




A New Technique to Uniquely Identify the Edges of a Graph

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Abstract: Graphs are useful for analysing the structure models in computer science, operations research, and sociology. The word metric dimension is the basis of the distance function, which has a symmetric property. Moreover, finding the resolving set of a graph is NP-complete, and the possibilities of finding the resolving set are reduced due to the symmetric behaviour of the graph. In this paper, we introduce the idea of the edge-multiset dimension of graphs. A representation of an edge is defined as the multiset of distances between it and the vertices of a set, $B \subseteq V(\Gamma)$. If the representation of two different edges is unequal, then B is an edge-multiset resolving a set of Γ . The least possible cardinality of the edge-multiset resolving a set is referred to as the edge-multiset dimension of Γ . This article presents preliminary results, special conditions, and bounds on the edge-multiset dimension of certain graphs. This research provides new insights into structure models in computer science, operations research, and sociology. They could have implications for developing computer algorithms, aircraft scheduling, and species movement between regions.

Keywords: resolving set; metric dimension; edge metric dimension; mixed metric dimension; line graph; paroline graph



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1. Introduction

The study of distance-related parameters in graph theory, especially the metric dimension, has long been a topic of interest among researchers. The metric dimension of a graph is the minimum number of vertices required to uniquely identify every vertex in the graph using distances between vertices. In everyday applications, the concept of the metric dimension is helpful in different fields, such as computer networks, pattern recognition, and network security. In computer networks, the metric dimension of a graph can be used to determine the node's location in the network and assess the network's reliability and performance. In recent years, the study of metric dimensions has become ever more prevalent due to the growth of internet relationships and the increasing demand for robust and secure networks. The theory of metric dimensions was proposed by Slater in 1975 [1], and independently by Harary and Melter in 1976 [2]. However, the notion of metric dimensions in metric spaces generally dates back to 1953 [3]. Some recent studies on the metric dimension of graphs are given in the following articles [4–6]. The concept of metric dimensions is based on the distance function $d : V \times V \rightarrow R^+ \cup \{0\}$, which has the symmetric property. Determining the resolving set of a graph is an NP-complete problem, so it is computationally difficult to find the minimum size-resolving set for a graph. However, due to the graph's symmetric nature, finding the resolvability reduces the number of possibilities for the elements in the resolving set. The study of distance in graphs is a fast-expanding area of research in different science fields, especially mathematics, chemistry, and computer science. The concept of distances in graphs has several

applications in various fields. Researchers are constantly exploring new and innovative ways to measure and analyse graph distances and apply these methods to real-world problems. The field is marked by a high level of interdisciplinary collaboration, with computer science, mathematics, physics, and biology researchers all contributing to developing new techniques and algorithms.

In this manuscript, we define a new technique to identify the edges of a graph, termed the edge-multiset dimension and compared to the multiset dimension. Section 2 contains a comprehensive literature review and basic definitions. In Section 3, we give the methods used in the manuscript. Section 4 contains the main results of the multiset and edge-multiset dimensions of different families of graphs. This section also discusses the graphs having infinite, constant and edge-multiset dimension graphs depending on their order. In Section 5, we give different examples and compare multiset and edge-multiset dimensions. In the end we give the conclusions and open problems for future work.

2. Literature Review

Various distance-related parameters have been studied, such as the partition distance of graphs were studied in [7], Alhevaz et al. gave the sharp bounds for the generalized distance spectral radius of graphs [8], Wang studied distance bounds for generalized bicycle codes and Pryadko [9], Nadeem et al. found the fault-tolerant partition dimension of oxide interconnection networks [10]. Concerning metric dimensions, that have been of more interest to the research community, one could remark of a few of them (although possibly not all of the most remarkable ones): partition dimension [11], strong metric dimension [12], k -metric dimension [13], identifying codes [14], k -metric anti-dimension [15], local metric dimension [16], edge metric dimension [17] and multiset dimension [18] (see also [19] for the outer-multiset dimension). Each of these variations of the metric dimension mentioned above have been recently studied to a greater or lesser extent, and even some combinations between them have also appeared, including, for instance, k -partition dimension [20], or local edge dimension [21].

In their work [17], Kelenc et al. introduced the concept of the edge metric dimension of graphs and discussed various results, including comparisons with the metric dimension of graphs. Such variation has attracted the significant attention from several researchers, and we can find many papers on it. In [22], characterized the formation of the topful graph and some sufficient and important conditions of a graph to be topful were explained. The edge metric dimension of some convex polytopes and its relevant graphs were determined in [23]. The edge dimension of some generalized Petersen graphs was explained in [24]. The edge dimensions via integer linear programming and hierarchical products were given in [25], and many examples show that these methods can be used to obtain the edge dimensions for some graphs. The edge dimension of the two graphs' join, corona, and the lexicographic product was studied in [26]. The vertex, edge and mixed dimension of the dragon graph $T_{n,m}$, $L(T_{n,m})$, $L(S(T_{n,m}))$ and $L(L(T_{n,m}))$ have been computed [27]. Knor presented [28] a lot of graphs which proved that the edge dimension is less than the metric dimension, but it is impossible to bound the edge dimension of a graph by the vertex dimension. The approximation algorithm was presented for the edge dimension problem in [29]. On the other hand, Simanjuntak, et al. [18], presented the notion of using multisets for uniquely identifying the vertices of a graph, which allowed the birth of the multiset dimension of graphs in 2017. They presented some primary results, showed sufficient conditions for a graph to have a multiset dimension infinite, and computed the multiset dimension of some graphs. Some other results on this variant appeared in [30–32]. In connection with this, in [19], another version of the multiset dimension (called the outer-multiset dimension) was introduced as an attempt to avoid the existence of graphs with infinite multiset dimensions (that is, graphs for which the multiset dimension cannot be computed).

Based on the significance of these two latter variants of metric dimension (edge and multiset), we aim to introduce and begin the study of a natural combination of them, that is, the edge-multiset dimension of graphs. From now onward, we consider Γ a connected and

simple graph. The vertex set of Γ is $V(\Gamma)$ and edge set of Γ is $E(\Gamma)$. The distance between any pair of vertices α and $\beta \in V(\Gamma)$ is symbolized as $d_\Gamma(\alpha, \beta)$ (or $d(\alpha, \beta)$ for short), and is described as the number of edges of the least possible path between α and β . Assume $w \in V(\Gamma)$, resolves (recognizes or determines) the vertices α and β , if $d_\Gamma(w, \alpha) \neq d_\Gamma(w, \beta)$. For an ordered subset of t vertices $S = \{v_1, v_2, \dots, v_t\}$, the metric code or representation of a vertex u according to S is the ordered t -tuple of distances between u and the vertices in S , which is written as

$$r(u|S) = (d_\Gamma(u, v_1), d_\Gamma(u, v_2), \dots, d_\Gamma(u, v_t)).$$

If any pair of vertices of Γ have dissimilar metric codes or representations according to set S , then S is known as a resolving set for Γ . The metric dimension of Γ is defined as the number of vertices in the smallest resolving set for Γ , denoted by $dim(\Gamma)$. If S is a resolving set for Γ of cardinality $dim(\Gamma)$, then S is called a metric basis for Γ .

The distance between an edge $f = \alpha\beta$ and a vertex ρ is denoted as $d_\Gamma(f, \rho)$ (or $d(f, \rho)$ for short), and is defined as $d_\Gamma(f, \rho) = \min\{d_\Gamma(\alpha, \rho), d_\Gamma(\beta, \rho)\}$. The vertex p resolves (recognizes or determines) the edges f_1 and f_2 , if $d_\Gamma(f_1, p) \neq d_\Gamma(f_2, p)$. For an ordered subset of t vertices $S = \{v_1, v_2, \dots, v_t\}$, the metric code or representation of an edge f according to set S is the ordered t -tuple of distances between f and the vertices in S , which is written as

$$r(f|S) = (d_\Gamma(f, v_1), d_\Gamma(f, v_2), \dots, d_\Gamma(f, v_t)).$$

If any pair of edges of Γ have specific metric codes or representations according to S , then S is known as an edge-resolving set for Γ . The edge metric dimension of Γ is then defined as the number of vertices in the smallest edge resolving set for Γ , denoted by $dim_e(\Gamma)$. If S is an edge resolving set for Γ of cardinality $dim_e(\Gamma)$, then S is called an edge metric basis for Γ .

Multiset and Edge-Multiset Dimensions of Graphs

Let $A = \{a_1, a_2, \dots, a_l\}$ be a subset of $V(\Gamma)$. For any vertex u of Γ , the multiset code or representation of $u \in V(\Gamma)$ according to A is a multiset which is defined as

$$r_m(u|A) = \{d_\Gamma(u, a_1), d_\Gamma(u, a_2), \dots, d_\Gamma(u, a_l)\}.$$

Note that the notion of multisets allows repetitions of elements in $r_m(u|A)$. If $r_m(u|A) \neq r_m(v|A)$ for every pair of vertices u and $v \in V(\Gamma)$, then A is called a multiset resolving set of Γ . There could be graphs containing no multiset resolving sets. For instance, consider a graph with two vertices u and v , which are twins; that is, they share the same neighbours in Γ . In this sense, no matter which set of vertices D of Γ one could choose, it will always happen $r_m(u|D) = r_m(v|D)$. Hence, they will never be resolved by any set of vertices of Γ . In this sense, if a graph Γ contains at least one multiset resolving set, then a multiset resolving set containing the smallest possible number of vertices is called a multiset basis of Γ . In such a case, the cardinality of any multiset basis of Γ is called the multiset dimension of Γ , and is denoted as $md(\Gamma)$. On the contrary, if Γ does not possess a multiset resolving set, we agreed that $md(\Gamma) = \infty$. For some partial information on graphs satisfying this property, we suggest [33].

To give an example of the above concepts, assume graph G with $V(G) = \{v_1, v_2, \dots, v_8\}$ and $E(G) = \{a, b, c, d, e, f, g, h\}$ as given in Figure 1. We note that for instance, the set $A = \{v_2, v_3, v_7\}$ is a multiset resolving set of G .

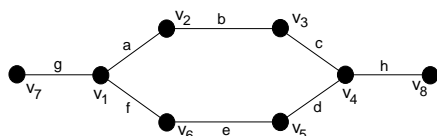


Figure 1. A graph with the $md(G) = 3$.

Next, we show Table 1, where the multiset codes or representations of $V(G)$ of G according to A are given. It can be easily verified that no multiset resolving set having cardinality less than three is a multiset resolving sets for G , which altogether leads to $md(G) = 3$.

Table 1. The multiset representations of the vertices of G with respect to the set $A = \{v_2, v_3, v_7\}$.

| | | | | |
|---------------|-----------|-----------|-----------|-----------|
| v_i | v_1 | v_2 | v_3 | v_4 |
| $r_m(v_i, A)$ | {1, 2, 1} | {0, 1, 2} | {1, 0, 3} | {2, 1, 4} |
| v_i | v_5 | v_6 | v_7 | v_8 |
| $r_m(v_i, A)$ | {3, 2, 3} | {2, 3, 2} | {2, 3, 0} | {3, 2, 5} |

Now, the multiset representation $r_{em}(e|B)$ of an edge $e = xy$ according to the set B , is defined as a multiset containing the distances between the edge e and the vertices in B . That is,

$$r_{em}(e|B) = \{d_\Gamma(e, b_1), d_\Gamma(e, b_2), \dots, d_\Gamma(e, b_j)\}.$$

If $r_{em}(e|B) \neq r_{em}(f|B)$ for any pair of dissimilar edges e and f , then B is called an edge-multiset resolving set for Γ . We again remark that there can be graphs containing no edge-multiset resolving sets. If Γ contains an edge-multiset resolving set, then the least possible edge-multiset resolving set is called an edge-multiset basis of Γ . The number of elements in an edge-multiset basis of Γ is known as the edge-multiset dimension, and is denoted as $md_e(\Gamma)$. If Γ does not contain an edge-multiset resolving set, then we assume that $md_e(\Gamma) = \infty$.

In a similar manner to the case of the multiset dimension, Table 2 shows the multiset representations of all the edges of the graph G given in Figure 1, according to $B = \{v_2, v_3, v_7\}$. It can be easily determined that no set with less than three vertices is an edge-multiset resolving set for G ; thus, $md_e(G) = 3$.

