symmetry

MDPI

Article

Evolution of Generalized Brans–Dicke Parameter within a Superbounce Scenario

Sunil Kumar Tripathy [1,*], Sasmita Kumari Pradhan [2,3], Biswakalpita Barik [1], Zashmir Naik [3] and B. Mishra [4]

[1] Department of Physics, Indira Gandhi Institute of Technology, Sarang, Dhenkanal 759146, Odisha, India; biswakalpitabarik1234@gmail.com
[2] Department of Physics, Centurion University of Technology and Management, Paralakhemundi 761211, Odisha, India; sasmita.gita91@gmail.com
[3] School of Physics, Sambalpur University, Jyoti Vihar, Sambalpur 768019, Odisha, India; zashmir@gmail.com
[4] Department of Mathematics, Birla Institute of Technology and Science-Pilani, Hyderabad Campus, Hyderabad 500078, Telangana, India; bivu@hyderabad.bits-pilani.ac.in
* Correspondence: tripathy_sunil@rediffmail.com

Abstract: We studied a superbounce scenario in a set up of the Brans–Dicke (BD) theory. The BD parameter was considered to be time-dependent and was assumed to evolve with the Brans–Dicke scalar field. In the superbounce scenario, the model bounced at an epoch corresponding to a Big Crunch provided the ekpyrotic phase continued until that time. Within the given superbounce scenario, we investigated the evolution of the BD parameter for different equations of state. We chose an axially symmetric metric that has an axial symmetry along the x-axis. The metric was assumed to incorporate an anisotropic expansion effect. The effect of asymmetric expansion and the anisotropic parameter on the evolving and non-evolving parts of the BD parameter was investigated.

Keywords: generalized Brans–Dicke theory; superbounce scenario; BD parameter

check for updates

Citation: Tripathy, S.K.; Pradhan, S.K.; Barik, B.; Naik, Z.; Mishra, B. Evolution of Generalized Brans–Dicke Parameter within a Superbounce Scenario. *Symmetry* **2023**, *15*, 790. https://doi.org/10.3390/sym15040790

Academic Editor: Muhammad Zubair

Received: 28 February 2023
Revised: 16 March 2023
Accepted: 20 March 2023
Published: 24 March 2023

1. Introduction

The standard cosmological model is quite successful in describing the evolution of the Universe at different phases of time. The standard cosmology provides useful information at the early evolutionary epochs in particular. However, it suffers from issues such as the flatness, horizon, and initial singularity problems. The inflationary model, described through a scalar field, solved some of these problems, including the flatness and cosmological horizon problems, and provided a causal theory of structure formation [1,2]. However, the long-standing issue of initial singularity remains unsolved.

In modern cosmology, there remains a fundamental question: whether our Universe had a beginning, perhaps in the form of an initial singularity leading to a breakdown of the space–time description, or whether the presently expanding phase of the Universe was preceded by a contraction phase. This may also be conceived of as the Universe undergoing phases of alternate contraction and expansion, suggesting a cyclic cosmology. The proposal of matter bounce scenarios came as a possible solution to the initial singularity issue [3–5]. Novello and Perez Bergliaffa emphasized the significance of a singularity-free Universe [6]. As possible alternatives to the standard cosmology, Battefeld et al. discussed some bouncing cosmological models [7]. The consequences of initial singularity issues and matter bounce scenarios as possible explanations were reviewed in [8,9]. Within the purview of scalar field cosmology, the Universe starts to contract with an increase in the kinetic energy of the scalar field. As it dominates, the Universe collapses, leading to a classical singular event. This situation may be avoided if an expansion occurs prior to the sudden collapse. This is what is assumed in a bouncing scenario, wherein the Universe undergoes a contraction phase primarily dominated by its matter content, followed by a non-singular bounce.

In a flat Universe, the cosmic matter content needs to violate the null energy condition (NEC) in order to experience a bouncing phase. In other words, the sum of pressure p

and energy density ρ has to be negative during the matter bounce, i.e., $\rho + p < 0$. With a cosmic fluid violating the NEC, it is possible for the Universe to switch from contraction to expansion, avoiding the singularity at the bouncing point. However, currently, no known matter forms violate the NEC, which suggests the presence of exotic matter forms. In general relativity theory (GR), singularity is unavoidable, but one may go beyond GR with the assumption of a new type of matter field that violates the key energy condition, raising obvious questions about the occurrence of non-singular bounces in nature. The violation of the null energy condition is seen in generalized Galileon theories, where non-singular cosmology may be witnessed [10–12].

An interesting aspect of non-singular bouncing cosmologies is that, in most cases, the models are unstable. However, it is possible to construct a stable bouncing cosmology beyond the Horndeski theory and effective field theory [13–18]. In recent times, many bouncing cosmological models have been presented either in modified gravity theories or scalar-field-mediated gravity theories. Some bouncing cosmological models have been presented by Bamba et al. [19], Chakraborty [20], and Amani [21] in the $f(R)$ theory of gravity. Mishra et al. [22], Tripathy et al. [23–26], and Singh et al. [27] constructed some stable bouncing models in the $f(R,T)$ theory. Agrawal et al. investigated the prospects of some bouncing models in the $f(Q,T)$ gravity theory [28]. The telleparallel $f(T)$ gravity theory provides a model that avoids the phenomenal Big Bang singularity and achieves a non-singular bouncing scenario [29]. In the setup of the $f(T, T_G)$ gravity theory, de la Cruz-Dombriz et al. reconstructed several bouncing scenarios [30]. Within general relativistic hydrodynamics, Saikh et al. obtained a class of bouncing cosmological models [31]. Tripathy et al. discussed the possibility of a bouncing scenario for an anisotropic and homogeneous Universe in the generalized Brans–Dicke theory [32]. The BD theory has been successful in dealing with many cosmological and astrophysical issues. Within the BD theory, Maurya et al. presented a charged anisotropic strange star model and showed that an increase in the BD parameter resulted in an enhancement in the superposition of the electric and scalar fields [33]. Zhang et al. obtained some bounds on the BD theory using the gravitational waves from inspiraling compact binaries [34]. Tirandri and Saaidi studied anisotropic inflation within BD gravity [35]. BD theory may be conceived as a unified model for dark matter and dark energy [36]. Durk and Clifton constructed discrete cosmological models within the BD theory [37]. The investigation of bouncing cosmologies has become interesting in the context of recent research trends. Additionally, bouncing cosmology is believed to emerge naturally in many early Universe scenarios [38,39].

