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Behavior as $t \to \infty$ of Solutions of a Mixed Problem for a Hyperbolic Equation with Periodic Coefficients on the Semi-Axis

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Abstract: In this paper, we consider the asymptotic behavior (as $t \to \infty$) of solutions as an initial boundary value problem for a second-order hyperbolic equation with periodic coefficients on the semi-axis ($x > 0$). The main approach to studying the problem under consideration is based on the spectral theory of differential operators, as well as on the properties of the spectrum ($\sigma(H_0)$) of the one-dimensional Schrödinger operator H_0 with periodic coefficients $p(x)$ and $q(x)$.

Keywords: asymptotic behavior; hyperbolic equation; periodic coefficients; initial boundary value problem; Schrödinger operator

MSC: 35B40; 35L10; 35L15

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1. Introduction

For some areas of theoretical physics, such as wave mechanics, the theory of oscillations, etc., the solution of problems is reduced to the problem of eigenvalues. In addition, the question of the unambiguous definition of a mechanical system, i.e., the Hamilton function, through the spectrum of eigenvalues of the linear differential equation associated with it is important.

In the case in which the string is vibrating and the boundary conditions are natural, it was shown in [1] that the spectrum of eigenvalues uniquely determines the differential equation, which, in Schrödinger's theory, is called the "amplitude equation".

The authors of [2] dealt with the problem of determining the Hill equation (or the one-dimensional Schrödinger equation) from its spectrum, as well as deriving the Hill equation from specific properties of its discriminant. A great deal is known about the analytic structure of the discriminant (see, for example, [3,4]).

The present article is devoted to the asymptotic behavior (as $t \to \infty$) of solutions to the initial boundary (mixed) problem for a one-dimensional second-order hyperbolic equation with periodic coefficients. The authors of [5,6] considered similar questions for the Cauchy problem with initial conditions, as in the case of a positive Hill operator ($H_0 > 0$) and in the case when the left end of the spectrum ($\sigma(H_0)$) of the operator Hill (H_0) is non-positive.

Consider as $t \to \infty$ the following initial-boundary value problem:

$$u_{tt}(x,t) - (p(x)\, u_x(x,t))_x + q(x)\, u(x,t) = 0, \quad x > 0,\, t > 0, \tag{1}$$

$$u(x,t)|_{t=0} = 0, \quad u_t(x,t)|_{t=0} = f(x), \quad x \geq 0, \tag{2}$$

$$u(x,t)|_{x=0} = 0, \quad t \geq 0, \tag{3}$$

where $p(x)$ and $q(x)$ are 1-periodic functions,

$$p(x+1) = p(x) \geq \text{const} > 0, \quad q(x+1) = q(x) \geq 0.$$

Here, we assume that the functions $p(x)$ and $q(x)$ are continuous or have a finite number of discontinuities of the first kind in the period $f \in C_0^\infty(\mathbb{R})$, $\operatorname{supp} f \subset [0,1]$.

We also consider the asymptotic behavior (as $t \to \infty$) of solutions as the initial boundary value problem for a one-dimensional second-order hyperbolic equation with periodic coefficients $p(x)$ and $q(x)$ on the semi-axis $x > 0$.

Similarly, these results extend to the symmetric case relative to the origin of the semi-axis, that is, when $x < 0$. Furthermore, the concept of symmetry is traced throughout the text in the process of solving the problem, for example, when we define the operators H^+ and H^-, for which the integration contours L_+ and L_- and Green's functions $\Gamma^+(x,\xi,k)$ and $\Gamma^-(x,\xi,k)$, respectively, are defined separately.

The asymptotic properties of solutions of exterior boundary value problems were studied in fundamental books and papers [7–9].

Let us point out one of the main books [7], which presents most of the methods for studying the asymptotic behavior (as $t \to \infty$) of solutions to problems of different formulations, including problems similar to (1)–(3) with $p(x) = 1$ and the corresponding multidimensional problems; provided that the potential differs from a constant by a finite value, the function tends toward a constant at infinity rather quickly. This book contains a large number of necessary references on the studied and actual field of modern science.

In [8], the asymptotics of the spectral function for equations in the whole space and the semiclassical asymptotics of the solution of the scattering problem and the scattering amplitude, as well as the asymptotic behavior with an unlimited increase in time of solutions of external mixed problems for hyperbolic equations, were studied in detail. A connection was also established between the departure of wave fronts and the decrease in local energy.

In [9], the asymptotic behavior for $t \to \infty$ and $|x| < a < \infty$ of a solution of the problem of the scattering of waves by periodically moving bodies, as well as the asymptotics of solutions of more general exterior mixed problems, periodic with respect to t, was studied.

In [10], the asymptotic behavior (as t) of the solution of a mixed problem for a hyperbolic equation in the following formulation was studied:

$$a(x)u_{tt}(x,t) = u_{xx}(x,t), \quad 0 < a_0 \leq a(x) \leq A < +\infty, \quad x > 0,\, t > 0,$$

$$u(x,t)|_{t=0} = f(x), \quad u_t(x,t)|_{t=0} = g(x), \quad x \geq 0,$$
$$u(x,t)|_{x=0} = 0, \quad t \geq 0,$$

It follows from the obtained asymptotic expansion that the solution to the problem under study uniformly decreases exponentially in x on any compact set as $t \to \infty$.

One of the most common methods for calculating the eigenvalues of the Sturm–Liouville problem of the type $(p(x)y'(x))' + (\lambda r(x) - q(x))y(x) = 0$ is the integration of a differential equation with trial values for λ. After applying the appropriate boundary condition at one end, the validity of the trial value λ is judged by how closely the corresponding function $y(x)$ matches the boundary condition at the other end.

The authors of [11] described an alternative procedure based on a differential equation satisfied by the phase function ϑ, where $\operatorname{tg} \vartheta = y(x)/(p(x)y'(x))$.

Mixed problems were considered in both bounded and unbounded domains, and the questions of existence, uniqueness, and stability were studied under various restrictions on the initial and boundary conditions, as well as under conditions characterizing the behavior of the solutions of these problems. In this regard, in [12], the first mixed problem for the wave equation in a cylindrical region was considered; using the method of characteristics, the authors obtained an explicit formula for the classical solution of this problem and found conditions for matching the original functions that guarantee sufficient smoothness of the solution in the entire region.

A huge number of both fundamental and applied scientific papers and books are devoted to the asymptotic behavior and spectral properties of the Schrödinger operator (see [13–22]).

In particular, in [13,14], the spectral properties of the Schrödinger operator in domains with infinite boundaries were studied, as well as the behavior of the solution in non-stationary problems as $t \to \infty$.

In [15], the problem of scattering by a one-dimensional periodic lattice ($p(x)$) with an impurity potential ($q(x)$) was considered. Using the asymptotics of scattered waves, in this paper, a stationary scattering matrix is constructed, and its properties are studied. In addition, it is shown that the stationary scattering matrix coincides with the non-stationary scattering operator, which is defined in a simple way in the quasi-momentum representation of the unperturbed operator (H_0). Here, the inverse scattering problem is also solved, i.e., the problem of recovering $q(x)$ from $p(x)$ and scattering data. To solve the inverse problem in the presence of a periodic lattice, a significant modification of the classical reasoning is necessary. As a result, in this paper, the conditions on the scattering data necessary with a finite second moment and sufficient for the existence of a single impurity potential with given scattering characteristics and a finite first moments are found.

In [16], the asymptotics of the Green's function as t are found for the one-dimensional diffusion equation both in the case when the potential is a function with compact support and in the case when the potential is a periodic function of coordinates. In the first case, the Green's function asymptotics can be represented by the elements of the scattering matrix of the corresponding Schrödinger operator for negative energies on the spectral plane, and in the second case, the asymptotics can be represented by the Floquet–Bloch function of the corresponding Hill operator for negative energy values on the spectral plane.

In [22], for a one-dimensional Schrödinger equation with a quasi-periodic potential analytic on its shell, it was shown that the Floquet representation can be used for almost any energy (E) in the upper part of the spectrum; it was also proven that the upper part of the spectrum is purely absolutely continuous, i.e., the Cantor set for the general potential. It was also shown that for a small potential, these results can be extended to the entire spectrum.

In the case of periodic $p(x)$ and $q(x)$, the first results of the Cauchy problem were published in [23,24] in the form of short communications, and as noted above, complete proofs were presented in [5,6]. Similar problems were considered in [25] with $p(x) = 1$, that is, in the case of a periodic potential ($q(x)$).

The authors of [26] proposed a different approach to the study of differential equations and related initial and boundary value problems. In particular, they presented some solutions to the 3D Laplace equation in terms of linear combinations of generalized hypergeometric functions in a prolate elliptic geometry that models current tokamak shapes. It was also proven that the obtained solutions are comparable with the solutions obtained in the standard toroidal geometry.

The main results of this paper were reported in [27] in the form of brief communications.

2. Preliminaries and Auxiliary Statements

2.1. Spectrum and Green's Function of the Schrödinger Operator with Periodic Coefficients on the Half-Axis

We continue the function $u(x,t)$ by zero in the region $t < 0$ and apply Fourier transform to the variable t:

$$y(x,k) = \int_0^\infty u(x,t)\, e^{ikt} dt.$$

Then, the mixed problem (1)–(3) becomes the next problem on the semi-axis

$$\begin{cases} (p(x)\, y'(x,k))' + (k^2 - q(x))\, y(x,k) = -f(x), \quad x > 0, \\ y(0) = 0. \end{cases} \tag{4}$$

In what follows, for an arbitrary function ($g(x,k)$) we denote its derivative with respect to x as g' and its derivative with respect to k as g_k.

Let the Equation (4) be homogeneous, i.e., $f(x) \equiv 0$, and let $\{y = \theta(x,k),\ y = \varphi(x,k)\}$ is the fundamental system of solutions of this equation such that

$$\begin{cases} \theta(0,k) = 1, & \theta'(0,k) = 0, \\ \varphi(0,k) = 0, & \varphi'(0,k) = 1. \end{cases}$$

It is known [17] that $\theta(x,k)$ and $\varphi(x,k)$ are entire functions in k on the real axis and, for $|k| \to \infty$, take the form:

$$\begin{cases} \theta(x,k) = & \cos kx + O(|k|^{-1}e^{|\tau|x}), \\ \varphi(x,k) = & \frac{1}{k}\sin kx + O(|k|^{-2}e^{|\tau|x}), \quad \tau = Im\, k. \end{cases} \tag{5}$$

uniformly in $x \in [0,1]$. Moreover, these expansions can be differentiated in x and in k.

