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# [$k$]-Roman Domination in Digraphs

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**Abstract:** Let $\mathscr{D} = (V(\mathscr{D}), A(\mathscr{D}))$ be a finite, simple digraph and $k$ a positive integer. A function $f : V(\mathscr{D}) \to \{0, 1, 2, \ldots, k+1\}$ is called a [$k$]-Roman dominating function (for short, [$k$]-RDF) if $f(AN^-[v]) \geq |AN^-(v)| + k$ for any vertex $v \in V(\mathscr{D})$, where $AN^-(v) = \{u \in N^-(v) : f(u) \geq 1\}$ and $AN^-[v] = AN^-(v) \cup \{v\}$. The weight of a [$k$]-RDF $f$ is $\omega(f) = \sum_{v \in V(\mathscr{D})} f(v)$. The minimum weight of any [$k$]-RDF on $\mathscr{D}$ is the [$k$]-Roman domination number, denoted by $\gamma_{[kR]}(\mathscr{D})$. For $k = 2$ and $k = 3$, we call them the double Roman domination number and the triple Roman domination number, respectively. In this paper, we presented some general bounds and the Nordhaus–Gaddum bound on the [$k$]-Roman domination number and we also determined the bounds on the [$k$]-Roman domination number related to other domination parameters, such as domination number and signed domination number. Additionally, we give the exact values of $\gamma_{[kR]}(P_n)$ and $\gamma_{[kR]}(C_n)$ for the directed path $P_n$ and directed cycle $C_n$.

**Keywords:** [$k$]-Roman dominating function; [$k$]-Roman domination number; domination number; signed domination number; Nordhaus–Gaddum bound

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## 1. Introduction and Terminology

Let $\mathscr{D} = (V(\mathscr{D}), A(\mathscr{D}))$ be a finite and simple digraph. The *order* of $\mathscr{D}$ is denoted by $n(\mathscr{D}) = |V(\mathscr{D})|$. For a vertex $w$ in $V(\mathscr{D})$, its out-neighbourhood (resp. in-neighbourhood) is $N^+_{\mathscr{D}}(w) = \{u \in V(\mathscr{D}) : (w, u) \in A(\mathscr{D})\}$ (resp. $N^-_{\mathscr{D}}(w) = \{u \in V(\mathscr{D}) : (u, w) \in A(\mathscr{D})\}$). The closed out-neighbourhood (resp. closed in-neighbourhood) of $w$ is the set $N^+_{\mathscr{D}}[w] = N^+_{\mathscr{D}}(w) \cup \{w\}$ (resp. $N^-_{\mathscr{D}}[w] = N^-_{\mathscr{D}}(w) \cup \{w\}$). For a vertex subset $\mathscr{Y}$ of $V(\mathscr{D})$, its out-neighbourhood (resp. in-neighbourhood) is $N^+_{\mathscr{D}}(\mathscr{Y}) = \bigcup_{w \in \mathscr{Y}} N^+_{\mathscr{D}}(w)$ (resp. $N^-_{\mathscr{D}}(\mathscr{Y}) = \bigcup_{w \in \mathscr{Y}} N^-_{\mathscr{D}}(w)$). Its closed out-neighbourhood (resp. closed in-neighbourhood) is $N^+_{\mathscr{D}}[\mathscr{Y}] = \bigcup_{w \in \mathscr{Y}} N^+_{\mathscr{D}}[w]$ (resp. $N^-_{\mathscr{D}}[\mathscr{Y}] = \bigcup_{w \in \mathscr{Y}} N^-_{\mathscr{D}}[w]$). For a vertex $w$ in $V(\mathscr{D})$, its out-degree (resp. in-degree) is $d^+_{\mathscr{D}}(w) = |N^+_{\mathscr{D}}(w)|$ (resp. $d^-_{\mathscr{D}}(w) = |N^-_{\mathscr{D}}(w)|$). We usually omit the subscript $\mathscr{D}$ when the digraph is known from the context. The symbols $\Delta^+(\mathscr{D})$, $\Delta^-(\mathscr{D})$, $\delta^+(\mathscr{D})$ and $\delta^-(\mathscr{D})$ denote the maximum out-degree, maximum in-degree, minimum out-degree and minimum in-degree of the digraph $\mathscr{D}$, respectively [1].

For a set $\mathscr{X} \subseteq V(\mathscr{D})$, the subdigraph of $\mathscr{D}$ induced by $\mathscr{X}$ is denoted by $\mathscr{D}[\mathscr{X}]$. Additionally, for two disjoint vertex subsets $\mathscr{X}$ and $\mathscr{Y}$ of $\mathscr{D}$, we define $A[\mathscr{X}, \mathscr{Y}]$ as the arc set satisfying that every arc $(u, v) \in A[\mathscr{X}, \mathscr{Y}]$ with $u \in \mathscr{X}$, $v \in \mathscr{Y}$. The distance from $u$ to $v$, denoted by $d(u, v)$, is the length of the shortest $u$-$v$ path.

A digraph $\mathscr{D}$ is empty if the number of arcs in $\mathscr{D}$ is 0. We write the path of order $n$ as $P_n$ and the cycle of order $n$ as $C_n$.

A connected digraph is a rooted tree if there is one vertex $r$ with $d^-(r) = 0$, called the root, and for any other vertex $v$ distinct from $r$, $d^-(v) = 1$. Let $\mathscr{T}$ be a rooted tree; its height is the maximum distance from the root to any vertex in $\mathscr{D}$. If every vertex of $\mathscr{D}$ has exactly one in-neighbor, we say $\mathscr{D}$ is contrafunctional. The complement $\overline{\mathscr{D}}$ of a digraph $\mathscr{D}$ is a digraph with vertex set $V(\mathscr{D})$ in which $(u,v) \in A(\overline{\mathscr{D}})$ if and only if $(u,v) \notin A(\mathscr{D})$. Please refer to reference [1] for notations and terminology that are not defined here.

A vertex set $\mathscr{S}$ of a digraph $\mathscr{D}$ is called a dominating set if $N^+[\mathscr{S}] = V(\mathscr{D})$. The domination number is $\gamma(\mathscr{D}) = \min\{|\mathscr{S}| : \mathscr{S}$ is a dominating set of $\mathscr{D}\}$. A $\gamma(\mathscr{D})$-set is a dominating set of $\mathscr{D}$ with the cardinality $\gamma(\mathscr{D})$. The concept of $\gamma(\mathscr{D})$ has been widely studied; see [1–3].

Let $\mathscr{G}$ be a bipartite graph and $(\mathcal{L}, \mathcal{R})$ the bipartition of $\mathscr{G}$. If there is a neighbour $y \in \mathscr{S}$ and $\mathscr{S} \subseteq \mathcal{R}$ for every $x \in \mathcal{L}$, then we say $\mathscr{S}$ is a left dominating set. $\gamma_{\mathcal{L}}(\mathscr{G}) = \min\{|\mathscr{S}| : \mathscr{S}$ is a left dominating set of $\mathscr{G}\}$ is the left domination number of $\mathscr{G}$. A $\gamma_{\mathcal{L}}(\mathscr{G})$-set is a left dominating set of $\mathscr{G}$ with the cardinality $\gamma_{\mathcal{L}}(\mathscr{G})$. For a vertex $v$ in $\mathcal{L}$, $\delta_{\mathcal{L}}(\mathscr{G})$ stands for its minimum degree. For more results about the left dominating set, see [4].

A signed dominating function (for short, SDF) in a digraph $\mathscr{D}$ is a function $\varphi : V(\mathscr{D}) \to \{-1, 1\}$ such that $\varphi(N^-[v]) = \sum_{x \in N^-[v]} \varphi(x) \geq 1$ for every vertex $v \in V(\mathscr{D})$. The signed domination number is $\gamma_{\mathscr{S}}(\mathscr{D}) = \min\{\omega(\varphi) : \varphi$ is an SDF of $\mathscr{D}\}$, where $\omega(\varphi) = \sum_{v \in V(\mathscr{D})} \varphi(v)$ is the weight of $\varphi$. If the weight of $\varphi$ is exactly $\gamma_{\mathscr{S}}(\mathscr{D})$, then $\varphi$ is a $\gamma_{\mathscr{S}}(\mathscr{D})$-function. This was further studied in [5].

In 2004, Cockayne and others proposed Roman domination based on Stewart's strategy of defending the Roman Empire. Initially, the study of Roman domination was inspired by the strategies used to defend the Roman Empire during the reign of Emperor Constantine the Great, from 274 to 337 A.D. He decreed that no more than two legions could be stationed in all cities of the Roman Empire. Moreover, if a place was not attacked by legions, then it must be near at least one city where two legions were stationed, so that one of them could be sent to defend the attacked city. The mathematical concept of Roman domination was first defined and discussed by Stewart in 1999 and ReVelle and Rosing in 2000, and was derived from this history of the Roman Empire.

The domination strategy also has many practical applications; for example, it is used in computer science, coding theory, optimal design of connecting networks, etc. Four different types of interconnected components (sink, standby station, power supply substation and power supply station) make up some electrical networks. The sinks need to be connected with the components of a powerful supply station or two less powerful substations. Reserve stations must be connected to the supply element, and because it must be used as electricity storage, at least one reserve station must not be connected to any sink.