In the present work, we considered a generalized Brans–Dicke (GBD) theory with a dynamically varying BD parameter and investigated a superbounce scenario. Assuming that a superbounce scenario occurs for a violation of the NEC, the time evolution of the dynamical BD parameter was studied. The article is organized as follows. In Section 2, for a homogeneous and anisotropic LRS Bianchi I (LRSBI) metric, the basic field equations for the GBD theory are obtained. In Section 3, a superbounce scenario is assumed, wherein the bounce occurs at an epoch corresponding to a Big Crunch. For different equations of state (EoS) between pressure and energy density, we investigate the evolution of the BD parameter in Section 4. We summarize our results in Section 5. The units chosen for this work are: $8\pi G_0 = c = 1$, with c being the light speed in a vacuum and G_0 representing the Newtonian gravitational constant at the present time.

2. Basic Equations

In the Jordan frame, the action of the generalized Brans–Dicke theory with an evolving BD parameter $\omega(\varphi)$ is given by [40,41]:

$$S = \int d^4x\sqrt{-g}\left[\varphi R - \frac{\omega(\varphi)}{\varphi}\varphi^{,\mu}\varphi_{,\mu} + \mathcal{L}_m\right]. \tag{1}$$

Here, R is the Ricci scalar, and $\mathcal{L}_m$ denotes the matter Lagrangian. In string theory, supergravity theory, and Kaluza–Klein theory, the GBD theory naturally involves a dynamical BD parameter [42,43]. The literature includes several interesting investigations on different cosmological and astrophysical issues in GBD theory [32,44–49]. Variations in action in the GBD theory lead to the following field equations [50,51]:

$$G_{\mu\nu} - \frac{\omega(\varphi)}{\varphi^2}\left[\varphi_{,\mu}\varphi_{,\nu} - \frac{1}{2}g_{\mu\nu}\varphi_{,\alpha}\varphi^{,\alpha}\right] - \frac{1}{\varphi}\left[\varphi_{,\mu;\nu} - g_{\mu\nu}\Box\varphi\right] = \frac{T_{\mu\nu}}{\varphi}. \tag{2}$$

The Klein–Gordon equation illustrates the evolution of the BD scalar field φ and is written as follows:

$$\Box\phi = \frac{T}{2\omega(\varphi)+3} - \frac{\frac{\partial\omega(\varphi)}{\partial\varphi}\varphi_{,\mu}\varphi^{,\mu}}{2\omega(\varphi)+3}, \tag{3}$$

where T is the trace of the energy–momentum tensor $T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}$.

For an LRSBI Universe [44]

$$ds^2 = -dt^2 + a_1^2 dx^2 + a_2^2(dy^2 + dz^2), \tag{4}$$

the GBD field equations become [32,44]

$$3(2-\xi)\xi H^2 - \frac{\omega(\varphi)}{2}\left(\frac{\dot\varphi}{\varphi}\right)^2 + 3H\left(\frac{\dot\varphi}{\varphi}\right) = \frac{\rho}{\varphi}, \tag{5}$$

$$2\xi\dot H + 3\xi^2 H^2 + \frac{\omega(\varphi)}{2}\left(\frac{\dot\varphi}{\varphi}\right)^2 + 2\xi H\left(\frac{\dot\varphi}{\varphi}\right) + \frac{\ddot\varphi}{\varphi} = -\frac{p}{\varphi}, \tag{6}$$

$$(3-\xi)\dot H + 3(\xi^2 - 3\xi + 3)H^2 + \frac{\omega(\varphi)}{2}\left(\frac{\dot\varphi}{\varphi}\right)^2 + (3-\xi)H\left(\frac{\dot\varphi}{\varphi}\right) + \frac{\ddot\varphi}{\varphi} = -\frac{p}{\varphi}. \tag{7}$$

Here, a_1 and a_2 are the time-dependent directional scale factors. We define $\xi = \frac{3}{k+2}$ as an anisotropic parameter, where k fixes the anisotropic relationship between the directional expansion rates: $\frac{\dot a_1}{a_1} = k\frac{\dot a_2}{a_2}$. The Hubble parameter for the LRSBI Universe is defined as $H = \frac{1}{3}\left(\frac{\dot a_1}{a_1} + 2\frac{\dot a_2}{a_2}\right)$. Within this assumption, the Hubble rate becomes $H = \frac{1}{\xi}\frac{\dot a_2}{a_2}$. One may note that, for a constant scalar field φ and $\xi = 1$, the above field equations reduce to the usual GR field equations with isotropic behavior.

The Klein–Gordon wave equation becomes

$$\frac{\ddot\varphi}{\varphi} + 3H\frac{\dot\varphi}{\varphi} = \frac{\rho - 3p}{2\omega(\varphi)+3} - \frac{\frac{\partial\omega(\varphi)}{\partial\varphi}\dot\varphi^2}{2\omega(\varphi)+3}. \tag{8}$$

From Equations (6) and (7), it may be reasoned that

$$\frac{\dot\phi}{\phi} + \frac{\dot H + 3H^2}{H} = 0, \tag{9}$$

which may be expressed as

$$\frac{\dot\varphi}{\varphi} + (2-q)H = 0, \tag{10}$$

where $q = -1 - \frac{\dot H}{H^2}$ is the deceleration parameter. Equation (10) ensures a power law relation between the scalar field and the scale factor a for a constant deceleration parameter.

3. Superbounce Scenario

We considered a superbounce scenario within the GBD formalism evolving through the following scale factor [52,53]:

$$a(t) \simeq \left(\frac{t_s - t}{t_0}\right)^{2/n^2}, \tag{11}$$

where $n > \sqrt{6}$ is a constant parameter; $t_0 > 0$ is an arbitrary time; and t_s is the time frame in which the model bounces, and it may correspond to the time of the Big Crunch if the ekpyrotic phase were to continue until that time. One may note that the scale factor has a unitary value at $t = t_s + t_0$.

The Hubble parameter for this scenario may be expressed as

$$H = -\frac{2}{n^2}(t_s - t)^{-1}. \quad (12)$$

The variation in the scale factor $a(t)$ for the superbounce scenario is shown in Figure 1. We considered the parameter space of the scale factor to be $t_s = 1, t_0 = 1$, and $n^2 = 6.0491$. The scale factor $a(t)$ was observed to decrease from a higher positive value up to $t = 1$; the bounce occurred at $t = 1$; and, beyond the epoch $t = 1$, the scale factor started increasing from a lower positive value to a higher positive value. As is apparent from the figure, the decrement in the scale factor prior to the bounce and the increment in the scale factor after the bounce were almost linear for the parameter space used in the present work. In Figure 2, the Hubble parameter is shown for the given superbounce scenario. The Hubble parameter had a singularity at $t = t_s$ and, while it became positive for $t > t_s$, it was negative for the time zone $t < t_s$.

We could obtain the slope and curvature of the Hubble parameter as

$$\dot{H} = -\frac{2}{n^2}(t_s - t)^{-2} = -\frac{n^2}{2}H^2, \quad (13)$$

$$\ddot{H} = -\frac{4}{n^2}(t_s - t)^{-3} = \frac{n^4}{2}H^3. \quad (14)$$

One should note that the superbounce scenario included a singular Hubble parameter at the bounce that reversed the sign in the pre- and post-bounce epoch. Additionally, it satisfied the bouncing conditions.

