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Some New Estimates of Hermite–Hadamard, Ostrowski and Jensen-Type Inclusions for *h*-Convex Stochastic Process via Interval-Valued Functions

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Abstract: Mathematical programming and optimization problems related to fluid dynamics are heavily influenced by stochastic processes associated with integral and variational inequalities. Furthermore, symmetry and convexity are intrinsically related. Over the last few years, both have become increasingly interconnected so that we can learn from one and apply it to the other. The objective of this note is to convert ordinary stochastic processes into interval stochastic processes due to the wide range of applications in various disciplines. We have developed Hermite–Hadamard ($\mathbb{H}.\mathbb{H}$), Ostrowski-, and Jensen-type inequalities using interval *h*-convex stochastic processes. Our main results can be applied to a variety of new and well-known outcomes as specific situations. The results of this study are expected to stimulate future research on inequalities using fractional and fuzzy integral operators. Furthermore, we validate our main findings by providing some non-trivial examples. To demonstrate their general properties, we illustrate the connections between the examined results and those that have already been published. The results discussed in this article can be seen as improvements and refinements to results that have already been published. This is a fascinating subject that can be investigated in the future to identify equivalent inequalities for various convexity types.

Keywords: Hermite–Hadamard inequality; Ostrowski inequality; Jensen inequality; stochastic process; interval-valued functions; stochastic systems

MSC: 05A30; 26D10; 26D15

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1. Introduction

Stochastic processes are mathematical representations of systems that vary randomly. Probability theory and related fields describe stochastic processes as random groups of variables. The stochastic process can be defined broadly and has piqued the interest of many academics due to its numerous applications in fields such as physics, mathematics, finance, and engineering. Convexity and symmetry are important characteristics of stochastic processes in a variety of nonlinear disciplines, including control problems, optimization, and nonlinear dynamics; see Refs. [1,2]. Various stochastic models have been proposed in the past in reliability theory to describe replacement policies of system components. It is most suitable to study such situations using stochastic models, which are both robust in their specification and flexible in their manipulation. The relevation transform is a well-known model in this field, and it describes the overall lifetime of a component that is replaced at its random failure time by another component of the same age, whose lifetime distribution may differ. Furthermore, since generalised nonlinear regression models of uneven/even-aged stands were first developed, modelling growth and yield in a forest stand have advanced quickly, moving on to stochastic differential equations models and artificial neural network models. Optimization, particularly optimal design, relies heavily on the convexity of stochastic processes, and it can also be used for numerical approximation when a probabilistic quantity in the literature is usually considered a time focus. Transforming stochastic processes into numerical models of systems can change over time, such as the problem related to Newton's law of cooling, the finance model, or the theory of electrical circuits. Stochastic optimization is presented under constraints in a general framework that covers models for finance, reinsurance, and portfolios with large investors; see Ref. [3]. An algorithm using constrained stochastic successive convex approximation is used for finding fixed points for nonconvex stochastic optimization problems that involve expectations over random states; see Ref. [4]. Here is more information about the applications of convex stochastic processes; see Refs. [5–7].

The study of intervals in the context of mathematical analysis and topology is the focus of interval analysis, a subset of set-valued analysis. It was created to address interval uncertainty, which is present in many mathematical or computer models of deterministic real-world systems. Archimedes' method for calculating the circumference of a circle is a historical example of an interval enclosure. A series of lower bounds for the area of a disc derived from the circumscribing and inscribed polygons of a circle with radius 1 were increased, while the upper bounds of the corresponding disc were decreased. The results are frequently skewed when specific numbers are used to describe uncertainty problems. First, the interval arithmetic explains how intervals are defined arithmetically and how to solve problems algebraically, integrally, and differentially. A recent increase in interest for this topic has been attributed to the application of specific tools, such as Julia and C++, and also to the implementation of computational systems like Maple and Mathematica; see Refs. [8–14]. A great deal of research is being conducted on the calculus for set-valued mappings these days, especially in connection with the calculus for fuzzy version of convex mappings, which has applications in almost all disciplines of mathematics, physics, and engineering. Among the papers that contribute to this area are ones on gH-differentiability and some on interval and fuzzy optimization, as well as multidimensional convex optimization; see Refs. [15–18].

The presence of inequalities has a significant impact on many areas of science, including mathematics, physics, engineering, and economics. Understanding a variety of problems in various branches of mathematics depends heavily on mathematical inequalities. One of the most well-known is the Hermite–Hadamard inequality, which had a significant influence on not only mathematics, but also other fields that were connected to it. Convex functions are well known for their significance and excellent applications in a number of fields, especially in integral inequalities, variational inequalities, and optimization. It is fascinating to look into the integral problem and the concept of convexity. Many inequalities have thus been presented as convex function applications. There have been many inequalities established for convex functions, and one of the most famous is $\mathbb{H}.\mathbb{H}$ and Jensen's inequality, because of its geometrical significance; see Refs. [19,20]. Nikodem defined convex stochastic processes in 1980 and also defined some classical properties of convex functions; see Ref. [21]. Later, Skowronski extended the findings and created a number of new convex stochastic process properties; see Ref. [22]. As a result of these notions, Kotrys presented a method for calculating the lower and upper bounds of the theses inequality for convex stochastic processes using integral operators; see Refs. [23,24]. The $\mathbb{H}.\mathbb{H}$ inequality for h-convex stochastic processes was extended by Li and Hao; see Ref. [25]. Budak et al. [26] further extended their findings by presenting inequalities in a more comprehensive manner using the idea of h-convexity. Furthermore, various authors used various notions of convex classes to develop these inequalities for different integral operators and order relations; see Refs. [27–32]. For the h-convex function, Tunc devised the Ostrowski-type inequality; see Ref. [33]. Later, Gonzales et al. [34] extended the Ostrowski inequality results and converted them to a stochastic process for various forms. Based on the development and use of interval analysis in diverse fields, the following authors developed proposed inequalities based on the results of inequalities related to intervals; see Ref. [35]. Mohan et al. [36] developed some interesting properties for the preinvex class of convexity. Chalco–Cano et al. [37] developed Ostrowski-type inequalities for interval-valued functions using a generalized Hukuhara derivative. Budak et al. [38] developed a fractional version of Ostrowski-type inequalities. Khan et al. [39–41] used fuzzy calculus to create some new variants of these inequalities using different classes of convexity. Using this concept, Afzal et al. [42] connected the stochastic process with interval analysis and provided some properties of Jensen and $\mathbb{H}.\mathbb{H}$ inequalities for using the h-Godunova–Levin class of convex mappings. For some recent advancements in these inequalities for interval-valued functions ($\mathbb{IVFS}$), see Refs. [43–57].

As a result of the numerous connections between stochastic processes and real-life phenomena in recent years, interval analysis has been linked in a precise manner with stochastic processes. The study also proves to be novel, since various inequalities play a very important role in ensuring the regularity, stability, and uniqueness of numerous interval stochastic mathematical models' solutions, which is why we connected ordinary stochastic processes with interval stochastic processes. Through this, we were able to explore a whole new dimension of inequalities in relation to interval analysis. The following are some recent developments in various disciplines related to interval stochastic processes; see Refs. [58–61].

We were inspired by the strong collection of literature and specific articles [25,26,34,42], as we introduced the concept of the h-convex stochastic process and developed $\mathbb{H}.\mathbb{H}$, Ostrowski- and Jensen-type inclusions. In addition, to demonstrate the validity of the main results, we provide some numerically non-trivial examples. The article is organised as follows: after reviewing the necessary and pertinent information regarding interval-valued analysis in Section 2, we provide some introduction related to the stochastic process under Section 3. In Section 4, we discuss our main results. Section 5 examines a succinct conclusion.

2. Preliminaries and Background

Although these concepts are not defined here, they are employed in this paper; see Ref. [27]. The interval $\mathfrak{P}$ is closed and bounded, and therefore can be defined as follows:

$$\mathfrak{P} = [\underline{\mathfrak{P}}, \overline{\mathfrak{P}}] = \{\mathfrak{r} \in \mathbb{R} : \underline{\mathfrak{P}} \leq \mathfrak{r} \leq \overline{\mathfrak{P}}\},$$

$\underline{\mathfrak{P}}, \overline{\mathfrak{P}} \in \mathbb{R}$ are the terminal points of $\mathfrak{P}$. When $\underline{\mathfrak{P}} = \overline{\mathfrak{P}}$, then the interval $\mathfrak{P}$ is called to be degenerated. When $\underline{\mathfrak{P}} > 0$ or $\overline{\mathfrak{P}} < 0$, we say that it is positive or negative, respectively. Therefore, we are referring to the collection of all intervals in $\mathbb{R}$ by $\mathbb{R}_{\mathfrak{I}}$ and positive intervals by $\mathbb{R}_{\mathfrak{I}}{}^{+}$. A commonly used Hausdorff separation is as follows for $\mathfrak{P}$ and $\mathfrak{Z}$:

$$\mathbb{D}(\mathfrak{P}, \mathfrak{Z}) = \mathbb{D}\big([\underline{\mathfrak{P}}, \overline{\mathfrak{P}}], [\underline{\mathfrak{Z}}, \overline{\mathfrak{Z}}]\big) = \max\big\{|\underline{\mathfrak{P}} - \underline{\mathfrak{Z}}|, |\overline{\mathfrak{P}} - \overline{\mathfrak{Z}}|\big\}.$$

It is obvious that $(\mathbb{R}_{\mathfrak{I}}, \mathbb{D})$ is a complete metric space.

