vertex signed graph G of order n, if* $\Delta(G) = n - 1$*, then* $\gamma_{sb}(G) = 0$.

**Proof.** Since $\Delta(G) = n - 1$, there exists $v \in V(G)$ such that $d(v) = n - 1$. If $G$ is a non-zero vertex signed graph; let $f$ be a maximum non-zero vertex signed dominating function of $G$, that is, $f(N[v]) = 0$ holds for every vertex $v \in V(G)$. Note that $d(v) = n - 1$, which implies that $f(V(G)) = f(N[v])$, that is, $\gamma_{sb}(G) = f(V(G)) = f(N[v]) = 0$.

This completes the proof. □

## 2. Some Upper Bounds on Non-Zero Vertex Signed Domination Number

In this section, we give some upper bounds on the non-zero vertex signed domination number of graphs.

**Theorem 1.** *For any non-zero vertex signed graph G of order n, if* $m = |E(G)|$*, then*

$$\gamma_{sb}(G) \le 2\left\lfloor n - \frac{\sqrt{n+2m}}{2} \right\rfloor - n.$$

**Proof.** Let $f$ be a maximum non-zero vertex signed dominating function of $G$, that is, $\gamma_{sb}(G) = f(V)$. Let $A = \{v \in V(G) | f(v) = 1\}$, $B = \{v \in V(G) | f(v) = -1\}$, $|A| = t$, we have $|B| = n - t$ and $\gamma_{sb}(G) = |A| - |B| = 2t - n$.

Let $|E(G[B])| = s$. Since $f(N[v]) = 0$ for any $v \in V(G)$, each vertex in $B$ must be adjacent to at least one vertex in $A$. If all vertices in $B$ are not adjacent to each other, which implies that there is no edge of $G[B]$, then $|E(A, B)| = n - t$. If one edge is added to $B$, we have two edges that are added to $E(A, B)$. That is the only way to make $f(N[v_i]) = 0$ for any $v_i \in B$. Note that $|B| = n - t$ and $|E(G[B])| = s$, which implies that $|E(A, B)| = 2s + n - t$. Similarly, if every vertex in $A$ removes an edge that is connected to a vertex in $B$, half of the remaining edges in $E(A, B)$ is the number of edges in $G[A]$. Note that $|A| = t$. We have

$$|E(G[A])| = \frac{|E(A, B)| - t}{2} = s - t + \frac{n}{2}.$$

It is easy to see that

$$|E(G[A])| + |E(G[B])| + |E(A, B)| = |E(G)| = m,$$

$$\left(s - t + \frac{n}{2}\right) + s + (2s + n - t) = m,$$

which means

$$t = \frac{3n}{4} + 2s - \frac{m}{2}.$$

Note that the number of edges in $G[B]$ does not exceed $\binom{n-t}{2}$, which implies that $2s \le (n - t)(n - t - 1)$, then

$$t = \frac{3n}{4} + 2s - \frac{m}{2} \le \frac{3n}{4} + (n - t)(n - t - 1) - \frac{m}{2},$$

$$t \le n - \frac{\sqrt{n+2m}}{2}.$$

It is easy to see that $t$ is an integer and $\gamma_{sb}(G) = 2t - n$, then

$$\gamma_{sb}(G) \leq 2\left\lfloor n - \frac{\sqrt{n+2m}}{2} \right\rfloor - n.$$

This completes the proof. □

**Theorem 2.** *For any non-zero vertex signed graph $G$ of order $n$, if $\delta$ and $\Delta$ denote the minimum degree and maximum degree of the graph, respectively. Then*

$$\gamma_{sb}(G) \leq \frac{(\Delta - \delta)n}{\delta + \Delta + 2}.$$

**Proof.** Let $f$ be a maximum non-zero vertex signed dominating function of graph $G$, that is, $\gamma_{sb}(G) = f(V)$.