Table 2. The multiset representations of the edges of G according to the set $B = \{v_2, v_3, v_7\}$.

| | | | | |
|--------------------|-----------|-----------|-----------|-----------|
| Edges | a | b | c | d |
| $r_{em}(\cdot, B)$ | {0, 1, 1} | {0, 0, 2} | {1, 0, 3} | {2, 1, 3} |
| Edges | e | f | g | h |
| $r_{em}(\cdot, B)$ | {2, 2, 2} | {1, 2, 1} | {1, 2, 0} | {2, 1, 4} |

After introducing the multiset dimension, it is natural to ask what the representation of edges will be with respect to the multiset resolving set. We introduce and study a new concept called the edge-multiset dimension of graphs, which combines two significant variants of the metric dimension (edge and multiset).

3. Methodology

There are many steps in this research known as determining the graph Γ , defining the set of vertices and the set of edges, determining the set $A \subseteq E(\Gamma)$, determining the multiset representations of vertices or edges of graph Γ , and determining the least possible resolving set for Γ .

Next, we present on the following known results for the multiset dimension of graphs that are interesting for this exposition.

Theorem 1 ([18]). *The multiset dimension of a graph Γ is one if, and only if, Γ is a path.*

Theorem 2 ([18]). *Let Γ be a graph other than a path. Then $md(\Gamma) \geq 3$.*

Theorem 3 ([18]). *Let $n \geq 6$. The multiset dimension of the cycle C_n is 3.*

Theorem 4 ([18]). Let $m \geq 3$ and $n \geq 2$. The multiset dimension of the grid graph $P_n \square P_m$ is 3.

Theorem 5 ([18]). If Γ is a non-path graph of diameter at most 2, then $md(\Gamma) = \infty$.

Consequently, from the above-known results, there is no graph Γ with a multiset dimension of 2. Since the graphs with a multiset dimension are characterized in Theorem 1, it would be desirable to characterize (at least partially) the graphs with a multiset dimension of 3. In concordance with this. Figure 2 shows the flow chart of the research methodology used in this paper.

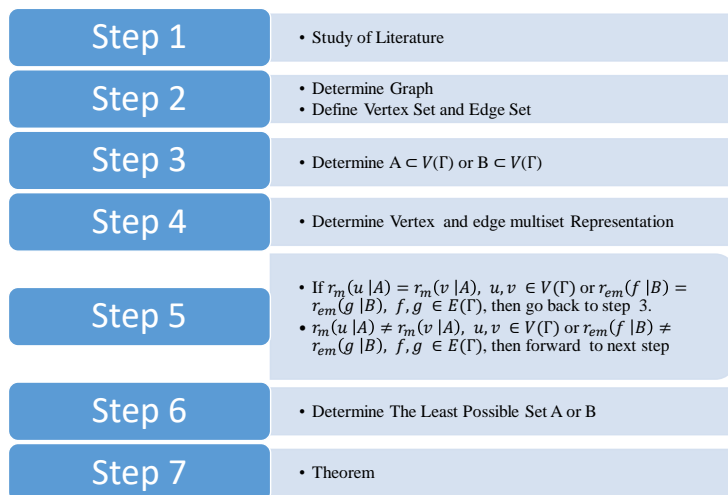


Figure 2. Diagram for the multiset and edge-multiset dimension.

4. Results on the Multiset and Edge-Multiset Dimensions of Graphs

Next, we give some families of graphs, kayak paddle, dragon and comb products of two path graphs with a multiset dimension of 3.

4.1. Kayak Paddle Graph

The kayak paddle graph, denoted by $KP(\vartheta, \lambda, \mu)$ is obtained from two cycles of length $\vartheta \geq 3$ and $\lambda \geq 3$ by joining one vertex of one cycle with a vertex of degree one in a path of length $\mu \geq 2$, and another vertex of the other cycle with the other vertex of degree one in the path of length μ . See Figure 3 for an example. We can write the vertex set of $KP(\vartheta, \lambda, \mu)$ as $V(KP(\vartheta, \lambda, \mu)) = \{\alpha_1, \alpha_2, \dots, \alpha_\vartheta\} \cup \{\beta_1, \beta_2, \dots, \beta_\lambda\} \cup \{\gamma_1, \dots, \gamma_{\mu-1}\}$ and the edge set as $E(KP(\vartheta, \lambda, \mu)) = \{\alpha_i \alpha_{i+1} : 1 \leq i \leq \vartheta\} \cup \{\beta_j \beta_{j+1} : 1 \leq j \leq \lambda\} \cup \{\gamma_k \gamma_{k+1} : 1 \leq k \leq \mu - 2\} \cup \{\alpha_1 \gamma_1, \gamma_{\mu-1} \beta_1\}$, where $\alpha_{\vartheta+1} = \alpha_1$ and $\beta_{\lambda+1} = \beta_1$.

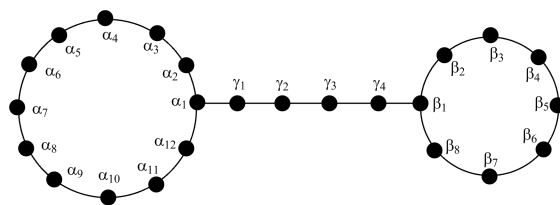


Figure 3. The kayak paddle graph $KP(12, 8, 5)$.

In Theorem 6 we determine the exact value of the multiset dimension of the kayak paddle graph.

Theorem 6. If $KP(\vartheta, \lambda, \mu)$ is a kayak paddle graph with $\vartheta, \lambda, \mu \geq 4$, then

$$md(KP(\vartheta, \lambda, \mu)) = 3.$$

Proof. Therefore, by using Theorem 2, we have that $md(KP(\vartheta, \lambda, \mu)) \geq 3$. Set $A = \{\alpha_2, \beta_2, \gamma_1\}$ is a multiset resolving set for the graph $KP(\vartheta, \lambda, \mu)$.

If $\vartheta \geq 4$ with $t = \lfloor \frac{\vartheta}{2} \rfloor + 1$, then the vertices represented according to A are given in Table 3:

Table 3. The multiset representations of the vertices of $KP(\vartheta, \lambda, \mu)$, for $\vartheta \geq 4$ with $t = \lfloor \frac{\vartheta}{2} \rfloor + 1$ according to the set $A = \{\alpha_2, \beta_2, \gamma_1\}$.

| $r_m(\cdot, \cdot)$ | $A = \{\alpha_2, \beta_2, \gamma_1\}$ |
|---|---|
| α_1 | $\{1, 1, \mu + 1\}$ |
| $\alpha_i: 2 \leq i \leq t$ | $\{i - 2, i, \mu + i\}$ |
| $\alpha_i: i = t + 1$ | $\{i - 2, \vartheta - i + 2, \vartheta + \mu - i + 2\}$ |
| $\alpha_i: t + 2 \leq i \leq \vartheta$ | $\{\vartheta - i + 2, \vartheta - i + 2, \vartheta + \mu - i + 2\}$ |

If $\lambda \geq 4$ with $t' = \lfloor \frac{\lambda}{2} \rfloor + 1$, then the vertices represented according to A are given in Table 4:

Table 4. The multiset representations of the vertices of $KP(\vartheta, \lambda, \mu)$, for $\lambda \geq 4$ with $t' = \lfloor \frac{\lambda}{2} \rfloor + 1$ according to the set $A = \{\alpha_2, \beta_2, \gamma_1\}$.

| $r_m(\cdot, \cdot)$ | $A = \{\alpha_2, \beta_2, \gamma_1\}$ |
|---------------------------------------|---|
| β_1 | $\{1, \mu - 1, \mu + 1\}$ |
| $\beta_j: 2 \leq j \leq t'$ | $\{j - 2, \mu + j - 2, \mu + j\}$ |
| $\beta_j: j = t' + 1$ | $\{j - 2, \lambda + \mu - j + 2, \lambda + \mu - j\}$ |
| $\beta_j: t' + 2 \leq j \leq \lambda$ | $\{\lambda - j + 2, \lambda + \mu - j + 2, \lambda + \mu - j\}$ |

If $\mu \geq 4$, then the vertices represented according to A are given in Table 5:

Table 5. The multiset representations of the vertices of $KP(\vartheta, \lambda, \mu)$, for $\mu \geq 4$, according to the set $A = \{\alpha_2, \beta_2, \gamma_1\}$.

| $r_m(\cdot, \cdot)$ | $\gamma_k: 1 \leq k \leq \mu - 1$ |
|---------------------------------------|-----------------------------------|
| $A = \{\alpha_2, \beta_2, \gamma_1\}$ | $\{k + 1, k - 1, \mu - k + 1\}$ |

Therefore, any two vertices do not have the same multiset code or representation according to A , as shown in Tables 3–5. This implies that $md(KP(\vartheta, \lambda, \mu)) \leq 3$. Hence, $md(KP(\vartheta, \lambda, \mu)) = 3$. □

4.2. Dragon Graph

The dragon graph $T_{n,m}$ is obtained by joining a vertex v_n of a cycle graph C_n with a vertex u_1 of a path graph P_m with a bridge. See Figure 4 for an example. The vertex set of a dragon graph $T_{n,m}$ is $V(T_{n,m}) = \{v_i, u_j | 1 \leq i \leq n, 1 \leq j \leq m\}$ and the edge set is $E(T_{n,m}) = \{v_i v_{i+1} : 1 \leq i \leq n - 1\} \cup \{u_j u_{j+1} : 1 \leq j \leq m - 1\} \cup \{v_1 v_n, v_n u_1\}$.

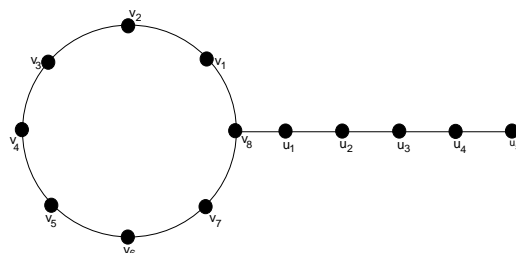


Figure 4. The dragon graph $T_{8,5}$.

In Theorem 7 we determine the exact value of the multiset dimension of a dragon graph.

Theorem 7. Let $T_{n,m}$ be a dragon graph with $n \geq 4$ and $m \geq 3$. Then $md(T_{n,m}) = 3$.

Proof. Therefore, by Theorem 2, $md(T_{n,m}) \geq 3$. We next show that the set $A = \{v_1, v_2, u_m\}$ is a multiset resolving set for $T_{n,m}$.

If $n \geq 4$ is even with $t = \frac{n}{2}$, then the vertices represented according to A are given in Table 6:

Table 6. The multiset representations of the vertices of $T_{n,m}$, for $n \geq 4$, even and $t = \frac{n}{2}$ according to the set $A = \{v_1, v_2, u_m\}$.

| $r_m(\cdot, \cdot)$ | $A = \{v_1, v_2, u_m\}$ |
|----------------------------|---------------------------------------|
| v_1 | $\{0, 1, m + 1\}$ |
| $v_i: 2 \leq i \leq t$ | $\{i - 1, i - 2, m + i\}$ |
| v_{t+1} | $\{t, t - 1, m + t - 1\}$ |
| $v_i: t + 2 \leq i \leq n$ | $\{n - i + 1, n - i + 2, n + m - i\}$ |

If $n \geq 4$ is odd with $t = \lfloor \frac{n}{2} \rfloor$, then the vertices represented according to A are given in Table 7:

Table 7. The multiset representations of the vertices of $T_{n,m}$, for $n \geq 4$, odd and $t = \lfloor \frac{n}{2} \rfloor$ according to the set $A = \{v_1, v_2, u_m\}$.

| $r_m(\cdot, \cdot)$ | $A = \{v_1, v_2, u_m\}$ |
|----------------------------|---------------------------------------|
| v_1 | $\{0, 1, m + 1\}$ |
| $v_i: 2 \leq i \leq t$ | $\{i - 1, i - 2, m + i\}$ |
| v_{t+1} | $\{t, t - 1, m + t\}$ |
| v_{t+2} | $\{t, t, m + t - 1\}$ |
| $v_i: t + 3 \leq i \leq n$ | $\{n - i + 1, n - i + 2, n + m - i\}$ |

If $m \geq 4$ then the vertices represented according to A are given in Table 8:

Table 8. The multiset representations of the vertices of $T_{n,m}$, for $m \geq 4$ according to the set $A = \{v_1, v_2, u_m\}$.

| $r_m(\cdot, \cdot)$ | $A = \{v_1, v_2, u_m\}$ |
|------------------------|---------------------------|
| $u_j: 1 \leq j \leq m$ | $\{j + 1, j + 2, m - j\}$ |

Therefore, any two vertices do not have the same multiset code or representation according to A , as shown in Tables 6–8. This implies that $md(T_{n,m}) \leq 3$. Hence, $md(T_{n,m}) = 3$. \square

4.3. Comb Products Graph

Let G and H be two graphs. Then the comb product, symbolically written as $G \triangleright_o H$, is produced by picking $|V(G)|$ copies of H and one copy of G and identifying the i th copy of graph H at vertex o to the i th vertex of graph G . Such a product is also called a rooted product graph, as introduced in [34], or a hierarchical product graph as defined in [35]. Several studies related to metric dimension parameters of such graphs are found in the literature, for a couple of recent ones, see [36,37]. A fairly representative example of a comb product graph of P_5 and P_4 appears in Figure 5.