The following are the definitions of the basic interval arithmetic operations for $\mathfrak{P}$ and $\mathfrak{Z}$:

$$\begin{aligned}
&\mathfrak{P} - \mathfrak{Z} = [\underline{\mathfrak{P}} - \overline{\mathfrak{Z}}, \overline{\mathfrak{P}} - \underline{\mathfrak{Z}}],\\
&\mathfrak{P} + \mathfrak{Z} = [\underline{\mathfrak{P}} + \underline{\mathfrak{Z}}, \overline{\mathfrak{P}} + \overline{\mathfrak{Z}}],\\
&\mathfrak{P} \cdot \mathfrak{Z} = [\min \mathfrak{O}, \max \mathfrak{O}] \quad \text{where } \mathfrak{O} = \{\underline{\mathfrak{P}}\,\underline{\mathfrak{Z}}, \underline{\mathfrak{P}}\,\overline{\mathfrak{Z}}, \overline{\mathfrak{P}}\underline{\mathfrak{Z}}, \overline{\mathfrak{P}}\,\overline{\mathfrak{Z}}\},\\
&\mathfrak{P}/\mathfrak{Z} = [\min \mathfrak{P}, \max \mathfrak{P}] \quad \text{where } \mathfrak{P} = \{\underline{\mathfrak{P}}/\underline{\mathfrak{Z}}, \underline{\mathfrak{P}}/\overline{\mathfrak{Z}}, \overline{\mathfrak{P}}/\underline{\mathfrak{Z}}, \overline{\mathfrak{P}}/\overline{\mathfrak{Z}}\} \text{ and } 0 \notin \mathfrak{Z}.
\end{aligned}$$

Scalar multiplication can be used for the interval $\mathfrak{P}$ by

$$\mathfrak{L}[\underline{\mathfrak{P}},\overline{\mathfrak{P}}] = \begin{cases} [\mathfrak{L}\underline{\mathfrak{P}},\mathfrak{L}\overline{\mathfrak{P}}], & \mathfrak{L} > 0; \\ \{0\}, & \mathfrak{L} = 0;. \\ [\mathfrak{L}\overline{\mathfrak{P}},\mathfrak{L}\underline{\mathfrak{P}}], & \mathfrak{L} < 0 \end{cases}$$

When the algebraic characteristics of its quasilinear nature are clarified, it will be possible to explain its algebraic characteristics on $\mathbb{R}_{\mathfrak{I}}$. In general, they can be categorized as follows:

- (Associativie w.r.t addition) $(\mathfrak{P}+\mathfrak{Z})+\mathfrak{L} = \mathfrak{P}+(\mathfrak{Z}+\mathfrak{L})\ \forall\ \mathfrak{P},\mathfrak{Z},\mathfrak{L} \in \mathbb{R}_{\mathfrak{I}}$
- (Commutative w.r.t addition) $\mathfrak{P}+\mathfrak{L} = \mathfrak{L}+\mathfrak{P}\ \forall\ \mathfrak{P},\mathfrak{L} \in \mathbb{R}_{\mathfrak{I}}$,
- (Additive element) $\mathfrak{P}+0 = 0+\mathfrak{P}\ \forall\ \mathfrak{P} \in \mathbb{R}_{\mathfrak{I}}$,
- (Law of Cancellation) $\mathfrak{L}+\mathfrak{P} = \mathfrak{L}+\mathfrak{L} \Rightarrow \mathfrak{P} = \mathfrak{L}\ \forall\ \mathfrak{P},\mathfrak{L} \in \mathbb{R}_{\mathfrak{I}}$,
- (Associative w.r.t multiplication) $(\mathfrak{P}\cdot\mathfrak{L})\cdot\mathfrak{L} = \mathfrak{P}\cdot(\mathfrak{L}\cdot\mathfrak{L})\ \forall\ \mathfrak{P},\mathfrak{L} \in \mathbb{R}_{\mathfrak{I}}$,
- (Commutative w.r.t multiplication) $\mathfrak{P}\cdot\mathfrak{L} = \mathfrak{L}\cdot\mathfrak{P}\ \forall\ \mathfrak{P},\mathfrak{L} \in \mathbb{R}_{\mathfrak{I}}$,
- (Unity element) $\mathfrak{P}\cdot 1 = 1\cdot\mathfrak{P}\ \forall\ \mathfrak{P} \in \mathbb{R}_{\mathfrak{I}}$,

A set's inclusion $\subseteq$ is another property that is given by

$$\mathfrak{P} \subseteq \mathfrak{Z} \quad \Longleftrightarrow \quad \underline{\mathfrak{Z}} \leq \underline{\mathfrak{P}} \quad \text{and} \quad \overline{\mathfrak{P}} \leq \overline{\mathfrak{Z}}.$$

We obtain the following relationship when we combine inclusion and arithmetic operations. Let $\odot$ be used to represent the basic arithmetic operations. If $\mathfrak{P},\mathfrak{Z},\mathfrak{L}$ and $\mathfrak{W}$ are intervals, then

$$\mathfrak{P} \subseteq \mathfrak{Z} \quad \text{and} \quad \mathfrak{L} \subseteq \mathfrak{W};$$

then, the following relation is valid

$$\mathfrak{P} \odot \mathfrak{L} \subseteq \mathfrak{Z} \odot \mathfrak{W}.$$

The preservation of inclusion in scalar multiplication is the subject of this proposition.

Proposition 1. *Let $\mathfrak{P}$ and $\mathfrak{Z}$ be intervals and $\mathfrak{L} \in \mathbb{R}$, then $\mathfrak{L}\mathfrak{P} \subseteq \mathfrak{L}\mathfrak{Z}$.*

The concepts discussed below lay the groundwork for this section's discussion of the integral for $\mathbb{IVFS}$:

A function $\mathfrak{F}$ is known as $\mathbb{IVF}$ at $\mathfrak{P}_a \in [\mathfrak{P}_1,\mathfrak{P}_2]$, if it gives each a nonempty interval $\mathfrak{P}_a \in [\mathfrak{P}_1,\mathfrak{P}_2]$

$$\mathfrak{F}(\mathfrak{P}_a) = \left[\underline{\mathfrak{F}}(\mathfrak{P}_a),\overline{\mathfrak{F}}(\mathfrak{P}_a)\right].$$

A partition of any arbitrary subset $\mathbb{P}$ of $[\mathfrak{P}_1,\mathfrak{P}_2]$ can be represented as:

$$\mathbb{P} : \mathfrak{P}_1 = \mathfrak{P}_a < \mathfrak{P}_b < \cdots < \mathfrak{P}_m = \mathfrak{P}_2.$$

The mesh of $\mathbb{P}$ is represented by

$$\text{Mesh}(\mathbb{P}) = \max\{\mathfrak{P}_i - \mathfrak{P}_{i-1} : i = 1,2,\ldots,m\}.$$

All partitions of $[\mathfrak{P}_1,\mathfrak{P}_2]$ can be represented by $\mathbb{P}([\mathfrak{P}_1,\mathfrak{P}_2])$. Let $\mathbb{P}(\Lambda,[\mathfrak{P}_1,\mathfrak{P}_2])$ be the pack of all $\mathbb{P} \in \mathbb{P}([\mathfrak{P}_1,\mathfrak{P}_2])$ satisfying this mesh$(\mathbb{P}) < \Lambda$ for any arbitrary point in intervals; then, the sum is denoted by:

$$\mathbb{S}(\mathfrak{F},\mathbb{P},\Lambda) = \sum_{i=1}^{n} \mathfrak{F}(\mathfrak{P}_{ii})[\mathfrak{P}_i - \mathfrak{P}_{i-1}],$$

where $\mathfrak{F} : [\mathfrak{P}_1,\mathfrak{P}_2] \to \mathbb{R}_{\mathfrak{I}}$. We say that $\mathbb{S}(\mathfrak{F},\mathbb{P},\Lambda)$ is a sum of $\mathfrak{F}$ with reference to $\mathbb{P} \in \mathbb{P}(\Lambda,[\mathfrak{P}_1,\mathfrak{P}_2])$.