Let $A = \{v \in V(G) | f(v) = 1\}$, $B = \{v \in V(G) | f(v) = -1\}$, $|A| = t$, we have $|B| = n - t$ and $\gamma_{sb}(G) = |A| - |B| = 2t - n$.

Let $|E(G[B])| = s$. Since $f(N[v]) = 0$ holds for any $v \in V(G)$, each vertex in $B$ must be adjacent to at least one vertex in $A$. If all vertices in $B$ are not adjacent each other, which implies that there is no edge of $G[B]$, then $|E(A, B)| = n - t$. If one edge is added to $B$, we have two edges are added to $E(A, B)$, that is the only way to make $f(N[v_i]) = 0$ holds for any $v_i \in B$. Note that $|B| = n - t$ and $|E(G[B])| = s$, which implies that $|E(A, B)| = 2s + n - t$. We have $|E(A, B)| \geq n - t$. It is easy to see that

$$|N[v] \cap A| - |N[v] \cap B| = 0,$$

$$|N[v] \cap A| + |N[v] \cap B| = |N[v]| \leq \Delta + 1,$$

holds for each vertex $v \in V(G)$, which means

$$|N[v] \cap A| \leq \frac{\Delta + 1}{2},$$

$$\begin{aligned} \sum_{v_i \in B} |N[v_i] \cap A| &= |E(A, B)|, \\ &\leq \sum_{v_i \in B} \left(\frac{\Delta + 1}{2}\right), \\ &= \left(\frac{\Delta + 1}{2}\right)(n - t), \\ &= \frac{(\Delta + 1)(n - t)}{2}. \end{aligned}$$

Note that $|A| = t$, which implies that there is at least one vertex $u \in A$, and $u$ is at most adjacent to $\frac{(\Delta+1)(n-t)}{2t}$ vertices in $B$. Then $u$ is adjacent to $\frac{(\Delta+1)(n-t)}{2t} - 1$ vertices in $A$. Thus,

$$\delta \leq d(u) \leq 2\frac{(\Delta + 1)(n - t)}{2t} - 1,$$

$$t \leq \frac{(\Delta + 1)n}{\delta + \Delta + 2},$$

$$\gamma_{sb}(G) = 2t - n \leq \frac{(\Delta - \delta)n}{\delta + \Delta + 2}.$$

This completes the proof. □

The next results are an immediate consequence of Theorem 2.

**Corollary 1.** *For any k-regular non-zero vertex signed graph G, we have* $\gamma_{sb}(G) = 0$.

**Corollary 2.** *Any complete graph* $K_n$ *of even order is a non-zero vertex signed graph, and* $\gamma_{sb}(K_n) = 0$.

By reference [9], we know that $\gamma_b(G) \le n + 1 - \sqrt{1+4n}$. Hence, the following corollary can be obtained.

**Corollary 3.** *For any non-zero vertex signed graph G of order n , we have*

$$\gamma_{sb}(G) \le n - 2\left\lfloor \frac{-1+\sqrt{1+4n}}{2} \right\rfloor.$$

**Theorem 3.** *For any non-zero vertex signed graph G of order n , with degree sequence is* $d_1 \le d_2 \le \cdots \le d_n$, *if k is the largest positive integer, that makes the following formula true:*

$$d_{k+1} + d_{k+2} + \cdots + d_n - (d_1 + d_2 + \cdots + d_k) \ge 2k - n,$$

Then

$$\gamma_{sb}(G) \le 2k - n.$$

**Proof.** Let $f$ be a maximum non-zero vertex signed dominating function of graph $G$; we have $\gamma_{sb}(G) = f(V)$. Let $A = \{v \in V(G) | f(v) = 1\}$, $B = \{v \in V(G) | f(v) = -1\}$, $|A| = t$. We have $|B| = n - t$, and $\gamma_{sb}(G) = |A| - |B| = 2t - n$. It is easy to see that $\sum\limits_{v \in V} f(N[v]) = 0$ for every vertex $v$ in $G$, which means