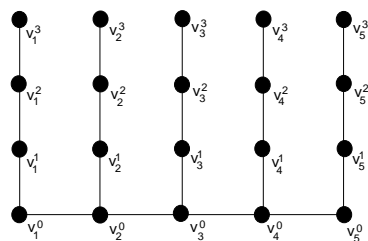


Figure 5. The comb product of P_5 and P_4 , $P_5 \triangleright_{\circ} P_4$.

In Theorem 8 we determine the exact value of the multiset dimension of the comb product of P_n and P_m .

Theorem 8. *If $P_n \triangleright_{\circ} P_m$ is the comb product of P_n and P_m with $n, m \geq 4$, then $md(P_n \triangleright_{\circ} P_m) = 3$.*

Proof. Since $P_n \triangleright_{\circ} P_m$ is not a path, by Theorem 2, $md(P_n \triangleright_{\circ} P_m) \geq 3$. Assume $P_n \triangleright_{\circ} P_m$ has $V(P_n \triangleright_{\circ} P_m) = \{v_i^j : 1 \leq i \leq n, 0 \leq j \leq m - 1\}$ and $E(P_n \triangleright_{\circ} P_m) = \{v_i^0 v_{i+1}^0 : 1 \leq i \leq n - 1\} \cup \{v_i^j v_i^{j+1} : 1 \leq i \leq n; 0 \leq j \leq m - 2\}$. Let us show that the set $A = \{v_1^{m-1}, v_1^0, v_n^{m-1}\}$ is a multiset resolving set for the graph $P_n \triangleright_{\circ} P_m$. The vertices represented according to A are given in Table 9.

Table 9. The multiset representations of the vertices of $P_n \triangleright_{\circ} P_m$ according to the set $A = \{v_1^{m-1}, v_1^0, v_n^{m-1}\}$.

| $r_m(\cdot, \cdot)$ | $A = \{v_1^{m-1}, v_1^0, v_n^{m-1}\}$ |
|--|---|
| $v_1^j : 1 \leq j \leq m - 1$ | $\{m - j - 1, j, n + m + j - 2\}$ |
| $v_i^0 : 1 \leq i \leq n$ | $\{m + i - 2, i - 1, n + m - i - 1\}$ |
| $v_n^j : 1 \leq j \leq m - 1$ | $\{n + m + j - 2, n + j - 1, m - j - 1\}$ |
| $v_i^j : 2 \leq i \leq n - 1, 1 \leq j \leq m - 1$ | $\{m + i + j - 2, i + j - 1, n + m + j - i - 1\}$ |

Therefore, any two vertices do not have the same multiset code or representation according to A , as shown in Table 9. This implies that $md(P_n \triangleright_{\circ} P_m) \leq 3$. Hence, $md(P_n \triangleright_{\circ} P_m) = 3$. □

4.4. Results on the Edge-Multiset Dimension of Graphs

By definition of the edge-multiset dimension and the edge metric dimension, It is clear that the edge-multiset dimension of a connected graph Γ is at least the edge metric dimension of graph Γ which we prove in Lemma 1.

Lemma 1. *Let Γ be a connected graph. Then $dim_e(\Gamma) \leq md_e(\Gamma)$.*

Proof. Let B be an edge resolving set for graph Γ . If we have different edges f and g which have representation with respect to B as $r_e(f|B) = (a, b, c)$ and $r_e(g|B) = (b, a, c)$. For the edge metric dimension, the $r_e(f|B) \neq r_e(g|B)$. This satisfies the properties of the edge metric dimension. However, if we focus on the edge-multiset distance, which leads to $\{a, b, c\} = \{b, a, c\}$, this gives the same edge-multiset representation of edges f and g according to B , $r_{em}(f|B) = r_{em}(g|B) = \{a, b, c\}$. This does not satisfy the properties of the edge-multiset dimension. On the other hand, if we have two edges f_1 and g_1 which have representation according to B , $r_e(f_1|B) = (a, b, c)$ and $r_e(g_1|B) = (b, a, d)$, respectively, then this satisfies the properties of the edge metric dimension. Furthermore, $\{a, b, c\} \neq \{b, a, d\}$ shows that f_1 and g_1 have distinct edge-multiset representation with respect to B , that is $r_{em}(f_1|B) \neq r_{em}(g_1|B)$. Thus, we concludes that $dim_e(\Gamma) \leq md_e(\Gamma)$. □

To give an example of Lemma 1, assume graph Γ with $V(\Gamma) = \{a_1, a_2, \dots, a_9\}$ and $E(\Gamma) = \{f_1, f_2, \dots, f_9\}$ as given in Figure 6. Let the set $B = \{a_3, a_6, a_8\}$ be an edge resolving set for graph Γ and $\dim_e(\Gamma) = 3$.

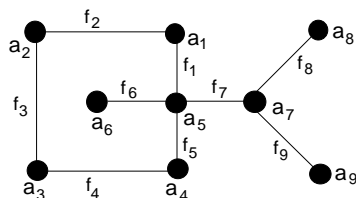


Figure 6. The graph Γ .

If two different edges f_2 and f_9 which have representation with respect to B as $r_e(f_2|B) = (1, 2, 3)$ and $r_e(f_9|B) = (3, 2, 1)$, respectively. For the edge metric dimension, $r_e(f_3|B) \neq r_e(f_9|B)$. This satisfies the properties of the edge metric dimension. However, if we focus on the edge-multiset distance, which leads to $\{1, 2, 3\} = \{3, 2, 1\}$, this gives the same edge-multiset representation of edges f_2 and f_9 according to B , $r_{em}(f_2|B) = r_{em}(f_9|B) = \{1, 2, 3\}$. This does not satisfy the properties of the edge-multiset dimension. Thus, we conclude that $\dim_e(\Gamma) \leq md_e(\Gamma)$.

Theorem 9. Let Γ be a graph. Then $md_e(\Gamma) = 1$, if, and only if, $\Gamma = P_n$.

Proof. Let $V(P_n) = \{w_i : 1 \leq i \leq n\}$ and $E(P_n) = \{e_i = w_i w_{i+1} : 1 \leq i \leq n - 1\}$. In one direction, it is clear that the set $B = \{w_1\}$ is an edge-multiset basis for P_n , that is, $md_e(P_n) = 1$.

For the converse, assume that Γ is a connected graph with $md_e(\Gamma) = 1$. Let $B = \{w_1\}$ be an edge-multiset basis of a graph Γ . Hence, $r_{em}(e|B) \neq r_{em}(f|B)$ for every pair of edges e and $f \in E(\Gamma)$. Since the representations of all edges of Γ , according to B , are distinct, there must exist an edge g such that $r_{em}(g|B) = n - 2$, where $n - 1$ is the size of the graph G . Thus, the diameter of Γ is $n - 1$, which implies that Γ is the path P_n . \square

To give an example of Theorem 9, assume graph $\Gamma = P_7$ with $V(P_7) = \{a, b, c, d, e, f, g\}$ and $E(P_7) = \{e_1, e_2, \dots, e_6\}$ as given in Figure 7.

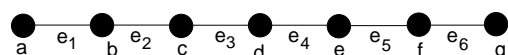


Figure 7. The graph Γ .

It is noted, the set $B = \{a\}$ or $B = \{g\}$ is an edge-multiset resolving set for $\Gamma = P_7$. The set B gives unique edge-multisets representations of the edges of $\Gamma = P_7$.

In the Lemma 2, we show that no graph Γ has an edge-multiset dimension of 2.

Lemma 2. Let Γ be a connected graph. Then $md_e(\Gamma) \neq 2$.

Proof. If Γ is a path, then $md_e(\Gamma) = 1$. So, we may assume that $\Gamma \neq P_n$. Assume that $md_e(\Gamma) = 2$ for some graph Γ and let $B = \{a, b\}$ be an edge resolving set for Γ . If there are two edges $e = ax$ and $f = yb$ (it is possible that $x = y$) such that the vertices x and y lie on the shortest path between a and b , then $d(e, b) = d(f, a)$. Thus, $r_{em}(e|B) = \{0, d(e, b)\} = \{0, d(f, a)\} = r_{em}(f|B)$ is a contradiction. Now, if there does not exist such a pair of edges, then a and b are adjacent. Since Γ is not a path, Γ cannot be P_2 , and so, one of the following possibilities must occur:

- (i) there are two edges incident to a or to b ;
- (ii) there is one edge incident to a and one edge incident to b .

In both cases, we observe that two edges have the same multiset representation, which is impossible. Therefore, no graph Γ has an edge-multiset dimension equal to 2. \square

To give an example of Lemma 2, assume graph $\Gamma = L(T_{7,3})$ with $V(L(T_{7,3})) = \{a_1, a_2, \dots, a_7, b_1, b_2, b_3\}$ and $E(L(T_{7,3})) = \{f_1, f_2, \dots, f_{11}\}$, as given in Figure 8. Let the set $B = \{a_2, b_3\}$ be an edge-multiset resolving set for $L(T_{7,3})$.

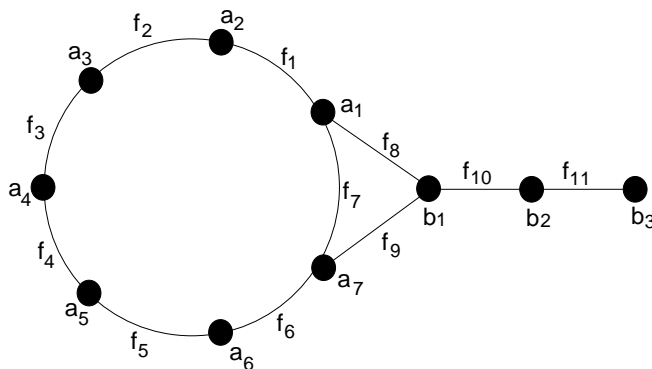


Figure 8. The graph $L(T_{7,3})$.

There are two edges $f_1 = a_1a_2$ and $f_{11}=b_2b_3$ such that the vertices a_1 and b_2 lie on the shortest path between a_2 and b_3 , then $d(a_1, b_3) = d(b_2, a_2) = 3$. Thus, $r_{em}(f_1|B) = \{0, d(a_1, b_3)\} = \{0, 3\} = \{0, d(b_2, a_2)\} = r_{em}(f_{11}|B)$ is a contradiction.

As we know from Lemma 2, every connected graph different from the path has $md_e(\Gamma) \geq 3$.

Theorem 10. For any connected graph Γ other than a path, $md_e(\Gamma) \geq 3$.

In view of the previous result, it is interesting to characterize the graphs having an edge-multiset dimension of 3 as we know that $dime(\Gamma) = 1$, if, and only if, $\Gamma = P_n$ and we obtain the same result for the edge-multiset dimension.

4.5. Graphs Having an Infinite Edge-Multiset Dimension

In this section, we present a few conditions a graph needs to satisfy to have an infinite edge-multiset dimension. In Lemma 3, we show that if the distance between a vertex and an edge is no more than 2, then the graph Γ does not have an edge-multiset resolving set.

Lemma 3. Let Γ be a graph with a set of vertices and edges, V and E , respectively, where $|V| \geq 2$. If the distance between a vertex and an edge is at most 2, then Γ does not contain an edge-multiset resolving set.