Definition 1 (see [27]). *A function $\mathfrak{F} : [\mathfrak{P}_1,\mathfrak{P}_2] \to \mathbb{R}_{\mathfrak{I}}$ is known as Riemann-integrable for $\mathbb{IVF}$, or it can be represented by $(\mathbb{IR})$ on $[\mathfrak{P}_1,\mathfrak{P}_2]$, if $\exists\ \tau \in \mathbb{R}_{\mathfrak{I}}$ such that, for every $\mathfrak{P}_2 > 0\ \exists\ \Lambda > 0$,*

$$d\big(\mathbb{S}(\mathfrak{F},\mathbb{P},\Lambda),\tau\big) < \mathfrak{P}_2$$

for each Riemann sum $\mathbb{S}$ of $\mathfrak{F}$ with reference to $\mathbb{P} \in \mathbb{P}(\Lambda, [\mathfrak{P}_1, \mathfrak{P}_2])$ and unrelated to the choice of $\mathfrak{P}_{ii} \in [\mathfrak{P}_{i-1}, \mathfrak{P}_{\Im}]$, $\forall\, 1 \le i \le m$. In this scenario, τ is known as the $(\mathbb{IR})$-integral of $\mathfrak{F}$ on $[\mathfrak{P}_1, \mathfrak{P}_2]$ and is represented by

$$\tau = (\mathbb{IR}) \int_{\mathfrak{P}_1}^{\mathfrak{P}_2} \mathfrak{F}(\mathfrak{P}_a)\, d\mathfrak{P}_a.$$

The pack of all $(\mathbb{IR})$-integral functions of $\mathfrak{F}$ on $[\mathfrak{P}_1, \mathfrak{P}_2]$ can be represented by $\mathbb{IR}_{([\mathfrak{P}_1, \mathfrak{P}_2])}$.

Theorem 1 (see [27]). *Let $\mathfrak{F} : [\mathfrak{P}_1, \mathfrak{P}_2] \to \mathbb{R}_{\Im}$ be an $\mathbb{IVF}$ defined as $\mathfrak{F}(\mathfrak{P}_a) = [\underline{\mathfrak{F}}(\mathfrak{P}_a), \overline{\mathfrak{F}}(\mathfrak{P}_a)]$. $\mathfrak{F} \in \mathbb{IR}_{([\mathfrak{P}_1, \mathfrak{P}_2])}$ iff $\underline{\mathfrak{F}}(\mathfrak{P}_a), \overline{\mathfrak{F}}(\mathfrak{P}_a) \in \mathbb{R}_{([\mathfrak{P}_1, \mathfrak{P}_2])}$ and*

$$(\mathbb{IR}) \int_{\mathfrak{P}_1}^{\mathfrak{P}_2} \mathfrak{F}(\mathfrak{P}_a)\, d\mathfrak{P}_a = \left[(\mathbb{R}) \int_{\mathfrak{P}_1}^{\mathfrak{P}_2} \underline{\mathfrak{F}}(\mathfrak{P}_a)\, d\mathfrak{P}_a, (\mathbb{R}) \int_{\mathfrak{P}_1}^{\mathfrak{P}_2} \overline{\mathfrak{F}}(\mathfrak{P}_a)\, d\mathfrak{P}_a \right],$$

where $\mathbb{R}_{([\mathfrak{P}_1, \mathfrak{P}_2])}$ represent the bunch of all $\mathbb{R}$-integrable functions. If $\mathfrak{F}(\mathfrak{P}_a) \subseteq \mathfrak{G}(\mathfrak{P}_a)$ for all $\mathfrak{P}_a \in [\mathfrak{P}_1, \mathfrak{P}_2]$, then this holds

$$(\mathbb{IR}) \int_{\mathfrak{P}_1}^{\mathfrak{P}_2} \mathfrak{F}(\mathfrak{P}_a)\, d\mathfrak{P}_a \subseteq (\mathbb{IR}) \int_{\mathfrak{P}_1}^{\mathfrak{P}_2} \mathfrak{G}(\mathfrak{P}_a)\, d\mathfrak{P}_a.$$

3. Stochastic Process

Definition 2. *A mapping $\mathfrak{F} : \Lambda \to \mathbb{R}$ on probability space $(\Lambda, \mathbb{A}, \mathbb{P})$ is known as a random variable if they obey the properties of the $\mathbb{A}$-measurable. A function $\mathfrak{F} : \Im \times \Lambda \to \mathbb{R}$ where $\Im \subseteq \mathbb{R}$ is called a stochastic process if $\forall\, \mathfrak{P} \in \Im$, the function $\mathfrak{F}(\mathfrak{P}, \cdot)$, is a random variable.*

Properties of the Stochastic Process

A stochastic process $\mathfrak{F} : \Im \times \Lambda \to \mathbb{R}$ is

- continuous over interval $\Im$ if $\forall\, \mathfrak{P}_o \in \Im$, one has

$$p - \lim_{\mathfrak{P} \to \mathfrak{P}_o} \mathfrak{P}(\mathfrak{P}, .) = \mathfrak{P}(\mathfrak{P}_o, .)$$

  where the probability space limit is represented by $p - \lim$.
- For the continuity in mean square sense over interval $\Im$, if $\forall\, \mathfrak{P}_o \in \Im$, one has

$$\lim_{\mathfrak{P} \to \mathfrak{P}_o} \mathcal{E}\left[(\mathfrak{F}(\mathfrak{P}, .) - \mathfrak{F}(\mathfrak{P}_o, .))^2 \right] = 0,$$

  where $\mathcal{E}[\mathfrak{F}(\mathfrak{P}, \cdot)]$ represent the random variable's expected value.

- For the differentiability in mean square sense at any arbitrary point $\mathfrak{P}$, if there is a random variable $\mathfrak{F}' : \Im \times \Lambda \to \mathbb{R}$, then this is true.

$$\mathfrak{F}'(\mathfrak{P}, \cdot) = p - \lim_{\mathfrak{P} \to \mathfrak{P}_o} \frac{\mathfrak{F}(\mathfrak{P}, \cdot) - \mathfrak{F}(\mathfrak{P}_o, \cdot)}{\mathfrak{P} - \mathfrak{P}_o}.$$

- For the mean-square integral over $\Im$, if $\forall\, \mathfrak{P} \in \Im$, and $\mathcal{E}[\mathfrak{F}(\mathfrak{P}_1, \cdot)] < \infty$. Let $[\mathfrak{P}_1, \mathfrak{P}_2] \subseteq \Im$, $\mathfrak{P}_1 = u_o < u_1 < u_2 \ldots < u_s = \mathfrak{P}_2$ is a partition of $[\mathfrak{P}_1, \mathfrak{P}_2]$. Let $\mathfrak{F}_p \in [u_{p-1}, u_p]$, $\forall\, p = 1, \ldots, s$. A random variable $S : \Lambda \to \mathbb{R}$ is mean-square integrable over $[\mathfrak{P}_1, \mathfrak{P}_2]$, and if this holds true,

$$\lim_{s \to \infty} \mathcal{E}\left[\left(\sum_{p=1}^{s} \mathfrak{F}(\mathfrak{F}_p, .)(u_p - u_{p-1}) - W(.) \right)^2 \right] = 0.$$

  In that case, it would be written as

$$W(\cdot)=\int_{\mathfrak{P}_1}^{\mathfrak{P}_2}\mathfrak{F}(\mathfrak{f},\cdot)d\mathfrak{f} \quad \text{(a.e).}$$

By using the mean-square integral as a definition, we can easily deduce the following for each $\mathfrak{f}\in[\mathfrak{P}_1,\mathfrak{P}_2]$; where the inequality $\mathfrak{F}(\mathfrak{f},\cdot)\leq W(\mathfrak{f},\cdot)$ (a.e) holds, then

$$\int_{\mathfrak{P}_1}^{\mathfrak{P}_2}\mathfrak{F}(\mathfrak{f},\cdot)d\mathfrak{f}\leq\int_{\mathfrak{P}_1}^{\mathfrak{P}_2}W(\mathfrak{f},\cdot)d\mathfrak{f} \quad \text{(a.e.).}$$

Afzal et al. [27] developed the following results using interval calculus for the stochastic process.