Let $\theta(k) = \theta(1,k)$, $\theta'(k) = \theta'(1,k)$, $\varphi(k) = \varphi(1,k)$, $\varphi'(k) = \varphi'(1,k)$ and $F(k) \equiv \theta(k) + \varphi'(k)$. The functions $\theta(k)$, $\theta'(k)$, $\varphi(k)$, $\varphi'(k)$ and $F(k)$ are on the real axis of the complex plane of the variable k.

The one-dimensional Schrödinger operator (or Hill operator) is the differential operator

$$H_0 := -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x),$$

generated in the Hilbert space ($L^2(\mathbb{R})$) by the operation

$$\Lambda_0 y := -(p(x)\,y')' + q(x)\,y.$$

Let H^+ (or H^-) be the operator generated by the differential operation

$$\Lambda_0 y := -(p(x)\,y')' + q(x)\,y$$

with the boundary condition $y(0) = 0$ in the space $L^2(0;+\infty)$ (or $L^2(-\infty;0)$).

We formulate some statements as important information about the spectrum of the operators H^+ (or H^-), the proof of which, in most cases, can be found in [3,17,19,28].

The spectrum ($\sigma(H_0)$) of the Schrödinger operator (H_0) is absolutely continuous and is a finite or infinite sequence of isolated segments (zones) separated by lacunae to infinity.

Let us provide a more detailed characterization of the spectrum ($\sigma(H_0)$) of the Schrödinger operator (H_0). To this end, we consider the following Sturm–Liouville problems.

Let $\hat{v}(x,\lambda_n)$ be an eigenfunction of the periodic problem:

$$\begin{cases} -(p(x)y')' + q(x)y = \lambda_n y, \quad x \in [0,1], \\ y(0) = y(1), \quad y'(0) = y'(1), \end{cases} \tag{6}$$

and let $\hat{v}(x,\mu_n)$ be an eigenfunction of the antiperiodic problem:

$$\begin{cases} -(p(x)y')' + q(x)y = \mu_n y, \quad x \in [0,1], \\ y(0) = -y(1), \quad y'(0) = -y'(1). \end{cases} \tag{7}$$

Let both of these functions be normalized in $L^2([0,1])$. The eigenvalues of the corresponding problems ($\lambda_n = \lambda_n^2$ and $\mu_n = \mu_n^2$, $n = 0,1,2,\ldots$) are numbered in ascending order, taking into account the multiplicity.

Extending the function $\hat{v}(x,\lambda_n)$ (or $\hat{v}(x,\mu_n)$) to the entire real axis in a periodic (or antiperiodic) way, we obtain the function denoted by $v(x,\lambda_n)$ (or $v(x,\mu_n)$).

It is known [17] (§ 21.4) that if the Schrödinger operator (H_0) is positive, then all eigenvalues of the periodic (antiperiodic) Sturm–Liouville problem are positive. Moreover, between the numbers $\lambda_n = \lambda_n^2$ and $\mu_n = \mu_n^2$, $n = 0, 1, 2, \ldots$, there is a relation,

$$\lambda_0 < \mu_0 \leq \mu_1 < \lambda_1 \leq \lambda_2 < \mu_2 \leq \mu_3 < \lambda_3 \leq, \ldots. \tag{8}$$

The Schrödinger operator has only a continuous spectrum, which is located on the real axis and is semibounded on the left [17].

Let us replace the spectral parameter (λ) with k^2 and consider the spectra of the operators H^+ and H^- on the complex plane of the variable k.

The continuous spectra of the operators H^+ and H^- are the same and coincide with the continuous spectrum ($\sigma(H_0)$) of the Schrödinger operator (H_0).

Let the Schrödinger operator (H_0) be positive: $H_0 > 0$. Then, the continuous spectra of the operators H^+ and H^- on the complex plane of the variable k coincide with the sequence of segments on the real axis extending in both directions to infinity

$$[-\lambda_{2n+1}, -\mu_{2n+1}], [-\mu_{2n}, -\lambda_{2n}], [\lambda_{2n}, \mu_{2n}], [\mu_{2n+1}, \lambda_{2n+1}], n = 0, 1, 2, \ldots.$$

The set of points ($\pm\lambda_n$) coincides with the set of roots of the equation $F(k) = 2$ (or $\pm\mu_n$ with the set of roots of the equation $F(k) = -2)$, $n = 0, 1, 2, \ldots$.

In addition to the continuous spectrum, the operators H^+ and H^- have eigenvalues that are determined by the zeros of the function $\varphi(k)$.

For gaps in the spectrum, that is, segments not included in the spectrum,

$$[-\lambda_{2n}, -\lambda_{2n-1}],\ [\lambda_{2n-1}, \lambda_{2n}],\ n \geq 1, \quad [-\mu_{2n+1}, -\mu_{2n}],\ [\mu_{2n}, \mu_{2n+1}],\ n \geq 0,$$

$\lambda_{2n-1} \neq \lambda_{2n}$ and $\mu_{2n} \neq \mu_{2n+1}$ are denoted as lacunae.

The function $\varphi(k)$ has one simple zero in these intervals and no other zeros.

If $\lambda_n = \lambda_n^2$ (or $\mu_n = \mu_n^2$) is the end of a lacuna, then (8) implies that $\pm\lambda_n$ is a simple root of the equation $F(k) = 2$ (or $\pm\mu_n$ is the root of the equation $F(k) = -2$), $n = 0, 1, 2, \ldots$, ([18]).

As is known [17], if $\lambda_n = \lambda_n^2$ (or $\mu_n = \mu_n^2$) is the end of a lacuna, then λ_n (or μ_n) is the simple eigenvalue of the periodic (or antiperiodic) Sturm–Liouville problem (6) (or (7)).

The eigenvalues of the operator H^+ are the numbers ($\lambda_n = k_n^2$) for which the following conditions are satisfied

$$\varphi(k_n) = 0, \quad \operatorname{sign} \sqrt{F^2(k_n) - 4}\,(\varphi'(k_n) - \theta(k_n)) = -1. \tag{9}$$

Similarly, the eigenvalues of the operator H^- are the numbers ($\lambda_n = k_n^2$) for which the following conditions are satisfied

$$\varphi(k_n) = 0, \quad \operatorname{sign} \sqrt{F^2(k_n) - 4}\,(\varphi'(k_n) - \theta(k_n)) = 1. \tag{10}$$

The eigenvalues of the operators H^+ and H^- on the k plane correspond to the points that are strictly inside the lacunae, since the ends of the lacunae are determined by the condition $F^2(k_n) - 4 = 0$:

$$(\varphi'(k) - \theta(k))^2 - (F^2(k) - 4) = -4\varphi'(k)\theta(k), \tag{11}$$

It follows from Equation (11) that at the zeros of the function ($\varphi(k)$) are located at the ends of the lacunae ($\varphi'(k) - \theta(k) = 0$).

It also follows from Equation (11) that if inside the lacunae, the function $\varphi(k) = 0$, then the condition of Equation (9) (or (10)) is satisfied, which means that the point $\lambda = k^2$ is an eigenvalue of the operator H^+ (or H^-).

Let γ_ν, $\nu = 1, 2, \ldots$ (or σ_ν, $\nu = 1, 2, \ldots$) be the set of zeros of the function $\varphi(k)$ located inside the lacunae in which the conditions of Equation (9) (or (10)) are satisfied.

Let $\gamma_{-\nu} = -\gamma_\nu$, $\nu = 1, 2, \ldots$ (or $\sigma_{-\nu} = -\sigma_\nu$, $\nu = 1, 2, \ldots$).

Since the functions $\varphi(k)$, $\varphi'(k)-\theta(k)$ and $\sqrt{F^2(k_n)-4}$ are even, then on the complex plane of the variable k, the discrete spectrum of the operator H^+ (or H^-) coincides with the set $\{\gamma_\nu\}$, $\nu=\pm1,\pm2,\ldots$ (or $\{\sigma_\nu\}$, $\nu=\pm1,\pm2,\ldots$).

Let us consider the following eigenvalue problem

$$\begin{cases} -(p(x)y')'+q(x)y=k^2y, \quad 0<x<1, \\ y(0)=y(1). \end{cases} \tag{12}$$

It is known [17,19] that if the Schrödinger operator (H_0) is positive, then all eigenvalues $(\lambda=k^2)$ of the problem described in Equation (12) are positive and simple. We enumerate the eigenvalues of the problem described in Equation (12) in ascending order and let $\{\kappa_n\}$, $n=1,2,\ldots$ be the set of eigenvalues.

Furthermore, $v(x,\kappa_n)$ denotes the eigenfunction corresponding to the eigenvalue $\kappa_n=\kappa_n^2$ normalized by the condition $||v;L^2([0,1])||=1$.

If we introduce the notation $\kappa_{-\nu}=-\kappa_\nu$, then the set $\{\kappa_\nu\}$, $\nu=\pm1,\pm2,\ldots$ coincides with the set of all zeros of the function $\varphi(k)$, and the following inequalities hold:

$$\mu_{2n}\le\kappa_{2n+1}\le\mu_{2n+1}, \quad \lambda_{2n+1}\le\kappa_{2n+2}\le\lambda_{2n+2},\ n=0,1,2,\ldots \tag{13}$$

The sets $\{\gamma_\nu\}$, $\nu=\pm1,\pm2,\ldots$ and $\{\sigma_\nu\}$, $\nu=\pm1,\pm2,\ldots$ are disjoint subsets of the set $\{\kappa_\nu\}$, $\nu=\pm1,\pm2,\ldots$.

Generally speaking, they do not exhaust all $\{\kappa_\nu\}$, $\nu=\pm1,\pm2,\ldots$, since the last set may contain zeros of the function $\varphi(k)$ located on the boundaries of the lacuna in which $\varphi'(k)-\theta(k)=0$.

Since the function $\varphi(x,\kappa_n)$ satisfies the equation and boundary conditions of the problem described in Equation (12), then

$$v(x,\kappa_n)=\frac{\varphi(x,\kappa_n)}{\sqrt{\int_0^1|\varphi(x,\kappa_n)|^2dx}}=\frac{\varphi(x,-\kappa_n)}{\sqrt{\int_0^1|\varphi(x,\kappa_n)|^2dx}},$$

and the set $\{v(x,\kappa_n)\}$, $n=1,2,\ldots$ is complete in the space $L^2([0,1])$.