Proof. On the contrary, suppose that every pair of edges in E is of distance at most 2 and V is an edge-multiset resolving set of Γ . We represent the vertices in V by v_1, v_2, \dots, v_p , where $p \geq 2$ and edges in E by e_1, e_2, \dots, e_n , where $n \geq p - 1$. For $i = 1, 2, \dots, n$, we have $r_{em}(e_i|V) = \{0, 0, 1^{m_1}, 2^{m_2}\}$ where $m_1 + m_2 = p - 2$. Therefore, we have p vertices in V and n edges in E . All edges have different representations with respect to V and their representations should be of the form $\{0, 0, 1^{p-2}\}, \{0, 0, 1^{p-3}, 2\}, \{0, 0, 1^{p-4}, 2^2\}, \dots, \{0, 0, 2^{p-2}\}$. Without loss of generality, we assume that the edge having the representation $\{0, 0, 1^{p-2}\}$ is $e_1 = v_1v_x$ and the edge having the representation $\{0, 0, 2^{p-2}\}$ is $e_n = v_pv_y$. Since $r_{em}(e_1|V) = \{0, 0, 1^{p-2}\}$, it follows that $r_{em}(e_1 = v_1v_x|V \setminus \{v_1, v_x\}) = \{1^{p-2}\}$ which is in contradiction to $r_{em}(e_n|V) = \{0, 0, 2^{p-2}\}$. Hence, V is not an edge-multiset resolving set of Γ . \square

To give an example of Lemma 3, assume graph Γ with $V(\Gamma) = \{a_1, a_2, \dots, a_5\}$ and $E(\Gamma) = \{f_1, f_2, \dots, f_5\}$, as given in Figure 9. The distance between a vertex a and an edge e_3 (or e_4) is 2.

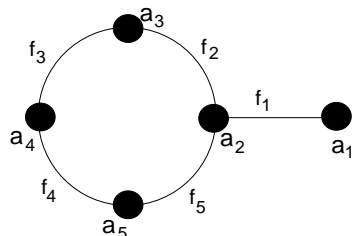


Figure 9. The graph Γ .

Therefore, by Theorem 10, $md_e(\Gamma) \geq 3$. We note that the set $B = \{a, b, c\}$ is not an edge-multiset resolving set for Γ as $r_{em}(e_1|B) = r_{em}(e_2|B) = \{0, 0, 1\}$. If the set $B_1 = \{a, b, c, d\}$ is an edge-multiset resolving set for Γ , then $r_{em}(e_1|B_1) = r_{em}(e_3|B_1) = \{0, 0, 1, 2\}$, which means that the set B_1 is not an edge-multiset resolving set for Γ . If the set $B_2 = \{a, b, c, d, e\}$ is an edge-multiset resolving set for Γ , then $r_{em}(e_1|B_2) = r_{em}(e_3|B_2) = r_{em}(e_4|B_2) = \{0, 0, 1, 1, 2\}$, and $r_{em}(e_2|B_2) = r_{em}(e_5|B_2) = \{0, 0, 1, 1, 1\}$, which means that the set B_2 is not an edge-multiset resolving set for Γ . It is clear that an edge-multiset resolving set for Γ does not exist.

In Theorem 11, we classify the graphs whose edge-multiset dimension is not finite.

Theorem 11. *If any graph Γ has a vertex adjacent to at least three pendant vertices, then $md_e(\Gamma) = \infty$.*

Proof. Assume that v_1, v_2 and v_3 are three pendant vertices adjacent to vertex v_x in Γ . Let $e_1 = v_1v_x, e_2 = v_2v_x$ and $e_3 = v_3v_x$ be three edges incident on pendant vertices v_1, v_2 and v_3 , respectively. Let B be any edge-multiset resolving set of graph Γ , then either at least two of the pendant vertices v_i or v_j where $i, j = 1, 2, 3$ and $i \neq j$, are in B or at least two of the pendant vertices are not in B . In both cases, edges e_i and e_j where $i, j = 1, 2, 3$ and $i \neq j$, cannot be resolved because $r_{em}(e_i, v) = r_{em}(e_j, v)$ where $v \in V(\Gamma)$. Hence, Γ contains a vertex adjacent to at least three pendant vertices. Thus, $md_e(\Gamma) = \infty$. \square

To give an example of Theorem 11, assume graph Γ with $V(\Gamma) = \{a, b, c, d, e, f, g\}$ and $E(\Gamma) = \{e_1, e_2, \dots, e_6\}$, as given in Figure 10.

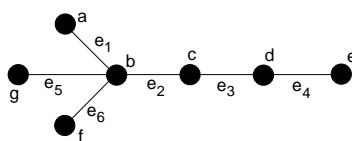


Figure 10. The graph Γ .

There are three pendant vertices a, g and f . We note that the set $B = \{b, c, d, e\}$ is not an edge-multiset resolving set for Γ as $r_{em}(e_1|B) = r_{em}(e_5|B) = r_{em}(e_6|B) = \{0, 1, 2, 3\}$. If the set $B_1 = \{a, b, c, d, e\}$ is an edge-multiset resolving set for Γ , then $r_{em}(e_5|B_1) = r_{em}(e_6|B_1) = \{0, 1, 1, 2, 3\}$, which means that the set B_1 is not an edge-multiset resolving set for Γ . If the set $B_2 = \{a, b, c, d, e, f\}$ is an edge-multiset resolving set for Γ , then $r_{em}(e_1|B_2) = r_{em}(e_6|B_2) = \{0, 0, 1, 1, 2, 3\}$, which means that the set B_2 is not an edge-multiset resolving set for Γ . If the set $B_3 = \{a, b, c, d, e, f, g\}$ is an edge-multiset resolving set for Γ , then $r_{em}(e_1|B_3) = r_{em}(e_5|B_3) = r_{em}(e_6|B_3) = \{0, 0, 1, 1, 1, 2, 3\}$, which means that the set B_2 is not an edge-multiset resolving set for Γ . It is clear that the edge-multiset resolving set for Γ does not exist. Hence, Γ contains a vertex adjacent to three pendant vertices, then $md_e(\Gamma) = \infty$.

Example 1 give some graphs of infinite edge-multiset dimensions.

Example 1. The following families of graphs have infinite edge-multiset dimensions. Complete graph, star graph, friendship graph, wheel graph, the Peterson graph, fan graph, complete bipartite graph and cycle graph with at most 6 vertices.

The edge-multiset resolving set is determined for only some connected graphs. If Γ is a connected graph for which $md_e(\Gamma)$ is decided, then each and every edge-multiset resolving set for Γ is an edge resolving set for Γ , and so $1 \leq dim_e(\Gamma) \leq md_e(\Gamma)$

4.6. Graphs Having a Constant Edge-Multiset Dimension

Next, we present some families of graphs, cycle, kayak paddle, comb product of two paths, caterpillar and lobster which have constant edge-multiset dimensions. In Theorem 12, we show that C_n has a constant edge-multiset dimension.

Theorem 12. Let C_n be a cycle graph with $n \geq 7$. Then $md_e(C_n) = 3$.

Proof. Therefore, by Theorem 10, $md_e(C_n) \geq 3$. Assume $C_n = v_0v_1 \dots v_{n-1}v_0$. Let us show that the set $B = \{v_0, v_1, v_3\}$ is an edge-multiset resolving set for C_n .

If $n = 2t$ with $t > 3$, then the edge representations according to B are represented in Tables 10 and 11.

Table 10. The edge-multiset representations of the edges of C_n according to the set $B = \{v_0, v_1, v_3\}$.

| | | | |
|-------------------------|---------------------------|---------------------------|---------------------------|
| $r_{em}(\cdot, \cdot)$ | e_0 | e_1 | e_2 |
| $B = \{v_0, v_1, v_3\}$ | $\{0, 0, 2\}$ | $\{0, 1, 1\}$ | $\{0, 1, 2\}$ |
| $r_{em}(\cdot, \cdot)$ | e_3 | e_t | e_{t+1} |
| $B = \{v_0, v_1, v_3\}$ | $\{0, 2, 3\}$ | $\{t - 3, t - 1, t - 1\}$ | $\{t - 2, t - 2, t - 1\}$ |
| $r_{em}(\cdot, \cdot)$ | e_{t+2} | e_{t+3} | |
| $B = \{v_0, v_1, v_3\}$ | $\{t - 3, t - 2, t - 1\}$ | $\{t - 4, t - 3, t - 1\}$ | |

Table 11. The edge-multiset representations of the edges of C_n according to the set $B = \{v_0, v_1, v_3\}$.

| | | |
|-------------------------|-----------------------|-----------------------------------|
| $r_{em}(\cdot, \cdot)$ | $e_i: 3 < i < t$ | $e_{t+i}: 3 < i < t$ |
| $B = \{v_0, v_1, v_3\}$ | $\{i - 3, i - 1, i\}$ | $\{t - i - 1, t - i, t - i + 2\}$ |

If $n = 2t + 1$ with $t > 3$, then the edge representations according to B are represented in Tables 12 and 13.

Table 12. The edge-multiset representations of the edges of C_n according to the set $B = \{v_0, v_1, v_3\}$.

| | | | | |
|-------------------------|-----------------------|-----------------------|---------------------------|-----------------------|
| $r_{em}(\cdot, \cdot)$ | e_0 | e_1 | e_2 | e_3 |
| $B = \{v_0, v_1, v_3\}$ | $\{0, 0, 2\}$ | $\{0, 1, 1\}$ | $\{0, 1, 2\}$ | $\{0, 2, 3\}$ |
| $r_{em}(\cdot, \cdot)$ | e_t | e_{t+1} | e_{t+2} | e_{t+3} |
| $B = \{v_0, v_1, v_3\}$ | $\{t - 3, t - 1, t\}$ | $\{t - 2, t - 1, t\}$ | $\{t - 2, t - 1, t - 1\}$ | $\{t - 3, t - 2, t\}$ |

Table 13. The edge-multiset representations of the edges of C_n according to the set $B = \{v_0, v_1, v_3\}$.

| | | |
|-------------------------|-----------------------|-----------------------------------|
| $r_{em}(\cdot, \cdot)$ | $e_i: 3 < i < t$ | $e_{t+i}: 3 < i < t$ |
| $B = \{v_0, v_1, v_3\}$ | $\{i - 3, i - 1, i\}$ | $\{t - i, t - i + 1, t - i + 3\}$ |

Therefore, any two edges do not have the same multiset code or representation according to B , as shown in Tables 10–13. This implies that $md_e(C_n) \leq 3$. Hence, $md_e(C_n) = 3$. \square

In Theorem 13 we show that kayak paddle graph with $\vartheta, \lambda, \mu \geq 4$ has a constant edge-multiset dimension.

Theorem 13. *If $KP(\vartheta, \lambda, \mu)$ is a kayak paddle graph with $\vartheta, \lambda, \mu \geq 4$, then*

$$md_e(KP(\vartheta, \lambda, \mu)) = 3.$$

Proof. By Theorem 10, $md_e(KP(\vartheta, \lambda, \mu)) \geq 3$. Let $V(KP(\vartheta, \lambda, \mu)) = \{\alpha_1, \alpha_2, \dots, \alpha_\vartheta\} \cup \{\beta_1, \beta_2, \dots, \beta_\lambda\} \cup \{\gamma_1, \dots, \gamma_{\mu-1}\}$ and $E(KP(\vartheta, \lambda, \mu)) = \{\alpha_i \alpha_{i+1} : 1 \leq i \leq \vartheta\} \cup \{\beta_j \beta_{j+1} : 1 \leq j \leq \lambda\} \cup \{\gamma_k \gamma_{k+1} : 1 \leq k \leq \mu - 2\} \cup \{\alpha_1 \gamma_1, \gamma_{\mu-1} \beta_1\}$, where $\alpha_{\vartheta+1} = \alpha_1$ and $\beta_{\lambda+1} = \beta_1$. All edges are labelled as follows.

$$e_i = \alpha_i \alpha_{i+1} : 1 \leq i \leq \vartheta \quad e'_j = \beta_j \beta_{j+1} : 1 \leq j \leq \lambda$$

$$e''_k = \gamma_k \gamma_{k-1} : 2 \leq k \leq \mu - 1 \quad e''_1 = \alpha_1 \gamma_1, e''_\mu = \gamma_{\mu-1} \beta_1$$

We show that the set $B = \{\alpha_2, \beta_2, \gamma_1\}$ is an edge-multiset resolving set of $KP(\vartheta, \lambda, \mu)$.