Theorem 2 (See [27])**.** *Let $h:[0,1]\rightarrow\mathbb{R}^+$ and $h\neq 0$. A function $\mathfrak{F}:\Im\times\Lambda\rightarrow\mathbb{R}^+_{\Im}$ is h-Godunova–Levin $(\mathbb{G}.\mathbb{L})$ stochastic process for mean square integrable $\mathbb{IVFS}$. For each $\mathfrak{P}_1,\mathfrak{P}_2\in\Im$, if $\mathfrak{F}\in SGPX(h,\Im,\mathbb{R}^+_{\Im})$ and $\mathfrak{F}\in\mathbb{R}^+_{\Im}$. Almost everywhere, the following inclusion is satisfied:*

$$\frac{h\left(\frac{1}{2}\right)}{2}\mathfrak{F}\left(\frac{\mathfrak{P}_1+\mathfrak{P}_2}{2},\cdot\right)\supseteq\frac{1}{\mathfrak{P}_2-\mathfrak{P}_1}\int_{\mathfrak{P}_1}^{\mathfrak{P}_2}\mathfrak{F}(\mathfrak{r},\cdot)d\mathfrak{r}\supseteq[\mathfrak{F}(\mathfrak{P}_1,\cdot)+\mathfrak{F}(\mathfrak{P}_2,\cdot)]\int_0^1\frac{d\mathfrak{r}}{h(\mathfrak{r})}. \quad (1)$$

Theorem 3 (See [27])**.** *Let $g_p\in\mathbb{R}^+$. If h is non-negative and $\mathfrak{F}:\Im\times\Lambda\rightarrow\mathbb{R}$ is a non-negative h–Godunova–Levin stochastic process for $\mathbb{IVFS}$ almost everywhere, the following inclusion is valid:*

$$\mathfrak{F}\left(\frac{1}{G_k}\sum_{p=1}^{k}g_p\mathfrak{r}_p,\cdot\right)\supseteq\sum_{p=1}^{k}\left[\frac{\mathfrak{F}(\mathfrak{r}_p,\cdot)}{h\left(\frac{g_p}{G_k}\right)}\right]. \quad (2)$$

Definition 3 (See [27])**.** *Let $h:[0,1]\rightarrow\mathbb{R}^+$. Then, $\mathfrak{F}:\Im\times\Lambda\rightarrow\mathbb{R}^+$ is known as a h-convex stochastic process, or that $\mathfrak{F}\in SPX(h,\Im,\mathbb{R}^+)$, if $\forall\,\mathfrak{P}_1,\mathfrak{P}_2\in\Im$ and $\mathfrak{r}\in[0,1]$, we have*

$$\mathfrak{F}(\mathfrak{r}\mathfrak{P}_1+(1-\mathfrak{r})\mathfrak{P}_2,\cdot)\leq\ h(\mathfrak{r})\mathfrak{F}(\mathfrak{P}_1,\cdot)+h(1-\mathfrak{r})\mathfrak{F}(\mathfrak{P}_2,\cdot). \quad (3)$$

In (3), if "$\leq$" is reverse, then we call it a h-concave stochastic process or $\mathfrak{F}\in SPV(h,\Im,\mathbb{R}^+)$.

Definition 4 (See [27])**.** *Let $h:(0,1)\rightarrow\mathbb{R}^+$. The stochastic process $\mathfrak{F}=[\underline{\mathfrak{F}},\overline{\mathfrak{F}}]:\Im\times\Lambda\rightarrow\mathbb{R}^+_{\Im}$, where $[\mathfrak{P}_1,\mathfrak{P}_2]\subseteq\Im$ is known as a $(\mathbb{G}.\mathbb{L})$ stochastic process for $\mathbb{IVFS}$ or that $\mathfrak{F}\in SGPX(h,[\mathfrak{P}_1,\mathfrak{P}_2],\mathbb{R}^+_{\Im})$, if $\forall\,\mathfrak{P}_1,\mathfrak{P}_2\in\Im$ and $\mathfrak{r}\in(0,1)$, one has*

$$\mathfrak{F}(\mathfrak{r}\mathfrak{P}_1+(1-\mathfrak{r})\mathfrak{P}_2,\cdot)\supseteq\frac{\mathfrak{F}(\mathfrak{P}_1,\cdot)}{h(\mathfrak{r})}+\frac{\mathfrak{F}(\mathfrak{P}_2,\cdot)}{h(1-\mathfrak{r})}. \quad (4)$$

In (4), if "$\supseteq$" is reverse, then we call it a $(\mathbb{G}.\mathbb{L})$ concave stochastic process for $\mathbb{IVFS}$ or $\mathfrak{F}\in SGPV(h,[\mathfrak{P},\mathfrak{J}_2],\mathbb{R}^+_{\Im})$.

4. Main Results

In light of the literature and previously noted definitions, we are now able to describe a new class of stochastic processes that are convex.

Definition 5. *Let $h:[0,1]\rightarrow\mathbb{R}^+$. Then the stochastic process $\mathfrak{F}=[\underline{\mathfrak{F}},\overline{\mathfrak{F}}]:\Im\times\Lambda\rightarrow\mathbb{R}^+_{\Im}$, where $[\mathfrak{P}_1,\mathfrak{P}_2]\subseteq\Im$ is known as a h-convex stochastic process for $\mathbb{IVFS}$, or that $\mathfrak{F}\in SPX(h,[\mathfrak{P}_1,\mathfrak{P}_2],\mathbb{R}^+_{\Im})$, if $\forall\,\mathfrak{P}_1,\mathfrak{P}_2\in\Im$ and $\mathfrak{a}\in[0,1]$, we have*

$$\mathfrak{F}(\mathfrak{r}\mathfrak{P}_1+(1-\mathfrak{r})\mathfrak{P}_2,\cdot)\supseteq\ h(\mathfrak{r})\mathfrak{F}(\mathfrak{P}_1,\cdot)+h(1-\mathfrak{r})\mathfrak{F}(\mathfrak{P}_2,\cdot). \quad (5)$$

In (4), if "$\supseteq$" is reverse with "$\subseteq$", then we call it a h-concave stochastic process for $\mathbb{IVFS}$ or $\mathfrak{F}\in SPV(h,[\mathfrak{P}_1,\mathfrak{P}_2],\mathbb{R}^+_{\Im})$.

Remark 1.

(i) *If $h = 1$, Definition 5 incorporates the output in the sense of a stochastic process for the P-function.*

(ii) *If $h(\mathfrak{r}) = \frac{1}{h(\mathfrak{r})}$, Definition 5 incorporates the output in the sense of a stochastic process for the $(\mathbb{G}.\mathbb{L})$ function.*

(iii) *If $h(\mathfrak{r}) = \mathfrak{a}$, Definition 5 incorporates the output in the sense of a stochastic process for the usual convex function.*

(iv) *If $h = \mathfrak{a}^s$, Definition 5 incorporates the output in the sense of a stochastic process for the s-convex function.*

4.1. Stochastically Hermite–Hadamard Inclusions

Theorem 4. *Let $h : (0,1) \to \mathbb{R}^+$ and $h \neq 0$. A function $\mathfrak{H} : \Im \times \Omega \to \mathbb{R}^+_{\Im}$ is a h-convex stochastic process as well as mean square integrable for $\mathbb{IVFS}$. For every $\mathfrak{P}_1, \mathfrak{P}_2 \in [\mathfrak{P}_1, \mathfrak{P}_2] \subseteq \Im$, if $\mathfrak{H} \in SPX(h, [\mathfrak{P}_1, \mathfrak{P}_2], \mathbb{R}^+_{\Im})$ and $\mathfrak{H} \in \mathbb{R}^+_{\Im}$. Almost everywhere, the following inequality is satisfied:*

$$\frac{1}{2\left[h\left(\frac{1}{2}\right)\right]}\mathfrak{H}\left(\frac{\mathfrak{P}_1+\mathfrak{P}_2}{2},\cdot\right) \supseteq \frac{1}{\mathfrak{P}_2-\mathfrak{P}_1}\int_{\mathfrak{P}_1}^{\mathfrak{P}_2}\mathfrak{H}(\mathfrak{f},\cdot)d\mathfrak{f} \supseteq [\mathfrak{H}(\mathfrak{P}_1,\cdot)+\mathfrak{H}(\mathfrak{P}_2,\cdot)]\int_0^1 h(\mathfrak{r})d\mathfrak{r}. \tag{6}$$