Let $\mathscr{D}$ be a finite, simple digraph and $k$ a positive integer. A function $\varphi : V(\mathscr{D}) \to \{0, 1, 2, \ldots, k+1\}$ is called a $[k]$-Roman dominating function (for short, $[k]$-RDF) if $\varphi(AN^-[w]) \geq |AN^-(w)| + k$ for any vertex $w \in V(\mathscr{D})$, where $AN^-(w) = \{u \in N^-(w) : \varphi(u) \geq 1\}$ and $AN^-[w] = AN^-(w) \cup \{w\}$. Its weight $\omega(\varphi)$ is the sum of $\varphi(w)$ for every vertex $w \in V(\mathscr{D})$. The $[k]$-Roman domination number is the minimum weight of $\varphi$ (for short, $[k]$-RD-number), denoted by $\gamma_{[kR]}(\mathscr{D})$. A $[k]$-RDF of $\mathscr{D}$ with the weight $\gamma_{[kR]}(\mathscr{D})$ is called a $\gamma_{[kR]}(\mathscr{D})$-function. In particular, when $k = 1$, the $[k]$-RDF is exactly the Roman dominating function, which has been studied extensively; see [6–9]. When $k = 2$ and $k = 3$, we call them the double Roman dominating function and the triple Roman dominating function; these denotations were introduced in [10–13].

In 2019, G. Hao, X. Chen and L. Volkmann presented the Nordhaus–Gaddum bound on the double Roman domination number in [13]. In 2021, in [11], H.A. Ahangar et al. determined the bounds of the triple Roman domination number related to other domination parameters, such as domination number and signed domination number. As we know, the symmetry of a digraph is significant in theoretical and practical problems. A few digraphs with symmetrical structure, for example the Roman domination of the Kautz digraph and de Bruijn digraph, have been thoroughly studied by authors in [14,15]. Due to their many excellent properties such as small diameter and symmetry, the Kautz digraph and de Bruijn digraph are widely used in computer drum design, VLSI structure design and other fields. At the same time, the de Bruijn digraph and hypercube are considered to be the interconnection network of the real large-scale next-generation multi-computer system. In this paper, the above results are extended to $[k]$-Roman domination numbers of all integers with $k \geq 3$. The contributions of this paper are summarized as follows.

(a) In Section 2, we investigate the $k$-RD-number of the connected digraph with $\delta^-(\mathscr{D}) \geq 1$.

(b) In Section 3, we provide some general bounds for the $k$-RD-number.

(c) In Section 4, we present the Nordhaus–Gaddum bound for $\gamma_{[kR]}(\mathscr{D}) + \gamma_{[kR]}(\overline{\mathscr{D}})$.

(d) In Section 5, we give the bounds of the $[k]$-RD-number related to the domination number and the signed domination number.

(e) In Section 6, we obtain the exact values of $\gamma_{[kR]}(P_n)$ and $\gamma_{[kR]}(C_n)$ for the directed path $P_n$ and the directed cycle $C_n$.

## 2. The $[k]$-RD-Number of a Connected Digraph with $\delta^-(D) \geq 1$

In this section, we give the $[k]$-RD-number of a connected digraph with $\delta^-(\mathscr{D}) \geq 1$. To show the main results, we need a key observation, Proposition 1.

For a $[k]$-RDF, let $V_i = \{v \in V(\mathscr{D}) : \varphi(v) = i\}$ for $i \in \{0, 1, \ldots, k+1\}$. It is noted that there is a bijective correspondence between $\varphi : V(\mathscr{D}) \rightarrow \{0, 1, 2, \ldots, k+1\}$ and $(V_0, V_1, V_2, \ldots, V_{k+1})$ of $\mathscr{D}$. Therefore, we use $(V_0, V_1, V_2, \ldots, V_{k+1})$ to represent $\varphi$ throughout this paper.

**Proposition 1.** *If $\mathscr{D}$ is a directed graph, then there is a $\gamma_{[kR]}(\mathscr{D})$-function $\varphi = (V_0, V_1, \ldots, V_{k+1})$ such that $V_1 = \phi$ can be found.*

**Proof.** Let $\varphi = (V_0, V_1, \ldots, V_{k+1})$ be a $\gamma_{[kR]}(\mathscr{D})$-function such that the number of vertices assigned 1 by $\varphi$ is minimum. Suppose that $V_1 \neq \phi$, that is, there is a vertex $v$ such that $\varphi(v) = 1$. Then, we define a $[k]$-RDF $\tau$ follows: $\tau(v) = 0$, $\tau(u) = \varphi(u) + 1$ for a vertex $u \in AN^-(v)$, and $\tau(x) = \varphi(x)$ for any vertex $x \in V(\mathscr{D}) \backslash \{v, u\}$. This leads to a $\gamma_{[kR]}(\mathscr{D})$-function with fewer vertices assigned to 1, which contradicts the choice of $\varphi$. □

According to Proposition 1, there is a $[k]$-RDF $\varphi = (V_0, V_1, \ldots, V_{k+1})$ with no vertex assigned 1. Without loss of generality, we assume that no vertex is assigned to 1 under consideration when determining the $\gamma_{[kR]}(\mathscr{D})$-function for any digraph $\mathscr{D}$. In this case, arbitrarily $[k]$-RDF $\varphi$ on $\mathscr{D}$ can be written as $\varphi = (V_0, V_2, V_3, \ldots, V_{k+1})$.

**Proposition 2.** *Let $\mathscr{T}$ be a rooted tree with $h(\mathscr{T}) = 1$. Then, $\gamma_{[kR]}(\mathscr{T}) = k + 1$.*

**Proof.** Let $r$ be the root of $\mathscr{T}$. Define a function $\tau : V(\mathscr{D}) \rightarrow \{0, 2, 3, \ldots, k+1\}$ so that $\tau(r) = k + 1$ and $\tau(x) = 0$ otherwise. Then, $\tau$ is a $[k]$-RDF on $\mathscr{D}$, and so $\gamma_{[kR]}(\mathscr{D}) = \omega(\tau) = k + 1$. □

**Theorem 1.** *Let $\mathscr{T} \ncong P_3$ be a rooted tree of order $n \geq 2$. Then, $\gamma_{[kR]}(\mathscr{T}) \leq \frac{(2k+1)n-(k-1)}{3}$.*

**Proof.** We prove the theorem by induction on $n$. If $n = 2$, we have $\gamma_{[kR]}(\mathscr{T}) = k + 1 = \frac{(2k+1)n-(k-1)}{3}$ in accordance with Proposition 2. For the tree $\mathscr{T}'$ of order $m$, the theorem is assumed to be true, where $3 \leq m \leq n - 1$ and $n \geq 3$. If the height of $\mathscr{T}$ is 1, then $\gamma_{[kR]}(\mathscr{T}) = k + 1 < \frac{(2k+1)n-(k-1)}{3}$ in accordance with Proposition 2. Hence, assume that $h(\mathscr{T}) \geq 2$. Let $r$ be the root of $\mathscr{T}$, and $v$ a vertex for $d(r, v) = h(\mathscr{T}) - 1$ and $d^+(v) \geq 1$. Let $\mathscr{T}_1$ be the connected component of $\mathscr{T} - v$ containing the root $r$ and $\mathscr{T}_2 = \mathscr{T} - \mathscr{T}_1$. Because the distance from $r$ to $v$ is $h(T) - 1$, $h(\mathscr{T}_2) = 1$. From Proposition 2, we find that $\gamma_{[kR]}(\mathscr{T}_2) = k + 1 \leq \frac{(2k+1)|V(\mathscr{T}_2)|-(k-1)}{3}$. If $|V(\mathscr{T}_1)| = 1$, then $|V(\mathscr{T}_2)| \geq 3$ by $\mathscr{T} \ncong$

$P_3$. Let $\tau : V(\mathscr{D}) \to \{0,2,3,\ldots,k+1\}$ be defined as follows: $\tau(v) = k+1$, $\tau(r) = k$ and $\tau(x) = 0$ otherwise. Then, $\tau$ is a $[k]$-RDF on $\mathscr{T}$, and so $\gamma_{[kR]}(\mathscr{T}) \le \omega(\tau) = 2k+1 < \frac{(2k+1)n-(k-1)}{3}$. Now we assume that $|V(\mathscr{T}_1)| \ge 2$. If $\mathscr{T}_1 \ncong P_3$, then we have $\gamma_{[kR]}(\mathscr{T}_1) \le \frac{(2k+1)|V(\mathscr{T}_1)|-(k-1)}{3}$ given the induction hypotheses. Furthermore, $\gamma_{[kR]}(\mathscr{T}) \le \gamma_{[kR]}(\mathscr{T}_1) + \gamma_{[kR]}(\mathscr{T}_2) < \frac{(2k+1)|V(\mathscr{T}_1)|-(k-1)}{3} + \frac{(2k+1)|V(\mathscr{T}_2)|-(k-1)}{3} < \frac{(2k+1)n-(k-1)}{3}$. If $\mathscr{T}_1 \cong P_3$, then $\gamma_{[kR]}(\mathscr{T}_1) = 2k+1 = \frac{(2k+1)|V(\mathscr{T}_1)|}{3}$. Hence, $\gamma_{[kR]}(\mathscr{T}) \le \gamma_{[kR]}(\mathscr{T}_1) + \gamma_{[kR]}(\mathscr{T}_2) \le \frac{(2k+1)|V(\mathscr{T}_1)|}{3} + \frac{(2k+1)|V(\mathscr{T}_2)|-(k-1)}{3} = \frac{(2k+1)n-(k-1)}{3}$. The proof is completed. □

**Theorem 2** ([16])**.** *Let $\mathscr{D}$ be a contrafunctional digraph that is connected. Then, $\mathscr{D}$ has a unique directed cycle.*

For a directed graph $\mathscr{D}$ that is connected and contrafunctional, given Theorem 2 and the definition of the rooted tree $\mathscr{T}_{\mathscr{D}}$, it is clear that $\gamma_{[kR]}(\mathscr{D}) \le \gamma_{[kR]}(\mathscr{T}_{\mathscr{D}})$. Combined with Theorem 1, these facts will lead to the following conclusion.