If $\vartheta \geq 4$ with $t = \lfloor \frac{\vartheta}{2} \rfloor$, then the edge representations according to B are represented as follows in Table 14.

Table 14. The edge-multiset representations of the edges of $KP(\vartheta, \lambda, \mu)$ according to the set $B = \{\alpha_2, \beta_2, \gamma_1\}$.

| $r_{em}(\cdot, \cdot)$ | $B = \{\alpha_2, \beta_2, \gamma_1\}$ |
|------------------------------------|---|
| e_1 | $\{0, 1, \mu + 1\}$ |
| $e_i: 2 \leq i \leq t$ | $\{i - 2, i, \mu + i\}$ |
| $e_i: i = t + 1$ | $\{i - 2, \vartheta - i + 1, \vartheta + \mu - i + 1\}$ |
| $e_i: t + 2 \leq i \leq \vartheta$ | $\{\vartheta - i + 1, \vartheta - i + 1, \vartheta + \mu - i + 1\}$ |

If $\lambda \geq 4$ with $t' = \lfloor \frac{\lambda}{2} \rfloor$, then the edge representations according to B are represented in Table 15.

Table 15. The edge-multiset representations of the edges of $KP(\vartheta, \lambda, \mu)$ according to the set $B = \{\alpha_2, \beta_2, \gamma_1\}$.

| $r_{em}(\cdot, \cdot)$ | $B = \{\alpha_2, \beta_2, \gamma_1\}$ |
|------------------------------------|---|
| e'_1 | $\{0, \mu - 1, \mu + 1\}$ |
| $e'_j: 2 \leq j \leq t'$ | $\{j - 2, \mu + j - 2, \mu + j\}$ |
| $e'_j: j = t' + 1$ | $\{j - 2, \lambda + \mu - j - 1, \lambda + \mu - j + 1\}$ |
| $e'_j: t' + 2 \leq j \leq \lambda$ | $\{\lambda - j + 1, \lambda + \mu - j - 1, \lambda + \mu - j + 1\}$ |

If $\mu \geq 4$, then the edge representations according to B are represented in Table 16.

Table 16. The edge-multiset representations of the edges of $KP(\vartheta, \lambda, \mu)$ according to the set $B = \{\alpha_2, \beta_2, \gamma_1\}$.

| $r_{em}(\cdot, \cdot)$ | e''_1 | $e''_k: 2 \leq k \leq \mu$ |
|---------------------------------------|-------------------------|-----------------------------|
| $B = \{\alpha_2, \beta_2, \gamma_1\}$ | $\{0, 1, \mu - k + 1\}$ | $\{k - 2, k, \mu - k + 1\}$ |

Therefore, any two edges do not have the same multiset code or representation according to B , as shown in Tables 14–16. This implies that $md_e(KP(\theta, \lambda, \mu)) \leq 3$. Hence, $md_e(KP(\theta, \lambda, \mu)) = 3$. \square

In the next theorem we determine the exact value of the multiset dimension of the dragon graph.

Theorem 14. *If $T_{n,m}$ is a dragon graph with $n \geq 4$ and $m \geq 3$, then $md_e(T_{n,m}) = 3$.*

Proof. By Theorem 10, $md_e(T_{n,m}) \geq 3$. Let $V(T_{n,m}) = \{v_i, u_j | 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(T_{n,m}) = \{v_i v_{i+1} : 1 \leq i \leq n\} \cup \{u_j u_{j+1} : 1 \leq j \leq m-1\} \cup \{v_n u_1\}$, where $v_{n+1} = v_1$. The edges of $T_{n,m}$ are labelled as follows.

$$f_i = \{v_i v_{i+1} : 1 \leq i \leq n\}$$

$$f'_{j+1} = \{u_j u_{j+1} : 1 \leq j \leq m-1\}, f'_1 = v_n u_1$$

Now, we prove that $B = \{v_1, v_2, u_m\}$ is an edge-multiset resolving set for $T_{n,m}$.

If $n \geq 4$ is even with $t = \frac{n}{2}$, then the edge representations according to B are represented in Table 17.

Table 17. The edge-multiset representations of the edges of $T_{n,m}$ according to the set $B = \{v_1, v_2, u_m\}$.

| $r_{em}(\cdot, \cdot)$ | $B = \{v_1, v_2, u_m\}$ |
|----------------------------|---------------------------|
| f_1 | $\{0, 0, m+1\}$ |
| $f_i: 2 \leq i \leq t-1$ | $\{i-1, i-2, m+i\}$ |
| f_t | $\{t-1, t-2, m+t-1\}$ |
| f_{t+1} | $\{t, t, m+t-2\}$ |
| $f_i: t+2 \leq i \leq n-1$ | $\{n-i, n-i+1, n+m-i-1\}$ |
| f_n | $\{0, 1, m\}$ |

If $n \geq 4$ is odd with $t = \lceil \frac{n}{2} \rceil$, then the edge representations according to B are represented in Table 18.

Table 18. The edge-multiset representations of the edges of $T_{n,m}$ according to the set $B = \{v_1, v_2, u_m\}$.

| $r_{em}(\cdot, \cdot)$ | $B = \{v_1, v_2, u_m\}$ |
|----------------------------|---------------------------|
| f_1 | $\{0, 0, m+1\}$ |
| $f_i: 2 \leq i \leq t-1$ | $\{i-1, i-2, m+i\}$ |
| f_t | $\{t-1, t-2, m+t-2\}$ |
| f_{t+1} | $\{t-1, t-2, m+t-3\}$ |
| $f_i: t+2 \leq i \leq n-1$ | $\{n-i, n-i+1, n+m-i-1\}$ |
| f_n | $\{0, 1, m\}$ |

If $m \geq 4$, then the edge representations according to B are represented in Table 19.

Table 19. The edge-multiset representations of the edges of $T_{n,m}$ according to the set $B = \{v_1, v_2, u_m\}$.

| $r_{em}(\cdot, \cdot)$ | $B = \{v_1, v_2, u_m\}$ |
|-------------------------|-------------------------|
| $f'_j: 1 \leq j \leq m$ | $\{j, j+1, m-j\}$ |

Therefore, any two edges do not have the same multiset code or representation according to B , as shown in Tables 17–19. We have $md_e(T_{n,m}) \leq 3$. Hence, $md_e(T_{n,m}) = 3$. \square

Theorem 15 determines the exact value of the edge-multiset dimension of a comb product of two paths P_n and P_m .

Theorem 15. *If $P_n \triangleright_{\circ} P_m$ is the comb product of two paths P_n and P_m , with $n, m \geq 4$, then $md_e(P_n \triangleright_{\circ} P_m) = 3$.*

Proof. By Theorem 10, $md_e(P_n \triangleright_{\circ} P_m) \geq 3$. Let $V(P_n \triangleright_{\circ} P_m) = \{v_i^j : 1 \leq i \leq n, 0 \leq j \leq m - 1\}$ and the edges are labelled as follows.

$$e_i^j = \begin{cases} v_i^{j-1}v_i^j, & j \neq 0; \\ v_i^0a_{i+1}^0 & j = 0. \end{cases} \quad 1 \leq i \leq n; 0 \leq j \leq m - 1$$

We show that the set $B = \{v_1^{m-1}, v_1^0, v_n^{m-1}\}$ is an edge-multiset resolving set for $P_n \triangleright_{\circ} P_m$. The edge representations according to B are represented in Table 20.

Table 20. The edge-multiset representations of the edges of $P_n \triangleright_{\circ} P_m$ according to the set $B = \{v_1^{m-1}, v_1^0, v_n^{m-1}\}$.

| $r_{em}(\cdot, \cdot)$ | $B = \{v_1^{m-1}, v_1^0, v_n^{m-1}\}$ |
|---|---|
| $e_1^j: 1 \leq j \leq m - 1$ | $\{m - j - 1, j - 1, n + m + j - 3\}$ |
| $e_i^0: 1 \leq i \leq n$ | $\{m + i - 2, i - 1, n + m - i - 2\}$ |
| $e_n^j: 1 \leq j \leq m - 1$ | $\{n + m + j - 3, n + j - 2, m - j - 1\}$ |
| $e_i^j: 2 \leq i \leq n - 1, 1 \leq j \leq m - 1$ | $\{m + i + j - 3, i + j - 2, n + m + j - i - 2\}$ |

Therefore, any two edges do not have the same multiset code or representation according to B , as shown in Table 20. We thus deduced that $md_e(P_n \triangleright_{\circ} P_m) \leq 3$. Hence, $md_e(P_n \triangleright_{\circ} P_m) = 3$. \square

4.7. Caterpillar Graph

A caterpillar is a tree. A path graph is obtained if all a caterpillar graph’s leaves are removed. Throughout the article, we denote a caterpillar graph by CT_n with a diametral path $P = (a_1, a_2, \dots, a_n)$. CT_n is a caterpillar with $deg(a_i) \leq 3$ where $2 \leq i \leq n - 1$. Let $D = \{i | 2 \leq i \leq n - 1\}$. Let b_i be the pendent vertices of CT_n of degree 1 joined to a_i . Thus, the vertex set of a caterpillar graph CT_n is $V(CT_n) = \{a_i | 1 \leq i \leq n\} \cup \{b_i | i \in D\}$ and the edge set of a caterpillar graph CT_n is $E(CT_n) = \{a_i a_{i+1} | 1 \leq i \leq n - 1\} \cup \{a_i b_i | i \in D\}$. If $i \notin D$, then a_i is a non-leg or gap vertex. A leg vertex has degree 3, a gap vertex a_i , $i \notin D$, and $deg(a_i) \leq 2$. See Figure 11 for an example.

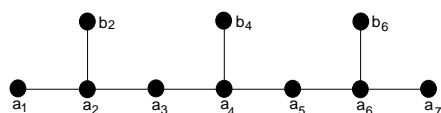


Figure 11. Caterpillar graph CT_7 with $D = \{2, 4, 6\}$.

Theorem 16 shows that the edge-multiset dimension of a caterpillar graph is 3.

Theorem 16. *If CT_n is a caterpillar graph with $n \geq 5$, then $md_e(CT_n) = 3$.*

Proof. Therefore, by Theorem 10, $md_e(CT_n) \geq 3$. If n is even and the set $B = \{a_1, a_{n-2}, a_n\}$ is an edge-multiset resolving set for the graph CT_n , then the edge representations according to B are represented in Table 21.

Table 21. The edge-multiset representations of the edges of CT_n according to the set $B = \{a_1, a_{n-2}, a_n\}$.

| | |
|------------------------------------|-----------------------------------|
| $r_{em}(\cdot, \cdot)$ | $B = \{a_1, a_{n-2}, a_n\}$ |
| a_1a_2 | $\{0, n - i - 1, n - 4\}$ |
| $a_i a_{i+1}: 2 \leq i \leq n - 3$ | $\{i - 1, n - i - 1, n - i - 3\}$ |
| $a_{n-2}a_{n-1}$ | $\{i - 1, n - i - 1, 0\}$ |
| $a_{n-1}a_n$ | $\{i - 1, n - i - 1, 1\}$ |

If $i \in D$, then the edge representations according to B are represented in Table 22.

Table 22. The edge-multiset representations of the edges of CT_n according to the set $B = \{a_1, a_{n-2}, a_n\}$.

| | | |
|-----------------------------|--------------------------------|-----------------------|
| $r_{em}(\cdot, \cdot)$ | $a_i b_i: 2 \leq i \leq n - 2$ | $a_{n-1} b_{n-1}$ |
| $B = \{a_1, a_{n-2}, a_n\}$ | $\{i - 1, n - i, n - i - 2\}$ | $\{i - 1, n - i, 1\}$ |

If n is odd, $i \in D$ and the set $B = \{a_1, a_2, a_n\}$ is an edge-multiset resolving set for the graph CT_n , then the edge representations according to B are represented in Table 23.