$\mathbb{C}'$ denotes the complex plane of the variable k with cuts along the vertical rays lying in the lower half-plane and starting at the ends of the lacunae.

Furthermore,

$$m_1(k)=\frac{\varphi'(k)-\theta(k)}{2\varphi(k)}+\frac{\sqrt{(\theta(k)+\varphi'(k))^2-4}}{2\varphi(k)}, \quad k\in\mathbb{C}',$$

$$m_2(k)=\frac{\varphi'(k)-\theta(k)}{2\varphi(k)}-\frac{\sqrt{(\theta(k)+\varphi'(k))^2-4}}{2\varphi(k)}, \quad k\in\mathbb{C}',$$

where the branch of the root is determined by the condition $\sqrt{F(k)^2-4}>0$ for $k=0$.

Note that the function $\sqrt{F(k)^2-4}$ branches only at the ends of the lacunae [17], so $m_1(k)$ and $m_2(k)$ are single-valued in $\mathbb{C}'$. Then, for any k, $Im\,k>0$

$$\begin{aligned} \psi_1(x,k)\equiv&\quad\theta(x,k)+m_1(k)\,\varphi(x,k)\in L^2(-\infty,0), \\ \psi_2(x,k)\equiv&\quad\theta(x,k)+m_2(k)\,\varphi(x,k)\in L^2(0,+\infty). \end{aligned} \tag{14}$$

For $Im\,k>0$ the Green's function of the operator H^+ is equal to

$$\Gamma^+(x,\xi,k)=\frac{\psi_2(x,k)\psi_2(\xi,k)}{m_1(k)-m_2(k)}+T(x,\xi,k). \tag{15}$$

Similarly, for $Im\, k > 0$, the Green's function of the operator H^- is equal to

$$\Gamma^-(x,\xi,k) = \frac{\psi_1(x,k)\psi_1(\xi,k)}{m_1(k) - m_2(k)} + T(x,\xi,k). \tag{16}$$

From the identities presented in Equation (14) and from the obvious consequence of the following equations,

$$\Lambda\psi_i = -(p(x)\psi_i')' + q(x)\psi_i \equiv 0, \quad i = 1,2,$$

it follows that the constructed functions, $\Gamma^+(x,\xi,k)$ and $\Gamma^-(x,\xi,k)$, are Green's functions of the operators H^+ and H^-, respectively, where $\psi_1 \in L^2(-\infty,0)$ and $\psi_2 \in L^2(0,+\infty)$ for $Im\, k > 0$. Then,

$$\psi_i\big|_{x=0} = 1, \quad (\psi_i)'\big|_{x=0} = m_i, \quad i = 1,2.$$

The function

$$T(x,\xi,k) = \frac{1}{m_2(k) - m_1(k)} \begin{cases} \psi_1(\xi,k)\,\psi_2(x,k) & \text{for} \quad \xi < x, \\ \psi_1(x,k)\,\psi_2(\xi,k) & \text{for} \quad \xi > x, \end{cases}$$

coincides with the Green's function $\Gamma(x,\xi,k)$ of the Hill operator H_0:

$$\Gamma(x,\xi,k) = \begin{cases} \frac{\psi_1(\xi,k)\psi_2(x,k)}{m_2(k)-m_1(k)} & \text{for} \quad \xi < x, \\ \frac{\psi_1(x,k)\psi_2(\xi,k)}{m_2(k)-m_1(k)} & \text{for} \quad \xi > x, \end{cases}$$

Given the identities presented in Equation (14) and the following equation,

$$\theta(x,k)\,\varphi'(x,k) - \theta'(x,k)\,\varphi(x,k) = 1 \quad \text{for} \quad x \in \mathbb{R},$$

we obtain

$$\Gamma(x,\xi,k) = \begin{cases} -\frac{h(x,\xi,k)}{\sqrt{(\varphi'(k)+\theta(k))^2-4}} + \frac{1}{2}(\theta(\xi,k)\,\varphi(x,k) - \theta(x,k)\,\varphi(\xi,k)) & \text{for} \quad \xi < x, \\ -\frac{h(x,\xi,k)}{\sqrt{(\varphi'(k)+\theta(k))^2-4}} + \frac{1}{2}(\theta(x,k)\,\varphi(\xi,k) - \theta(\xi,k)\,\varphi(x,k)) & \text{for} \quad \xi > x, \end{cases}$$

with

$$h(x,\xi,k) = \varphi(k)\,\theta(x,k)\,\theta(\xi,k) - \theta'(k)\,\varphi(\xi,k)\varphi(x,k) +$$
$$+ \frac{\varphi'(k)-\theta(k)}{2}(\theta(\xi,k)\,\varphi(x,k) + \theta(x,k)\,\varphi(\xi,k)).$$

In Equations (15) and (16), instead of the functions ψ_i and m_i, $i = 1,2$, we write their expressions in terms of the functions φ and θ and obtain:

$$\frac{\psi_2(x,k)\psi_2(\xi,k)}{m_1(k) - m_2(k)} = \frac{h(x,\xi,k)}{\sqrt{(\varphi'(k)+\theta(k))^2 - 4}} - \frac{1}{2}(\theta(x,k)\,\varphi(\xi,k) + \theta(\xi,k)\,\varphi(x,k)) -$$

$$-\frac{\varphi'(k) - \theta(k) - \sqrt{(\varphi'(k)+\theta(k))^2 - 4}}{2\varphi(k)} \cdot \varphi(x,k)\varphi(\xi,k).$$

Therefore, the Green's function of the operator H^+ is equal to

$$\Gamma^+(x,\xi,k) = -\frac{\varphi'(k) - \theta(k) - \sqrt{(\varphi'(k)+\theta(k))^2 - 4}}{2\varphi(k)} \cdot \varphi(x,k)\varphi(\xi,k) - \begin{cases} \theta(x,k)\,\varphi(\xi,k) \text{ for } \xi < x, \\ \theta(\xi,k)\,\varphi(x,k) \text{ for } \xi \geq x. \end{cases} \tag{17}$$

Similarly, for the Green's function of the operator H^-, we obtain

$$\Gamma^{-}(x,\xi,k) = -\frac{\varphi'(k)-\theta(k)+\sqrt{(\varphi'(k)+\theta(k))^2-4}}{2\varphi(k)}\cdot\varphi(x,k)\varphi(\xi,k)+\begin{cases}\theta(x,k)\,\varphi(\xi,k) \text{ for } \xi\geq x,\\ \theta(\xi,k)\,\varphi(x,k) \text{ for } \xi<x.\end{cases} \tag{18}$$

In Equations (17) and (18), the single-valued branch of the root

$$\sqrt{G(k)} = \sqrt{(\varphi'(k)+\theta(k))^2-4}$$

is determined by the condition $\sqrt{G(k)}\big|_{k=0} > 0$. The singularities of the function $\Gamma^{+}(x,\xi,k)$ (or $\Gamma^{-}(x,\xi,k)$) on the complex plane of the variable k are branch points that coincide with ends of lacunae in the spectrum of the Schrödinger operator (H_0), that is, the points where $(\varphi'(k)+\theta(k))^2-4=0$, and poles located at points γ_ν (or σ_ν). Therefore, the functions $\Gamma^{+}(x,\xi,k)$ (or $\Gamma^{-}(x,\xi,k)$) can be extended metaphoricallyto the domain $\mathbb{C}'$.

2.2. Auxiliary Statements

The solution to the problem presented in Equations (1)–(3) is expressed in terms of the function $\Gamma^{+}(x,\xi,k)$ by the following formula

$$u(x,t) = -\frac{1}{2\pi}\int_{Im\,k=a}\int_0^1 \Gamma^{+}(x,\xi,k)\,f(\xi)\,e^{-ikt}d\xi\,dk, \tag{19}$$

where a is a positive constant.

The Green's function ($\Gamma^{+}(x,\xi,k)$ (or $\Gamma^{-}(x,\xi,k)$)) for each $x,\xi\in[0,1]$ continues analytically to $\mathbb{C}'$.

To study the properties of the Equation (19) integral, we introduce the following notation:

$$L_{+} := \{k : Im\,k = a,\ a>0\},\quad L_{-} = \{k : Im\,k=-d,\ d>0\},$$

and through q_l, the segment $Re\,k = l\pi+\frac{\pi}{3}, -d\leq Im\,k\leq a$, l is any real number.

Proposition 1. *For the integrals*

$$\begin{aligned} J_1 &\equiv \int\limits_{L+}\int\limits_0^1 \theta(\xi,k)\,\varphi(x,k)\,f(\xi)\,e^{-ikt}d\xi\,dk,\quad x\in[0,1],\\ J_2 &\equiv \int\limits_{L+}\int\limits_0^1 \theta(x,k)\,\varphi(\xi,k)\,f(\xi)\,e^{-ikt}d\xi\,dk,\quad x\in[0,1]. \end{aligned} \tag{20}$$

the following estimates hold:

$$\begin{aligned} |J_1| &\leq Ce^{-td}||f;L^2||,\quad x\in[0,1],\\ |J_2| &\leq Ce^{-td}||f;L^2||.\quad x\in[0,1]. \end{aligned} \tag{21}$$

Proof. From Equation (5), it follows that

$$\int_{q_l}\int_0^x \theta(\xi,k)\,\varphi(x,k)\,f(\xi)\,e^{-ikt}d\xi\,dk \to 0 \quad \text{for} \quad |l|\to\infty,$$

Moreover, $|l|$ can tend toward infinity in any way, so in Equation (20), the straight line L_{+} can be replaced by L_{-}. From Equation (5), it also follows that

$$\theta(\xi,k)\,\varphi(x,k) = S_1(x,\xi,k)+S_2(x,\xi,k),$$

where

$$S_1(x,\xi,k) = \frac{1}{k}\cos k\xi\,\sin kx$$

is an entire function ($k \in \mathbb{C}'$) for each $x, \xi \in [0,1]$, and the function $S_2(x,\xi,k)$ for $k \to \infty$ uniformly in $x, \xi \in [0,1]$ has the form

$$S_2(x,\xi,k) = O\Big(|k|^{-2} e^{|\tau|(x+\xi)}\Big).$$

Hence,

$$J_1 = J_1^{(1)} + J_1^{(2)},$$

with

$$J_1^{(1)} = -\int_{L_-} \int_0^1 \frac{1}{k} \cos k\xi \, \sin kx \, f(\xi)\, e^{-ikt} d\xi \, dk, \quad J_1^{(2)} = -\int_{L_-} \int_0^1 S_2(x,\xi,k)\, f(\xi)\, e^{-ikt} d\xi \, dk.$$