**Corollary 1.** *Let $\mathscr{D}$ be a directed graph that is connected and contrafunctional. Then, $\gamma_{[kR]}(\mathscr{D}) = 2k+1$ if $\mathscr{D} \cong C_3$, and $\gamma_{[kR]}(\mathscr{D}) \le \frac{(2k+1)n-(k-1)}{3}$ otherwise, where $|V(\mathscr{D})| = n$.*

**Theorem 3.** *If $\mathscr{D} \ncong C_3$ is a connected digraph of order $n \ge 3$ with minimum in-degree $\delta^-(\mathscr{D}) \ge 1$, then $\gamma_{[kR]}(\mathscr{D}) \le \frac{(2k+1)n-(k-1)}{3}$.*

**Proof.** We prove the theorem by induction on $n$. If $n = 3$, because $\mathscr{D} \ncong C_3$ and $\delta^-(\mathscr{D}) \ge 1$, it is easy to see that $\gamma_{[kR]}(\mathscr{D}) = k+1 < \frac{(2k+1)n-(k-1)}{3}$. Suppose $n \ge 4$. For every vertex of $\mathscr{D}$, there is an incoming arc that can be chosen by $\delta^-(\mathscr{D}) \ge 1$. All such arcs induce a spanning subdigraph $\mathscr{H}$ of $\mathscr{D}$, and $\mathscr{H}$ consists of some connected components, which are denoted as $\mathscr{H}_1, \mathscr{H}_2, \ldots, \mathscr{H}_l$. In addition, because the in-degree of each vertex in $\mathscr{H}_i$ is 1 for $i \in \{1,2,\ldots,l\}$, this implies that $\mathscr{H}_i$ is a subdigraph of $\mathscr{D}$ that is connected and contrafunctional.

Now, we consider that not all $\mathscr{H}_i$ are isomorphic to $C_3$. In general, we may assume that there exist $m$ connected components not isomorphic to $C_3$ and $l-m$ connected components isomorphic to $C_3$, which are denoted by $\mathscr{H}_i$ for $i \in \{1,2,\ldots,m\}$ and $\mathscr{H}_j$ for $j \in \{m+1, m+2,\ldots,l\}$, respectively. According to Corollary 1, we have $\gamma_{[kR]}(\mathscr{H}_i) \le \frac{(2k+1)|V(\mathscr{H}_i)|-(k-1)}{3}$ for any $\mathscr{H}_i \ncong C_3$ and $\gamma_{[kR]}(\mathscr{H}_j) = \frac{(2k+1)\left|V(\mathscr{H}_j)\right|}{3}$ for any $\mathscr{H}_j \cong C_3$. Hence,

$$\begin{aligned}
\gamma_{[kR]}(\mathscr{D}) \le \gamma_{[kR]}(\mathscr{H}) &= \sum_{i=1}^{l} \gamma_{[kR]}(\mathscr{H}_i) \\
&\le \sum_{i=1}^{m} \frac{(2k+1)|V(\mathscr{H}_i)|-(k-1)}{3} + \sum_{i=m+1}^{l} \frac{(2k+1)|V(\mathscr{H}_i)|}{3} \\
&\le \frac{(2k+1)n-(k-1)}{3}.
\end{aligned}$$

Next, we consider that all $\mathscr{H}_i$ are isomorphic to $C_3$ for $i \in \{1,2,\ldots,l\}$. $l \ge 2$ because of $n \ge 4$. Notice that $\mathscr{D}$ is connected and $\mathscr{H}$ is not connected; this implies that there is at least one arc that is in $\mathscr{D}$ but $\mathscr{H}$. If we take the arc in $A(\mathscr{D}) \backslash A(\mathscr{H})$ and add it to $\mathscr{H}$, then, as shown in Figure 1, it is easy to verify that $\mathscr{D}$ has a $[k]$-RD-number which is strictly smaller than $\mathscr{H}$ by $k$. Therefore, we find that $\gamma_{[kR]}(\mathscr{D}) \le \gamma_{[kR]}(\mathscr{H}) - k = \sum_{i=1}^{l} \gamma_{[kR]}(\mathscr{H}_i) - k = \sum_{i=1}^{l} \frac{(2k+1)|V(\mathscr{H}_i)|}{3} - k < \frac{(2k+1)n-(k-1)}{3}$ given Corollary 1. □

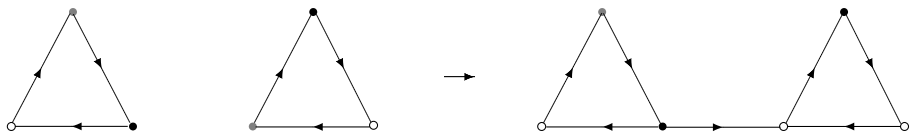

**Figure 1.** Black circles denote the vertices in $V_{k+1}$, grey circles denote the vertices in $V_k$, and white circles denote the vertices in $V_0$.

### 3. Some Bounds of the [$k$]-RD-Number

In this section, some general bounds for $\gamma_{[kR]}(\mathscr{D})$ are presented. We first provide the upper bounds of $\gamma_{[kR]}(\mathscr{D})$.

**Proposition 3.** *If $\mathscr{D}$ is a directed graph with $|V(\mathscr{D})| = n$, then $\gamma_{[kR]}(\mathscr{D}) \leq kn$. Furthermore, $\gamma_{[kR]}(\mathscr{D}) = kn$ if and only if there is no arc in $\mathscr{D}$.*

**Proof.** Define a function $\tau : V(\mathscr{D}) \to \{0, 2, 3, \dots, k+1\}$ such that $\tau(v) = k$ for each vertex of $\mathscr{D}$. Then, $\tau$ is a [$k$]-RDF on $\mathscr{D}$ and hence $\gamma_{[kR]}(\mathscr{D}) \leq \omega(\tau) = kn$. The sufficiency is obvious, so here we only show the necessity. Suppose, to the contrary, that there are two vertices $u$, $v$ such that $(u, v) \in A(\mathscr{D})$. Define a function $g : V(\mathscr{D}) \to \{0, 2, 3, \dots, k+1\}$ such that $g(v) = 0$, $g(u) = k+1$ and $g(x) = k$ for other vertices. Then, $g$ is a [$k$]-RDF with $\omega(g) = kn - k + 1 < kn$. Thus, $\gamma_{[kR]}(\mathscr{D}) \leq \omega(g) = kn - k + 1 < kn = \gamma_{[kR]}(\mathscr{D})$, a contradiction. □

**Theorem 4.** *Let $\mathscr{D}$ be a directed graph with $|V(\mathscr{D})| = n$ and $A(\mathscr{D}) \neq 0$. Then, $\gamma_{[kR]}(\mathscr{D}) \leq kn - k + 1$. Furthermore, $\gamma_{[kR]}(\mathscr{D}) = kn - k + 1$ iff there is exactly one nontrivial connected component $\mathscr{H}$ in $\mathscr{D}$, where $2 \leq |V(\mathscr{H})| \leq 3$, and when $\mathscr{H}$ has three vertices, $\mathscr{H}$ is $P_3$-path or $C_3$-cycle.*

**Proof.** Because $\mathscr{D}$ is a non-empty digraph, we have $\gamma_{[kR]}(\mathscr{D}) \leq kn - 1$ according to Proposition 3. Suppose, to the contrary, that $kn - k + 2 \leq \gamma_{[kR]}(\mathscr{D}) \leq kn - 1$. Because $|A(\mathscr{D})| \geq 1$, there are two vertices $u$, $v$ such that $(u, v) \in A(\mathscr{D})$. Define a function $g : V(\mathscr{D}) \to \{0, 2, 3, \dots, k+1\}$ such that $g(v) = 0$, $g(u) = k+1$ and $g(x) = k$ for other vertices $x$. Then, $g$ is a [$k$]-RDF with $\omega(g) = kn - k + 1$. Hence, $\gamma_{[kR]}(\mathscr{D}) \leq \omega(g) = kn - k + 1 < kn - k + 2$, a contradiction. Thus, $\gamma_{[kR]}(\mathscr{D}) \leq kn - k + 1$.

($\Rightarrow$) To prove the necessity, assume that $\gamma_{[kR]}(\mathscr{D}) = kn - k + 1$. First, suppose to the contrary that $\mathscr{D}$ contains at least two nontrivial connected components. Then, we can choose two arcs, say $(u, z)$ and $(s, t)$, from two distinct connected components. Define a function $\tau_1 : V(\mathscr{D}) \to \{0, 2, 3, \dots, k+1\}$ such that $\tau_1(u) = \tau_1(s) = k+1$, $\tau_1(z) = \tau_1(t) = 0$ and $\tau_1(x) = k$ for other vertices $x$. Then, $\tau_1$ is a [$k$]-RDF on $\mathscr{D}$, and so $\gamma_{[kR]}(\mathscr{D}) \leq \omega(\tau_1) = kn - 2(k-1) < kn - k + 1 = \gamma_{[kR]}(\mathscr{D})$, a contradiction. Therefore, $\mathscr{D}$ has exactly one nontrivial connected component, say $\mathscr{H}$.

Now we show that the unique nontrivial component is $\mathscr{H}$ with no more than three vertices. If there are more than three vertices in $\mathscr{H}$, we can obtain the contradiction by distinguishing three cases as follows:

Case 1: There are four distinct vertices $u, z, s, t$ such that $\{(u, z), (s, t)\} \subseteq A(\mathscr{D})$.

With the same method as above, there is a contradiction.

Case 2: There are three different vertices $u, z, t$ such that $\{(u, z), (u, t)\} \subseteq A(\mathscr{D})$.

Define a function $\tau_2 : V(\mathscr{D}) \to \{0, 2, 3, \dots, k+1\}$ such that $\tau_2(u) = k+1$, $\tau_2(z) = \tau_2(t) = 0$ and $\tau_2(x) = k$ for other vertices $x$. Then, $\tau_2$ is a [$k$]-RDF on $\mathscr{D}$, and so $\gamma_{[kR]}(\mathscr{D}) \leq \omega(\tau_2) = kn - 2k + 1 < kn - k + 1 = \gamma_{[kR]}(\mathscr{D})$, a contradiction.

Case 3: There are three different vertices $u, v, s$ such that $\{(u, z), (s, z)\} \subseteq A(\mathscr{D})$.