$$\sum_{v \in V} (d(v) + 1) f(v) = 0,$$

and $\sum\limits_{v \in A} (d(v) + 1) - \sum\limits_{v \in B} (d(v) + 1) = 0$. Note, also, that

$$d_{t+1} + d_{t+2} + \cdots + d_n + n - t \ge \sum_{v \in B} (d(v) + 1),$$

$$d_1 + d_2 + \cdots + d_t + t \le \sum_{v \in A} (d(v) + 1).$$

Then

$$d_{t+1} + d_{t+2} + \cdots + d_n - (d_1 + d_2 + \cdots + d_t) + (n - t) - t \ge \sum_{v \in B} (d(v) + 1) - \sum_{v \in A} (d(v) + 1), \tag{1}$$

$$d_{t+1} + d_{t+2} + \cdots + d_n - (d_1 + d_2 + \cdots + d_t) \ge 2t - n. \tag{2}$$

It can be known from the definition of $k$, we have $k \ge t$, which means

$$\gamma_{sb}(G) \le 2k - n. \tag{3}$$

This completes the proof. □

## 3. Some Special Non-Zero Vertex Signed Graphs

In this section, we determine some classes of non-zero vertex signed graphs.

A tree $T = S(t_1, t_2, \cdots, t_n)$ is said to be a caterpillar tree if a path $P_n = (v_1, v_2, \cdots, v_n)$ can be obtained by removing all leaf vertices from $T$. The number of leaf vertices adjacent to $v_i$ is $t_i$. Among them $n \geq 2$, $t_1 \neq 0$ and $t_n \neq 0$. Caterpillar tree $S(2,3,2,2,0,2)$ is shown in Figure 1.

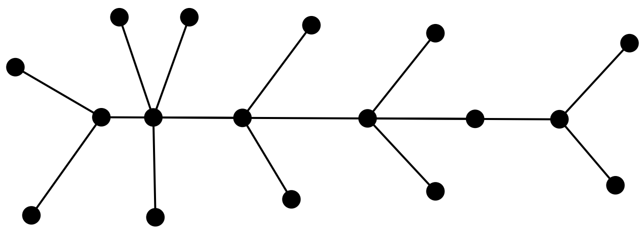

**Figure 1.** Caterpillar tree.

Further, we need the following preliminary results.

**Lemma 4.** *Let $G$ be a connected graph, $|V(G)| = n(n \geq 2)$ and $|E(G)| = m$. Then*

$$\gamma_b(G) \leq \frac{2m - n}{2}$$

The next result is an immediate consequence of Lemma 4.

**Corollary 4.** *For any non-zero vertex signed graph $T$ of order $n$ $(n \geq 2)$, then*

$$\gamma_{sb}(T) \leq \frac{n-2}{2}.$$

**Proof.** Note that any non-zero vertex signed dominating function of $T$ is a balanced dominating function of it, which implies that $\gamma_{sb}(T) \leq \gamma_b(T)$. Let $|E(T)| = m$, then it can be known from lemma 4 $\gamma_b(T) \leq \frac{2m-n}{2}$ and $m = n - 1$. Thus, $\gamma_{sb}(T) \leq \frac{n-2}{2}$.

Let caterpillar tree $T = \left(2, \underbrace{3, 3, \cdots, 3}_{k}, 2\right)$ be the given labeling in Figure 2, and $V(T) = 4k + 6$. The functional value of these vertices with no labels is equal to "+1".

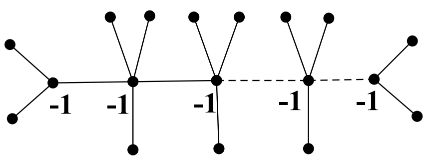


**Figure 2.** Caterpillar tree.1.

Then $\gamma_{sb}(G) = 2(k+1) = \frac{n-2}{2}$, and thus, the existence of an infinite number of caterpillar trees makes the corollary valid.