To investigate the integrals $J_1^{(1)}$ and $J_1^{(2)}$, we use the technique developed in [5,6]. Let $k = \sigma - id$ for $k \in L_-$; then,

$$J_1^{(1)} = -\int_{-\infty}^{+\infty} \frac{1}{\sigma - id} \sin(\sigma - id)x \, e^{-i\sigma t} e^{-dt} \Phi(\sigma, x)\, d\sigma, \quad x \in [0,1], \tag{22}$$

where

$$\Phi(\sigma,x) \equiv \int_0^x \cos(\sigma - id)\xi \, f(\xi)\, d\xi = \frac{1}{2} \int_0^x e^{i\sigma\xi} e^{d\xi} f(\xi)\, d\xi + \frac{1}{2} \int_0^x e^{-i\sigma\xi} e^{-d\xi} f(\xi)\, d\xi. \tag{23}$$

To investigate the first term on the right side of Equation (23), we single out the function

$$w(x,\xi) = \begin{cases} e^{d\xi} f(\xi) & \text{for} \quad \xi \le x, \\ 0 & \text{for} \quad \xi > x. \end{cases}$$

For any fixed $x \in [0,1]$, $w \in L^2(-\infty, +\infty)$, and

$$||w; L^2|| = \left(\int_0^x e^{2d\xi} f^2(\xi)\, d\xi\right)^{1/2} \le \left(\int_0^1 e^{2d\xi} f^2(\xi)\, d\xi\right)^{1/2} \le C_1 ||f; L^2||,$$

where C_1 does not depend on f and x.

According to the Parseval equality for the Fourier transform,

$$|| \int_0^x e^{i\sigma\xi} e^{d\xi} f(\xi)\, d\xi; L^2(\mathbb{R}^1_\sigma)|| = \sqrt{2\pi} ||w; L^2(\mathbb{R}^1_\xi)|| \le C_1 \sqrt{2\pi} ||f; L^2||, \quad x \in [0,1].$$

Similarly, the second term of the right-hand side of Equation (23) is investigated. Consequently,

$$||\Phi(\sigma,x); L^2(\mathbb{R}^1_\sigma)|| \le C_2 ||f; L^2||, \text{ for any fixed } x \in [0,1], \tag{24}$$

where C_2 does not depend on f and x.

Due to the Cauchy–Schwarz inequality and the inequality of Equation (24) from the expression (22), we obtain

$$|J_1^{(1)}| \le C_3 e^{-td} ||f; L^2||, \quad C_3 = \text{const} > 0. \tag{25}$$

To study $J_1^{(2)}$, note that

$$J_1^{(2)} = -\int_{L_-} \int_0^1 S_2(x,\xi,k)\, f(\xi)\, e^{-ikt} d\xi \, dk =$$

$$= -\int_{-\infty}^{+\infty} \frac{1}{\sigma - id} e^{-i\sigma t} e^{-dt} \left(\int_0^x f(\xi) O\left(\frac{e^{d(x+\xi)}}{|\sigma - id|}\right) d\xi\right) d\sigma.$$

We can easily show that

$$\left| \int_0^x f(\xi) O\left(\frac{e^{d(x+\xi)}}{|\sigma - id|} \right) d\xi \right|^2 \leq \frac{C}{|\sigma - id|^2} ||f; L^2||.$$

Next, by applying the Cauchy–Schwarz inequality, we obtain the estimate

$$|J_1^{(2)}| \leq C_5 e^{-td} ||f; L^2||, \quad C_4 = \text{const} > 0. \tag{26}$$

The estimates (25) and (26) for $J_1^{(1)}$ and $J_1^{(2)}$ imply the first inequality (21):

$$|J_1| \leq C e^{-td} ||f; L^2||.$$

Similarly, for the second integral (20)

$$J_2 \equiv \int_{L+} \int_0^1 \theta(x,k)\, \varphi(\xi,k)\, f(\xi)\, e^{-ikt} d\xi\, dk, \quad x \in [0,1],$$

we obtain the second inequality (21):

$$|J_2| \leq C e^{-td} ||f; L^2||.$$

Thus, the integrals J_1 and J_2 decay exponentially as $t \to \infty$. □

Let δ be some finite contour in $\mathbb{C}'$, and J_δ denotes the integral

$$J_\delta = \int_\delta \int_0^1 m_2(k)\, \varphi(x,k)\, \varphi(\xi,k)\, f(\xi)\, e^{-ikt} d\xi\, dk, \quad x \in [0,1].$$

Now, let the contour (Δ) be unbounded. Furthermore,

$$J_\Delta = \lim_{j \to \infty} J_{\Delta \cap \{k:\, |Rek| \leq \pi j + \frac{\pi}{2}\}}, \quad j \in N.$$

Proposition 2. *The solution to the problem expressed in Equations* (1)–(3) *has the form*

$$u(x,t) = \frac{1}{2\pi} J_L - i J_3 + v_1(x,t), \quad x \in [0,1],\ t > 0, \tag{27}$$

where

$$J_3 = \int_0^1 \left(\sum_{\nu=-\infty,\, \nu \neq 0} \text{res}_{k=\gamma_\nu}\ m_2(k)\, \varphi(x,k)\, \varphi(\xi,k)\, e^{-ikt} \right) f(\xi)\, d\xi,$$

while the function $v_1(x,t)$ for $x \in [0,1]$ and $t > 0$ satisfies the following estimate

$$|v_1(x,t)| \leq C\, e^{-td} ||f; L^2||, \quad C = \text{const} > 0. \tag{28}$$

Proof. From the formulas (17), (19) and the estimates (20), it follows that

$$u(x,t) = \frac{1}{2\pi} J_{L_+} + v_1(x,t),$$

where the estimate (28) is valid for the function v_1.

Based on Equation (5), it is easy to show that

$$J_{q_j} \equiv \int_{q_j} \int_0^1 m_2(k)\, \varphi(x,k)\, \varphi(\xi,k)\, f(\xi)\, e^{-ikt} d\xi\, dk \to 0 \quad \text{for} \quad |j| \to \infty,\ j \in N. \tag{29}$$

The features of the function $m_2(k)$ are the points where $(\varphi'(k)+\theta(k))^2-4=0$, that is, the branch points that coincide with the ends of the lacunae in the spectrum of the operator H_0, as well as points where $\varphi(k)=0$, that is, simple poles located at points γ_ν. The function $m_2(k)$ has no other singular points. According to Equation (29), the validity of the Proposition follows. □

From the point $(k=p)$ lying on the real axis, we draw a vertical cut into the lower half-plane of the variable k. Consider a contour consisting of the left edge of this cut from point $p-id$, $d>0$ to point p, then from the right edge of the same cut from point p to point $p-id$; this contour is denoted by l_p.

Consider on the plane $\mathbb{C}'$ a contour (L), which is the union of the following three contours

$$L = L_1 \cup L_2 \cup L_3, \tag{30}$$

where

$$L_1 = \left(\bigcup_{n=0}^{\infty} l_{\lambda_n}\right) \bigcup \left(\bigcup_{n=0}^{\infty} l_{-\lambda_n}\right), \quad L_2 = \left(\bigcup_{n=0}^{\infty} l_{\mu_n}\right) \bigcup \left(\bigcup_{n=0}^{\infty} l_{-\mu_n}\right),$$
$$L_3 = L_- \cap \mathbb{C}',$$

Moreover, if $\lambda_{j+1}=\lambda_j$ (or $\mu_{j+1}=\mu_j$) for some non-negative integer (j), then these unions do not include l_{λ_j}, $l_{\lambda_{j+1}}$, $l_{-\lambda_j}$, $l_{-\lambda_{j+1}}$ (or l_{μ_j}, $l_{\mu_{j+1}}$, $l_{-\mu_j}$, $l_{-\mu_{j+1}}$).

Taking into account Equation (30), we write the integral J_L as

$$J_L = J_{L_1} + J_{L_2} + J_{L_3},$$

with

$$J_{L_1} = \sum_{n=0}^{\infty}(J_{l_{\lambda_n}} + J_{l_{-\lambda_n}}), \quad J_{L_2} = \sum_{n=0}^{\infty}(J_{l_{\mu_n}} + J_{l_{-\mu_n}}). \tag{31}$$

Here, summation is carried out only over those n values for which the points λ_n (or μ_n) are the ends of lacunae.

Proposition 3. *The following estimate takes place*

$$|J_{L_3}| \le C\, e^{-td}||f;L^2|| \quad \textit{for} \quad x\in[0,1],\, t>0. \tag{32}$$

Proof. According to Equation (5) in the strip $-d \le Im\, k \le a$ for $|k| \to \infty$ uniformly in $x,\xi \in [0,1]$, the following expression holds

$$\varphi(x,k)\,\varphi(\xi,k) = \frac{1}{k^2}\Big(\sin kx \sin k\xi + O\big(|k|^{-1}\big)\Big). \tag{33}$$

Taking into account Equation (33), we rewrite J_{L_3} as

$$\begin{aligned} J_{L_3} &= \\ &= \frac{1}{2}\int_{L_-}\int_0^1 \frac{\varphi'(k)-\theta(k)-\sqrt{(\varphi'(k)+\theta(k))^2-4}}{\frac{1}{k}\big(\sin k + O\big(|k|^{-1}\big)\big)} \cdot \frac{1}{k^2}\big(\sin kx \sin k\xi + O\big(|k|^{-1}\big)\big)\, f(\xi)\, e^{-ikt} d\xi\, dk = \\ &= \frac{1}{2}\int_{L_-}\int_0^1 \frac{1}{k}\cdot \frac{\varphi'(k)-\theta(k)-\sqrt{(\varphi'(k)+\theta(k))^2-4}}{\sin k + O\big(|k|^{-1}\big)} \cdot \sin kx \sin k\xi\, f(\xi)\, e^{-ikt} d\xi\, dk + \\ &+ \frac{1}{2}\int_{L_-}\int_0^1 \frac{\varphi'(k)-\theta(k)-\sqrt{(\varphi'(k)+\theta(k))^2-4}}{\sin k + O\big(|k|^{-1}\big)} \cdot O\big(|k|^{-2}\big) f(\xi)\, e^{-ikt} d\xi\, dk \end{aligned} \tag{34}$$

According to the Equation (5), it follows that there is a constant $(C_1>0)$ such that

$$\left|\varphi'(k) - \theta(k) - \sqrt{(\varphi'(k)+\theta(k))^2-4}\right| \le C_1 \quad \text{for} \quad k \in L_-.$$

Since there exists $C_2 > 0$ such that $|\sin\ k| \geq C_2 > 0$ for $k \in L_-$, then

$$|\sin\ k + O\left(|k|^{-1}\right)| \geq C_3 > 0 \quad \text{for} \quad k \in L_- \quad \text{and} \quad k \gg 1. \tag{35}$$

Furthermore, the function $\varphi(k)$ does not vanish on L_-, that is, it has only real zeros.