Define a function $\tau_3 : V(\mathscr{D}) \to \{0,2,3,\ldots,k+1\}$ such that $\tau_3(z) = 0$ and $\tau_3(x) = k$ otherwise. Thus, $\tau_3$ is a $[k]$-RDF on $\mathscr{D}$, and so $\gamma_{[kR]}(\mathscr{D}) \le \omega(\tau_3) = kn - k < kn - k + 1 = \gamma_{[kR]}(\mathscr{D})$, a contradiction.

Consequently, $2 \le |V(\mathscr{H})| \le 3$. Furthermore, following the arguments of Case 2 and Case 3, we find that $\mathscr{H}$ is $P_3$-path or $C_3$-cycle when $|V(\mathscr{H})| = 3$.

$(\Leftarrow)$ Assume that $\mathscr{D}$ contains exactly one nontrivial connected component $\mathscr{H}$ with $2 \le |V(\mathscr{H})| \le 3$, and $\mathscr{H}$ is $P_3$-path or $C_3$-cycle when there are three vertices in $\mathscr{H}$. If there are two vertices in $\mathscr{H}$, $\gamma_{[kR]}(\mathscr{D}) = \gamma_{[kR]}(\mathscr{D}[V(\mathscr{D})\backslash V(\mathscr{H})]) + \gamma_{[kR]}(\mathscr{H}) = k(n-2) + (k+1) = kn - k + 1$. If there are three vertices in $\mathscr{H}$ and $\mathscr{H}$ is $P_3$-path or $C_3$-cycle, $\gamma_{[kR]}(\mathscr{D}) = \gamma_{[kR]}(\mathscr{D}[V(\mathscr{D})\backslash V(\mathscr{H})]) + \gamma_{[kR]}(\mathscr{H}) = k(n-3) + (k+1+k) = kn - k + 1$. $\square$

**Lemma 1.** *Let $\mathscr{D}$ be a digraph with $|V(\mathscr{D})| = n$ and maximum out-degree $\Delta^+(\mathscr{D})$. Then, $\gamma_{[kR]}(\mathscr{D}) \le k(n - \Delta^+(\mathscr{D})) + 1$.*

**Proof.** Let $w$ be a vertex with the maximum out-degree $\Delta^+(\mathscr{D})$. Define a function $\tau : V(\mathscr{D}) \to \{0,2,3,\ldots,k+1\}$ such that $\tau(w) = k+1$, $\tau(z) = 0$ for any vertex $z \in N^+(w)$ and $\tau(u) = k$ for other vertices $u$. Then, $\tau$ is a $[k]$-RDF on $\mathscr{D}$. Hence, $\gamma_{[kR]}(\mathscr{D}) \le \omega(\tau) = (k+1) + k(n - 1 - \Delta^+(\mathscr{D})) = k(n - \Delta^+(\mathscr{D})) + 1$. $\square$

**Theorem 5.** *Let $\mathscr{D}$ be a digraph of order $n$. Then*

$$\gamma_{[kR]}(\mathscr{D}) \le \lfloor \frac{(k+1)n}{\delta^-(\mathscr{D})+1}\big(\ln \frac{k(\delta^-(\mathscr{D})+1)}{k+1} + 1\big)\rfloor.$$

**Proof.** Let $\mathscr{U}$ be a vertex set of $\mathscr{D}$ satisfying the possibility that the vertices in any $\mathscr{U}$ are independently selected is $p$, where $0 \le p \le 1$. Thus, the expected size of $\mathscr{U}$, denoted by $\mathbf{E}[|\mathscr{U}|]$, is $np$. Let $\mathscr{W} = V(\mathscr{D})\backslash N^+[\mathscr{U}]$. Then

$$\begin{aligned} P[v \in \mathscr{W}] &= P[v \notin \mathscr{U} \text{ and } u \notin \mathscr{U} \text{ for } u \in N^-(v)] \\ &= (1-p)(1-p)^{d^-(v)} \\ &= (1-p)^{d^-(v)+1} \\ &\le (1-p)^{\delta^-(\mathscr{D})+1}. \end{aligned}$$

Hence, $\mathbf{E}[|\mathscr{W}|] = n(1-p)^{d^-(v)+1} \le n(1-p)^{\delta^-(\mathscr{D})+1}$. Let $\tau : V(\mathscr{D}) \to \{0,2,3,\ldots,k+1\}$ be defined as follows: $\tau(w) = k+1$ for any vertex $w \in \mathscr{U}$, $\tau(z) = k$ for any vertex $z \in \mathscr{W}$, and $\tau(x) = 0$ for any vertex $x \in N^+(\mathscr{U})$. Then, the expected size of $\tau$ is

$$\mathbf{E}[\omega(\tau)] = \mathbf{E}[(k+1)|\mathscr{U}| + k|\mathscr{W}|] \le (k+1)np + kn(1-p)^{\delta^-(D)+1}.$$

Because $1 - p \le e^{-p}$ when $0 \le p \le 1$, we have $\mathbf{E}[\omega(\tau)] \le (k+1)np + kne^{-p(\delta^-(\mathscr{D})+1)}$. We can further know that the upper bound of $\mathbf{E}[\omega(\tau)]$ is at its minimum when $p = \frac{1}{\delta^-(\mathscr{D})+1} \ln \frac{k(\delta^-(\mathscr{D})+1)}{k+1}$, therefore $\mathbf{E}[\omega(\tau)] \le \frac{(k+1)n}{\delta^-(\mathscr{D})+1}\left(\ln \frac{k(\delta^-(\mathscr{D})+1)}{k+1} + 1\right)$. This implies that $\gamma_{[kR]}(\mathscr{D}) \le \left\lfloor \frac{(k+1)n}{\delta^-(\mathscr{D})+1}\left(\ln \frac{k(\delta^-(\mathscr{D})+1)}{k+1} + 1\right)\right\rfloor$. $\square$

We now establish the lower bound of $\gamma_{[kR]}(\mathscr{D})$.

**Theorem 6.** *Let $\mathscr{D}$ be a connected digraph with $|V(\mathscr{D})| = n$. Then, $\gamma_{[kR]}(\mathscr{D}) \ge \left\lceil \frac{2n(2+k)}{2+k+2\Delta^+(\mathscr{D})} \right\rceil$.*

**Proof.** Let $\tau = (V_0, V_2, V_3, \ldots, V_{k+1})$ be a $\gamma_{[kR]}(\mathscr{D})$-function and $|V_0| = n_0$. We consider two cases:

Case 1: $V_{k+1} = \phi$.

If $n_0 = 0$, then $\gamma_{[kR]}(\mathscr{D}) = \sum_{i=2}^{k} i|V_i| = 2\sum_{i=2}^{k} |V_i| + \sum_{i=3}^{k}(i-2)|V_i| \geq 2n \geq \left\lceil \frac{2n(2+k)}{2+k+2\Delta^+(\mathscr{D})} \right\rceil$.

If $n_0 \neq 0$, then $\gamma_{[kR]}(\mathscr{D}) = \sum_{i=2}^{k} i|V_i| = 2\sum_{i=2}^{k} |V_i| + \sum_{i=3}^{k}(i-2)|V_i| \geq 2(n-n_0)$. Because $\tau = (V_0, V_2, V_3, \ldots, V_{k+1})$ is a $\gamma_{[kR]}(\mathscr{D})$-function, we have $\sum_{u \in N^-(v)} \tau(u) \geq |AN^-(v)| + k \geq 2+k$ for each vertex $v \in V_0$. Then, $\gamma_{[kR]}(\mathscr{D}) = \omega(\tau) \geq \frac{(2+k)n_0}{\Delta^+(\mathscr{D})}$, implying that

$$\frac{2+k}{2}\gamma_{[kR]}(\mathscr{D}) \geq (2+k)n - (2+k)n_0 \geq (2+k)n - \Delta^+(\mathscr{D})\gamma_{[kR]}(\mathscr{D}).$$

Hence, $\gamma_{[kR]}(\mathscr{D}) \geq \left\lceil \frac{2n(2+k)}{2+k+2\Delta^+(\mathscr{D})} \right\rceil$.

Case 2: $V_{k+1} \neq \phi$.

If $n_0 = 0$, then $\gamma_{[kR]}(\mathscr{D}) = \sum_{i=2}^{k+1} i|V_i| = 2\sum_{i=2}^{k+1} |V_i| + \sum_{i=3}^{k+1}(i-2)|V_i| \geq 2n + (k-1)|V_{k+1}| \geq 2n + k - 1 > \left\lceil \frac{2n(2+k)}{2+k+2\Delta^+(\mathscr{D})} \right\rceil$. If $n_0 \neq 0$, then $\gamma_{[kR]}(\mathscr{D}) = \sum_{i=2}^{k+1} i|V_i| = 2\sum_{i=2}^{k+1} |V_i| + \sum_{i=3}^{k+1}(i-2)|V_i| \geq 2(n-n_0) + (k-1)|V_{k+1}| \geq 2(n-n_0) + (k-1)$. Because $\tau = (V_0, V_2, V_3, \ldots, V_{k+1})$ is a $\gamma_{[kR]}(\mathscr{D})$-function, we have $\sum_{u \in N^-(v)} \tau(u) \geq |AN^-(v)| + k \geq 1+k$ for each vertex $v \in V_0$. Then, $\gamma_{[kR]}(\mathscr{D}) = \omega(\tau) \geq \frac{(1+k)n_0}{\Delta^+(\mathscr{D})}$, implying that

$$\begin{aligned}\frac{1+k}{2}\gamma_{[kR]}(\mathscr{D}) &\geq (1+k)n - (1+k)n_0 + \frac{1+k}{2}(k-1) \geq (1+k)n \\ &\quad - \Delta^+(\mathscr{D})\gamma_{[kR]}(\mathscr{D}) + \frac{1+k}{2}(k-1).\end{aligned}$$

Hence, $\gamma_{[kR]}(\mathscr{D}) \geq \left\lceil \frac{(1+k)(2n+k-1)}{1+k+2\Delta^+(\mathscr{D})} \right\rceil > \left\lceil \frac{2n(2+k)}{2+k+2\Delta^+(\mathscr{D})} \right\rceil$. □

## 4. Nordhaus–Gaddum Bounds on the [$k$]-RD-Number

In this part, we establish Nordhaus–Gaddum bounds for $\gamma_{[kR]}(\mathscr{D}) + \gamma_{[kR]}(\overline{\mathscr{D}})$.