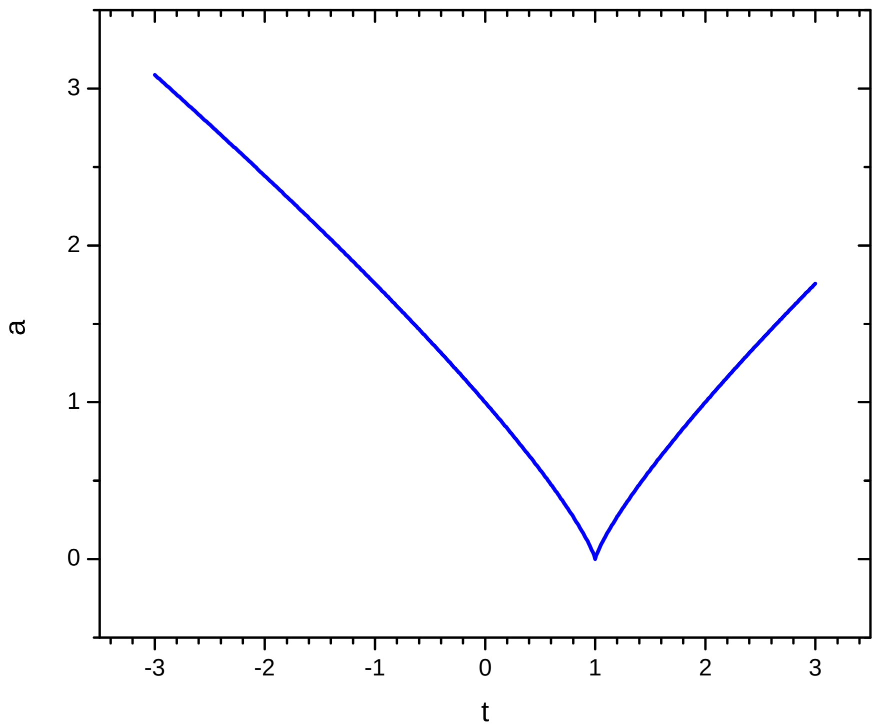

Figure 1. The scale factor in the superbounce scenario. We considered the parameter space for the scale factor to be $t_s = 1, t_0 = 1$, and $n^2 = 6.0491$.

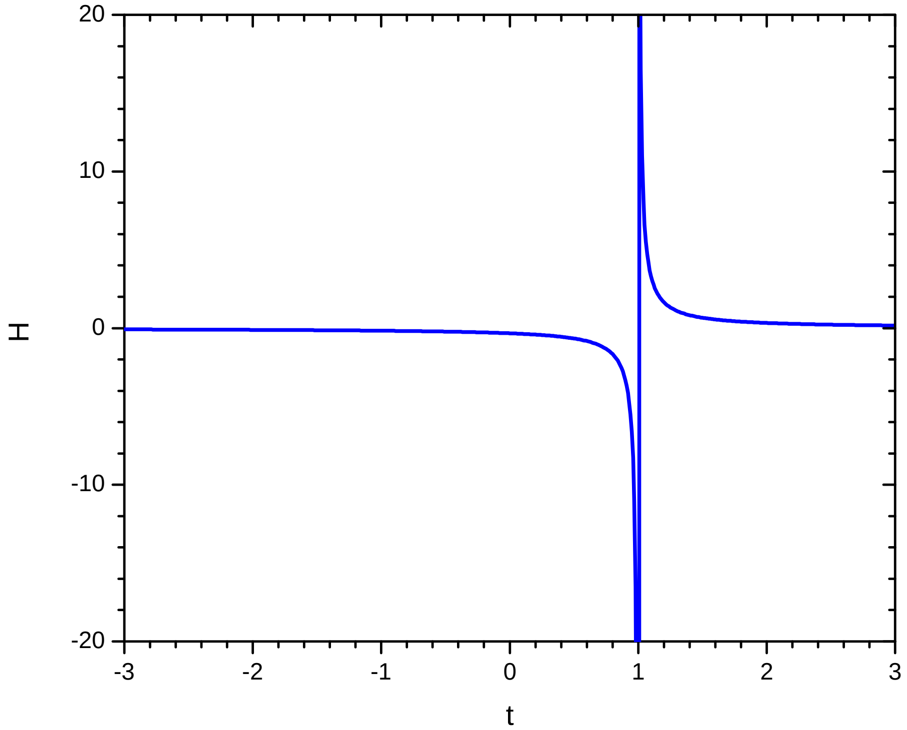

Figure 2. Hubble parameter in the superbounce scenario.

For the given superbounce scenario, the deceleration parameter becomes

$$q = \frac{n^2}{2} - 1, \tag{15}$$

which is a constant quantity that depends only on the parameter n.

With this value of q, Equation (10) reduces to

$$\frac{\dot{\varphi}}{\varphi} = \left(\frac{n^2}{2} - 3\right)H, \tag{16}$$

which on integration gives the BD scalar field

$$\varphi = \varphi_0 \left(\frac{H}{H_0}\right)^{\frac{6}{n^2} - 1}, \tag{17}$$

where φ_0 and H_0 are the present epoch values of the scalar field and Hubble parameter, respectively.

Since $\frac{a}{a_0} = \left(\frac{H}{H_0}\right)^{-\frac{2}{n^2}}$, the BD scalar field may be expressed as

$$\frac{\varphi}{\varphi_0} = \left(\frac{a}{a_0}\right)^{(\frac{n^2}{2} - 3)}, \tag{18}$$

where a_0 represents the present epoch value of the scale factor. As usual, the scalar field depends on the scalar factor, increasing correspondingly. The time variation of the Brans–Dicke scalar field is shown in Figure 3. In the pre-bounce epoch, the BD scalar field decayed slowly with time up to the bounce. At the bounce, $t = 1$, the BD scalar field suddenly presented a sharp dip and then bounced with the growth in cosmic time.