Table 23. The edge-multiset representations of the edges of CT_n according to the set $B = \{a_1, a_2, a_n\}$.

| | | | |
|-------------------------|-------------------|------------------------------------|--------------------------------|
| $r_{em}(\cdot, \cdot)$ | $a_1 a_2$ | $a_i a_{i+1}: 2 \leq i \leq n - 1$ | $a_i b_i: 2 \leq i \leq n - 1$ |
| $B = \{a_1, a_2, a_n\}$ | $\{0, n - 2, 0\}$ | $\{i - 1, i - 2, n - i - 1\}$ | $\{i - 1, i - 2, n - i\}$ |

Therefore, no two edges have the same multiset code or representation according to B , as shown in Tables 21–23. We thus deduce that $md_e(CT_n) \leq 3$. Hence, $md_e(CT_n) = 3$. □

4.8. Lobster Graph

Throughout the article, we denote a lobster graph by L_n with a diametral path $P = (x_1, x_2, \dots, x_n)$. Obviously, P_n is the spine of L_n . Let L_n be a lobster with $deg(x_i) \leq 3$ where $3 \leq i \leq n - 2$. Let $D = \{i | 3 \leq i \leq n - 2\}$. If $i \in D$, then x_i is a base-leg vertex. For $i \in D$, let y_i be the vertex of L_n of degree 2 joined to x_i , and z_i be the pendent vertex of L_n joined to y_i . Thus, the vertex set of lobster graph L_n is $V(L_n) = \{x_i | 1 \leq i \leq n\} \cup \{y_i | i \in D\} \cup \{z_i | i \in D\}$ and the edge set of lobster graph L_n is $E(L_n) = \{x_i x_{i+1} | 1 \leq i \leq n - 1\} \cup \{x_i y_i | i \in D\} \cup \{y_i z_i | i \in D\}$. If $i \notin D$, then x_i is a gap vertex or a non-leg vertex. Thus, a leg vertex is of degree 3, a gap vertex x_i , and $i \notin D$ is of degree $deg(x_i) \leq 2$. See Figure 12 for an example.

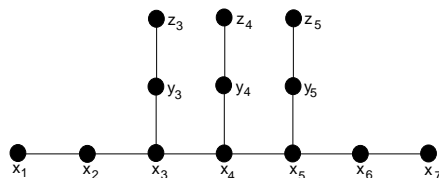


Figure 12. Lobster graph L_7 with $D = \{3, 4, 5\}$.

Theorem 17 shows that a lobster graph also has a constant edge-multiset dimension.

Theorem 17. If L_n is a lobster graph with $n \geq 5$, then $md_e(L_n) = 3$.

Proof. Therefore, by Theorem 10, $md_e(L_n) \geq 3$. If n is even and the set $B = \{x_1, x_2, x_{n-1}\}$ is an edge-multiset resolving set for the graph L_n , then the edge representations according to B are represented in Table 24.

Table 24. The edge-multisets representations of the edges of L_n according to the set $B = \{x_1, x_2, x_{n-1}\}$.

| $rem(\cdot, \cdot)$ | x_1x_2 | $x_ix_{i+1}: 2 \leq i \leq n-2$ | $x_{n-1}x_n$ |
|-----------------------------|-----------------|---------------------------------|-------------------|
| $B = \{x_1, x_2, x_{n-1}\}$ | $\{0, 0, n-3\}$ | $\{i-1, i-2, n-i-2\}$ | $\{i-1, i-2, 0\}$ |

If $i \in D$ then the edge representations according to B are represented in the Table 25.

Table 25. The edge-multisets representations of the edges of L_n according to the set $B = \{x_1, x_2, x_{n-1}\}$.

| $rem(\cdot, \cdot)$ | $x_iy_i: 3 \leq i \leq n-2$ | $y_iz_i: 3 \leq i \leq n-2$ |
|-----------------------------|-----------------------------|-----------------------------|
| $B = \{x_1, x_2, x_{n-1}\}$ | $\{i-1, i-2, n-i-1\}$ | $\{i, i-1, n-i\}$ |

If n is odd and the set $B = \{x_1, x_2, x_n\}$ is an edge-multiset resolving set for the graph L_n , then the edge representations according to B are represented in the Table 26.

Table 26. The edge-multisets representations of the edges of L_n according to the set $B = \{x_1, x_2, x_n\}$.

| $rem(\cdot, \cdot)$ | x_1x_2 | $x_ix_{i+1}: 2 \leq i \leq n-1$ |
|-------------------------|-----------------|---------------------------------|
| $B = \{x_1, x_2, x_n\}$ | $\{0, 0, n-2\}$ | $\{i-1, i-2, n-i-1\}$ |

If $i \in D$, then the edge representations according to B are represented in Table 27.

Table 27. The edge-multisets representations of the edges of L_n according to the set $B = \{x_1, x_2, x_n\}$.

| $rem(\cdot, \cdot)$ | $x_iy_i: 3 \leq i \leq n-2$ | $y_iz_i: 3 \leq i \leq n-2$ |
|-------------------------|-----------------------------|-----------------------------|
| $B = \{x_1, x_2, x_n\}$ | $\{i-1, i-2, n-i\}$ | $\{i, i-1, n-i+1\}$ |

Therefore, no two edges have the same multiset code or representation according to B as shown in Tables 24–27. We thus deduce that $md_e(L_n) \leq 3$. Hence, $md_e(L_n) = 3$. \square

4.9. Graphs Having Dependent Edge-Multiset Dimension on Their Order

In this section, Lemma 4 characterizes the trees of order n and diameter 3.

Lemma 4. Let T_n be a tree with order n and diameter 3. If $md_e(T_n) \neq \infty$, then $md_e(T_n) \leq n-2$.

Proof. Therefore, if $diam(T_n) = 3$ there exist two vertices a and b of T_n , such that $2 \leq deg(a)$, $deg(b) \leq 3$. We shall thus obtain the following three cases. See Figure 13 for visualization.

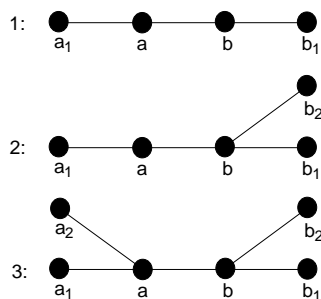


Figure 13. Three cases of graph T_n with diameter 3.

- If $deg(a) = deg(b) = 2$, then $T_n \equiv P_4$ and $md_e(T_n) = 1 = n-3$;
- If $deg(a) = 2$ and $deg(b) = 3$. Let $N(a) = \{a_1, b\}$ and $N(b) = \{a, b_1, b_2\}$. Then we have $B = \{a_1, a, b_1\}$ as an edge-multiset resolving set for T_n , which means $md_e(T_n) = 3 = n-2$;

- If $deg(a) = deg(b) = 3$. Let $N(a) = \{a_1, a_2, b\}$ and $N(b) = \{a, b_1, b_2\}$. Then we do not have an edge-multiset resolving set for T_n , which means $md_e(T_n) = \infty$.
□

Theorem 18 gives the bounds of the edge-multiset dimension of trees with order n , which is not a path.

Theorem 18. Let T_n be a tree with order n and T_n as not a path. If $md_e(T_n) \neq \infty$, then $3 \leq md_e(T_n) \leq n - 2$.

Proof. As we know, if graph Γ is not a path graph then the edge-multiset dimension of any graph Γ is $md_e(\Gamma) \geq 3$ by Theorem 10. T_n is a tree graph but not a path graph so, $md_e(T_n) \geq 3$ which is the lower bound for T_n . It is clear that the edge-multiset dimension of T_n is $md_e(T_n) \leq n - 2$ by Lemma 4, so, we obtain $3 \leq md_e(T_n) \leq n - 2$. □

In Theorem 19, the exact value of the edge-multiset dimension of a caterpillar graph is obtain.

Theorem 19. If CT_n is a caterpillar graph with $n \geq 5$, $deg(x_i) \leq 4$ where $3 \leq i \leq n - 2$ and $D = \{i | 3 \leq i \leq n - 2\}$, then $md_e(L_n) = 3 + |D|$.

Proof. Let CT_n be a caterpillar graph with a spine $P_n = (x_1, x_2, \dots, x_n)$. Here, $deg(x_i) \leq 4$ with $3 \leq i \leq n - 2$. Let $D = \{i | 3 \leq i \leq n - 2\}$. For $i \in D$, let y_i^1 and y_i^2 be the pendant vertices of CT_n of degree 1. Thus, the vertex set of a caterpillar graph CT_n is $V(CT_n) = \{x_i | 1 \leq i \leq n\} \cup \{y_i^1, y_i^2 | i \in D\}$ and the edge set of caterpillar graph CT_n is $E(CT_n) = \{x_i x_{i+1} | 1 \leq i \leq n - 1\} \cup \{x_i y_i^1, x_i y_i^2 | i \in D\}$. If $i \notin D$, x_i is called a gap vertex. Thus, a leg vertex is of degree 4, a gap vertex x_i , and $i \notin D$, is of degree $deg(x_i) \leq 2$. See Figure 14 for an example.

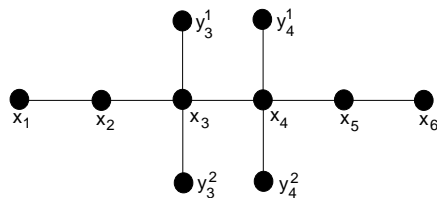


Figure 14. Caterpillar graph CT_6 with $D = \{3, 4\}$.

Let the set $B = \{x_1, x_2, x_n, y_i^1 | 3 \leq i \leq n - 2\}$ by an edge-multiset resolving set for the graph CT_n . Suppose a vertex y_i^1 with $3 \leq i \leq n - 2$ is not an element of the set B . Then the edges $x_i y_i^1, x_i y_i^2$ have the same multiset distance from the elements of B , which is a contradiction. Therefore y_i^1 , with $i \in D, 3 \leq i \leq n - 2$, are elements of set B . If vertex x_1 is not an element of B , then the edges $x_2 x_3$ and $x_3 y_3^1$ have the same multiset distance from the elements of B , which is a contradiction. If vertex x_2 is not an element of B , then the edges $x_i x_{i+1}$ and $x_{n-i} x_{n-i+1}$, where $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$, have the same multiset distance from the elements of B , which is a contradiction. If vertex x_n is not an elements of B , then the edges $x_{n-2} x_{n-1}$ and $x_{n-2} y_{n-2}^2$ have the same multiset distance from the elements of B , which is a contradiction. Therefore, $md_e(CT_n) \geq 3 + |D|$.

If $W = \{x_1, x_2, x_n\}$ is a subset of $V(CT_n)$, then the edge representations are presented in Table 28:

Table 28. The edge-multisets representations of some edges of CT_n according to the set $W = \{x_1, x_2, x_n\}$.

| $r_{em}(\cdot, \cdot)$ | $x_1 x_2$ | $x_i x_{i+1}: 2 \leq i \leq n - 1$ | $x_i y_i^1, x_i y_i^2: 3 \leq i \leq n - 2$ |
|-------------------------|-------------------|------------------------------------|---|
| $W = \{x_1, x_2, x_n\}$ | $\{0, 0, n - 2\}$ | $\{i - 1, i - 2, n - i - 1\}$ | $\{i - 1, i - 2, n - i\}$ |

It is observed that the multiset distance of edges according to W is $r_{em}(x_i y_i^1 | W) = r_{em}(x_i y_i^2 | W)$ so, the vertices y_i^1 or y_i^2 , where $i \in D, 3 \leq i \leq n - 2$, are included in the set W . Therefore, $md_e(CT_n) \leq 3 + |D|$. Hence, $md_e(CT_n) = 3 + |D|$. \square

In Theorem 20, the exact value of the edge-multiset dimension of a lobster graph is obtain.

Theorem 20. *If L_n is a lobster graph with $n > 5, deg(x_i) = 4$, where $i \in D = \{i | 3 \leq i \leq n - 2\}$, then $md_e(L_n) = 3 + |D|$.*

Proof. Let L_n be a lobster graph with a spine $P_n = (x_1, x_2, \dots, x_n)$. Here, $deg(x_i) \leq 4$ and $3 \leq i \leq n - 2$. Let $D = \{i | 3 \leq i \leq n - 2\}$. For all x_i with $3 \leq i \leq n - 2$ there exist $i \in D$, such that y_i^1 and y_i^2 are the vertices of L_n of degree 2 joined to x_i . z_i^1 and z_i^2 are the pendant vertices of L_n joined to y_i^1 and y_i^2 , respectively. Thus, the vertex set of a lobster graph L_n is $V(L_n) = \{x_i | 1 \leq i \leq n\} \cup \{y_i^1, y_i^2 | i \in D\} \cup \{z_i^1, z_i^2 | i \in D\}$ and the edge set of the lobster graph L_n is $E(L_n) = \{x_i x_{i+1} | 1 \leq i \leq n - 1\} \cup \{x_i y_i^1, x_i y_i^2 | i \in D\} \cup \{y_i^1 z_i^1, y_i^2 z_i^2 | i \in D\}$. See Figure 15 for an example.