**Theorem 7.** *Let $\mathscr{D}$ be a digraph of order $n \geq k+1$ for $k \geq 3$. Then, $\gamma_{[kR]}(\mathscr{D}) + \gamma_{[kR]}(\overline{\mathscr{D}}) \leq kn + k + 1$.*

**Proof.** Because $d^+_{\mathscr{D}}(v) + d^+_{\overline{\mathscr{D}}}(v) = n-1$ for each $v \in V(\mathscr{D})$, we see that $\Delta^+(\overline{\mathscr{D}}) = n - 1 - \delta^+(\mathscr{D})$. According to Lemma 1, we have

$$\begin{aligned}\gamma_{[kR]}(\mathscr{D}) + \gamma_{[kR]}(\overline{\mathscr{D}}) &\leq (k(n - \Delta^+(\mathscr{D})) + 1) + (k(n - \Delta^+(\overline{\mathscr{D}})) + 1) \\ &= kn - k\Delta^+(\mathscr{D}) + k\delta^+(\mathscr{D}) + k + 2 \\ &\leq kn + k + 2.\end{aligned}$$

Now assume that $\gamma_{[kR]}(\mathscr{D}) + \gamma_{[kR]}(\overline{\mathscr{D}}) = kn + k + 2$, then $\Delta^+(\mathscr{D}) = \delta^+(\mathscr{D})$, given the above inequality chain. Let $\Delta^+(\mathscr{D}) = \delta^+(\mathscr{D}) = m$, then $\Delta^+(\overline{\mathscr{D}}) = \delta^+(\overline{\mathscr{D}}) = n - 1 - m$. Furthermore, we have that $\gamma_{[kR]}(\mathscr{D}) = k(n-m) + 1$ and $\gamma_{[kR]}(\overline{\mathscr{D}}) = k(m+1) + 1$ by $\gamma_{[kR]}(\mathscr{D}) + \gamma_{[kR]}(\overline{\mathscr{D}}) = kn + k + 2$. Let $v \in V(\mathscr{D})$ be arbitrary.

Claim 1: For every vertex $u \in V(\mathscr{D}) \backslash N^+_{\mathscr{D}}[v]$, it must be that $N^+_{\mathscr{D}}(u) \subseteq N^+_{\mathscr{D}}[v]$.

**Proof.** Proving by contradiction, assume that there exists a vertex $u \in V(\mathscr{D}) \backslash N^+_{\mathscr{D}}[v]$ such that $w \in N^+_{\mathscr{D}}[u] \backslash N^+_{\mathscr{D}}[v]$ (see Figure 2a). Let $g_1 : V(\mathscr{D}) \to \{0, 2, 3, \ldots, k+1\}$ be defined as follows: $g_1(v) = k+1$, $g_1(x) = 0$ for any vertex $x \in N^+_{\mathscr{D}}(v)$, $g_1(w) = 1$, and $g_1(y) = k$

otherwise. Then $g_1$ is a $[k]$-RDF on $\mathscr{D}$. Thus, $\gamma_{[kR]}(\mathscr{D}) \leq \omega(g_1) = k(n-m) - k + 2 < k(n-m) + 1 = \gamma_{[kR]}(\mathscr{D})$, a contradiction. □

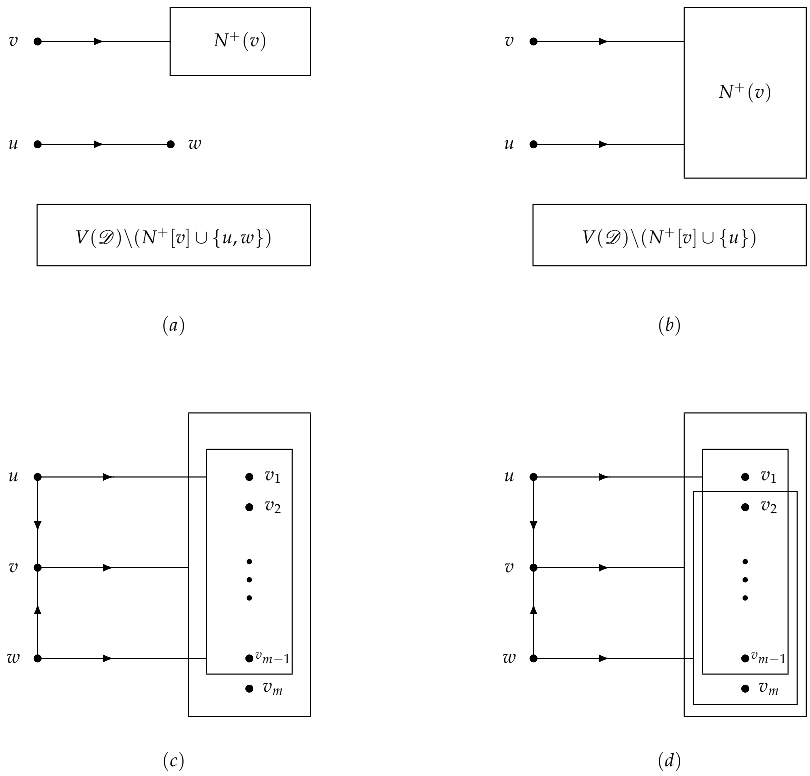


**Figure 2.** The counterexample of Claims 1–4 in Theorem 7.

Claim 2: For any vertex $u \in V(\mathscr{D})\backslash N^+_{\mathscr{D}}[v]$, it must be that $(u, v) \in A(\mathscr{D})$.

**Proof.** Proving by contradiction, assume that there exists a vertex $u \in V(\mathscr{D})\backslash N^+_{\mathscr{D}}[v]$ such that $(u, v) \notin A(\mathscr{D})$. Because $\Delta^+(\mathscr{D}) = \delta^+(\mathscr{D}) = m$, given Claim 1, $N^+_{\mathscr{D}}(u) = N^+_{\mathscr{D}}(v)$ (see Figure 2b). Let $g_2 : V(\mathscr{D}) \to \{0, 2, 3, \ldots, k+1\}$ be defined as follows: $g_2(x) = 0$ for any vertex $x \in N^+_{\mathscr{D}}(v)$ and $g_2(y) = k$ otherwise. Then, $g_2$ is a $[k]$-RDF on $\mathscr{D}$, and so $\gamma_{[kR]}(\mathscr{D}) \leq \omega(g_2) = k(n-m) < k(n-m) + 1 = \gamma_{[kR]}(\mathscr{D})$, a contradiction. □

Claim 3: $|V(\mathscr{D})\backslash N^+_{\mathscr{D}}[v]| \leq 1$.

**Proof.** Proving by contradiction, assume that there are two vertices $\{u, w\} \subseteq V(\mathscr{D})\backslash N^+_{\mathscr{D}}[v]$. Given Claim 2, we have $\{(u, v), (w, v)\} \subseteq A(\mathscr{D})$. Because $\Delta^+(\mathscr{D}) = \delta^+(\mathscr{D}) = m$, by Claim 1, we find that either $N^+_{\mathscr{D}}(u) = N^+_{\mathscr{D}}(w)$ or $|N^+_{\mathscr{D}}(v)\backslash(N^+_{\mathscr{D}}(u) \cap N^+_{\mathscr{D}}(w))| = 2$. Let $N^+_{\mathscr{D}}(v) = \{v_1, v_2, \ldots, v_m\}$. If $N^+_{\mathscr{D}}(u) = N^+_{\mathscr{D}}(w)$, then we may assume $N^+_{\mathscr{D}}(u) = N^+_{\mathscr{D}}(w) = \{v, v_1, v_2, \ldots, v_{m-1}\}$ (see Figure 2c). Define a function $g_3 : V(\mathscr{D}) \to \{0, 2, 3, \ldots, k+1\}$ such that $g_3(x) = 0$ for any vertex $x \in \{v, v_1, v_2, \ldots, v_{m-1}\}$ and $g_3(y) = k$ otherwise. Then, $g_3$ is a $[k]$-RDF on $\mathscr{D}$ and $\gamma_{[kR]}(\mathscr{D}) \leq \omega(g_3) = k(n-m) < k(n-m) + 1 = \gamma_{[kR]}(\mathscr{D})$, a contradiction. If $|N^+_{\mathscr{D}}(v)\backslash(N^+_{\mathscr{D}}(u) \cap N^+_{\mathscr{D}}(w))| = 2$, without loss of generality, let $N^+_{\mathscr{D}}(u) = \{v, v_1, v_2, \ldots, v_{m-1}\}$ and $N^+_{\mathscr{D}}(w) = \{v, v_2, v_3, \ldots, v_m\}$ (see Figure 2d). Define a function $g_4 : V(\mathscr{D}) \to \{0, 2, 3, \ldots, k+1\}$ such that $g_4(x) = 0$ for any vertex $x \in \{v, v_2, v_3, \ldots, v_{m-1}\}$ and $g_4(y) = 1$ for any vertex $y \in \{v_1, v_m\}$ and $g_4(z) = k$ otherwise. Then, $g_4$ is a $[k]$-RDF on $\mathscr{D}$ and $\gamma_{[kR]}(\mathscr{D}) \leq \omega(g_4) = k(n-m) - k + 2 < k(n-m) + 1 = \gamma_{[kR]}(\mathscr{D})$, a contradiction. □

From Claim 3, we have $m = d^+_{\mathscr{D}}(v) \geq n-2$ for any vertex $v \in V(\mathscr{D})$. Notice that the discussion is symmetrical for $\mathscr{D}$ and $\overline{\mathscr{D}}$. Without loss of generality, we may assume $m \leq \frac{n-1}{2}$. Thus, $n-2 \leq m \leq \frac{n-1}{2}$, which means that $n \leq 3$, a contradiction. Consequently, $\gamma_{[kR]}(\mathscr{D}) \leq kn+k+1$. □

## 5. Relations between the $[k]$-RD-Number and Other Domination Parameters

In this part, we give relations including $\gamma_{[kR]}(\mathscr{D})$ with $\gamma_{[(k-1)R]}(\mathscr{D})$, $\gamma(\mathscr{D})$ and $\gamma_{\mathscr{S}}(\mathscr{D})$. We begin with the relationship between $\gamma_{[kR]}(\mathscr{D})$ and $\gamma_{[(k-1)R]}(\mathscr{D})$.