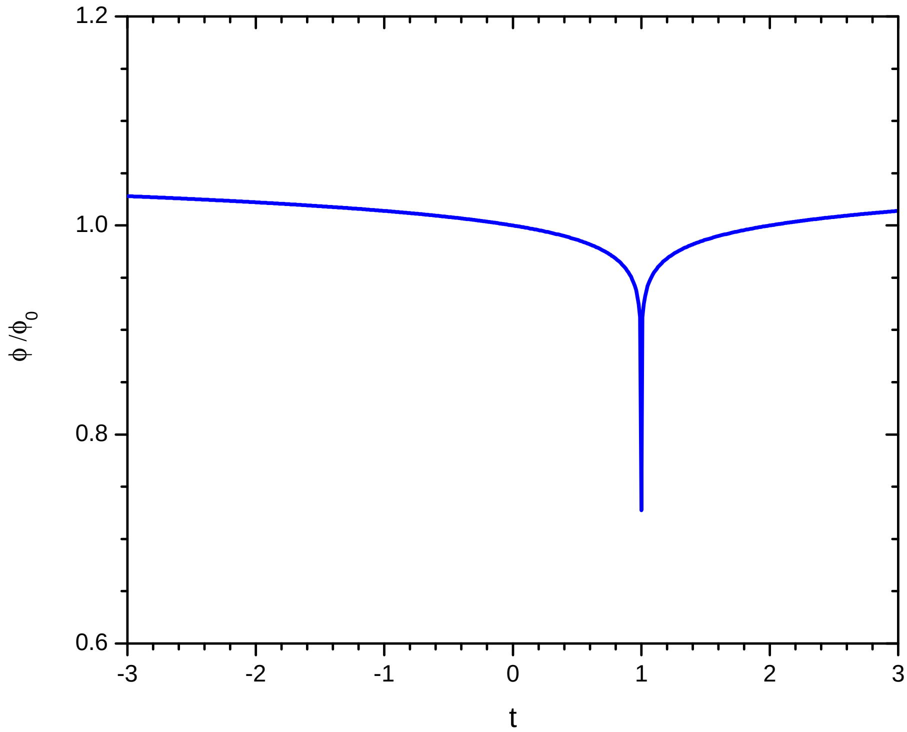

Figure 3. Evolution of the Brans–Dicke scalar field in the superbounce scenario.

4. Evolution of the Brans–Dicke Parameter

The dynamic aspect of the BD parameter as function of the BD scalar field may be obtained from the GBD field Equations (5) and (6) as follows:

$$\omega(\varphi) = \left(\frac{\dot{\varphi}}{\varphi}\right)^{-2}\left[-\frac{\rho+p}{\varphi} - \frac{\ddot{\varphi}}{\varphi} + (3-2\xi)H\frac{\dot{\varphi}}{\varphi} - 2\xi\dot{H} + 6(1-\xi)\xi H^2\right]. \tag{19}$$

However, from Equations (5)–(7), we obtain

$$\omega(\varphi) = \left(\frac{\dot{\varphi}}{\varphi}\right)^{-2}\left[-\frac{\rho+p}{\varphi} - \frac{\ddot{\varphi}}{\varphi} + \xi H\frac{\dot{\varphi}}{\varphi} - (3-\xi)\dot{H} + 3(5\xi - 2\xi^2 - 3)\xi H^2\right]. \tag{20}$$

Obviously, the above two expressions (19) and (20) are consistent for $\xi = 1$. However, for $\xi \neq 1$, we could infer from these two expressions a consistency condition for the BD scalar field, as given in (9).

For a given superbounce scenario and a given cosmic anisotropy, the evolution of the BD parameter could be obtained once we knew the equation of state (EoS) parameter, defined as the ratio of the pressure to energy density, i.e., $\omega_D = \frac{p}{\rho}$. In this work, two different cases of the EoS parameter were chosen. In the first case, we considered a constant EoS parameter, and in the second case, a unified dark fluid simulating a dynamic EoS parameter was chosen.

We considered $\frac{\dot{\varphi}}{\varphi} = \left(\frac{n^2}{2} - 3\right)H$, which on differentiation provided

$$\frac{\ddot{\varphi}}{\varphi} = -\frac{3}{2}(n^2-6)H^2. \tag{21}$$

Using Equations (16) and (21), the BD parameter could be reduced to

$$\omega(\varphi) = \omega_1(\varphi) + \omega_0, \tag{22}$$

where

$$\omega_0 = \frac{12}{(n^2-3)^2}\left[\left(n^2-6\right)+2(2-\xi)\xi\right], \tag{23}$$

is the non-evolving part of $\omega(\varphi)$, which depends on the anisotropic parameter ξ and the parameter n.

$$\omega_1(\varphi) = -\left(\frac{\rho+p}{\varphi}\right)\frac{1}{\left(\frac{n^2}{2}-3\right)^2 H^2} \tag{24}$$

represents the evolutionary aspect of $\omega(\varphi)$ and could be expressed in terms of the EoS parameter ω_D as

$$\omega_1(\varphi) = -\frac{(1+\omega_D)}{\varphi}\frac{\rho}{\left(\frac{n^2}{2}-3\right)^2 H^2}. \tag{25}$$

The above expression implies that the evolutionary aspect of the BD parameter depends on the evolutionary aspect of the EoS parameter, besides being a function of the scalar field.

For the given superbounce scenario, one may note that the BD parameter splits into two parts. There is a non-evolving part ω_0 that only depends on the choice of the anisotropic parameter ξ and is independent of the choice of the scale factor parameters t_s, t_0. The other part of the BD parameter, i.e., $\omega_1(\varphi)$, depends on the equation of state $p = p(\rho)$ and the evolutionary behavior of the BD scalar field derived from the superbounce scenario. $\omega_1(\varphi)$ is the evolving part of the BD parameter $\omega(\varphi)$ and decides its evolution. In the following, we considered some specific equations of state $p = p(\rho)$ and investigated the evolution of the BD parameter (focusing on $\omega_1(\varphi)$) within the given superbounce scenario.

4.1. Case 1

Let us now consider the EoS as

$$p = \omega_D \rho, \tag{26}$$

with the EoS parameter ω_D being a constant.

The energy–momentum conservation equation is given by $\dot{\rho} + 3H(\rho + p) = 0$, which reduces to

$$\frac{\dot{\rho}}{\rho} = -3H(1+\omega_D). \tag{27}$$

Equation (27) could be integrated to obtain the energy density as

$$\frac{\rho}{\rho_0} = \left(\frac{a}{a_0}\right)^{-3(1+\omega_D)}, \tag{28}$$

where ρ_0 is the energy density at the present epoch. For a given superbounce scenario, $\frac{a}{a_0} = \left(\frac{H_0}{H}\right)^{2/n^2}$ and $\frac{\varphi}{\varphi_0} = \left(\frac{H}{H_0}\right)^{\frac{5}{n^2}}$. Consequently, the energy density becomes

$$\rho = \rho_0 \left(\frac{H_0}{H}\right)^{-\frac{6(1+\omega_D)}{n^2}}, \tag{29}$$

so that

$$\rho + p = (1+\omega_D)\rho = (1+\omega_D)\rho_0 \left(\frac{H_0}{H}\right)^{-\frac{6(1+\omega_D)}{n^2}}. \tag{30}$$

The evolving part of the BD parameter now becomes

$$\omega_1 = \frac{\rho_0}{\varphi_0}(1+\omega_D)\left(\frac{H}{H_0}\right)^{\frac{6(1+\omega_D)+5}{n^2}}. \quad (31)$$

It is obvious that the evolution of ω_1 and ω for a given anisotropic parameter k depends on the evolution of the Hubble parameter. In the pre-bounce period, the Hubble parameter is a negative quantity, and it decreases further with an increase in cosmic time up until the bounce occurs, where the Hubble parameter has a large value. However, after the bounce, the Hubble parameter becomes a positive quantity. We may assume the the EoS parameter ω_D is a negative quantity with values close to -1. Such an assumption is in accordance with the observation of the late cosmic acceleration phenomena. Some recent estimates for ω_D are $\omega_D < -1$ [54], $\omega_D = -1.073^{+0.090}_{-0.089}$ [55], and $\omega_D = -1.084 \pm 0.063$ [55]. Constraints from the Supernova cosmology project provided $\omega_D = -1.035^{+0.055}_{-0.059}$ [56], from Planck 2018 results $\omega_D = -1.03 \pm 0.03$ [57], and from Pantheon data $\omega_D = -1.006 \pm 0.04$ [58]. In view of these estimates, the behavior of ω_1 could be assessed. In the pre-bounce epochs, the magnitude of ω_1 decreases with time.