Proof. Since $\mathfrak{H} \in SPX(h, [\mathfrak{P}_1, \mathfrak{P}_2], \mathbb{R}^+_{\Im})$, and consequently, integrates over $(0,1)$, we have

$$\begin{aligned}
&\frac{1}{\left[h\left(\frac{1}{2}\right)\right]}\mathfrak{H}\left(\frac{\mathfrak{P}_1+\mathfrak{P}_2}{2},\cdot\right) \supseteq \mathfrak{H}(\mathfrak{r}\mathfrak{P}_1+(1-\mathfrak{r})\mathfrak{P}_2,\cdot)+\mathfrak{H}((1-\mathfrak{r})\mathfrak{P}_1+\mathfrak{r}\mathfrak{P}_2,\cdot)\\
&\frac{1}{\left[h\left(\frac{1}{2}\right)\right]}\mathfrak{H}\left(\frac{\mathfrak{P}_1+\mathfrak{P}_2}{2},\cdot\right) \supseteq \left[\int_0^1 \mathfrak{H}(\mathfrak{r}\mathfrak{P}_1+(1-\mathfrak{r})\mathfrak{P}_2,\cdot)d\mathfrak{f}+\int_0^1 \mathfrak{H}((1-\mathfrak{r})\mathfrak{P}_1+\mathfrak{r}\mathfrak{P}_2,\cdot)d\mathfrak{f}\right]\\
&=\left[\int_0^1 \underline{\mathfrak{H}}(\mathfrak{r}\mathfrak{P}_1+(1-\mathfrak{r})\mathfrak{P}_2,\cdot)d\mathfrak{f}+\int_0^1 \underline{\mathfrak{H}}((1-\mathfrak{r})\mathfrak{P}_1+\mathfrak{r}\mathfrak{P}_2,\cdot)d\mathfrak{f},\right.\\
&\left.\int_0^1 \overline{\mathfrak{H}}(\mathfrak{r}\mathfrak{P}_1+(1-\mathfrak{r})\mathfrak{P}_2,\cdot)d\mathfrak{f}+\int_0^1 \overline{\mathfrak{H}}((1-\mathfrak{r})\mathfrak{P}_1+\mathfrak{r}\mathfrak{P}_2,\cdot)d\mathfrak{f}\right]\\
&=\left[\frac{2}{\mathfrak{P}_2-\mathfrak{P}_1}\int_{\mathfrak{P}_1}^{\mathfrak{P}_2}\underline{\mathfrak{H}}(\mathfrak{f},\cdot)d\mathfrak{f},\frac{2}{\mathfrak{P}_2-\mathfrak{P}_1}\int_{\mathfrak{P}_1}^{\mathfrak{P}_2}\overline{\mathfrak{H}}(\mathfrak{f},\cdot)d\mathfrak{f}\right]\\
&=\frac{2}{\mathfrak{P}_2-\mathfrak{P}_1}\int_{\mathfrak{P}_1}^{\mathfrak{P}_2}\mathfrak{H}(\mathfrak{f},\cdot)d\mathfrak{f}.
\end{aligned} \tag{7}$$

By Definition 5, we have

$$\mathfrak{H}(\mathfrak{r}\mathfrak{P}_1+(1-\mathfrak{r})\mathfrak{P}_2,\cdot) \supseteq h(\mathfrak{r})\mathfrak{H}(\mathfrak{P}_1,\cdot)+h(1-\mathfrak{r})\mathfrak{H}(\mathfrak{P}_2,\cdot).$$

Integrating this, we have

$$\int_0^1 \mathfrak{H}(\mathfrak{r}\mathfrak{P}_1+(1-\mathfrak{r})\mathfrak{P}_2,\cdot)d\mathfrak{r} \supseteq \mathfrak{H}(\mathfrak{P}_1,\cdot)\int_0^1 h(\mathfrak{r})d\mathfrak{r}+\mathfrak{H}(\mathfrak{P}_2,\cdot)\int_0^1 h(1-\mathfrak{r})d\mathfrak{r}.$$

Accordingly,

$$\frac{1}{\mathfrak{P}_2-\mathfrak{P}_1}\int_{\mathfrak{P}_1}^{\mathfrak{P}_2}\mathfrak{H}(\mathfrak{f},\cdot)d\mathfrak{f} \supseteq [\mathfrak{H}(\mathfrak{P}_1,\cdot)+\mathfrak{H}(\mathfrak{P}_2,\cdot)]\int_0^1 h(\mathfrak{r})d\mathfrak{r}. \tag{8}$$

Now, utilizing (7) and (8), we have

$$\frac{1}{2\left[h\left(\frac{1}{2}\right)\right]}\mathfrak{H}\left(\frac{\mathfrak{P}_1+\mathfrak{P}_2}{2},\cdot\right) \supseteq \frac{1}{\mathfrak{P}_2-\mathfrak{P}_1}\int_{\mathfrak{P}_1}^{\mathfrak{P}_2}\mathfrak{H}(\mathfrak{f},\cdot)d\mathfrak{f} \supseteq [\mathfrak{H}(\mathfrak{P}_1,\cdot)+\mathfrak{H}(\mathfrak{P}_2,\cdot)]\int_0^1 h(\mathfrak{r})d\mathfrak{r}.$$

□

Example 1. *Consider* $[\mathfrak{P}_1, \mathfrak{P}_2] = [0, 2], h(\mathfrak{r}) = \mathfrak{r}, \forall\ \mathfrak{r} \in [0, 1]$. *If* $\mathfrak{H} : [\mathfrak{P}_1, \mathfrak{P}_2] \to \mathbb{R}_{\mathfrak{I}}{}^{+}$ *is defined as*

$$\mathfrak{H}(\mathfrak{f}, \cdot) = [\mathfrak{f}^2, 10 - e^{\mathfrak{f}}], \quad \mathfrak{f} \in [0, 2].$$

Then,

$$\frac{1}{2\left[h\left(\frac{1}{2}\right)\right]}\mathfrak{H}\left(\frac{\mathfrak{P}_1 + \mathfrak{P}_2}{2}, \cdot\right) = [1, 10 - e],$$

$$\frac{1}{\mathfrak{P}_2 - \mathfrak{P}_1}\int_{\mathfrak{P}_1}^{\mathfrak{P}_2} \mathfrak{H}(\mathfrak{f}, \cdot)d\mathfrak{f} = \left[\frac{4}{3}, \frac{-e^2 + 21}{2}\right],$$

$$[\mathfrak{H}(\mathfrak{P}_1, \cdot) + \mathfrak{H}(\mathfrak{P}_2, \cdot)]\int_0^1 h(\mathfrak{r})d\mathfrak{r} = \left[2, \frac{19 - e^2}{2}\right].$$

As a result,

$$[1, 10 - e] \supseteq \left[\frac{4}{3}, \frac{-e^2 + 21}{2}\right] \supseteq \left[2, \frac{19 - e^2}{2}\right].$$

The theorem is proved.

Theorem 5. *Let* $h : (0, 1) \to \mathbb{R}^+$ *and* $h \neq 0$. *A function* $\mathfrak{H} : \mathfrak{I} \times \Omega \to \mathbb{R}_{\mathfrak{I}}^+$ *is a h-convex stochastic process as well as mean square integrable for* $\mathbb{IVFS}$. *For every* $\mathfrak{P}_1, \mathfrak{P}_2 \in [\mathfrak{P}_1, \mathfrak{P}_2] \subseteq \mathfrak{I}$, *if* $\mathfrak{H} \in SPX(h, [\mathfrak{P}_1, \mathfrak{P}_2], \mathbb{R}_{\mathfrak{I}}^+)$ *and* $\mathfrak{H} \in \mathbb{R}_{\mathfrak{I}}^+$. *Almost everywhere, the following inequality is satisfied:*

$$\frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2}\mathfrak{H}\left(\frac{\mathfrak{P}_1 + \mathfrak{P}_2}{2}, \cdot\right) \supseteq \triangle_1 \supseteq \frac{1}{\mathfrak{P}_2 - \mathfrak{P}_1}\int_{\mathfrak{P}_1}^{\mathfrak{P}_2} \mathfrak{H}(\mathfrak{f}, \cdot)d\mathfrak{f} \supseteq \triangle_2$$

$$\supseteq \left\{[\mathfrak{H}(\mathfrak{P}_1, \cdot) + \mathfrak{H}(\mathfrak{P}_2, \cdot)]\left[\frac{1}{2} + h\left(\frac{1}{2}\right)\right]\right\}\int_0^1 h(\mathfrak{r})d\mathfrak{r},$$

where

$$\triangle_1 = \frac{1}{4h\left(\frac{1}{2}\right)}\left[\mathfrak{H}\left(\frac{3\mathfrak{P}_1 + \mathfrak{P}_2}{4}, \cdot\right) + \mathfrak{H}\left(\frac{3\mathfrak{P}_2 + \mathfrak{P}_1}{4}, \cdot\right)\right],$$

$$\triangle_2 = \left[\mathfrak{H}\left(\frac{\mathfrak{P}_1 + \mathfrak{P}_2}{2}, \cdot\right) + \frac{\mathfrak{H}(\mathfrak{P}_1, \cdot) + \mathfrak{H}(\mathfrak{P}_2, \cdot)}{2}\right]\int_0^1 h(\mathfrak{r})d\mathfrak{r}.$$