**Lemma 2.** *Let $\mathscr{D}$ be a digraph and $f = (V_0, V_2, V_3, \ldots, V_k)$ a $\gamma_{[(k-1)R]}(\mathscr{D})$-function. Then, $\gamma_{[kR]}(\mathscr{D}) \leq (k+1)|V_k| + k\sum_{i=2}^{k-1}|V_i|$.*

**Proof.** Define a function $g : V(\mathscr{D}) \to \{0, 2, 3, \ldots, k+1\}$ by $g(u) = k+1$ for $u \in V_k$, $g(v) = 0$ for $v \in V_0$, and $g(x) = k$ otherwise. It is easy to verify that $g$ is a $[k]$-RDF of weight $\omega(g) = (k+1)|V_k| + k\sum_{i=2}^{k-1}|V_i|$. Hence, $\gamma_{[kR]}(\mathscr{D}) \leq (k+1)|V_k| + k\sum_{i=2}^{k-1}|V_i|$. □

**Theorem 8.** *Let $\mathscr{D}$ be any digraph. Then, $\gamma_{[(k-1)R]}(\mathscr{D}) + 1 \leq \gamma_{[kR]}(\mathscr{D}) \leq \frac{k}{2}\gamma_{[(k-1)R]}(\mathscr{D})$.*

**Proof.** Firstly, we prove the lower bound of the inequality. Let $\tau = (V_0, V_2, V_3, \ldots, V_{k+1})$ be a $\gamma_{[kR]}(\mathscr{D})$-function. Define a function $g : V(\mathscr{D}) \to \{0, 2, 3, \ldots, k\}$ by $g(v) = k$ for any vertex $v \in V_{k+1}$ and $g(x) = \tau(x)$ otherwise. Then, $g$ is a $[k-1]$-RDF, and so $\gamma_{[(k-1)R]}(\mathscr{D}) \leq \omega(g) = \omega(\tau) - |V_{k+1}| \leq \omega(\tau) = \gamma_{[kR]}(\mathscr{D})$. If $\gamma_{[(k-1)R]}(\mathscr{D}) = \gamma_{[kR]}(\mathscr{D})$, then $V_{k+1} = \phi$. Clearly, there is at least one non-empty set $V_i \in \{V_2, V_3, \ldots, V_k\}$. Choose a vertex $v \in V(\mathscr{D})$ with $2 \leq \tau(v) \leq k$ and let $h : V(\mathscr{D}) \to \{0, 2, 3, \ldots, k\}$ be a function defined by $h(v) = \tau(v) - 1$, $h(u) = \tau(u)$ otherwise. It is not difficult to see that $h(AN^-[x]) \geq |AN^-(x)| + k - 1$ for any vertex $x \in V(\mathscr{D})$. Then, $h$ is a $[k-1]$-RDF and $\omega(g) = \gamma_{[(k-1)R]}(\mathscr{D}) \leq \omega(h) = \omega(\tau) - 1 = \gamma_{[kR]}(\mathscr{D}) - 1 = \omega(g) - 1$, a contradiction. Thus, $\gamma_{[(k-1)R]}(\mathscr{D}) + 1 \leq \gamma_{[kR]}(\mathscr{D})$.

Furthermore, we prove the upper bound of the inequality. Let $l = (V_0^l, V_2^l, V_3^l, \ldots, V_k^l)$ be a $\gamma_{[(k-1)R]}(\mathscr{D})$-function. From Lemma 2, we have

$$\begin{aligned}
\gamma_{[kR]}(\mathscr{D}) &\leq (k+1)|V_k^l| + k\sum_{i=2}^{k-1}|V_i^l| \\
&= \frac{k}{2}\cdot 2|V_2^l| + \frac{k}{3}\cdot 3|V_3^l| + \cdots + \frac{k}{k-1}\cdot(k-1)|V_{k-1}^l| + \frac{k+1}{k}\cdot k|V_k^l| \\
&\leq \frac{k}{2}\left(\sum_{i=2}^{k} i|V_i^l|\right) \\
&= \frac{k}{2}\gamma_{[(k-1)R]}(\mathscr{D}).
\end{aligned}$$

□

Next we consider $\gamma_{[kR]}(\mathscr{D})$ and $\gamma(\mathscr{D})$.

**Theorem 9.** *Let $\mathscr{D}$ be a digraph. Then, $\gamma_{[kR]}(\mathscr{D}) \leq (k+1)\gamma(\mathscr{D})$. Furthermore, $\gamma_{[kR]}(\mathscr{D}) = (k+1)\gamma(\mathscr{D})$ if and only if there is a $\gamma_{[kR]}(\mathscr{D})$-function $f = (V_0, V_2, V_3, \ldots, V_{k+1})$ such that $V_2 = V_3 = \cdots = V_k = \phi$.*

**Proof.** Let $\mathscr{S}$ be a $\gamma(\mathscr{D})$-set. Define a function $g : V(\mathscr{D}) \to \{0, 2, 3, \ldots, k+1\}$ such that $g(v) = k+1$ for any vertex $v \in \mathscr{S}$ and $g(x) = 0$ otherwise. Then, $g$ is a $[k]$-RDF on $\mathscr{D}$ and $\gamma_{[kR]}(\mathscr{D}) \leq \omega(g) = (k+1)|\mathscr{S}| = (k+1)\gamma(\mathscr{D})$.

Below, we prove the necessity and sufficiency.

($\Rightarrow$) Suppose that $\gamma_{[kR]}(\mathscr{D}) = (k+1)\gamma(\mathscr{D})$. Let $\tau = (V_0, V_2, V_3, \ldots, V_{k+1})$ be defined as follows: $\tau(v) = k+1$ for any vertex $v \in \mathscr{S}$ and $\tau(x) = 0$ otherwise. Then, $\tau$ is a $[k]$-RDF on $\mathscr{D}$ with weight $\omega(\tau) = (k+1)|\mathscr{S}| = (k+1)\gamma(\mathscr{D}) = \gamma_{[kR]}(\mathscr{D})$. Thus, $\tau$ is a $\gamma_{[kR]}(\mathscr{D})$-function with $V_2 = V_3 = \cdots = V_k = \phi$.

($\Leftarrow$) Let $\tau = (V_0, V_2, V_3, \ldots, V_{k+1})$ be a $\gamma_{[kR]}(\mathscr{D})$-function with $V_2 = V_3 = \cdots = V_k = \phi$. Then, $V_{k+1}$ is a dominating set of $\mathscr{D}$. This implies that $|V_{k+1}| \geq \gamma(\mathscr{D})$, and so $\gamma_{[kR]}(\mathscr{D} = (k+1)|V_{k+1}| \geq (k+1)\gamma(\mathscr{D})$. On the other hand, $\gamma_{[kR]}(\mathscr{D}) \leq (k+1)\gamma(\mathscr{D})$, hence $\gamma_{[kR]}(\mathscr{D}) = (k+1)\gamma(\mathscr{D})$. □

**Theorem 10.** *Let $\mathscr{D}$ be a digraph of order $n$ with maximum out-degree $\Delta^+(\mathscr{D})$ and domination number $\gamma(\mathscr{D})$. Then, $\gamma_{[kR]}(\mathscr{D}) \geq \left\lceil \frac{2n+(\Delta^+(\mathscr{D})-1)\gamma(\mathscr{D})}{\Delta^+(\mathscr{D})} \right\rceil$.*

**Proof.** Let $\tau = (V_0, V_2, V_3, \ldots, V_{k+1})$ be a $\gamma_{[kR]}(\mathscr{D})$-function with $V_0 = V_0^2 \cup V_0^3 \cup \cdots \cup V_0^{k+1}$, where $V_0^{k+1} = V_0 \cap N^+(V_{k+1})$, $V_0^i = (V_0 \cap N^+(V_i)) - \bigcup\limits_{j=i+1}^{k+1} V_0^j$ for $i \in \{2, 3, \ldots, k\}$. Because the maximum out-degree of $\mathscr{D}$ is $\Delta^+(\mathscr{D})$, $v$ has at most $\Delta^+(\mathscr{D})$ out-neighbours in $V_0$ for any vertex $v \in V_{k+1}$. This means that $|V_0^{k+1}| \leq \Delta^+(\mathscr{D})|V_{k+1}|$. For any positive integer $2 \leq t \leq k$, we know that $m(t)|V_0^t| \leq \left| A\left[\bigcup\limits_{i=2}^{t} V_i, V_0^t\right]\right| = \sum\limits_{i=2}^{t} |A[V_i, V_0^t]|$ given the definition of $[k]$-RDF, where $m(t)$ is the minimum value of $|N^-(v) \cap (V_2 \cup V_3 \cup \cdots \cup V_t)|$ for $v \in V_0^t$. Clearly, $m(t) \geq 2$. When $t = k$, there exist at least two in-neighbours in $V_i \cup V_k$ for any vertex of $V_0^k$, $i \in \{2, 3, \ldots, k-1\}$. Thus, $2|V_0^k| \leq \left| A\left[\bigcup\limits_{i=2}^{k} V_i, V_0^k\right]\right| \leq \sum\limits_{i=2}^{k-1} |A[V_i, V_0^k]| + \Delta^+(\mathscr{D})|V_k|$. Finally, there exist at least $k$ in-neighbours in $V_2$ for any vertex $v \in V_0^2$, and thus $k|V_0^2| \leq |A[V_2, V_0^2]|$. Because $V_0 = V_0^2 \cup V_0^3 \cup \cdots \cup V_0^{k+1}$ and $m(t) \geq 2$, we have that