4.1.1. Case I

If we considered the ΛCDM model envisaging an accelerating Universe with a cosmological constant, the EoS could be set as

$$p = -\rho, \quad (32)$$

so that $\rho + p = 0$ and, consequently, the evolving part of the BD parameter $\omega_1(\varphi)$ vanishes. The BD parameter for this cosmological constant case turned out to be non-evolving with a value of $\omega = \omega_0$. However, the BD parameter depends on the anisotropic parameter ξ. In fact, in this case, the NEC is not violated, and we should not expect a superbounce scenario. In principle, for this case, a cosmic bounce cannot occur, as the curvatures may build up to a large extent to initiate the gravitational collapse.

4.1.2. Case II

For a Zeldovich stiff fluid,

$$p = \rho, \quad (33)$$

so that $\omega_D = 1$, and the evolving part of the BD parameter is

$$\omega_1(\varphi) = -\frac{(1+\omega_D)}{\left(\frac{n^2}{2}-3\right)^2\varphi_0}\rho_0 H_0^{-(\frac{6}{n^2}\omega_D+1)} H^{(\frac{6}{n^2}\omega_D-1)}. \quad (34)$$

For the Zeldovich fluid case, the BD parameter could be obtained as

$$\omega(\varphi) = \frac{-2\rho_0}{\left(\frac{n^2}{2}-3\right)^2\varphi_0} H^{\frac{6}{n^2}-1} H_0^{-(\frac{6}{n^2}+1)} + \omega_0(\varphi), \quad (35)$$

which evolves with cosmic time. One should note that, for the Zeldovich fluid case, $p + \rho > 0$, for which the bouncing scenario may not be viable. In Figure 4, we show the evolution of the BD parameter for the two different cosmic fluid cases with a constant anisotropy parameter $k = 1.1$. In addition to these cases, we also considered a cosmic fluid filled with dark energy and represented by an EoS parameter $\omega_D = -1.03$. One may note that, for the dark energy and the cosmological constant cases, the BD parameter remained almost constant throughout the cosmic period considered in this work. However, for the Zeldovich fluid case, the BD parameter remained almost constant but evolved near the bouncing epoch, showing a sort of singularity at the bounce. In Figure 5, considering a given constant EoS parameter $\omega_D = -1.03$ and three representative anisotropy parameters $k = 0.9, 1.1,$ and 1.3, we show that the cosmic anisotropy only affected the non-evolving part of the BD parameter.

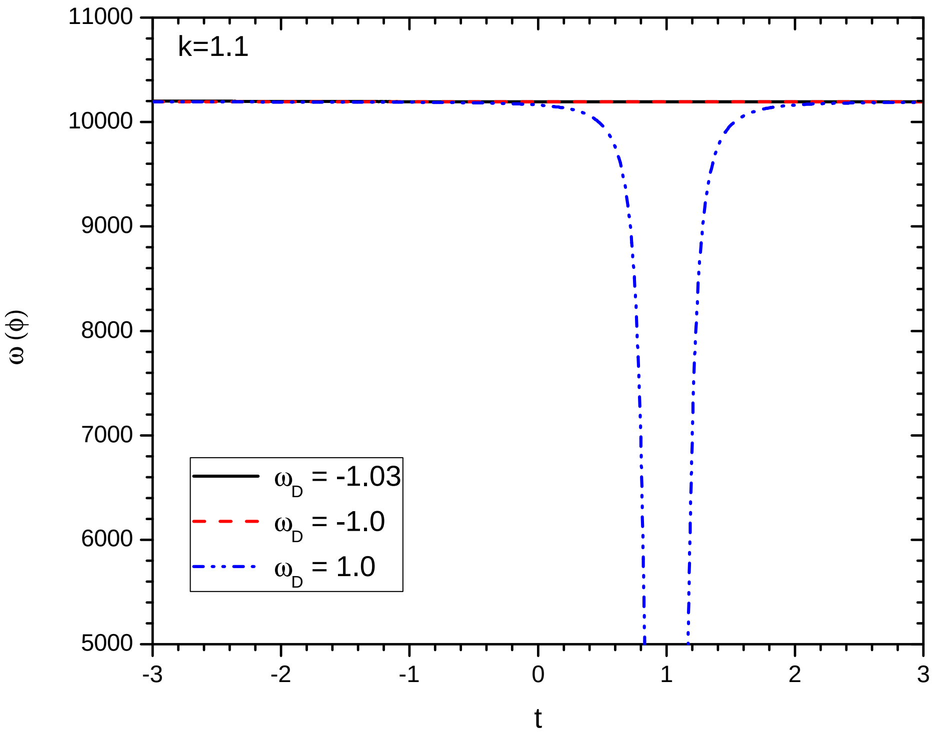

Figure 4. Evolution of the Brans–Dicke parameter in the superbounce scenario for constant EoS parameter.