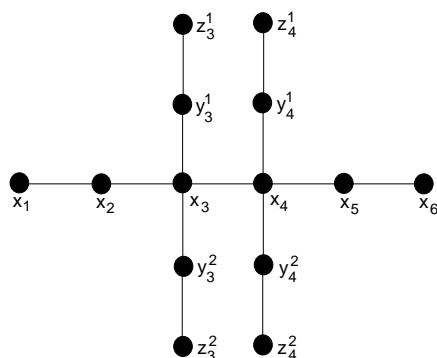


Figure 15. Lobster graph L_6 with $D = \{3, 4\}$.

Let the set $B = \{x_1, x_2, x_n, y_{n-2}^1, z_i^1 | 3 \leq i \leq n - 3\}$ is an edge-multiset resolving set for the graph L_n . Suppose that a vertex z_i^1 with $3 \leq i \leq n - 3$ is not an element of the set B . Then the edges $x_i y_i^1$ and $x_i y_i^2$ have the same multiset distances from the elements of B , and the edges $y_i^1 z_i^1$ and $y_i^2 z_i^2$ also have the same multiset distances from the elements of B , which is a contradiction. Therefore, y_i^1 with $i \in D, 3 \leq i \leq n - 3$ are elements of the set B . If vertex x_1 is not an element of B , then the edges $x_1 x_2$ and $y_{n-2}^1 y_{n-2}^2$ have the same multiset distances from the elements of B , and the edges $x_2 x_3$ and $x_{n-2} y_{n-2}^1$ also have the same multiset distances from the element of B , which is a contradiction. If vertex x_2 is not an element of B , then the edges $x_1 x_2$ and $y_3^1 y_3^2$ and the edges $x_2 x_3$ and $x_3 y_3^1$ have the same multiset distances from the elements of B , which is a contradiction. If vertex x_n is not an element of B , then the edges $x_{n-2} x_{n-1}$ and $x_{n-2} y_{n-2}^1$ and the edges $x_{n-1} x_n$ and $y_{n-2}^1 y_{n-2}^2$ have the same multiset distances from the elements of B , which is a contradiction. If the set B does not contain z_{n-2}^1 the edges $x_{n-2} y_{n-2}^1$ and $x_{n-2} y_{n-2}^2$ and the edges $y_{n-2}^1 z_{n-2}^1$ and $y_{n-2}^2 z_{n-2}^2$ have the same multiset distances from the elements of B , which is a contradiction. Therefore, $md_e(L_n) \geq 3 + |D|$.

If $W = \{x_1, x_2, x_n\}$ is a subset of $V(L_n)$, then the edge representations are presented in Table 29:

Table 29. The edge-multiset representations of some edges of L_n according to the set $W = \{x_1, x_2, x_n\}$.

| $r_{em}(\cdot, \cdot)$ | $x_1 x_2$ | $x_i x_{i+1}: 2 \leq i \leq n - 1$ | $x_i y_i^1, x_i y_i^2: 3 \leq i \leq n - 2$ | $y_i^1 z_i^1, y_i^2 z_i^2: 3 \leq i \leq n - 2$ |
|-------------------------|-------------------|------------------------------------|---|---|
| $W = \{x_1, x_2, x_n\}$ | $\{0, 0, n - 2\}$ | $\{i - 1, i - 2, n - i - 1\}$ | $\{i - 1, i - 2, n - i\}$ | $\{i, i - 1, n - i + 1\}$ |

It is clear that the multiset distance of edges according to W are $r_{em}(x_i y_i^1 | W) = r_{em}(x_i y_i^2 | W)$ and $r_{em}(y_i^1 z_i^1 | W) = r_{em}(y_i^2 z_i^2 | W)$. So, the vertices z_i^1 or z_i^2 , where $i \in D$, $3 \leq i \leq n - 3$ and the vertex y_{n-2}^1 , are included in the set W . Therefore, $md_e(L_n) \leq 3 + |D|$. Hence, $md_e(L_n) = 3 + |D|$. \square

In Theorem 21, we show that the edge-multiset dimension of a complete r -ary tree is finite if, and only if, r is either 1 or 2. If r is greater than 2, then the edge-multiset dimension of the complete r -ary tree is infinite.

4.10. Complete r -ary Tree Graph

Theorem 21. *The multiset dimension of a complete r -ary tree is finite if, and only if, $r = 1$ or 2. Moreover, if T is a complete binary tree of height h , then $md_e(T) = 2^h - 1$.*

Proof. Let T be a complete r -ary tree of height $h \geq 1$. If $r \geq 3$, by Theorem 11, then $md_e(T) = \infty$. If $r = 1$, then T is a path and $md_e(T) = 1$.

Let $r = 2$. If $h = 1$, then T is a path with two edges and $md_e(T) = 1$. Consider the binary trees of height $h \geq 2$. Let the set B be any edge-multiset resolving set of T . Examine the last level h of T containing 2^{h-1} pairs of pendant vertices of distance two and the edges are linked with these pendant vertices of distance 1. From the pair of these pendant vertices, one vertex is necessary for set B ; otherwise, the representation of edges linked with pendant edges is the same. Now, we examine the $h - 1$ level of T , having 2^{h-2} pair of edges of distance 1. Edges of each pair have the same multiset representations according to the vertices in the set B , which are in level h , and these edges have the same distance to any other vertex of T . So, these pair of edges cannot be resolved by any other vertices, which means that exactly one of the vertex linked with each pair of edges of $h - 1$ level is in the set B . Similarly, for the next level $h - 2, h - 3, \dots, 1$ we obtain $2^{h-1} + 2^{h-2} + \dots + 1 = 2^h - 1$ vertices that must be in the set B . Thus, $md_e(T) \geq 2^h - 1$.

For every level l of T , where $1 \leq l \leq h$, there are exactly 2^{l-1} pairs of edges of distance 1, these edges join the vertices of level $l - 1$ and level l . Let the set B contain exactly one vertex of level l from each such pair. So, the $|B| = \sum_{l=1}^h 2^{l-1} = 2^h - 1$. We prove that the set B is an edge-multiset resolving set. Let us show that B resolves any two edges f and g of T . We consider two cases.

- The edges f and g are in different levels, say p and q , respectively, where $0 \leq p < q \leq h$:

The distance between edge g joined to vertices of level q and $q - 1$ and 2^{h-2} vertices of the set B in level h is $h + q - 1$. There is no vertex in set B of distance $h + q - 1$ from edge f . Thus, the edges g and f have different representations.

- The edges f and g are in the same level (the edges f and g are linked the vertices of levels $p - 1$ and p), say p , where $0 \leq p \leq h$:

The edges f and g have different representations if exactly one vertex of level p is linked with the edges f or g is in the set B . If there is no vertex of level p linked with the edges f and g in the set B or two vertices (a vertex linked with edge f and a vertex linked with edge g) are in the set B , then let us denote by v the central vertex of the path connecting edges f and g (initial and final vertex of the path are in level q). This path has an even length, say $2n$, and then v is in level $q - n$. The vertex v is adjacent to two vertices, say v_1 and v_2 , in level $q - n + 1$. The vertices v_1 and v_2 belong to the path containing the edges f and g and one vertex from v_1 and v_2 are in the set B , thus obtaining $d(f, v_1) \neq d(g, v_2)$. It can be easily verified that the edges f and g have the same multiset code or representation according to set $B \setminus \{v_1\}$. Hence, the set B is an edge-multiset resolving set and $md_e(T) \leq 2^h - 1$. Hence, $md_e(T) = 2^h - 1$. \square

The complete 2-ary tree graph with $h = 4$, as given in Figure 16, is an example of Theorem 21. The red vertices are included in the edge-multiset resolving set for a complete 2-ary tree graph with $h = 4$.

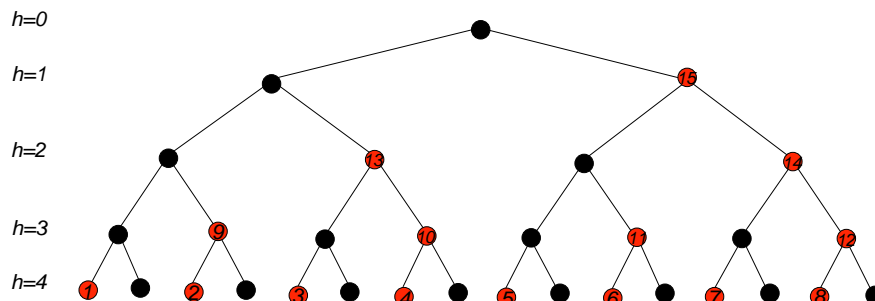


Figure 16. Complete 2-ary tree graph with $h = 4$.

5. Discussion on the Multiset and Edge-Multiset Dimensions of Graphs

It is already known from [17] that the edge metric and the metric dimensions of graphs are not comparable since there are graphs Γ for which $\dim(\Gamma) = \dim_e(\Gamma)$, $\dim(\Gamma) > \dim_e(\Gamma)$ or $\dim(\Gamma) < \dim_e(\Gamma)$. In [28], a lot of families of graphs were described for which the edge dimension was greater than the vertex dimension $\dim(\Gamma) > \dim_e(\Gamma)$, and an open problem was presented in [17].

In concordance with this, it is natural to consider comparing the multiset and edge-multiset dimensions of graphs by wondering if a similar situation happens to the metric and edge metric dimensions. We next see that this is precisely the case. For instance, Section 4 there were given some values for the multiset and edge-multiset dimensions of some graphs (comb product of paths, kayak paddle graphs and dragon graphs), respectively. This shows that the equality $md(\Gamma) = md_e(\Gamma)$ can be realized for some graphs Γ (another example could be paths and cycles). We next show that $md(\Gamma) < md_e(\Gamma)$ and $md(\Gamma) > md_e(\Gamma)$ can also occur.

5.1. Graphs G with $md(\Gamma) < md_e(\Gamma)$

To achieve our goal in this section, we consider Γ be the graph with vertex set $V(\Gamma) = \{u_1, u_2, \dots, u_{17}\}$ and edge set $E(\Gamma) = \{e_1, e_2, \dots, e_{18}\}$, as shown in Figure 17.

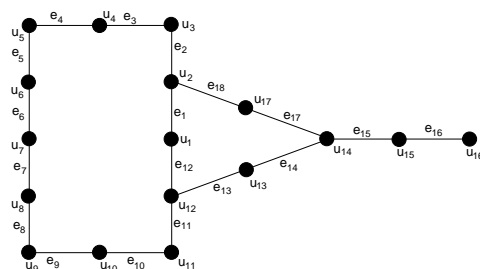


Figure 17. The graph Γ .

We note that, for instance, the set $A = \{u_3, u_{16}, u_{17}\}$ is a multiset resolving set for Γ . Since, the graph Γ is not a path graph, by Theorem 1, $md(\Gamma) \geq 3$.

Table 30 shows the multiset representations of all vertices of Γ according to $A = \{u_3, u_{16}, u_{17}\}$. Since no two vertices have the same multiset code or representation according to A , we deduce that $md(\Gamma) \leq 3$. So, by using Theorem 1 we have $md(\Gamma) = 3$.

Table 30. The multiset representations of the vertices of Γ according to the set $A = \{u_3, v_{16}, v_{17}\}$.

| | | | | | | |
|---------------|-----------|-----------|-----------|-----------|-----------|-----------|
| u_i | u_1 | u_2 | u_3 | u_4 | u_5 | u_6 |
| $r_m(u_i, A)$ | {2, 2, 5} | {1, 1, 4} | {0, 2, 5} | {1, 3, 6} | {2, 4, 7} | {3, 5, 8} |
| u_i | u_7 | u_8 | u_9 | u_{10} | u_{11} | u_{12} |
| $r_m(u_i, A)$ | {4, 6, 9} | {5, 7, 8} | {6, 6, 7} | {5, 5, 6} | {4, 4, 5} | {3, 3, 4} |
| u_i | u_{13} | u_{14} | u_{15} | u_{16} | u_{17} | |
| $r_m(u_i, A)$ | {2, 3, 4} | {1, 2, 3} | {1, 2, 4} | {0, 3, 5} | {0, 2, 3} | |

Now, we note that for instance, the set $B = \{u_1, u_7, u_{13}, u_{16}\}$ is an edge-multiset resolving set of Γ . Since, the graph Γ is not a path graph, by Theorem 10, $md_e(\Gamma) \geq 3$.