Hence, taking into account Equation (35) and the integral of Equation (34), the denominator $(\sin\ k + O(|k|^{-1}))$ equal to $k\,\varphi(k)$ is modulo-limited to L_-, that is,

$$|k\,\varphi(k)| = |\sin\ k + O\left(|k|^{-1}\right)| \geq C_4 > 0 \quad \text{for} \quad k \in L_-.$$

Furthermore, just as in deriving the estimates of Equations (25) and (26) for the integrals $J_1^{(1)}$ and $J_1^{(2)}$ in Proposition 1, we are convinced of the validity of this Proposition with the estimate of Equation (32) for the integral J_{L_3}. $\square$

Let $B(a)$ be the circle $B(a) = \{k:\ |k - \pi a| \leq \frac{\pi}{4}\}$ and $G(k) \equiv (\theta(k) + \varphi'(k))^2 - 4$.

Let there exist $n_1 \in N$ such that the following formulas hold for $n > n_1$ (see [17,19,28])

$$\begin{gathered} \lambda_{2n-1} = 2n\pi + O\left(\tfrac{1}{n}\right), \quad \lambda_{2n} = 2n\pi + O\left(\tfrac{1}{n}\right), \\ \mu_{2n+1} = (2n+1)\pi + O\left(\tfrac{1}{n}\right), \quad \mu_{2n+1} = (2n+1)\pi + O\left(\tfrac{1}{n}\right). \end{gathered} \tag{36}$$

The number $(d > 0)$ with the presence of which the contours L_-, $l_{\pm\lambda_i}$ and $l_{\pm\mu_i}$ are determined is chosen so that $d < \frac{\pi}{4}$.

Therefore, according to Equation (36), we conclude that there exists $n_2 > n_1, n_2 \in N$ such that for $n > n_2$, the contours $l_{\lambda_{2n}}$ and $l_{\lambda_{2n-1}}$ (or $l_{\mu_{2n}}$ and $l_{\mu_{2n+1}}$) belong to the circle $B(2n)$ (or the circle $B(2n+1)$).

In the same way, we conclude that the contours $l_{-\lambda_{2n}}$ and $l_{-\lambda_{2n-1}}$ (or $l_{-\mu_{2n}}$ and $l_{-\mu_{2n+1}}$) belong to the circle $B(-2n)$ (or the circle $B(-(2n+1))$).

Proposition 4. *The following representations are valid:*

$$\begin{cases} G(k) = (k - \lambda_{2m-1})(k - \lambda_{2m})\, g_{2m}(k),\ |g_{2m}(k)| \leq C_{2m} \quad \textit{for} \quad k \in l_{\lambda_{2m-1}} \cup l_{\lambda_{2m}}, \\ G(k) = (k + \lambda_{2m+1})(k + \lambda_{2m})\, g_{-2m}(k),\ |g_{-2m}(k)| \leq C_{-2m} \quad \textit{for} \quad k \in l_{-\lambda_{2m-1}} \cup l_{-\lambda_{2m}}, \\ m = 1,2,3,\ldots, \\ G(k) = (k - \mu_{2m})(k - \mu_{2m+1})\, g_{2m+1}(k),\ |g_{2m+1}(k)| \leq C_{2m+1} \quad \textit{for} \quad k \in l_{\mu_{2m}} \cup l_{\mu_{2m+1}}, \\ G(k) = (k + \mu_{2m})(k + \mu_{2m+1})\, g_{-(2m+1)}(k),\ |g_{-(2m+1)}(k)| \leq C_{-(2m+1)} \quad \textit{for}\, k \in l_{-\mu_{2m}} \cup l_{-\mu_{2m+1}}, \\ m = 0,1,2,\ldots, \end{cases}$$

where the constants $C_{\pm 2m}$ and $C_{\pm(2m+1)}$ depend only on m.

Proof. The validity of the first of the equalities follows from the fact that the entire function $(G(k))$ on the contours $l_{\lambda_{2m-1}}$ and $l_{\lambda_{2m}}$ has no zeros other than λ_{2m-1} and λ_{2m}.

The other equalities are proven similarly. $\square$

Proposition 5. *For sufficiently large $n > n_2$, the following equalities hold*

$$\begin{cases} G(k) = (k - \lambda_{2n-1})(k - \lambda_{2n})\, g_{2n}(k),\ |g_{2n}(k)| \leq C \quad \textit{for} \quad k \in B(2n), \\ G(k) = (k + \lambda_{2n-1})(k + \lambda_{2n})\, g_{-2n}(k),\ |g_{-2n}(k)| \leq C \quad \textit{for} \quad k \in B(-2n), \\ G(k) = (k - \mu_{2n})(k - \mu_{2n+1})\, g_{2n+1}(k),\ |g_{2n+1}(k)| \leq C \quad \textit{for} \quad k \in B(2n+1), \\ G(k) = (k + \mu_{2n})(k + \mu_{2n+1})\, g_{-(2n+1)}(k),\ |g_{-(2n+1)}(k)| \leq C \quad \textit{for} \quad k \in B(-(2n+1)), \end{cases}$$

where the constant (C) does not depend on n.

Proof. Here, we will prove the first equality. The case of other equalities is proven similarly.

Based on the definition of the number n_2 for $n > n_2$, we can conclude that the numbers λ_{2n-1} and λ_{2n} belong to the circle $B(2n)$, and in this circle, the function $G(k)$ has no other zeros [17]. Therefore, the function $G(k)$ for $k \in B(2n)$ can be represented as

$$G(k) = (k - \lambda_{2n-1})(k - \lambda_{2n})\, g_{2n}(k),$$

with $g_{2n}(k) \neq 0$ for $k \in B(2n)$.

In the circle $B(0) = \{k : |k| \leq \frac{\pi}{4}\}$, the function $h_0(k) = -\frac{4\sin^2 k}{k^2}$ has no zeros, so there exists $C_1 > 0$ such that $|h_0(k)| < C_1$ for $k \in B(0)$.

Having made the change of variable ($k = k' + 2n\pi$, $k' \in B(0)$), the functions $G(k)$ and $g_{2n}(k)$ become functions with new variables ($G_{2n}(k') = G(k' + 2n\pi)$ and $\tilde{g}_{2n}(k') = g_{2n}(k' + 2n\pi)$).

From the Formula (5), it follows that:

$$G(k) = -4\sin^2 k + O\left(|k|^{-1} e^{2|\tau|}\right) \quad \text{as} \quad |k| \to \infty,$$

$$G_{2n}(k') = -4\sin^2 k' + O\left(n^{-1} e^{2|\tau|}\right) \quad \text{as} \quad n \to \infty. \tag{37}$$

Hence,

$$\tilde{g}_{2n}(k') = g_{2n}(k' + 2n\pi) = \frac{G(k' + 2n\pi)}{(k' + 2n\pi - \lambda_{2n-1})(k' + 2n\pi - \lambda_{2n})}.$$

Based on Equations (36) and (37), we can conclude that on the circle $|k'| = \frac{\pi}{4}$, the sequence $\tilde{g}_{2n}(k')$ tends uniformly toward $h_0(k')$ as $n \to \infty$. □

Remark 1. *Since in a sufficiently small neighborhood of the contour l_{λ_0} (or $l_{-\lambda_0}$), the function $G(k)$ has a single zero $k = \lambda_0$ (or $k = -\lambda_0$), the following equalities hold*

$$G(k)| = (k - \lambda_0)\, g_0(k) \quad |g_0(k)| \leq C \quad \textit{for} \quad k \in l_{\lambda_0},$$

$$G(k)| = (k + \lambda_0)\, \tilde{g}_0(k) \quad |\tilde{g}_0(k)| \leq C \quad \textit{for} \quad k \in l_{-\lambda_0}.$$

Proposition 6. *The following representations are valid:*

$$\begin{cases} \varphi(k) = (k - \kappa_{2m})\, \varphi_{2m}(k),\ |\varphi_{2m}(k)| \geq C_{2m} > 0 \quad \textit{for} \quad k \in l_{\lambda_{2m-1}} \cup l_{\lambda_{2m}}, \\ \varphi(k) = (k + \kappa_{2m})\, \varphi_{-2m}(k),\ |\varphi_{-2m}(k)| \geq C_{-2m} > 0 \quad \textit{for} \quad k \in l_{-\lambda_{2m-1}} \cup l_{-\lambda_{2m}}, \\ m = 1, 2, 3, \ldots, \\ \varphi(k) = (k - \kappa_{2m+1})\, \varphi_{2m+1}(k),\ |\varphi_{2m+1}(k)| \geq C_{2m+1} > 0 \quad \textit{for} \quad k \in l_{\mu_{2m}} \cup l_{\mu_{2m+1}}, \\ \varphi(k) = (k + \kappa_{2m+1})\, \varphi_{-(2m+1)}(k),\ |\varphi_{-(2m+1)}(k)| \geq C_{-(2m+1)} > 0 \quad \textit{for} \quad k \in l_{-\mu_{2m}} \cup l_{-\mu_{2m+1}}, \\ m = 0, 1, 2, \ldots. \end{cases} \tag{38}$$

where the constants $C_{\pm 2m}$ and $C_{\pm(2m+1)}$ depend only on m.