Proof. Take $\left[\mathfrak{P}_1, \frac{\mathfrak{P}_1 + \mathfrak{P}_2}{2}\right]$, we have

$$\mathfrak{H}\left(\tfrac{3\mathfrak{P}_1 + \mathfrak{P}_2}{4}, \cdot\right) \supseteq h\left(\tfrac{1}{2}\right)\mathfrak{H}\left(\mathfrak{r}\mathfrak{P}_1 + (1 - \mathfrak{r})\tfrac{\mathfrak{P}_1 + \mathfrak{P}_2}{2}, \cdot\right) + h\left(\tfrac{1}{2}\right)\mathfrak{H}\left((1 - \mathfrak{r})\mathfrak{P}_1 + \mathfrak{I}\tfrac{\mathfrak{P}_1 + \mathfrak{P}_2}{2}, \cdot\right)$$

With integration over (0,1), we have

$$\begin{aligned}
\mathfrak{H}\left(\frac{3\mathfrak{P}_1 + \mathfrak{P}_2}{2}, \cdot\right) &\supseteq h\left(\frac{1}{2}\right)\left[\int_0^1 \mathfrak{H}\left(\mathfrak{r}\mathfrak{P}_1 + (1 - \mathfrak{r})\frac{\mathfrak{P}_1 + \mathfrak{P}_2}{2}, \cdot\right)d\mathfrak{r}\right. \\
&\quad \left. + \int_0^1 \mathfrak{H}\left(\mathfrak{I}\frac{\mathfrak{P}_1 + \mathfrak{P}_2}{2} + (1 - \mathfrak{r})\mathfrak{P}_2, \cdot\right)d\mathfrak{f}\right] \\
&= h\left(\frac{1}{2}\right)\left[\frac{2}{\mathfrak{P}_2 - \mathfrak{P}_1}\int_{\mathfrak{P}_1}^{\frac{\mathfrak{P}_1 + \mathfrak{P}_2}{2}} \mathfrak{H}(\mathfrak{f}, \cdot)d\mathfrak{f} + \frac{2}{\mathfrak{P}_2 - \mathfrak{P}_1}\int_{\mathfrak{P}_1}^{\frac{\mathfrak{P}_1 + \mathfrak{P}_2}{2}} \mathfrak{H}(\mathfrak{f}, \cdot)d\mathfrak{f}\right] \\
&= h\left(\frac{1}{2}\right)\left[\frac{4}{\mathfrak{P}_2 - \mathfrak{P}_1}\int_{\mathfrak{P}_1}^{\frac{\mathfrak{P}_1 + \mathfrak{P}_2}{2}} \mathfrak{H}(\mathfrak{f}, \cdot)d\mathfrak{f}\right]. \qquad (9)
\end{aligned}$$

Accordingly,

$$\frac{1}{4h\left(\frac{1}{2}\right)}\mathfrak{H}\left(\frac{3\mathfrak{P}_1+\mathfrak{P}_2}{2},\cdot\right) \supseteq \frac{1}{\mathfrak{P}_2-\mathfrak{P}_1}\int_{\mathfrak{P}_1}^{\frac{\mathfrak{P}_1+\mathfrak{P}_2}{2}} \mathfrak{H}(\mathfrak{f},\cdot)d\mathfrak{f}. \tag{10}$$

Similarly for interval $\left[\frac{\mathfrak{P}_1+\mathfrak{P}_2}{2},\mathfrak{P}_2\right]$, we have

$$\frac{1}{4h\left(\frac{1}{2}\right)}\mathfrak{H}\left(\frac{3\mathfrak{P}_2+\mathfrak{P}_1}{2},\cdot\right) \supseteq \frac{1}{\mathfrak{P}_2-\mathfrak{P}_1}\int_{\frac{\mathfrak{P}_1+\mathfrak{P}_2}{2}}^{\mathfrak{P}_2} \mathfrak{H}(\mathfrak{f},\cdot)d\mathfrak{f}. \tag{11}$$

Adding inclusions (10) and (11), we get

$$\triangle_1 = \frac{1}{4h\left(\frac{1}{2}\right)}\left[\mathfrak{H}\left(\frac{3\mathfrak{P}_1+\mathfrak{P}_2}{4},\cdot\right)+\mathfrak{H}\left(\frac{3\mathfrak{P}_2+\mathfrak{P}_1}{4},\cdot\right)\right] \supseteq \left[\frac{1}{\mathfrak{P}_2-\mathfrak{P}_1}\int_{\mathfrak{P}_1}^{\mathfrak{P}_2}\mathfrak{H}(\mathfrak{f},\cdot)d\mathfrak{f}\right].$$

Now

$$\begin{aligned}
&\frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2}\mathfrak{H}\left(\frac{\mathfrak{P}_1+\mathfrak{P}_2}{2},\cdot\right)\\
&= \frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2}\mathfrak{H}\left(\frac{1}{2}\left(\frac{3\mathfrak{P}_1+\mathfrak{P}_2}{4},\cdot\right)+\frac{1}{2}\left(\frac{3\mathfrak{P}_2+\mathfrak{P}_1}{4},\cdot\right)\right)\\
&\supseteq \frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2}\left[h\left(\frac{1}{2}\right)\mathfrak{H}\left(\frac{3\mathfrak{P}_1+\mathfrak{P}_2}{4},\cdot\right)+h\left(\frac{1}{2}\right)\mathfrak{H}\left(\frac{3\mathfrak{P}_2+\mathfrak{P}_1}{4},\cdot\right)\right]\\
&= \frac{1}{4h\left(\frac{1}{2}\right)}\left[\mathfrak{H}\left(\frac{3\mathfrak{P}_1+\mathfrak{P}_2}{4},\cdot\right)+\mathfrak{H}\left(\frac{3\mathfrak{P}_2+\mathfrak{P}_1}{4},\cdot\right)\right]\\
&= \triangle_1\\
&\supseteq \frac{1}{4h\left(\frac{1}{2}\right)}\left\{h\left(\frac{1}{2}\right)\left[\mathfrak{H}(\mathfrak{P}_1,\cdot)+\mathfrak{H}\left(\frac{\mathfrak{P}_1+\mathfrak{P}_2}{2},\cdot\right)\right]+h\left(\frac{1}{2}\right)\left[\mathfrak{H}(\mathfrak{P}_2,\cdot)+\mathfrak{H}\left(\frac{\mathfrak{P}_1+\mathfrak{P}_2}{2},\cdot\right)\right]\right\}\\
&= \frac{1}{2}\left[\frac{\mathfrak{H}(\mathfrak{P}_1,\cdot)+\mathfrak{H}(\mathfrak{P}_2,\cdot)}{2}+\mathfrak{H}\left(\frac{\mathfrak{P}_1+\mathfrak{P}_2}{2},\cdot\right)\right]\\
&\supseteq \left[\frac{\mathfrak{H}(\mathfrak{P}_1,\cdot)+\mathfrak{H}(\mathfrak{P}_2,\cdot)}{2}+\mathfrak{H}\left(\frac{\mathfrak{P}_1+\mathfrak{P}_2}{2},\cdot\right)\right]\int_0^1 h(\mathfrak{r})d\mathfrak{r}\\
&= \triangle_2\\
&\supseteq \left[\frac{\mathfrak{H}(\mathfrak{P}_1,\cdot)+\mathfrak{H}(\mathfrak{P}_2,\cdot)}{2}+h\left(\frac{1}{2}\right)\mathfrak{H}(\mathfrak{P}_1,\cdot)+h\left(\frac{1}{2}\right)\mathfrak{H}(\mathfrak{P}_2,\cdot)\right]\int_0^1 h(\mathfrak{r})d\mathfrak{r}\\
&\supseteq \left[\frac{\mathfrak{H}(\mathfrak{P}_1,\cdot)+\mathfrak{H}(\mathfrak{P}_2,\cdot)}{2}+h\left(\frac{1}{2}\right)[\mathfrak{H}(\mathfrak{P}_1,\cdot)+\mathfrak{H}(\mathfrak{P}_2,\cdot)]\right]\int_0^1 h(\mathfrak{r})d\mathfrak{r}\\
&\supseteq \left\{[\mathfrak{H}(\mathfrak{P}_1,\cdot)+\mathfrak{H}(\mathfrak{P}_2,\cdot)]\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\right\}\int_0^1 h(\mathfrak{r})d\mathfrak{r}.
\end{aligned}$$