This completes the proof. □

**Theorem 4.** *The product graph $C_m WP_n$ is a non-zero vertex signed graph if and only if $n = 2$, and $m$ is even.*

**Proof.** These vertices on the inner ring of the product graph are $V\left(C_m^{(1)}\right) = \{v_1, v_2, \cdots, v_m\}$, and these vertices on the outer ring of the product graph are $V\left(C_m^{(2)}\right) = \{u_1, u_2, \cdots, u_m\}$. The product graph is depicted in Figure 3.

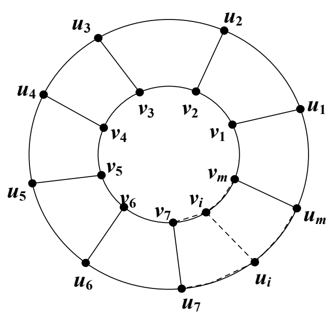


**Figure 3.** Product graph.

Case 1. $n \geq 3$. There exists a vertex $u$ in $C_m WP_n$, where $d(u)$ is even. Then $f(N[v]) \neq 0$ for every $v \in V(C_m WP_n)$, that is, the product graph is not a non-zero vertex signed graph.

Case 2. $n = 2$. The following is an analysis based on the parity of $m$.

Subcase 2.1. $m$ is an even number. The graph is described in Figure 4. It is easy to see that $f(N[v]) = 0$ for every vertex $v$ in $C_m WP_n$, and hence, if $m$ is even, we have $C_m WP_n$ is a non-zero vertex signed graph.

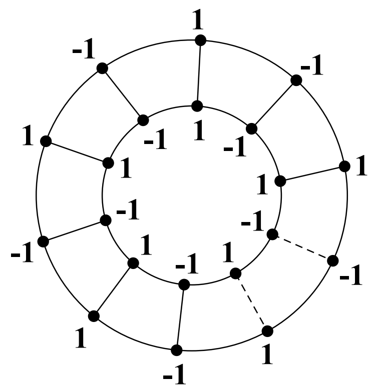

**Figure 4.** Product graph.1.

Subcase 2.2. $m$ is an odd number.

Suppose $C_m WP_n$ is a non-zero vertex signed graph. Then, it satisfies $f(N[v]) = 0$ holds for every vertex $v$ in $C_m WP_n$, we have

$$f(v_i) + f(v_{i+1}) + f(v_{i+2}) + f(u_{i+1}) = 0 \tag{4}$$

Then

$$\sum_{i=1}^{m} (f(v_i) + f(v_{i+1}) + f(v_{i+2}) + f(u_{i+1})) = 0 \tag{5}$$

In this case, the subscript of $v_i$ takes the minimum positive residue of modulo m. Formulas (6) and (7) can be obtained from Formula (5).

$$3f\left(V\left(C_m^{(1)}\right)\right) + f\left(V\left(C_m^{(2)}\right)\right) = 0. \tag{6}$$

$$f\left(V\left(C_m^{(1)}\right)\right) + 3f\left(V\left(C_m^{(2)}\right)\right) = 0 \tag{7}$$

Then

$$f\left(V\left(C_m^{(1)}\right)\right) = f\left(V\left(C_m^{(2)}\right)\right) = 0.$$

Therefore, there must be an even number of vertices in each circle, a contradiction. Therefore, if $m$ is an odd number, and then, the product graph is not a non-zero vertex signed graph.

In summary, the product graph $C_m \Box P_n$ is a non-zero vertex signed graph if and only if $n = 2$, and $m$ is an even number.