Proof. The validity of the first of the equalities follows from the fact that the entire function ($\varphi(k)$) on the contours $l_{\lambda_{2m-1}}$ and $l_{\lambda_{2m}}$ has no zeros other than κ_{2m}.

The case of the remaining equalities is proven in a similar way. □

Proposition 7. *For sufficiently large $n > n_2$, the following equalities hold*

$$\begin{cases} k\,\varphi(k) = (k-\kappa_{2n})\,\varphi_{2n}(k),\ |\varphi_{2n}(k)| \geq C > 0 \quad \textit{for} \quad k \in B(2n), \\ k\,\varphi(k) = (k+\kappa_{2n})\,\varphi_{-2n}(k),\ |\varphi_{-2n}(k)| \geq C > 0 \quad \textit{for} \quad k \in B(-2n), \\ k\,\varphi(k) = (k-\kappa_{2n+1})\,\varphi_{2n+1}(k),\ |\varphi_{2n+1}(k)| \geq C > 0 \quad \textit{for} \quad k \in B(2n+1), \\ k\,\varphi(k) = (k+\kappa_{2n+1})\,\varphi_{-(2n+1)}(k),\ |\varphi_{-(2n+1)}(k)| \geq C > 0 \quad \textit{for} \quad k \in B(-(2n+1)), \end{cases} \tag{39}$$

where the constant C does not depend on n.

Proof. We will prove the first equality. The case of the remaining equalities is proven in a similar way.

Based on the definition of the number n_2, for $n > n_2$, the numbers κ_{2n} belong to the circle $B(2n)$, and the function $k\,\varphi(k)$ has no other zeros in this circle [17]. This implies that the function $k\,\varphi(k)$ for $k \in B(2n)$ can be written as

$$k\,\varphi(k) = (k-\kappa_{2n})\,\varphi_{2n}(k),$$

where $\varphi_{2n}(k) \neq 0$ for $k \in B(2n)$.

The functions $\varphi(k)$ and $\theta'(k)$ each have one simple zero in the segments $[\lambda_{2n-1}, \lambda_{2n}]$, $n \geq 1$ and $[\mu_{2n}, \mu_{2n+1}]$, $n \geq 0$.

The zeros of the function $\varphi(k)$ lying in the segments $[\lambda_{2n-1}, \lambda_{2n}]$ are denoted by κ_{2n}, and the zeros lying in the segments $[\mu_{2n}, \mu_{2n+1}]$ are denoted by κ_{2n+1}.

Similarly, the zeros of the function $\theta'(k)$, which lies in the segments $[\lambda_{2n-1}, \lambda_{2n}]$ are denoted by σ_{2n}, and zeros lying in the segments $[\mu_{2n}, \mu_{2n+1}]$ are denoted by σ_{2n+1}.

Therefore, in the same way as for Proposition 5, we can prove that for sufficiently large $n \in N$ in the circle $(B(n))$, the equalities

$$k\,\varphi(k) = (k-k_n')\,\varphi_n(k) \quad \text{and} \quad \frac{1}{k}\,\theta'(k) = (k-k_n'')\,\theta'_n(k),$$

where k_n' and k_n'' are the zeros of the functions $\varphi(k)$ and $\theta'(k)$, respectively, and $|\varphi_n(k)| \geq C > 0$, $|\theta'_n(k)| \geq C > 0$ for $k \in B(n)$. □

Remark 2. *Since the function $\varphi(k)$ has no zeros in sufficiently small neighborhoods of the contours $(l_{\pm\lambda_0})$, the following inequality holds:*

$$|\varphi(k)| \geq C > 0 \quad \textit{for} \quad k \in l_{\lambda_0} \cup l_{-\lambda_0}.$$

Equalities in systems (38) (or (39)) are also valid for the function $\theta'(k)$ (or $\frac{1}{k}\theta'(k)$) if σ_i is substituted in for κ_i.

Lemma 1. *The following inequalities hold:*

$$\begin{array}{ll} |J_{L_1}| \leq \frac{C}{\sqrt{t}} \cdot ||f; L^2(\mathbb{R})||, & x \in [0,1],\ t > 1, \\ |J_{L_2}| \leq \frac{C}{\sqrt{t}} \cdot ||f; L^2(\mathbb{R})||, & x \in [0,1],\ t > 1. \end{array} \tag{40}$$

Proof. Let us write out in detail the first integral from Equation (31):

$$\begin{aligned} J_{L_1} &= \sum_{n=0}^{\infty} (J_{l_{\lambda_n}} + J_{l_{-\lambda_n}}) = \\ &= J_{l_{\lambda_0}} + J_{l_{-\lambda_0}} + \sum_{n=1}^{\infty} (J_{l_{\lambda_{2n}}} + J_{l_{\lambda_{2n-1}}}) + \sum_{n=1}^{\infty} (J_{l_{-\lambda_{2n}}} + J_{l_{-\lambda_{2n-1}}}). \end{aligned} \tag{41}$$

First, consider the sum

$$I = \sum_{n=1}^{\infty} J_{l_{\lambda_{2n}}}. \tag{42}$$

Note that based on the form of the contour $l_{\lambda_{2n}}$ and given that the function

$$\frac{\varphi'(k)-\theta(k)}{2\varphi(k)}\,\varphi(x,k)\varphi(\xi,k)e^{-ikt}$$

has no singular points on $l_{\lambda_{2n}}$, the following equality follows.

$$\int_{l_{\lambda_{2n}}}\int_0^1\frac{\varphi'(k)-\theta(k)}{2\varphi(k)}\,\varphi(x,k)\varphi(\xi,k)\,f(\xi)e^{-ikt}d\xi dk=0. \tag{43}$$

Let the number n_2 be the same as in Propositions 5 and 7. Taking into account Propositions 4–7, Equations (33) and (43) and the fact that $k=\lambda_{2n}+i\tau$, $-d\le\tau\le 0$ for $k\in l_{\lambda_{2n}}$, the sum (42) can be rewritten as

$$I=$$

$$=-i\sum_{n=1}^{\infty}e^{-i\lambda_{2n}t}\int_0^d\int_0^1\frac{\sqrt{(\varphi'(\lambda_{2n}-i\tau)+\theta(\lambda_{2n}-i\tau))^2-4}}{\varphi(\lambda_{2n}-i\tau)}\,\varphi(x,\lambda_{2n}-i\tau)\varphi(\xi,\lambda_{2n}-i\tau)\,f(\xi)e^{-t\tau}d\xi d\tau=$$

$$=I_1+I_2+I_3,$$

where

$$-I_1=$$

$$i\sum_{n=1}^{n_2}e^{-i\lambda_{2n}t}\int_0^d\int_0^1\frac{\sqrt{-i\tau((\lambda_{2n}-\lambda_{2n-1})-i\tau)\,g_{2n}(\lambda_{2n}-i\tau)}}{((\lambda_{2n}-\kappa_{2n})-i\tau)\varphi_{2n}(\lambda_{2n}-i\tau)}\,\varphi(x,\lambda_{2n}-i\tau)\varphi(\xi,\lambda_{2n}-i\tau)\,f(\xi)e^{-t\tau}d\xi d\tau,$$

$$-I_2=$$

$$i\sum_{n=n_2+1}^{\infty}\left(e^{-i\lambda_{2n}t}\int_0^d\frac{\sqrt{-i\tau((\lambda_{2n}-\lambda_{2n-1})-i\tau)\,g_{2n}(\lambda_{2n}-i\tau)}}{((\lambda_{2n}-\kappa_{2n})-i\tau)\varphi_{2n}(\lambda_{2n}-i\tau)}\cdot\frac{e^{-t\tau}}{\lambda_{2n}-i\tau}\sin(\lambda_{2n}-i\tau)x\left(\int_0^1 f(\xi)\sin(\lambda_{2n}-i\tau)\xi\,d\xi\right)d\tau\right),$$

$$-I_3=$$

$$i\sum_{n=n_2+1}^{\infty}\left(e^{-i\lambda_{2n}t}\int_0^d\frac{\sqrt{-i\tau((\lambda_{2n}-\lambda_{2n-1})-i\tau)\,g_{2n}(\lambda_{2n}-i\tau)}}{((\lambda_{2n}-\kappa_{2n})-i\tau)\varphi_{2n}(\lambda_{2n}-i\tau)}\cdot e^{-t\tau}\left(\int_0^1 f(\xi)O(|\lambda_{2n}-i\tau|^{-2})\,d\xi\right)d\tau\right).$$

According to Propositions 4 and 6 and the relations of Equation (13), for those n values over which summation is carried out in I_1, the following inequalities hold:

$$\left|\frac{\sqrt{-i\tau((\lambda_{2n}-\lambda_{2n-1})-i\tau)\,g_{2n}(\lambda_{2n}-i\tau)}}{((\lambda_{2n}-\kappa_{2n})-i\tau)\varphi_{2n}(\lambda_{2n}-i\tau)}\right|\le\frac{C_1}{\sqrt{\tau}}\quad\text{for}\quad 0<\tau\le d, \tag{44}$$

where the constant C_1 can be chosen independent of n.