Table 31 shows the edge-multiset representations of all edges of Γ according to B . Since any two edges do not have the identical edge-multiset code or representation according to B , we deduce that $md_e(\Gamma) \leq 4$. We must show that $md_e(\Gamma) \geq 4$.

Table 31. The edge-multiset representations of the edges of Γ according to the set $B = \{u_1, u_7, u_{13}, u_{16}\}$.

| | | | | | | |
|------------------|--------------|--------------|--------------|--------------|--------------|--------------|
| e_i | e_1 | e_2 | e_3 | e_4 | e_5 | e_6 |
| $r_{em}(e_i, B)$ | {0, 2, 5, 4} | {1, 3, 4, 4} | {2, 3, 4, 5} | {2, 3, 5, 6} | {1, 4, 6, 7} | {0, 5, 6, 8} |
| e_i | e_7 | e_8 | e_9 | e_{10} | e_{11} | e_{12} |
| $r_{em}(e_i, B)$ | {0, 5, 5, 8} | {1, 4, 4, 7} | {2, 3, 3, 6} | {2, 2, 3, 5} | {1, 1, 4, 4} | {0, 1, 4, 5} |
| e_i | e_{13} | e_{14} | e_{15} | e_{16} | e_{17} | e_{18} |
| $r_{em}(e_i, B)$ | {0, 1, 3, 5} | {0, 2, 2, 6} | {1, 1, 3, 7} | {0, 2, 4, 8} | {1, 2, 2, 6} | {1, 2, 3, 5} |

On the contrary, suppose that $md_e(\Gamma) = 3$; no two edges have the same edge-multiset representation.

- If $B_1 = \{u_1, v_{13}, v_{16}\}$ is an edge-multiset resolving set, then $r_{em}(e_1|B_1) = r_{em}(e_{16}|B_1) = \{0, 2, 4\}$ which is in contradiction to our supposition.
- If $B_2 = \{u_7, u_{13}, u_{16}\}$ is an edge-multiset resolving set, then $r_{em}(e_{10}|B_2) = r_{em}(e_{18}|B_2) = \{2, 3, 5\}$ which is in contradiction to our supposition.
- If $B_3 = \{u_1, u_7, u_{13}\}$ is an edge-multiset resolving set, then $r_{em}(e_{12}|B_3) = r_{em}(e_{13}|B_3) = \{0, 1, 5\}$ which is in contradiction to our supposition.
- If $B_4 = \{u_1, v_7, v_{16}\}$ is an edge-multiset resolving set, then $r_{em}(e_i|B_4) = r_{em}(e_{13-i}|B_4)$ for $1 \leq i \leq 6$, $r_{em}(e_{13}|B_4) = r_{em}(e_{18}|B_4) = \{1, 3, 5\}$ and $r_{em}(e_{14}|B_4) = r_{em}(e_{17}|B_4) = \{2, 2, 6\}$ which is in contradiction to our supposition.

Now, it is clear that $md_e(\Gamma) \geq 4$. Hence, $md_e(\Gamma) = 4$.

Remark 1. The graph Γ given in Figure 17 satisfies the inequality $md(\Gamma) < md_e(\Gamma)$.

5.2. Graphs G with $md(\Gamma) > md_e(\Gamma)$

Let Γ be the graph with vertex set $V(\Gamma) = \{v_1, v_2, \dots, v_{19}\}$ and edge set $E(\Gamma) = \{e_1, e_2, \dots, e_{21}\}$, as shown in Figure 18.

We note that, for instance, the set $B = \{v_2, v_3, v_{12}\}$ is an edge-multiset resolving set for Γ . Since, the graph Γ is not a path graph, by Theorem 10, $md_e(\Gamma) \geq 3$.

Table 32 shows the edge-multiset representations of all the edges of Γ according to $B = \{v_2, v_3, v_{12}\}$. Since no two edges have the same edge-multiset code or representation according to B , we deduce that $md_e(\Gamma) \leq 3$. So, by using Theorem 10 we have $md_e(\Gamma) = 3$.

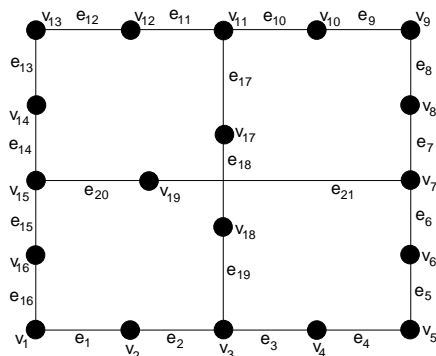


Figure 18. The graph Γ .

Table 32. The edge-multiset representations of the edges of Γ according to the set $B = \{v_2, v_3, v_{12}\}$.

| | | | | | | | |
|------------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| e_i | e_1 | e_2 | e_3 | e_4 | e_5 | e_6 | e_7 |
| $r_{em}(e_i, B)$ | $\{0, 1, 5\}$ | $\{0, 0, 4\}$ | $\{0, 1, 4\}$ | $\{1, 2, 5\}$ | $\{2, 3, 6\}$ | $\{3, 4, 5\}$ | $\{4, 4, 5\}$ |
| e_i | e_8 | e_9 | e_{10} | e_{11} | e_{12} | e_{13} | e_{14} |
| $r_{em}(e_i, B)$ | $\{3, 5, 6\}$ | $\{2, 4, 5\}$ | $\{1, 3, 4\}$ | $\{0, 3, 4\}$ | $\{0, 4, 5\}$ | $\{1, 4, 5\}$ | $\{2, 3, 4\}$ |
| e_i | e_{15} | e_{16} | e_{17} | e_{18} | e_{19} | e_{20} | e_{21} |
| $r_{em}(e_i, B)$ | $\{2, 3, 3\}$ | $\{1, 2, 4\}$ | $\{1, 2, 3\}$ | $\{1, 2, 2\}$ | $\{0, 1, 3\}$ | $\{3, 3, 4\}$ | $\{4, 4, 4\}$ |

Now, we note that, for instance, the set $A = \{v_2, v_4, v_9, v_{19}\}$ is a multiset resolving set for Γ . Since the graph Γ is not a path graph, by Theorem 1, $md(\Gamma) \geq 3$.

Table 33 shows the multiset representations of all vertices of Γ according to A . Since no two vertices have the same multiset code or representation according to A , we deduce that $md(\Gamma) \leq 4$. We must show that $md(\Gamma) \geq 4$. On the contrary, suppose that $md(\Gamma) = 3$; no two vertices have the same multiset representation.

Table 33. The multiset representations of the vertices of Γ according to the set $A = \{v_2, v_4, v_9, v_{19}\}$.

| | | | | | |
|---------------|------------------|------------------|------------------|------------------|------------------|
| v_i | v_1 | v_2 | v_3 | v_4 | v_5 |
| $r_m(v_i, A)$ | $\{1, 3, 3, 6\}$ | $\{0, 2, 4, 6\}$ | $\{1, 1, 5, 5\}$ | $\{0, 2, 4, 5\}$ | $\{1, 3, 3, 4\}$ |
| v_i | v_6 | v_7 | v_8 | v_9 | v_{10} |
| $r_m(v_i, A)$ | $\{2, 2, 3, 4\}$ | $\{1, 2, 3, 5\}$ | $\{1, 2, 4, 6\}$ | $\{0, 3, 5, 6\}$ | $\{1, 4, 5, 5\}$ |
| v_i | v_{11} | v_{12} | v_{13} | v_{14} | v_{15} |
| $r_m(v_i, A)$ | $\{2, 4, 4, 5\}$ | $\{3, 4, 5, 5\}$ | $\{3, 4, 5, 6\}$ | $\{2, 4, 5, 6\}$ | $\{1, 3, 4, 5\}$ |
| v_i | v_{16} | v_{17} | v_{18} | v_{19} | |
| $r_m(v_i, A)$ | $\{2, 2, 4, 5\}$ | $\{3, 3, 3, 6\}$ | $\{2, 2, 4, 6\}$ | $\{0, 3, 4, 4\}$ | |

- If $A_1 = \{v_2, v_4, v_9\}$ is a multiset resolving set, then $r_m(v_{13}|A_1) = r_m(v_{14}|A_1) = \{4, 5, 6\}$, which is a contradiction to our supposition.
- If $A_2 = \{v_2, v_4, v_{19}\}$ is a multiset resolving set, then $r_m(v_1|A_2) = r_m(v_5|A_2) = \{1, 3, 3\}$, $r_m(v_2|A_2) = r_m(v_4|A_2) = \{0, 2, 4\}$, $r_m(v_6|A_2) = r_m(v_{16}|A_2) = \{2, 2, 4\}$, $r_m(v_7|A_2) = r_m(v_{15}|A_2) = \{1, 3, 5\}$, $r_m(v_8|A_2) = r_m(v_{14}|A_2) = \{2, 4, 6\}$, $r_m(v_9|A_2) = r_m(v_{13}|A_2) = \{3, 5, 6\}$ and $r_m(v_{11}|A_2) = r_m(v_{12}|A_2) = \{4, 5, 5\}$, which is a contradiction to our supposition.
- If $A_3 = \{v_2, v_9, v_{19}\}$ is a multiset resolving set, then $r_m(v_4|A_3) = r_m(v_{11}|A_3) = r_m(v_{14}|A_3) = \{2, 4, 5\}$, which is a contradiction to our supposition.

- If $A_4 = \{v_4, v_9, v_{19}\}$ is a multiset resolving set, then $r_m(v_1|A_4) = r_m(v_{17}|A_4) = \{3, 3, 6\}$, $r_m(v_2|A_4) = r_m(v_{18}|A_4) = \{2, 4, 6\}$ and $r_m(v_{11}|A_4) = r_m(v_{16}|A_4) = \{2, 4, 5\}$, which is a contradiction to our supposition.

It is clear that $md(\Gamma) \geq 4$. Hence, $md(\Gamma) = 4$.

Remark 2. The graph Γ given in Figure 18 satisfies the inequality $md(\Gamma) > md_e(\Gamma)$.

6. Conclusions

In this work, we defined the notion of the edge-multiset dimension of graphs. We have noted that the multiset dimension $md(\Gamma)$ and edge-multiset dimension $md_e(\Gamma)$ of graphs are generally not comparable. We have given examples of graphs Γ , where $md(\Gamma) > md_e(\Gamma)$, $md(\Gamma) < md_e(\Gamma)$, or $md(\Gamma) = md_e(\Gamma)$. In particular, we found the exact values of the multiset dimension and edge-multiset dimension of the kayak paddle, dragon, and comb product of P_n and P_m graphs. We have proven that they have the same multiset dimension and edge-multiset dimension. Furthermore, the edge-multiset dimension of a connected graph is at least the edge metric dimension. No graph has an edge-multiset dimension of 2, which means every connected graph (other than the path graph whose edge-multiset dimension is one) always has an edge-multiset dimension greater than or equal to three.

Further, we classify the graphs as having infinite edge-multiset dimensions. Some classes of graphs with constant edge-multiset dimensions are also discussed. Graphs with dependent edge-multiset dimensions on their order have also been studied. In the end, we compare the multiset and edge-multiset dimensions of the graph.

The less explored case is that of $md(\Gamma) > md_e(\Gamma)$. In connection with all our expositions, we remark on some possible future lines that could be explored on this topic.

Problem 1. What is the computational complexity of the edge-multiset dimension of a graph?

Problem 2. Let Γ be a graph such that $md(\Gamma)$ and $md_e(\Gamma)$ are finite. Can all graphs Γ for which $md(\Gamma) > md_e(\Gamma)$, $md(\Gamma) < md_e(\Gamma)$, or $md(\Gamma) = md_e(\Gamma)$ be characterized?

Problem 3. Is there any relationship between the multiset and edge-multiset dimensions with the classical metric and edge metric dimensions of a graph?

Problem 4. Let Γ be a graph with n vertices. If $md_e(\Gamma)$ is finite, can any upper and/or lower bounds with respect to n be found for $md_e(\Gamma)$?

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