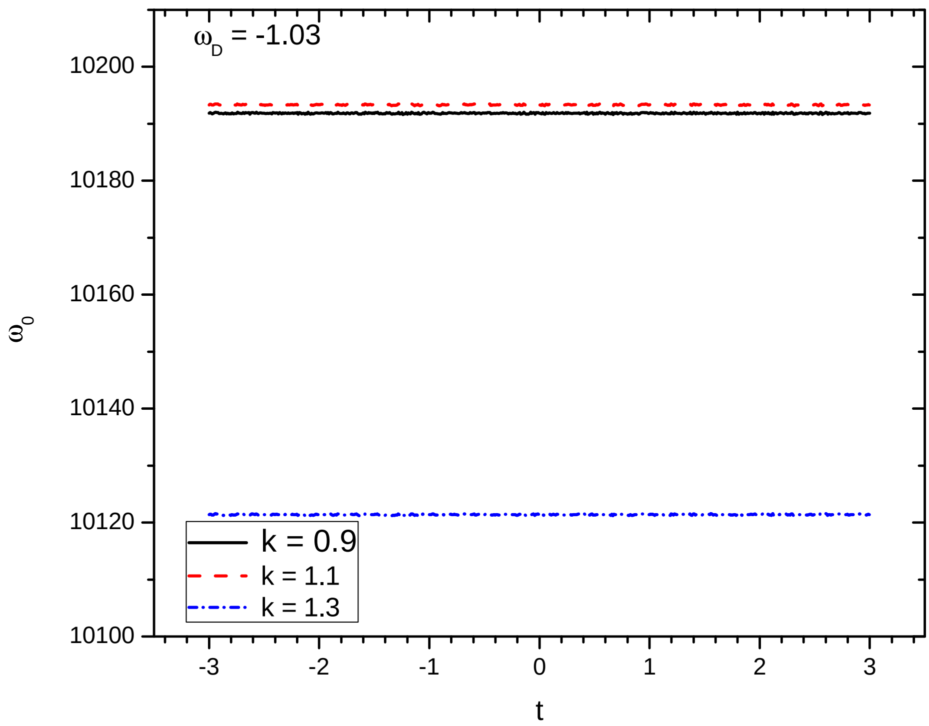

Figure 5. Variation in the non-evolving part of the Brans–Dicke scalar parameter in the superbounce scenario for $\omega_D = -1.03$.

4.2. Case 2

In this section, we consider a unified dark fluid EoS represented by [44]

$$p = \alpha(\rho - \rho_{ud}), \tag{36}$$

where α and ρ_{ud} are constants, and α may be identified with the adiabatic velocity of sound propagating within the cosmic fluid through the relation $C_s^2 = \alpha$. In view of this, the mechanical stability of the cosmic system requires a positive value of α.

The integration of the conservation equation for the unified dark fluid equation of state yielded [44]

$$\rho = \rho_X + \rho_\alpha \left(\frac{a}{a_0}\right)^{-3(1+\alpha)}, \tag{37}$$

$$p = -\rho_X + \alpha\rho_\alpha \left(\frac{a}{a_0}\right)^{-3(1+\alpha)}, \tag{38}$$

where $\rho_X = \frac{\alpha\rho_{ud}}{1+\alpha}$, $\rho_\alpha = \rho_0 - \rho_X$. For the unified dark fluid,

$$\rho + p = (1+\alpha)\rho_\alpha \left(\frac{a}{a_0}\right)^{-3(1+\alpha)}. \tag{39}$$

In the superbounce scenario, $\left(\frac{a}{a_0}\right) = \left(\frac{H_0}{H}\right)^{\frac{2}{n^2}}$. Consequently,

$$\rho + p = (1+\alpha)\rho_\alpha \left[\frac{H}{H_0}\right]^{\frac{6}{n^2}(1+\alpha)}. \tag{40}$$

The EoS parameter for the unified dark fluid could be obtained as

$$\omega_D = -1 + \frac{1+\alpha}{1 + \left(\frac{\rho_X}{\rho_\alpha}\right)(\frac{a}{a_0})^{3(1+\alpha)}}, \tag{41}$$

which could be expressed in terms of the redshift as

$$\omega_D = -1 + \frac{1+\alpha}{1 + \left(\frac{\rho_X}{\rho_\alpha}\right)(1+z)^{-3(1+\alpha)}}, \tag{42}$$

where $1 + z = \frac{a_0}{a}$. One may note that, near the bouncing epoch, for a given value of the ratio $\frac{\rho_X}{\rho_\alpha}$ within a mechanically stable cosmic fluid ($\alpha > 0$), the EoS parameter becomes $\omega_D \simeq \alpha$. In the pre- and post-bounce regimes, it evolves smoothly to coincide with the concordance ΛCDM value $\omega_D = -1$ at an infinite future epoch, where the scale factor becomes infinite.

The evolving part of the BD parameter for the unified dark fluid within the given superbounce scenario is obtained as

$$\omega_1(\varphi) = -(\alpha+1)\frac{H_0^{\frac{-6\alpha}{n^2}-1}}{\varphi_0(q-2)^2} H^{\frac{6\alpha}{n^2}-1}. \tag{43}$$

In this case, the evolving part of the BD parameter is dependent on the parameter α, and its evolutionary aspect is mostly decided by the evolution of the Hubble parameter. As usual, the non-evolving part depends on the anisotropy parameter and the exponent n.

5. Summary and Conclusions

The evolution of the BD parameter within a superbounce scenario was studied using a generalized Brans–Dicke theory. We considered an LRSBI Universe and thereby incorporated directional anisotropy in the expansion rates, which provided us with a more general approach compared to the FRW model. It is possible to recast the GBD theory coupled to a cosmological constant as a GR theory with an effective exotic dark-energy-dominated cosmic fluid. An effective theory for such a dark-energy-dominated cosmic fluid displays either phantom-like or quintessence-like behavior.

The superbounce scenario considered here bounced at an epoch corresponding to the time of the Big Crunch provided the ekpyrotic phase continued until that time. As expected, the Hubble parameter evolved in the pre-bounce phase with negative values and in the post-bounce regime with positive values. However, the evolution of the Hubble parameter was not continuous, and it suffered from a kind of singularity at the bounce. Such a scenario provided a constant deceleration parameter in the pre-bounce and post-bounce regimes. We obtained the evolutionary behavior of the BD scalar field for the superbounce scenario. In the pre-bounce phase, it decayed slowly with an increase in time up to the bouncing epoch. The BD field presented a sudden dip at the bounce and then increased with the growth in cosmic time. In the given superbounce scenario, the scale factor and the deceleration parameter were continuous through the bounce region. Additionally, the BD scalar field appeared to be continuous for the time zone spanning from the pre-bounce to the post-bounce phase and crossing through the bouncing epoch. Only the Hubble parameter presented discontinuity at the bounce, which was characteristics of the superbounce phenomenon. Our prime objective in the present study was to investigate the evolution of the dynamic BD parameter, which obviously depended on the Hubble parameter. Therefore, we observed the singular behavior of the BD parameter at the bounce epoch (Figure 4). In this context, we did not try to connect the solutions in terms of the Hubble parameter from the negative time zone to the positive time zone through the bounce. However it may be possible to extend the superbounce scale factor to the whole cosmological era, such that it gives a smooth unification from bounce to dark energy era. Such extension is also crucial to understand the generation era of perturbation [59–61].

In the present work, the BD parameter was assumed to vary with time and the scalar field. However, its time evolution was mostly model-dependent and depended on the choice of the equation of state. Within the given superbounce scenario, we considered different equations of state to investigate their effect on the evolution of the BD parameter. We also studied the effect of the anisotropic parameter on the evolution of the BD parameter. It was shown that, for the given scenario, only the non-evolving part of the BD parameter was affected by the anisotropic parameter. The superbounce scenario may be viable within the GBD theory, provided one has a dynamic EoS and the null energy condition is violated.