We also note that for $k\in\{k:|Rek|\le\lambda_{2n_2}\}\cap\{k:-d\le Imk\le 0\}$, the following estimate is valid

$$|\varphi(x,k)|\le C_2,\quad x\in[0,1]. \tag{45}$$

Based on Equations (44) and (45), is easy to obtain

$$|I_1|\le C_3||f;L^2||\int_0^d\frac{e^{-t\tau}}{\sqrt{\tau}}d\tau\le\frac{C}{\sqrt{t}}\cdot||f;L^2||,\quad x\in[0,1]. \tag{46}$$

Let us study the sum I_2. From Propositions 5 and 7, it follows

$$\left|\frac{\sqrt{-i\tau((\lambda_{2n}-\lambda_{2n-1})-i\tau)\,g_{2n}(\lambda_{2n}-i\tau)}}{((\lambda_{2n}-\kappa_{2n})-i\tau)\varphi_{2n}(\lambda_{2n}-i\tau)}\right|\le\frac{C_4}{\sqrt{\tau}}\quad\text{for}\quad 0<\tau\le d,\,n>n_2, \tag{47}$$

where the constant $C_4 > 0$ does not depend on n. Given $\lambda_{2n} = 2n\pi + O\left(\frac{1}{n}\right)$ for $n > n_2 > n_1$, then for $n > n_2$:

$$\begin{aligned}\int\limits_0^1 f(\xi)\sin(\lambda_{2n} - i\tau)\xi\, d\xi = \tfrac{1}{2i}\left(\int\limits_0^1 f(\xi)e^{i\lambda_{2n}\xi}e^{\tau\xi}\, d\xi - \int\limits_0^1 f(\xi)e^{-i\lambda_{2n}\xi}e^{-\tau\xi}\, d\xi\right) = \\ = \tfrac{1}{2i}\int\limits_0^1 f(\xi)e^{i2n\pi}e^{\tau\xi}\, d\xi - \tfrac{1}{2i}\int\limits_0^1 f(\xi)e^{-i2n\pi}e^{-\tau\xi}\, d\xi + \int\limits_0^1 f(\xi)O\left(\tfrac{1}{n}\right) d\xi.\end{aligned} \tag{48}$$

The number

$$\lambda_{2n} = \int\limits_0^1 f(\xi)e^{i2n\pi}e^{\tau\xi}\, d\xi \quad (\text{or} \quad \lambda_{-2n} = \int\limits_0^1 f(\xi)e^{-i2n\pi}e^{-\tau\xi}\, d\xi)$$

is the expansion coefficient of the function $f(x)e^{\tau\xi}$ (or $f(x)e^{-\tau\xi}$) in a Fourier series in the $\{e^{in\pi\xi}\}_{n=-\infty}^{\infty}$ system. It is clear that

$$\left|\int\limits_0^1 f(\xi)O\left(\frac{1}{n}\right) d\xi\right| \le \frac{C}{n} \cdot ||f; L^2||. \tag{49}$$

We also note that for $0 \le \tau \le d$ and $0 \le \xi \le 1$, the following inequalities hold:

$$\int\limits_0^1 \left|f(\xi)e^{\tau\xi}\right|^2 d\xi \le C||f; L^2||^2, \quad \int\limits_0^1 \left|f(\xi)e^{-\tau\xi}\right|^2 d\xi \le C||f; L^2||^2. \tag{50}$$

Since the functions $\sin(\lambda_{2n} - i\tau)x$ are uniformly bounded in n for $x \in [0,1]$ and $0 \le \tau \le d$, then based on the Cauchy–Schwartz inequality for an infinite sum and according to Equations (47)–(50), we obtain

$$\begin{aligned}|I_2| \le C_5 \int\limits_0^d \tfrac{e^{-t\tau}}{\sqrt{\tau}} \sqrt{\sum_{n=n_2+1}^{\infty} \lambda_{2n}^{1/2}} \sqrt{\sum_{n=n_2+1}^{\infty} \left(d_{2n}^2 + d_{-2n}^2 + \tfrac{1}{n^2} \cdot ||f; L^2||^2\right)}\, d\tau \le \\ \le C_6||f; L^2|| \int\limits_0^d \tfrac{e^{-t\tau}}{\sqrt{\tau}} d\tau \le \tfrac{C}{\sqrt{t}} \cdot ||f; L^2||, \quad x \in [0,1],\, t > 0.\end{aligned} \tag{51}$$

Similarly, we obtain

$$|I_3| \le \frac{C}{\sqrt{t}} \cdot ||f; L^2||, \quad x \in [0,1],\, t > 0. \tag{52}$$

It follows from Equations (46), (51) and (52) that

$$|I| = \left|\sum_{n=1}^{\infty} J_{I_{\lambda_{2n}}}\right| \le \frac{C}{\sqrt{t}} \cdot ||f; L^2||, \quad x \in [0,1],\, t > 0. \tag{53}$$

By analogous reasoning, we obtain

$$\left|\sum_{n=1}^{\infty} J_{I_{\lambda_{2n-1}}}\right| \le \frac{C}{\sqrt{t}} \cdot ||f; L^2||, \quad x \in [0,1],\, t > 0.$$

Hence,

$$\begin{aligned}\left|\sum_{n=1}^{\infty}\left(J_{l_{\lambda_{2n}}}+J_{l_{\lambda_{2n-1}}}\right)\right| &\le \tfrac{C}{\sqrt{t}}\cdot ||f;L^2||, \quad x\in[0,1],\ t>0,\\ \left|\sum_{n=1}^{\infty}\left(J_{l_{-\lambda_{2n}}}+J_{l_{-\lambda_{2n-1}}}\right)\right| &\le \tfrac{C}{\sqrt{t}}\cdot ||f;L^2||, \quad x\in[0,1],\ t>0.\end{aligned} \tag{54}$$

From Remark 2 and from the estimate (45), we can easily obtain

$$\begin{aligned}\left|J_{l_{\lambda_0}}\right| &\le \tfrac{C}{\sqrt{t}}\cdot ||f;L^2||, \quad x\in[0,1],\ t>0,\\ \left|J_{l_{-\lambda_0}}\right| &\le \tfrac{C}{\sqrt{t}}\cdot ||f;L^2||, \quad x\in[0,1],\ t>0.\end{aligned} \tag{55}$$

From Equations (41) and (53)–(55), the validity of the first inequality (40) follows. The second inequality (40) is proven similarly. Lemma 1 is completely proven. □

Lemma 2. *The following equality holds*

$$\begin{aligned}J_3 = \int_0^1\left(\sum_{\nu=-\infty,\nu\neq0}\operatorname{res}_{k=\gamma_\nu} m_2(k)\,\varphi(x,k)\,\varphi(\xi,k)\,e^{-ikt}\right)f(\xi)\,d\xi =\\ = i\sum_{\nu=1}^{\infty} b_\nu\, f_\nu\, v(x,n_\nu)\sin(\gamma_\nu t),\end{aligned} \tag{56}$$

where $v(x,n_\nu)$ is the normalized eigenfunction of the problem corresponding to the eigenvalue γ_ν^2,

$$b_\nu = -b_{-\nu} = -\frac{\left(\varphi'(k)-\theta(k)-\sqrt{(\varphi'(k)+\theta(k))^2-4}\right)\Big|_{k=\gamma_\nu}}{\varphi_k(\gamma_\nu)}\cdot\int_0^1|\varphi(x,\gamma_\nu)|^2dx. \tag{57}$$

For numbers b_ν with $n>n_2$, the following estimates hold:

$$\begin{aligned}|b_\nu| &\le C\,\tfrac{\lambda_{2n}-\lambda_{2n-1}}{\gamma_\nu} \quad \text{for} \quad \lambda_{2n-1}<\gamma_\nu<\lambda_{2n},\\ |b_\nu| &\le C\,\tfrac{\mu_{2n+1}-\mu_{2n}}{\gamma_\nu} \quad \text{for} \quad \mu_{2n}<\gamma_\nu<\mu_{2n+1}.\end{aligned} \tag{58}$$

Proof. Let γ_ν^2 be an eigenvalue of the operator H^+. Then, given that γ_ν is a simple zero of the function φ and

$$\varphi(x,\gamma_\nu) = v(x,n_\nu)\sqrt{\int_0^1|\varphi(x,\gamma_\nu)|^2dx}$$

where $v(x,n_\nu)$ is the normalized eigenfunction of Problem (12) corresponding to the eigenvalue γ_ν^2, we have

$$\begin{aligned}&\operatorname{res}_{k=\gamma_\nu} m_2(k)\,\varphi(x,k)\,\varphi(\xi,k)\,e^{-ikt} =\\ &= \operatorname{res}_{k=\gamma_\nu}\tfrac{\varphi'(k)-\theta(k)-\sqrt{(\varphi'(k)+\theta(k))^2-4}}{2\varphi(k)}\,\varphi(x,k)\,\varphi(\xi,k)\,e^{-ikt} =\\ &= -\tfrac{1}{2}b_\nu v(x,n_\nu)v(\xi,n_\nu)\,e^{-i\gamma_\nu t},\end{aligned} \tag{59}$$

where

$$b_\nu = -\frac{\left(\varphi'(k)-\theta(k)-\sqrt{(\varphi'(k)+\theta(k))^2-4}\right)\Big|_{k=\gamma_\nu}}{\varphi_k(\gamma_\nu)}\cdot\int_0^1|\varphi(x,\gamma_\nu)|^2dx$$

Carrying out a similar calculation algorithm for $k = -\gamma_\nu$, we obtain

$$\mathrm{res}_{k=-\gamma_\nu}\ m_2(k)\, \varphi(x,k)\, \varphi(\xi,k)\, e^{-ikt} = -\frac{1}{2} b_{-\nu} v(x,n_\nu) v(\xi,n_\nu)\, e^{i\gamma_\nu t}, \tag{60}$$

where

$$b_{-\nu} = -\frac{\left(\varphi'(k) - \theta(k) - \sqrt{(\varphi'(k)+\theta(k))^2-4}\right)\Big|_{k=-\gamma_\nu}}{\varphi_k(-\gamma_\nu)} \cdot \int_0^1 |\varphi(x,\gamma_\nu)|^2 dx.$$

Since $\varphi'(k) - \theta(k) - \sqrt{(\varphi'(k)+\theta(k))^2-4}$ and $\varphi(k)$ are even functions, we have

$$b_{-\nu} = -b_\nu. \tag{61}$$

Furthermore, according to Equations (59) and (60), it follows that

$$\int_0^1 \left(\mathrm{res}_{k=\gamma_\nu}\ m_2(k)\, \varphi(x,k)\, \varphi(\xi,k)\, e^{-ikt}\right) f(\xi)\, d\xi = -\tfrac{1}{2} b_\nu\, f_\nu\, v(x,n_\nu) e^{-i\gamma_\nu t},$$
$$\int_0^1 \left(\mathrm{res}_{k=-\gamma_\nu}\ m_2(k)\, \varphi(x,k)\, \varphi(\xi,k)\, e^{-ikt}\right) f(\xi)\, d\xi = -\tfrac{1}{2} b_{-\nu}\, f_\nu\, v(x,n_\nu) e^{i\gamma_\nu t},$$

where

$$f_\nu = \int_0^1 f(\xi)\, v(\xi,n_\nu)\, d\xi,$$

f_ν are the expansion coefficients of the function f in a Fourier series in the system $\{v(x,n)\}_{n=1}^{\infty}$.