$$
\begin{aligned}
|V_0| &= |V_0^2| + |V_0^3| + \cdots + |V_0^{k+1}| \\
&\leq \frac{|A[V_2, V_0^2]|}{k} + \frac{\sum\limits_{i=2}^{3} |A[V_i, V_0^3]|}{m(3)} + \cdots + \frac{\sum\limits_{i=2}^{k-1} |A[V_i, V_0^{k-1}]|}{m(k-1)} + \frac{\sum\limits_{i=2}^{k-1} |A[V_i, V_0^k]|}{2} \\
&\quad + \frac{\Delta^+(\mathscr{D})}{2}|V_k| + \Delta^+(\mathscr{D})|V_{k+1}| \\
&\leq \frac{\sum\limits_{j=2}^{k} |A[V_2, V_0^j]| + \sum\limits_{j=3}^{k} |A[V_3, V_0^j]| + \cdots + \sum\limits_{j=k-1}^{k} |A[V_{k-1}, V_0^j]|}{2} \\
&\quad + \frac{\Delta^+(\mathscr{D})}{2}|V_k| + \Delta^+(\mathscr{D})|V_{k+1}|.
\end{aligned}
\tag{1}
$$

Let $s = |V_2| + |V_3| + \cdots + |V_{k-1}|$. There exist at least $s$ in-neighbours of the whole vertex of $\bigcup\limits_{i=2}^{k-1} V_i$ included in $\bigcup\limits_{i=2}^{k+1} V_i$. Combined with the inequality (1), we can obtain the results shown in Table 1.

**Table 1.** The bound of $|V_0|$ when the $s$ in-neighbours originate from different sets.

| The Origin of the $s$ In-Neighbours | The Bound of $V_0$ |
|---|---|
| all in-neighbours from $\bigcup_{i=2}^{k-1} V_i$ | $|V_0| \leq \frac{\Delta^+(\mathscr{D})s-s}{2} + \frac{\Delta^+(\mathscr{D})}{2}|V_k| + \Delta^+(\mathscr{D})|V_{k+1}|$ |
| all in-neighbours from $V_k$ | $|V_0| \leq \frac{\Delta^+(\mathscr{D})}{2}s + \frac{\Delta^+(\mathscr{D})|V_k|-s}{2} + \Delta^+(\mathscr{D})|V_{k+1}|$ |
| all in-neighbours from $V_{k+1}$ | $|V_0| \leq \frac{\Delta^+(\mathscr{D})}{2}s + \frac{\Delta^+(\mathscr{D})}{2}|V_k| + \Delta^+(\mathscr{D})|V_{k+1}| - s$ |
| $p$ in-neighbours from $\bigcup_{i=2}^{k-1} V_i$, $q$ in-neighbours from $V_k$ and $s-p-q$ in-neighbours from $V_{k+1}$ where $1 \leq p \leq s-1, 1 \leq q \leq s-1$ | $|V_0| \leq \frac{\Delta^+(\mathscr{D})s-p}{2} + \frac{\Delta^+(\mathscr{D})|V_k|-q}{2} + \Delta^+(\mathscr{D})|V_{k+1}|$ |

From Table 1, it is not difficult to see that $|V_0| \leq \frac{\Delta^+(\mathscr{D})-1}{2}s + \frac{\Delta^+(\mathscr{D})}{2}|V_k| + \Delta^+(\mathscr{D})|V_{k+1}|$ wherever the in-neighbours originate from.

That is, $|V_0| \leq \frac{\Delta^+(\mathscr{D})-1}{2}(|V_2| + |V_3| + \cdots + |V_{k-1}|) + \frac{\Delta^+(\mathscr{D})}{2}|V_k| + \Delta^+(\mathscr{D})|V_{k+1}|$. This means that

$$\frac{2}{\Delta^+(\mathscr{D})}|V_0| \leq \frac{\Delta^+(\mathscr{D})-1}{\Delta^+(\mathscr{D})}\sum_{i=2}^{k-1}|V_i| + |V_k| + 2|V_{k+1}|. \tag{2}$$

It is obvious that, $V_2 \cup V_3 \cup \cdots \cup V_k \cup V_{k+1}$ is a dominating set of $\mathscr{D}$. Therefore,

$$\begin{aligned}
\gamma_{[kR]}(\mathscr{D}) &= 2|V_2| + 3|V_3| + \cdots + (k+1)|V_{k+1}| \\
&= \sum_{i=2}^{k+1}|V_i| + \frac{\Delta^+(\mathscr{D})-1}{\Delta^+(\mathscr{D})}\sum_{i=2}^{k-1}|V_i| + |V_k| + 2|V_{k+1}| \\
&\quad + \left(\frac{1}{\Delta^+(\mathscr{D})}\sum_{i=2}^{k-1}|V_i| + \sum_{i=3}^{k}(i-2)|V_i| + (k-2)|V_{k+1}|\right) \\
&\geq \sum_{i=2}^{k+1}|V_i| + \frac{2}{\Delta^+(\mathscr{D})}|V_0| + \left(\frac{1}{\Delta^+(\mathscr{D})}\sum_{i=2}^{k-1}|V_i| + \sum_{i=3}^{k}(i-2)|V_i|\right) \\
&\quad + (k-2)|V_{k+1}| \\
&= \sum_{i=2}^{k+1}|V_i| + \frac{2n - 2\sum_{i=2}^{k+1}|V_i|}{\Delta^+(\mathscr{D})} + \frac{1}{\Delta^+(\mathscr{D})}\sum_{i=2}^{k-1}|V_i| + \sum_{i=3}^{k}(i-2)|V_i| \\
&\quad + (k-2)|V_{k+1}| \\
&= \frac{2n + (\Delta^+(\mathscr{D})-1)\sum_{i=2}^{k+1}|V_i|}{\Delta^+(\mathscr{D})} + \sum_{i=3}^{k-1}(i-2)|V_i| + \left(k-2-\frac{1}{\Delta^+(\mathscr{D})}\right) \\
&\quad (|V_k| + |V_{k+1}|) \\
&\geq \frac{2n + (\Delta^+(\mathscr{D})-1)\gamma(\mathscr{D})}{\Delta^+(\mathscr{D})}
\end{aligned}$$

□

We are now in a position to relate $\gamma_{[kR]}(\mathscr{D})$ and $\gamma_{\mathscr{S}}(\mathscr{D})$, and here is a useful result from [4].

**Theorem 11** ([4])**.** *Let $\mathscr{G} = (\mathcal{L}, \mathcal{R})$ be a bipartite graph with $|V(\mathscr{D})| = n$. If $\delta_{\mathcal{L}}(\mathscr{G}) \geq 2$, then $\gamma_{\mathcal{L}}(\mathscr{G}) \leq \frac{n}{3}$.*

**Theorem 12.** *Let $\mathscr{D}$ be a digraph of order $n$. Then, $\gamma_{[kR]}(\mathscr{D}) \leq \frac{k}{2}\gamma_{\mathscr{S}}(\mathscr{D}) + \frac{(3k+2)n}{6}$.*

**Proof.** Let $\tau$ be a $\gamma_{\mathscr{S}}(\mathscr{D})$-function, $\mathcal{L}$ and $\mathcal{R}$ represent the vertex sets assigned as $-1$ and 1 under $f$, respectively. Then, we have $|\mathcal{L}| + |\mathcal{R}| = n$ and $\gamma_{\mathscr{S}}(\mathscr{D}) = \omega(f) = |\mathcal{R}| - |\mathcal{L}|$, which implies that $2|\mathcal{R}| = n + \gamma_{\mathscr{S}}(\mathscr{D})$.

If $\mathcal{L} = \phi$, then $\mathcal{R} = V(\mathscr{D})$. Define a function $g : V(\mathscr{D}) \to \{0, 2, 3, \ldots, k+1\}$ by $g(v) = k$ for each vertex of $\mathscr{D}$. Then, $g$ is a $[k]$-RDF on $\mathscr{D}$ and hence $\gamma_{[kR]}(\mathscr{D}) \leq \omega(g) = kn = k|\mathcal{R}| = k\gamma_{\mathscr{S}}(\mathscr{D}) < \frac{k}{2}\gamma_{\mathscr{S}}(\mathscr{D}) + \frac{(3k+2)n}{6}$.