Author Contributions: Formal analysis, Biswakalpita Barik; Investigation, Z.N.; Data curation, B.M.; Writing—original draft, S.K.T.; Writing—review & editing, S.K.P. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: S.K.T. and B.M. thank IUCAA, Pune (India) for providing support through the visiting Associateship programme.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Guth, A. Inflationary universe: A possible solution to the horizon and flatness problems. *Phys. Rev. D* **1981**, *23*, 347. [CrossRef]
2. Mukhanov, V.; Chibisov, G. Quantum fluctuations and a nonsingular universe. *JETP Lett.* **1981**, *33*, 532.
3. Brandenberger, R. Alternatives to the Inflationary Paradigm of structure formation. *Int. J. Mod. Phys. Conf. Ser.* **2011**, *1*, 67. [CrossRef]
4. Bars, I.; Chen, S.H.; Turok, N. Geodesically complete analytic solutions for a cyclic universe. *Phys. Rev. D* **2011**, *84*, 083513. [CrossRef]
5. Bars, I.; Chen, S.H.; Stenhardt, P.J.; Turok, N. Antigravity and the big crunch/big bang transition. *Phys. Lett. B* **2012**, *715*, 218. [CrossRef]
6. Novello, M.; Bergliaffa, S.E.P. Bouncing cosmologies. *Phys. Rep.* **2008**, *463*, 127. [CrossRef]
7. Battefeld, D.; Peter, P. A Critical Review of Classical Bouncing Cosmologies. *arXiv* **2014**, arXiv:1406.2790.
8. Brandenberger, R. The Matter Bounce Alternative to Inflationary Cosmology. *arXiv* **2012**, arXiv:1206.4196.

9. Brandenberger, R.; Peter, P. Bouncing Cosmologies: Progress and Problems. *Found. Phys.* **2017**, *47*, 797. [CrossRef]
10. Nicolis, A.; Rattazzi, R.; Trincherini, E. Galileon as a local modification of gravity. *Phys. Rev. D* **2009**, *79*, 064036. [CrossRef]
11. Deffayet, C.; Esposito-Farese, G.; Vikman, A. Covariant Galileon. *Phys. Rev. D* **2009**, *79*, 084003. [CrossRef]
12. Kobayashi, T. Generic instabilities of nonsingular cosmologies in Horndeski theory: A no-go theorem. *Phys. Rev. D* **2016**, *94*, 043511. [CrossRef]
13. Creminelli, P.; Pirtskhalava, D.; Santoni, L.; Trincherini, E. Stability of geodesically complete cosmologies. *J. Cosmol. Astropart. Phys.* **2016**, *11*, 047. [CrossRef]
14. Cai, Y.; Piao, Y. A covariant Lagrangian for stable nonsingular bounce. *JHEP* **2017**, *9*, 27. [CrossRef]
15. Cai, Y.; Wan, Y.; Li, H.-G.; Qiu, T.; Piao, Y.-S. The Effective Field Theory of nonsingular cosmology. *JHEP* **2017**, *01*, 090. [CrossRef]
16. Cai, Y.; Li, H.-G.; Qiu, T.; Piao, Y.-S. The effective field theory of nonsingular cosmology: II. *Eur. Phys. J. C* **2017**, *77*, 369. [CrossRef]
17. Ilyas, A.; Zhu, M.; Zheng, Y.; Cai, Y.; Saridakis, E.N. DHOST bounce. *J. Cosmol. Astropart. Phys.* **2020**, *09*, 002. [CrossRef]
18. Zhu, M.; Ilyas, A.; Zheng, Y.; Cai, Y.; Saridakis, E.N. Scalar and Tensor Perturbations in DHOST Bounce Cosmology. *J. Cosmol. Astropart. Phys.* **2021**, *11*, 045. [CrossRef]
19. Bamba, K.; Makarenko, A.N.; Myagky, A.N.; Nojiri, S.; Odintsov, S.D. Bounce cosmology from $F(R)$ gravity and F(R) bigravity. *J. Cosmol. Astropart. Phys.* **2014**, *01*, 008. [CrossRef]
20. Chakraborty, S. Reconstruction method of $f(R)$ gravity for isotropic and anisotropic spacetimes. *Phys. Rev. D* **2018**, *98*, 024009. [CrossRef]
21. Amani, A.R. The bouncing cosmology with F(R) gravity and its reconstructing. *Int. J. Mod. Phys. D* **2016**, *25*, 1650071. [CrossRef]
22. Mishra, B.; Ribeiro, G.; Moraes, P.H.R.S. De Sitter and bounce solutions from anisotropy in extended gravity cosmology. *Mod. Phys. Lett. A* **2019**, *34*, 1950321. [CrossRef]
23. Tripathy, S.K.; Khuntia, R.K.; Parida, P. Bouncing cosmology in an extended theory of gravity. *Eur. Phys. J. Plus* **2019**, *134*, 504. [CrossRef]
24. Tripathy, S.K.; Mishra, B.; Ray, S.; Sengupta, R. Bouncing universe models in an extended gravity theory. *Chin. J. Phys.* **2021**, *71*, 610. [CrossRef]
25. Agrawal, A.S.; Tello-Ortiz, F.; Mishra, B.; Tripathy, S.K. Bouncing Cosmology in Extended Gravity and Its Reconstruction as Dark Energy Model. *Forschritte Phys.* **2021**, *2021*, 2100065. [CrossRef]
26. Agrawal, A.S.; Tripathy, S.K.; Pal, S.; Mishra, B. Role of Extended Gravity Theory in Matter Bounce Dynamics. *Phys. Scr.* **2022**, *97*, 025002. [CrossRef]
27. Singh, J.K.; Bamba, K.; Nagpal, R.; Pacif, S.K.J. Bouncing cosmology in $f(R,T)$ gravity. *Phys. Rev. D* **2018**, *97*, 123536. [CrossRef]
28. Agrawal, A.S.; Pati, L.; Tripathy, S.K.; Mishra, B. Matter bounce scenario and the dynamical aspects in $f(Q,T)$ gravity. *Phys. Dark Univ.* **2021**, *33*, 100863. [CrossRef]
29. Cai, Y.F.; Chen, S.H.; Dent, J.B.; Dutta, S.; Saridakis, E.N. Matter bounce cosmology with the $f(T)$ gravity. *Class. Quantum Grav.* **2011**, *28*, 215011. [CrossRef]
30. De la Cruz-Dombriz, A.; Farrugia, G.; Said, J.L.; Gómez, D. Sáez-Chillón, Cosmological bouncing solutions in extended teleparallel gravity theories. *Phys. Rev. D* **2018**, *97*, 104040. [CrossRef]
31. Saikh, A.; Tarai, S.; Tripathy, S.K.; Mishra, B. Bouncing cosmological model with general relativistic hydrodynamics in extended gravity. *Int. J. Geom. Methods Phys.* **2022**, *19*, 2250193. [CrossRef]
32. Tripathy, S.K.; Pandey, S.; Sendha, A.P.; Behera, D. Bouncing scenario in Brans–Dicke theory. *Int. J. Geom. Methods Phys.* **2020**, *17*, 2050056. [CrossRef]
33. Tello-Ortiz, F.; Maurya, S.K.; Errehymy, A.; Singh, K.N.; Daoud, M. Anisotropic relativistic fluid spheres: An embedding class I approach. *Eur. Phys. J. C* **2019**, *79*, 885. [CrossRef]
34. Zhang, X.; Yu, J.; Liu, T.; Zhao, W.; Wang, A. Testing Brans-Dicke gravity using the Einstein telescope. *Phys. Rev. D* **2017**, *95*, 124008. [CrossRef]
35. Tirandari, M.; Saaidi, K. Anisotropic inflation in Brans-Dicke gravity. *Nucl. Phys. B* **2017**, *925*, 403. [CrossRef]
36. Kim, H. Brans-Dicke theory as a unified model for dark matter-dark energy. *Mon. Not. Roy. Astron. Soc.* **2005**, *364*, 813. [CrossRef]
37. Durk, J.; Clifton, T. Discrete Cosmological Models in the Brans-Dicke Theory of Gravity. *Class. Quant. Gravit.* **2019**, *36*, 185011. [CrossRef]
38. Bamba, K.; Nashed, G.G.L.; Hanafy, W.E.; Ibraheem, S.K. Bounce inflation in $f(T)$ cosmology: A unified inflaton-quintessence field. *Phys. Rev. D* **2016**, *94*, 083513. [CrossRef]
39. Hohmann, M.; Jarv, L.; Ualikhanova, U. Dynamical systems approach and generic properties of $f(T)$ cosmology. *Phys. Rev. D* **2017**, *96*, 043508. [CrossRef]
40. Nordtvedt, K., Jr. Post-Newtonian Metric for a General Class of Scalar-Tensor Gravitational Theories and Observational Consequences. *Astrophys. J.* **1970**, *161*, 1059. [CrossRef]
41. Wagoner, R.V. Scalar-Tensor Theory and Gravitational Waves. *Phys. Rev. D* **1970**, *1*, 3209. [CrossRef]
42. Freund, P.G.O. Kaluza-Klein cosmologies. *Nucl. Phys. B* **1982**, *209*, 146. [CrossRef]
43. Green, M.B.; Schwarz, J.H.; Witten, E. *Super String Theory*; Cambridge University Press: Cambridge, UK, 1987.
44. Tripathy, S.K.; Behera, D.; Mishra, B. Unified dark fluid in Brans–Dicke theory. *Eur. Phys. J. C* **2015**, *75*, 149. [CrossRef]
45. Sahoo, B.K.; Singh, L.P. Time Dependence of Brans-Dicke parameter ω for an expanding Universe. *Mod. Phys. Lett. A* **2002**, *17*, 2409. [CrossRef]