Taking into account Equation (61), we have

$$\begin{gathered} -\tfrac{1}{2} b_\nu\, f_\nu\, v(x,n_\nu) e^{-i\gamma_\nu t} - \tfrac{1}{2} b_{-\nu}\, f_\nu\, v(x,n_\nu) e^{i\gamma_\nu t} =, \\ = \tfrac{1}{2} b_\nu\, f_\nu\, v(x,n_\nu)\left(e^{i\gamma_\nu t} - e^{-i\gamma_\nu t}\right) = i b_\nu\, f_\nu\, v(x,n_\nu)\sin(\gamma_\nu t). \end{gathered} \tag{62}$$

This implies the validity of Equation (56) in the statement of Lemma, that is,

$$\int_0^1 \left(\sum_{\nu=-\infty,\, \nu\neq 0} \mathrm{res}_{k=\gamma_\nu}\ m_2(k)\, \varphi(x,k)\, \varphi(\xi,k)\, e^{-ikt}\right) f(\xi)\, d\xi = i \sum_{\nu=1}^{\infty} b_\nu\, f_\nu\, v(x,n_\nu)\sin(\gamma_\nu t).$$

To complete the proof of the Lemma, the validity of Equation (58) must be verified.

Let $\lambda_{2n-1} < \gamma_\nu < \lambda_{2n}$, $n > n_2$. Taking into account Propositions 5 and 7, the numbers b_ν can be written as

$$b_\nu = -\frac{\gamma_\nu\left((\varphi'(k) - \theta(k))\big|_{k=\gamma_\nu} - \sqrt{(\gamma_\nu - \lambda_{2n-1})(\gamma_\nu - \lambda_{2n}) g_{2n}(\gamma_\nu)}\right)}{\varphi_{2n}(\gamma_\nu)} \cdot \int_0^1 |\varphi(x,\gamma_\nu)|^2 dx \tag{63}$$

According to Equation (11),

$$|(\varphi'(k) - \theta(k))\big|_{k=\gamma_\nu}| = |\sqrt{(\varphi'(\gamma_\nu)+\theta(\gamma_\nu))^2-4}| = |\sqrt{(\gamma_\nu - \lambda_{2n-1})(\gamma_\nu - \lambda_{2n}) g_{2n}(\gamma_\nu)}|. \tag{64}$$

We also note that Equation (5) for $n > n_2$ yields

$$\int_0^1 |\varphi(x,\gamma_\nu)|^2 dx \le \frac{C}{\gamma_\nu^2}, \tag{65}$$

where the constant (C) can be chosen independent of n.

It is clear that

$$|\sqrt{(\gamma_\nu - \lambda_{2n-1})(\gamma_\nu - \lambda_{2n})}| \leq \lambda_{2n} - \lambda_{2n-1}. \tag{66}$$

The validity of the first inequality in Equation (58) follows from Equations (63)–(66) and Propositions 5 and 7. The second inequality in Equation (58) is obtained similarly. □

3. Main Results

In this section, we present the main results of this paper, the essence of which is the asymptotic behavior of the initial boundary value problem for a one-dimensional second-order hyperbolic equation with periodic coefficients.

All auxiliary statements and lemmas that used in the proof of the theorem are presented in the previous section.

Theorem 1. *If the one-dimensional Schrödinger operator (Hill operator) (H_0) is positive, i.e., $p(x) \geq \text{const} > 0$, $q(x) \geq 0$, then there is a compact operator*

$$M : L^2[0,1] \longmapsto L^2[0,1]$$

such that for $x \in [0,1]$ and $t > 0$, the solution to the initial boundary value problem, i.e., Equations (1)–(3), has the form

$$u(x,t) = u_1(x,t) + v(x,t),$$

where $u_1(x,t)$ is the solution to the following mixed problem

$$\begin{cases} u_{tt}(x,t) - (p(x)\, u_x(x,t))_x + q(x)\, u(x,t) = 0, \quad x \in [0,1],\, t > 0, \\ u(x,t)|_{t=0} = 0, \quad u_t(x,t)|_{t=0} = M[f(x)], \quad x \in [0,1], \\ u(x,t)|_{x=0} = u(x,t)|_{x=1} = 0, \quad t \geq 0, \end{cases}$$

while the function $v(x,t)$ for $x \in [0,1]$, $t > 0$ satisfies the estimate

$$|v(x,t)| \leq \frac{C}{t} ||f; L^2(\mathbb{R})||;$$

the function $u_1(x,t)$ has the form

$$u_1(x,t) = \sum_{\nu=1}^{\infty} b_\nu\, f_\nu\, v(x,n_\nu) \sin(\gamma_\nu t);$$

where $v(x,n_\nu)$ is the normalized eigenfunction of the problem presented in Equation (12) corresponding to the eigenvalue γ_ν^2,

$$f_\nu = \int_0^1 v(x,n_\nu) f(\xi)\, d\xi,\ \nu = 1,2,\ldots,$$

f_ν are the coefficients of the expansion of the function $f(x)$ in the Fourier series in the system $\{\hat{v}(x,n_\nu)\}_{n=1}^{\infty}$ and b_ν are constants of order $o(\frac{1}{\nu})$ as $\nu \to \infty$, as expressed by Equation (57).

Proof. By virtue of Proposition 2, the solution to the problem presented in Equations (1)–(3) has the form of Equation (27):

$$u(x,t) = \frac{1}{2\pi} J_L - iJ_3 + v_1(x,t),$$

where the function $v_1(x,t)$ for $x \in [0,1]$ and $t > 0$ satisfies the following estimate

$$|v_1(x,t)| \leq C\, e^{-td} ||f; L^2||.$$

According to Proposition 3, Lemmas 1 and 2,

$$\frac{1}{2\pi} J_L - iJ_3 = \sum_{\nu=1}^{\infty} b_\nu\, f_\nu\, v(x, n_\nu) \sin(\gamma_\nu t) + v_2(x,t), \tag{67}$$

where the function $v_2(x,t)$ for $x \in [0,1]$ and $t > 0$ satisfies the estimate

$$|v_2(x,t)| \le C\, e^{-td} ||f; L^2||.$$

Based on Equations (27) and (67), it follows that for $x \in [0,1]$ and $t > 1$, the solution to the problem presented in Equations (1)–(3) has the form

$$u(x,t) = u_1(x,t) + v(x,t),$$

where

$$u_1(x,t) = \sum_{\nu=1}^{\infty} b_\nu\, f_\nu\, v(x, n_\nu) \sin(\gamma_\nu t)$$

and the function $v(x,t)$ satisfies the estimate

$$|v(x,t)| \le \frac{C}{t} ||f; L^2(\mathbb{R})||, \quad x \in [0,1],\, t > 1;$$

where the constant C does not depend on the function f.

To complete the proof of the theorem, it remains to be observed that

$$Mf = \sum_{\nu=1}^{\infty} \gamma_\nu\, b_\nu\, f_\nu\, v(x, n_\nu),$$

and the compactness of the operator (M) follows from the estimate of Equation (58) for the numbers b_ν.

The representation $b_\nu = o(\frac{1}{\nu})$ with $\nu \to \infty$ follows from the fact that the set $\gamma_\nu, \nu = \pm 1, \pm 2, \ldots$, is a subset of the set $\kappa_\nu, \nu = \pm 1, \pm 2, \ldots$, and from relations in Equation (13). $\square$

4. Applications

Numerical study for the one-dimensional Schrödinger operator of the form

$$-\frac{1}{2}\partial_{xx} + \alpha q(x) \quad \text{with} \quad q(x) = \cos(x) + \varepsilon \cos(kx), \quad \alpha \in \mathbb{R}, \quad \varepsilon > 0$$

for irrational k was carried out in [29]. Furthermore, this determines the quantum wave function of an independent electron in a crystal lattice perturbed by impurities, the scattering of which induces only long-range order and which is transmitted using a quasi-periodic potential (q). The authors studied all the phenomena for different values of k and ε in detail and found that for $k > 1$ and $\varepsilon \ll 1$, i.e., when more than one impurity of the elementary cell of the original lattice appears inside “impurity bands”, they appear to be k-periodic. Furthermore, when $\varepsilon > 1$, the opposite occurs.

As an application, we also note [30], in which the authors investigated a simple one-dimensional model of an incommensurable “harmonic crystal” in terms of the spectrum of the corresponding Schrödinger equation. Here, it is shown that the lower spectrum of the operator is divided into “Cantor-like bands” and “impurity bands”, which correspond to critical and extended eigenstates, respectively. For the results obtained in the paper, numerical experiments were carried out, which were performed both for stationary and non-stationary problems.

As usual, the problem of the asymptotic behavior of solutions reduces to solving Sturm–Liouville eigenvalue problems and obtaining asymptotic estimates for the Sturm–Liouville spectrum. In this regard, the authors of [31] showed that the differential equation

$$y'' + q^2(x)y = 0$$

can be solved under suitable conditions, taking the solution of the form $y = A(x)\sin\varphi(x)$, where

$$\varphi'(x) = \sqrt{\lambda + q(x)} + \frac{1}{4}\frac{q'(x)}{\lambda + q(x)}\sin 2\varphi(x), \quad A'(x) = -\frac{A(x)}{2}\frac{q'(x)}{\lambda + q(x)}\cos^2\varphi(x).$$

When applying the boundary conditions, the use of the first equation leads to asymptotic estimates of the eigenvalues. In particular, in the case of the Hill equation, it is shown that the instability intervals vanish faster than any inverse power of k, as k is the order of the corresponding eigenvalues when $q(x)$ is an analytic function.

5. Conclusions

In this paper, an explicit formula for the asymptotic expansion of solutions to a mixed problem for a one-dimensional wave equation with periodic coefficients on the semi-axis for large values of the time parameter t was obtained. To study this initial boundary value problem with the corresponding Schrödinger operator, we applied the entire apparatus of spectral theory and the properties of the spectrum one-dimensionally to the Schrödinger operator for both finite and infinite segments of the real axis.

The next step in the development of these problems will be to consider the problem of finding the asymptotic behavior of a mixed problem in cases in which the left end of the spectrum of the Schrödinger operator is negative.

In addition, it is also important to establish the principle of the limiting amplitude of the Cauchy problem in Equations (1) and (2) for a hyperbolic equation with periodic coefficients $p(x)$ and $q(x)$ as $t \to \infty$.

As a basis for further applied research, we also note [32], in which the authors showed that a Fuchsian differential equation with five regular singular points admits solutions in terms of one generalized hypergeometric function for an infinite set of particular variants of the equation parameters. Each solution also assumes four restrictions imposed on the parameters.

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