If $\mathcal{L} \neq \phi$, let $\mathscr{D}_1 = (\mathcal{L}, \mathcal{R})$ be the bipartite spanning subdigraph of $\mathscr{D}$ satisfying that $A(\mathscr{D}_1) = \{(u, v) \in A(\mathscr{D}) : u \in \mathcal{R} \text{ and } v \in \mathcal{L}\}$. Because $\tau$ is a $\gamma_{\mathscr{S}}(\mathscr{D})$-function, we find that every vertex in $\mathcal{L}$ has at least two in-neighbours in $\mathcal{R}$ by the definition of SDF. Thus, $\delta^-_{\mathcal{L}}(\mathscr{D}_1) \geq 2$, where $\delta^-_{\mathcal{L}}(\mathscr{D}_1) = \min\{d^-_{\mathscr{D}_1}(v) : v \in \mathcal{L}\}$. Let $\mathscr{H}$ be the underlying graph of $\mathscr{D}_1$. Then, $\delta_{\mathcal{L}}(\mathscr{H}) = \delta^-_{\mathcal{L}}(\mathscr{D}_1) \geq 2$. Let $\mathcal{R}_2$ be a $\gamma_{\mathcal{L}}(\mathscr{H})$-set. From Theorem 11, we have $|\mathcal{R}_2| = \gamma_{\mathcal{L}}(\mathscr{H}) \leq \frac{n}{3}$. According to the definition of the left dominating set, every vertex in $\mathcal{L}$ has a neighbour in $\mathcal{R}_2$. Hence, every vertex in $\mathcal{L}$ has an in-neighbour in $\mathcal{R}_2$ for $\mathscr{D}_1$ and $\mathscr{D}$. Let $\mathcal{R}_1 = \mathcal{R} \backslash \mathcal{R}_2$, define a function $g : V(\mathscr{D}) \to \{0, 2, 3, \ldots, k+1\}$ such that $g(v) = k+1$ for any vertex $v \in \mathcal{R}_2$, $g(u) = k$ for any vertex $u \in \mathcal{R}_1$ and $g(x) = 0$ for any vertex $x \in \mathcal{L}$. Then $g$ is a $[k]$-RDF on $\mathscr{D}$. Then, we have

$$\begin{aligned}
\gamma_{[kR]}(\mathscr{D}) &\leq \omega(g) = k(|\mathcal{R}_1| + |\mathcal{R}_2|) + |\mathcal{R}_2| \\
&= k|\mathcal{R}| + |\mathcal{R}_2| \\
&= \frac{k}{2}(n + \gamma_{\mathscr{S}}(\mathscr{D})) + |\mathcal{R}_2| \\
&\leq \frac{k}{2}(n + \gamma_{\mathscr{S}}(\mathscr{D})) + \frac{n}{3} \\
&= \frac{k}{2}\gamma_{\mathscr{S}}(\mathscr{D}) + \frac{(3k+2)n}{6}.
\end{aligned}$$

The proof is completed. □

## 6. The [$k$]-RD-Numbers of the Directed Path and the Directed Cycle

In this section, we determine the exact values for the $[k]$-RD-numbers of $\overrightarrow{P_n}$ and $\overrightarrow{C_n}$.

**Proposition 4.** *Let $n \geq 2$ be a positive integer. Then*

$$\gamma_{[kR]}(\overrightarrow{P_n}) = \begin{cases} (k+1)\lfloor \frac{n}{2} \rfloor, & \text{if } n \equiv 0 \mod 2; \\ (k+1)\lfloor \frac{n}{2} \rfloor + k, & \text{if } n \equiv 1 \mod 2. \end{cases}$$

**Proof.** Let $\overrightarrow{P_n}$ be a directed path with $V(\overrightarrow{P_n}) = \{v_0, v_1, \ldots, v_{n-1}\}$. When $n \equiv 0 \mod 2$, let $\tau : V(\overrightarrow{P_n}) \to \{0, 2, 3, \ldots, k+1\}$ be defined as follows: $\tau(v_{2i}) = k+1$ and $\tau(v_{2i+1}) = 0$ for $0 \leq i \leq \frac{n-2}{2}$. Then, $\tau$ is a $[k]$-RDF on $\overrightarrow{P_n}$, and so $\gamma_{[kR]}(\overrightarrow{P_n}) \leq \omega(\tau) = (k+1)\lfloor \frac{n}{2} \rfloor$. When $n \equiv 1 \mod 2$, let $\tau : V(\overrightarrow{P_n}) \to \{0, 2, 3, \ldots, k+1\}$ be defined as follows: $\tau(v_{n-1}) = k$, $\tau(v_{2i}) = k+1$ and $\tau(v_{2i+1}) = 0$ for $0 \leq i \leq \frac{n-3}{2}$. Then, $\tau$ is a $[k]$-RDF on $\overrightarrow{P_n}$, and so $\gamma_{[kR]}(\overrightarrow{P_n}) \leq \omega(\tau) = (k+1)\lfloor \frac{n}{2} \rfloor + k$.

On the other hand, let $h : V(\overrightarrow{P_n}) \to \{0, 2, 3, \ldots, k+1\}$ be a $\gamma_{[kR]}(\overrightarrow{P_n})$-function. We prove the inverse inequality by induction on $n$. If $n = 2$, then $\gamma_{[kR]}(\overrightarrow{P_2}) = k + 1 = (k+1)\lfloor \frac{n}{2} \rfloor$. If $n = 3$, then $\gamma_{[kR]}(\overrightarrow{P_3}) = (k+1) + k = (k+1)\lfloor \frac{n}{2} \rfloor + k$. Suppose that the inverse inequality is true for each $\overrightarrow{P_m}$ of order $m$ with $4 \leq m \leq n-1$. When $n \equiv 0 \mod 2$, then $\gamma_{[kR]}(\overrightarrow{P_n}) = h\left(V(\overrightarrow{P_n})\right) \geq h\left(V(\overrightarrow{P_{n-2}})\right) + (k+1) \geq (k+1)\lfloor \frac{n-2}{2} \rfloor + (k+1) = (k+1)\lfloor \frac{n}{2} \rfloor$. When $n \equiv 1 \mod 2$, notice that $h(v_{n-2}) + h(v_{n-1}) = k+1$. Hence, $\gamma_{[kR]}(\overrightarrow{P_n}) = h\left(V(\overrightarrow{P_n})\right) = h\left(V(\overrightarrow{P_{n-2}})\right) + h(v_{n-2}) + h(v_{n-1}) \geq (k+1)\lfloor \frac{n-2}{2} \rfloor + k + (k+1) = (k+1)\lfloor \frac{n}{2} \rfloor + k$.

Consequently,

$$\gamma_{[kR]}(\overrightarrow{P_n}) = \begin{cases} (k+1)\lfloor \frac{n}{2} \rfloor, & \text{if } n \equiv 0 \mod 2; \\ (k+1)\lfloor \frac{n}{2} \rfloor + k, & \text{if } n \equiv 1 \mod 2. \end{cases}$$

□

**Proposition 5.** *Let $n \geq 3$ be a positive integer. Then*

$$\gamma_{[kR]}(\overrightarrow{C_n}) = \begin{cases} (k+1)\lfloor \frac{n}{2} \rfloor, & \text{if } n \equiv 0 \mod 2; \\ (k+1)\lfloor \frac{n}{2} \rfloor + \lceil \frac{k+1}{2} \rceil, & \text{if } n \equiv 1 \mod 2, \end{cases}$$

*where all subscripts are taking module $n$.*

**Proof.** Let $\overrightarrow{C_n}$ be a directed cycle with $V(\overrightarrow{C_n}) = \{v_0, v_1, \ldots, v_{n-1}\}$. When $n \equiv 0 \mod 2$, let $\tau : V(\overrightarrow{C_n}) \to \{0, 2, 3, \ldots, k+1\}$ be defined as follows: $\tau(v_{2i}) = k+1$ and $\tau(v_{2i+1}) = 0$ for $0 \leq i \leq \frac{n-2}{2}$. Then, $\tau$ is a $[k]$-RDF on $\overrightarrow{C_n}$, and so $\gamma_{[kR]}(\overrightarrow{C_n}) \leq \omega(\tau) = (k+1)\lfloor \frac{n}{2} \rfloor$. When $n \equiv 1 \mod 2$, let $\tau : V(\overrightarrow{C_n}) \to \{0, 2, 3, \ldots, k+1\}$ be defined as follows: $\tau(v_{2i}) = \left\lceil \frac{k+1}{2} \right\rceil$ and $\tau(v_{2i-1}) = \left\lfloor \frac{k+1}{2} \right\rfloor$ for $0 \leq i \leq \frac{n-1}{2}$. Then, $\tau$ is a $[k]$-RDF on $\overrightarrow{C_n}$, and so $\gamma_{[kR]}(\overrightarrow{C_n}) \leq \omega(\tau) = (k+1)\lfloor \frac{n}{2} \rfloor + \left\lceil \frac{k+1}{2} \right\rceil$.

On the other hand, let $h : V(\overrightarrow{C_n}) \to \{0, 2, 3, \ldots, k+1\}$ be a $\gamma_{[kR]}(\overrightarrow{C_n})$-function. When $n \equiv 0 \mod 2$, notice that $h(v_{2i}) + h(v_{2i+1}) = k+1$ for $0 \leq i \leq \frac{n-2}{2}$. Hence, $\gamma_{[kR]}(\overrightarrow{C_n}) = \omega(h) = (k+1)\lfloor \frac{n}{2} \rfloor$. When $n \equiv 1 \mod 2$, it is easy to see that there is one vertex $v_l$ such that $h(v_i) + h(v_{i+1}) = k+1$ for $i \notin \{l, l+1\}$, $h(v_l) + h(v_{l+1}) \geq k+1$ and $h(v_l) = h(v_{l+1})$. Without loss of generality, we assume that $h(v_i) + h(v_{i+1}) = k+1$ for $1 \leq i \leq n-2$, $h(v_{n-1}) + h(v_0) \geq k+1$ and $h(v_{n-1}) = h(v_0)$. This means that $h(v_{n-1}) \geq \left\lceil \frac{k+1}{2} \right\rceil$. Hence, $\gamma_{[kR]}(\overrightarrow{C_n}) = \omega(h) = h(v_{n-1}) + \sum_{i=0}^{n-2} h(v_i) \geq \left\lceil \frac{k+1}{2} \right\rceil + (k+1)\lfloor \frac{n}{2} \rfloor$.

Consequently,

$$\gamma_{[kR]}(\overrightarrow{C_n}) = \begin{cases} (k+1)\lfloor \frac{n}{2} \rfloor, & \text{if } n \equiv 0 \mod 2; \\ (k+1)\lfloor \frac{n}{2} \rfloor + \lceil \frac{k+1}{2} \rceil, & \text{if } n \equiv 1 \mod 2. \end{cases}$$

□

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