This completes the proof. □

**Theorem 5.** *$G = S(t_1, t_2, \cdots, t_n)$ is a caterpillar tree. Then, $G$ is a non-zero vertex signed graph, and the necessary and sufficient conditions are as follows:*

(1) $t_1 = t_n = 2$;
(2) $t_i \in \{1, 3\}(2 \le i \le n-1)$;
(3) *Let $S = \{v \in V(G) | \text{vertex } v \text{ is adjacent to a leaf vertex }\}$. The number of vertices in each branch of $G[S]$ is even.*

**Proof.** After removing all leaf vertices from the caterpillar tree, the road $P_n$ is obtained. The vertices on $P_n$ are $V(P_n) = \{v_1, v_2, \cdots, v_n\}$. The number of leaf vertices adjacent to $v_i$ is $t_i$. Let $v_{ij}$ be the $j$-th leaf vertex adjacent to $v_i$.

**Necessity.** *Note that $G$ is a non-zero vertex signed graph, let $f$ be a non-zero vertex signed dominating function of $G$.*

Case 1. $n = 2$. Note that $f\left(N\left[v_{1j}\right]\right) = 0$, which implies that $f\left(v_{1j}\right) = -f(v_1)$, and because of $f(N[v_1]) = 0$, then $(1 - t_1)f(v_1) + f(v_2) = 0$. If $f(v_1) = f(v_2)$, that is, $t_1 = 2$. Otherwise, $t_1 = 0$. It is easy to see that $t_1 \neq 0$ and $t_n \neq 0$, which means $t_1 = 2$. Similarly, we have $t_n = 2$.

Case 2. $n \ge 3$. It can be seen from case 1 and lemma 2, we have $t_1 = t_n = 2$ and $t_i$ must be an odd number. Note that $f\left(N\left[v_{1j}\right]\right) = 0$, which implies that $f\left(v_{ij}\right) = -f(v_i)$. It is easy to see that $f(N[v_1]) = 0$, which means $(1 - t_i)f(v_i) + f(v_{i-1}) + f(v_{i+1}) = 0$. Note that $f(v_i) \in \{+1 -1\}$ and $t_i$ is a positive integer, then $t_i \in \{+1, -1\}$.

Case 3. Assume to the contrary, the number of vertices of a branch of $G[S]$ is an odd number. Let $|V(S_1)| = 2k + 1$, $k = 0, 1, 2, \cdots, \frac{n-3}{2}$, and $S_1$ is a branch between vertices $v_i$ and $v_{i+2k+3}$, where $t_i \neq 1$ and $t_{i+2k+3} \neq 1$. Might as well make $t_i = t_{i+2k+3} = 3$. Note that $f(N[v]) = 0$ for every vertex $v$ in $G$, which implies that $f(v_i) = f(v_{i+1})$. Might as well make $f(v_i) = f(v_{i+1}) = -1$, because of $f(N[v_{i+1}]) = 0$, there is $f(v_i) = f(v_{i+1})$. Similarly, it can be obtained $f\left(v_{i+4j}\right) = f\left(v_{i+1+4j}\right) = -1$, $f\left(v_{i+2+4j}\right) = f\left(v_{i+3+4j}\right) = 1$. Note that the functional value of the vertices loop in even multiples, but the branch has an odd number of vertices, which implies that $f(v_{i+2k+2}) \neq f(v_{i+2k+3})$, then $f(N[v_{i+2k+3}]) \neq 0$. Thus, the caterpillar tree is not a non-zero vertex signed graph, a contradiction.

**Sufficiency.** It is easy to see that $t_1 = t_n = 2$. Then the number of vertices in each branch of $G[S]$ is even, and $t_i = \{1, 3\}(2 \le i \le n-1)$, where $S = \{v \in V(G) | \textit{vertex } v \textit{ is adjacent to a leaf vertex}\}$.Let $S_1 = S \cup \{v_1, v_n\} = \{v_1{}', v_2{}', \cdots, v_{2g}{}'\}$, where $v_1 = v_1{}', v_n = v_{2g}{}'$. Therefore, when $t_i = 3$, $v_i$ must be between $v_{2k+1}{}'$ and $v_{2k+2}{}'$. Let $f(v_i) = f(v_{2k+1}{}')$, $f(v_{ij}) = -f(v_i)$, $f(v_{1+4a}{}') = f(v_{2+4a}{}') = -1$, $f(v_{3+4a}{}') = f(v_{4+4a}{}') = +1$, where $k = 1, 2, \cdots, \frac{n-2}{2}$, $a = 0, 1, 2, \cdots, \frac{n-4}{4}$, then $f(N[v]) = 0$ holds for every $v$ in $G$. Therefore, according to the definition 2, $G$ is a non-zero vertex signed graph and $f$ is a non-zero vertex signed dominating function of $G$.