46. De Felice, A.; Tsujikawa, S. Conditions for the cosmological viability of the most general scalar-tensor theories and their applications to extended Galileon dark energy models. *J. Cosmol. Astropart. Phys.* **2012**, *1202*, 007. [CrossRef]
47. De Felice, A.; Tsujikawa, S. $f(R)$ Theories. *Living Rev. Relativ.* **2010**, *13*, 3. [CrossRef]
48. Tahmasebzadeh, B.; Karami, K. Generalized Brans–Dicke inflation with a quartic potential. *Nucl. Phys. B* **2017**, *918*, 1. [CrossRef]
49. Tripathy, S.K.; Pradhan, S.K.; Naik, Z.; Behera, D.; Mishra, B. Unified Dark Fluid and Cosmic Transit Models in Brans-Dicke Theory. *Phys. Dark Univ.* **2020**, *30*, 100722. [CrossRef]
50. Sharif, M.; Waheed, S. Cosmic Acceleration and Brans-Dicke Theory. *J. Exptl. Theor. Phys.* **2012**, *115*, 599. [CrossRef]
51. Chakraborty, W.; Debnath, U. Role of Brans-Dicke Theory with or without Self-Interacting Potential in Cosmic Acceleration. *Int. J. Theor. Phys.* **2009**, *48*, 232. [CrossRef]
52. Koehn, M.; Lehners, J.L.; Ovrut, B.A. Cosmological super-bounce. *Phys. Rev. D* **2014**, *90*, 025005. [CrossRef]
53. Oikonomou, V.K. Superbounce and Loop Quantum Cosmology Ekpyrosis from Modified Gravity. *arXiv* **2014**, arXiv:1412.4343.
54. Tripathi, A.; Sangwan, A.; Jassal, H.K. Dark energy equation of state parameter and its evolution at low redshift. *J. Cosmol. Astropart. Phys.* **2017**, *2017*, 012. [CrossRef]
55. Hinshaw, G.; Larson, D.; Komatsu, E.; Spergel, D.N.; Bennett, C.L.; Dunkley, J.; Nolta, M.R.; Halpern, M.; Hill, R.S.; Odegard, N.; et al. Nine-year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Parameter Results. *Astrophys. J. Suppl. Ser.* **2013**, *208*, 19. [CrossRef]
56. Amanullah, R.; Lidman, C.; Rubin, D.; Aldering, G.; Astier, P.; Barbary, K.; Burns, M.S.; Conley, A.; Dawson, K.S.; Deustua, S.E.; et al. Spectra and Hubble Space Telescope Light Curves of Six Type Ia Supernovae at $0.511 < z < 1.12$ and the Union2 Compilation. *Astrophys. J.* **2010**, *716*, 712.
57. Aghanim, N.; Akrami, Y.; Ashdown, M.; Aumont, J.; Baccigalupi, C.; Ballardini, M.; Banday, A.J.; Barreiro, R.B.; Bartolo, N.; Basak, S.; et al. Planck 2018 results. VI. Cosmological parameters. *arXiv* **2018**, arXiv:1807.06209.
58. Goswami, G.K.; Yadav, A.K.; Mishra, B.; Tripathy, S.K. Modeling of Accelerating Universe with Bulk Viscous Fluid in Bianchi V Space-Time. *Fortscr. Phys.* **2021**, *2021*, 2100007. [CrossRef]
59. Odintsov, S.D.; Paul, T. Bouncing Universe with finite time singularity. *Universe* **2022**, *8*, 292. [CrossRef]
60. Nojiri, S.; Odintsov, S.D.; Paul, T. Towards a smooth unification from an ekpyrotic bounce to the dark energy era. *Phys. Dark Univ.* **2022**, *35*, 100984. [CrossRef]
61. Odintsov, S.D.; Paul, T.; Banerjee, I.; Myrzakulov, R.; SenGupta, S. Unifying an asymmetric bounce to the dark energy in Chern–Simons $F(R)$ gravity. *Phys. Dark Univ.* **2021**, *33*, 100864. [CrossRef]