□

Example 2. *Recall the Example 1, where we have*

$$\frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2}\mathfrak{H}\left(\frac{\mathfrak{P}_1+\mathfrak{P}_2}{2},\cdot\right) = [1, 10-e],$$

$$\triangle_1 = \left[\frac{5}{4}, 10-\frac{\sqrt{e}(1+e)}{2}\right],$$

$$\triangle_2 = \left[\frac{3}{2}, \frac{39}{4}-\frac{e}{2}-\frac{e^2}{4}\right]$$

and

$$\left\{[\mathfrak{H}(\mathfrak{P}_1,\cdot)+\mathfrak{H}(\mathfrak{P}_2,\cdot)]\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\right\}\int_0^1 h(\mathfrak{r})d\mathfrak{r}=\left[2,\frac{19}{2}-\frac{e^2}{2}\right].$$

Thus, we obtain

$$[1,10-e]\supseteq\left[\frac{5}{4},10-\frac{\sqrt{e}(1+e)}{2}\right]\supseteq\left[\frac{4}{3},\frac{21}{2}-\frac{e^2}{2}\right]\supseteq\left[\frac{3}{2},\frac{39}{4}-\frac{e}{2}-\frac{e^2}{4}\right]\supseteq\left[2,\frac{19}{2}-\frac{e^2}{2}\right].$$

This verifies Theorem 5.

Theorem 6. *Let $h_1,h_2:(0,1)\to\mathbb{R}^+$ and $h_1,h_2\neq 0$. Two functions $\mathfrak{H},\mathfrak{C}:\Im\times\Omega\to\mathbb{R}^+_{\Im}$ are mean square integrable h-convex stochastic processes for $\mathbb{IVFS}$. For every $\mathfrak{P}_1,\mathfrak{P}_2\in\Im$, if $\mathfrak{H}\in SPX(h_1,[\mathfrak{P}_1,\mathfrak{P}_2],\mathbb{R}^+_{\Im})$, $\mathfrak{C}\in SPX(h_2,[\mathfrak{P}_1,\mathfrak{P}_2],\mathbb{R}^+_{\Im})$ and $\mathfrak{H},\mathfrak{C}\in\mathbb{IR}_{\Im}$. Almost everywhere, the following inequality is satisfied*

$$\frac{1}{\mathfrak{P}_2-\mathfrak{P}_1}\int_{\mathfrak{P}_1}^{\mathfrak{P}_2}\mathfrak{H}(\mathfrak{f},\cdot)\mathfrak{C}(\mathfrak{f},\cdot)d\mathfrak{f}\supseteq C(\mathfrak{P}_1,\mathfrak{P}_2)\int_0^1 h_1(\mathfrak{r})h_2(\mathfrak{r})d+D(\mathfrak{P}_1,\mathfrak{P}_2)\int_0^1 h_1(\mathfrak{r})h_2(1-\mathfrak{r})d,$$

where

$$C(\mathfrak{P}_1,\mathfrak{P}_2)=\mathfrak{H}(\mathfrak{P}_1,\cdot)\mathfrak{C}(\mathfrak{P}_1,\cdot)+\mathfrak{H}(\mathfrak{P}_2,\cdot)\mathfrak{C}(\mathfrak{P}_2,\cdot),$$
$$D(\mathfrak{P}_1,\mathfrak{P}_2)=\mathfrak{H}(\mathfrak{P}_1,\cdot)\mathfrak{C}(\mathfrak{P}_2,\cdot)+\mathfrak{H}(\mathfrak{P}_2,\cdot)\mathfrak{C}(\mathfrak{P}_1,\cdot).$$

Proof. Consider $\mathfrak{H}\in SPX(h_1,[\mathfrak{P}_1,\mathfrak{P}_2],\mathbb{R}^+_{\Im})$, $\mathfrak{C}\in SPX(h_2,[\mathfrak{P}_1,\mathfrak{P}_2],\mathbb{R}^+_{\Im})$ then, we have

$$\mathfrak{H}(\mathfrak{P}_1\mathfrak{r}+(1-\mathfrak{r})\mathfrak{P}_2,\cdot)\supseteq h_1(\mathfrak{r})\mathfrak{H}(\mathfrak{P}_1,\cdot)+h_1(1-\mathfrak{r})\mathfrak{H}(\mathfrak{P}_2,\cdot),$$

$$\mathfrak{C}(\mathfrak{P}_1\mathfrak{r}+(1-\mathfrak{r})\mathfrak{P}_2,\cdot)\supseteq h_2(\mathfrak{r})\mathfrak{C}(\mathfrak{P}_1,\cdot)+h_2(1-\mathfrak{r})\mathfrak{C}(\mathfrak{P}_2,\cdot).$$

Then,

$$\begin{aligned}&\mathfrak{H}(\mathfrak{P}_1\mathfrak{r}+(1-\mathfrak{r})\mathfrak{P}_2,\cdot)\mathfrak{C}(\mathfrak{P}_1\mathfrak{r}+(1-\mathfrak{r})\mathfrak{P}_2,\cdot)\\&\supseteq(h(1-\mathfrak{r})\mathfrak{H}(\mathfrak{P}_1,\cdot)+h(\mathfrak{r})\mathfrak{H}(\mathfrak{P}_2,\cdot))(h(1-\mathfrak{r})\mathfrak{C}(\mathfrak{P}_1,\cdot)+h(\mathfrak{r})\mathfrak{C}(\mathfrak{P}_2,\cdot)).\end{aligned}$$

With integration over (0,1), we have

$$\begin{aligned}&\int_0^1\mathfrak{H}(\mathfrak{P}_1\mathfrak{r}+(1-\mathfrak{r})\mathfrak{P}_2,\cdot)\mathfrak{C}(\mathfrak{P}_1\mathfrak{r}+(1-\mathfrak{r})\mathfrak{P}_2,\cdot)d\Im\\&=\left[\int_0^1\underline{\mathfrak{H}}(\mathfrak{P}_1\mathfrak{r}+(1-\mathfrak{r})\mathfrak{P}_2,\cdot)\underline{\mathfrak{C}}(\mathfrak{P}_1\mathfrak{r}+(1-\mathfrak{r})\mathfrak{P}_2,\cdot)d\mathfrak{f},\right.\\&\quad\left.\int_0^1\overline{\mathfrak{H}}(\mathfrak{P}_1\mathfrak{r}+(1-\mathfrak{r})\mathfrak{P}_2,\cdot)\overline{\mathfrak{C}}(\mathfrak{P}_1\mathfrak{r}+(1-\mathfrak{r})\mathfrak{P}_2,\cdot)d\mathfrak{f}\right]\\&=\left[\frac{1}{\mathfrak{P}_2-\mathfrak{P}_1}\int_{\mathfrak{P}_1}^{\mathfrak{P}_2}\underline{\mathfrak{H}}(\mathfrak{f},\cdot)\underline{\mathfrak{C}}(\mathfrak{f},\cdot)d\mathfrak{f},\frac{1}{\mathfrak{P}_2-\mathfrak{P}_1}\int_{\mathfrak{P}_1}^{\mathfrak{P}_2}\overline{\mathfrak{H}}(\mathfrak{f},\cdot)\overline{\mathfrak{C}}(\mathfrak{f},\cdot)d\mathfrak{f}\right]\\&=\frac{1}{\mathfrak{P}_2-\mathfrak{P}_1}\int_{\mathfrak{P}_1}^{\mathfrak{P}_2}\mathfrak{H}(\mathfrak{f},\cdot)\mathfrak{C}(\mathfrak{f},\cdot)d\mathfrak{f}\\&\supseteq C(\mathfrak{P}_1,\mathfrak{P}_2)\int_0^1 h_1(\mathfrak{r})h_2(\mathfrak{r})d+D(\mathfrak{P}_1,\mathfrak{P}_2)\int_0^1 h_1(\mathfrak{r})h_2(1-\mathfrak{r})d\mathfrak{r}.\end{aligned}$$

It follows that

$$\frac{1}{\mathfrak{P}_2-\mathfrak{P}_1}\int_{\mathfrak{P}_1}^{\mathfrak{P}_2}\mathfrak{H}(\mathfrak{f},\cdot)\mathfrak{C}(\mathfrak{f},\cdot)d\mathfrak{f}\supseteq C(\mathfrak{P}_1,\mathfrak{P}_2)\int_0^1 h_1(\mathfrak{r})h_2(\mathfrak{r})d+D(\mathfrak{P}_1,\mathfrak{P}_2)\int_0^1 h_1(\mathfrak{r})h_2(1-\mathfrak{r})d\mathfrak{r}.$$