This completes the proof. □

**Theorem 6.** *If the join graph $G$ of any two regular graphs is a non-zero vertex signed graph, then one of the following conditions must be satisfied:*

(1) *Both regular graphs are non-zero vertex signed graph,*
(2) *Join graph $G$ is a complete graph of even order.*

**Proof.** Let $G_1 = (V_1, E_1)$ be a $k_1$-regular graph of order $n_1$, $G_2 = (V_2, E_2)$ be a $k_2$-regular graph of order $n_2$. If $G = G_1 \vee G_2$ is a non-zero vertex signed graph, let $f$ be a non-zero vertex signed dominating function of graph $G$. Note that $f(N[v]) = 0$ holds for every vertex $v$ in $G_1$, then

$$f(V_2) + f\left(N_{G_1}[v]\right) = 0. \quad (8)$$

And $f(N[v]) = 0$ holds for every vertex $u$ in $G_2$, then

$$f\left(N_{G_2}[u]\right) + f(V_1) = 0.$$

Note, also, that $\sum_{v \in V_1} f\left(N_{G_i}[v]\right) = n_i f(V_i) = (k_i + 1) f(V_i)$, $i = 1, 2$, then

$$n_2 f(V_1) + (k_2 + 1) f(V_2) = 0,$$

which implies that

$$[n_1 n_2 - (k_1 + 1)(k_2 + 1)] f(V_2) = 0.$$

Case 1. $f(V_2) = 0$. It can be known from formula (8), $f\left(N_{G_1}[v]\right) = 0$ holds for any $v$ in $G_1$, then $G_1$ is a non-zero vertex signed graph. Similarly, $G_2$ is also a non-zero vertex signed graph.

Case 2. $f(V_2) \neq 0$. Then $n_1 n_2 = (k_1 + 1)(k_2 + 1)$. Note that $n_1 \geq (k_1 + 1)$ and $n_2 \geq (k_2 + 1)$, if $n_1 > (k_1 + 1)$, we have $n_2 = \frac{(k_1+1)(k_2+1)}{n_1} < (k_2 + 1)$, a contradiction. Hence, $n_1 = (k_1 + 1)$ and $n_2 = (k_2 + 1)$, which implies that $G$ is a complete graph of even order.

This completes the proof. □

## 4. Some Open Problems

In this paper, we give some properties on the non-zero vertex signed graphs and obtain some upper bounds on the non-zero vertex signed domination number, and the non-zero vertex signed domination number of some special graphs are determined. For all these, it is difficult to obtain the conditions to achieve this upper bound. In addition, the general conditions for determining whether a graph is a non-zero vertex signed graph have not been obtained yet. Therefore, we pose some problems as follows:

**Problem 1.** *Characterize all non-zero vertex signed graphs with*

$$\gamma_{sb}(G) = 2\left\lfloor n - \frac{\sqrt{n + 2m}}{2} \right\rfloor - n. \quad (9)$$

**Problem 2.** *Characterize all non-zero vertex signed graphs with*

$$\gamma_{sb}(G) = n - 2\left\lfloor \frac{-1 + \sqrt{1 + 4n}}{2} \right\rfloor. \quad (10)$$

**Problem 3.** *In Theorem 6, we have characterized the non-zero vertex signed caterpillar tree; how to characterize all non-zero vertex signed trees?*

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