The theorem is proved. □

Example 3. *Let* $[\mathfrak{P}_1, \mathfrak{P}_2] = [0,1], h_1(\mathfrak{r}) = \mathfrak{r}, h_2(\mathfrak{r}) = 1 \, \forall \, \mathfrak{r} \in (0,1)$. *If* $\mathfrak{H}, \mathfrak{C} : [\mathfrak{P}_1, \mathfrak{P}_2] \subseteq \Im \to \mathbb{R}_{\Im}{}^{+}$ *are defined as*

$$\mathfrak{H}(\mathfrak{f}, \cdot) = [\mathfrak{f}^2, 8 - e^{\mathfrak{f}}] \text{ and } \mathfrak{C}(\mathfrak{f}, \cdot) = [\mathfrak{f}, 7 - \mathfrak{f}^2].$$

Then, we have

$$\frac{1}{\mathfrak{P}_2 - \mathfrak{P}_1} \int_{\mathfrak{P}_1}^{\mathfrak{P}_2} \mathfrak{H}(\mathfrak{f}, \cdot)\mathfrak{C}(\mathfrak{f}, \cdot) d\mathfrak{f} = \left[\frac{1}{4}, -6e + \frac{175}{3}\right],$$
$$C(\mathfrak{P}_1, \mathfrak{P}_2) \int_0^1 h_1(\mathfrak{r}) h_2(\mathfrak{r}) d\mathfrak{r} = \left[\frac{1}{2}, \frac{17 - 2e}{2}\right]$$

and

$$D(\mathfrak{P}_1, \mathfrak{P}_2) \int_0^1 h_1(\mathfrak{r}) h_2(1-\mathfrak{r}) d\mathfrak{r} = \left[0, \frac{18 - 3e}{4}\right].$$

Since

$$\left[\frac{1}{4}, -6e + \frac{175}{3}\right] \supseteq \left[\frac{1}{2}, \frac{32 - 7e}{4}\right],$$

consequently, Theorem 6 is verified.

Theorem 7. *Let* $h_1, h_2 : (0,1) \to \mathbb{R}^+$ *and* $h_1, h_2 \neq 0$. *Two functions* $\mathfrak{H}, \mathfrak{C} : \Im \times \Omega \to \mathbb{R}^+_{\Im}$ *are mean square integrable h-convex stochastic processes for* $\mathbb{IVFS}$. *For each* $\mathfrak{P}_1, \mathfrak{P}_2 \in \Im$, *if* $\mathfrak{H} \in SPX(h_1, [\mathfrak{P}_1, \mathfrak{P}_2], \mathbb{R}^+_{\Im})$, $\mathfrak{C} \in SPX(h_2, [\mathfrak{P}_1, \mathfrak{P}_2], \mathbb{R}^+_{\Im})$ *and* $\mathfrak{H}, \mathfrak{C} \in \mathbb{IR}_{\Im}$ *with* $h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) = \lambda$. *Almost everywhere, the following inequality is satisfied:*

$$\begin{aligned}
&\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} \mathfrak{H}\left(\frac{\mathfrak{P}_1 + \mathfrak{P}_2}{2}, \cdot\right)\mathfrak{C}\left(\frac{\mathfrak{P}_1 + \mathfrak{P}_2}{2}, \cdot\right) \\
&\supseteq \frac{1}{\mathfrak{P}_2 - \mathfrak{P}_1} \int_{\mathfrak{P}_1}^{\mathfrak{P}_2} \mathfrak{H}(\mathfrak{f}, \cdot)\mathfrak{C}(\mathfrak{f}, \cdot) d\mathfrak{f} \\
&+ C(\mathfrak{P}_1, \mathfrak{P}_2) \int_0^1 h_1(\mathfrak{r}) h_2(1-\mathfrak{r}) d\mathfrak{r} + D(\mathfrak{P}_1, \mathfrak{P}_2) \int_0^1 h_1(\mathfrak{r}) h_2(\mathfrak{r}) d\mathfrak{r}.
\end{aligned}$$

Proof. Since $\mathfrak{H} \in SPX(h_1, [\mathfrak{P}_1, \mathfrak{P}_2], \mathbb{R}^+_{\Im})$, $\mathfrak{C} \in SPX(h_2, [\mathfrak{P}_1, \mathfrak{P}_2], \mathbb{R}^+_{\Im})$, we have

$$\begin{aligned}
\mathfrak{H}\left(\frac{\mathfrak{P}_1 + \mathfrak{P}_2}{2}, \cdot\right) &\supseteq h_1\left(\frac{1}{2}\right)\mathfrak{H}(\mathfrak{P}_1\mathfrak{r} + (1-\mathfrak{r})\mathfrak{P}_2, \cdot) + h_1\left(\frac{1}{2}\right)\mathfrak{H}(\mathfrak{P}_1(1-\mathfrak{r}) + \mathfrak{r}\mathfrak{P}_2, \cdot), \\
\mathfrak{C}\left(\frac{\mathfrak{P}_1 + \mathfrak{P}_2}{2}, \cdot\right) &\supseteq h_2\left(\frac{1}{2}\right)\mathfrak{C}(\mathfrak{P}_1\mathfrak{r} + (1-\mathfrak{r})\mathfrak{P}_2, \cdot) + h_2\left(\frac{1}{2}\right)\mathfrak{C}(\mathfrak{P}_1(1-\mathfrak{r}) + \mathfrak{r}\mathfrak{P}_2, \cdot). \qquad (12)
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{H}\left(\frac{\mathfrak{P}_1 + \mathfrak{P}_2}{2}, \cdot\right)\mathfrak{C}\left(\frac{\mathfrak{P}_1 + \mathfrak{P}_2}{2}, \cdot\right) \\
&\supseteq \lambda[\mathfrak{H}(\mathfrak{P}_1\mathfrak{r} + (1-\mathfrak{r})\mathfrak{P}_2, \cdot)\mathfrak{C}(\mathfrak{P}_1\mathfrak{r} + (1-\mathfrak{r})\mathfrak{P}_2, \cdot) + \mathfrak{H}(\mathfrak{P}_1(1-\mathfrak{r}) + \mathfrak{r}\mathfrak{P}_2, \cdot)\mathfrak{C}(\mathfrak{P}_1(1-\mathfrak{r}) + \mathfrak{r}\mathfrak{P}_2, \cdot)] \\
&+ \lambda[\mathfrak{H}(\mathfrak{P}_1\mathfrak{r} + (1-\mathfrak{r})\mathfrak{P}_2, \cdot)\mathfrak{C}(\mathfrak{P}_1(1-\mathfrak{r}) + \mathfrak{r}\mathfrak{P}_2, \cdot) + \mathfrak{H}(\mathfrak{P}_1(1-\mathfrak{r}) + \mathfrak{r}\mathfrak{P}_2, \cdot)\mathfrak{C}(\mathfrak{P}_1\mathfrak{r} + (1-\mathfrak{r})\mathfrak{P}_2, \cdot)] \\
&\supseteq \lambda[\mathfrak{H}(\mathfrak{P}_1\mathfrak{r} + (1-\mathfrak{r})\mathfrak{P}_2, \cdot)\mathfrak{C}(\mathfrak{P}_1\mathfrak{r} + (1-\mathfrak{r})\mathfrak{P}_2, \cdot) + \mathfrak{H}(\mathfrak{P}_1(1-\mathfrak{r}) + \mathfrak{r}\mathfrak{P}_2, \cdot)\mathfrak{C}(\mathfrak{P}_1(1-\mathfrak{r}) + \mathfrak{r}\mathfrak{P}_2, \cdot)] \\
&+ \lambda[(h_1(\mathfrak{r})\mathfrak{H}(\mathfrak{P}_1, \cdot) + h_1(1-\mathfrak{r})\mathfrak{H}(\mathfrak{P}_2, \cdot))(h_2(1-\mathfrak{r})\mathfrak{C}(\mathfrak{P}_1, \cdot) + h_2(\mathfrak{r})\mathfrak{C}(\mathfrak{P}_2, \cdot))] \\
&+ [(h_1(1-\mathfrak{r})\mathfrak{H}(\mathfrak{P}_1, \cdot) + h_1(\mathfrak{r})\mathfrak{H}(\mathfrak{P}_2, \cdot))(h_2(\mathfrak{r})\mathfrak{C}(\mathfrak{P}_1, \cdot) + h_2(1-\mathfrak{r})\mathfrak{C}(\mathfrak{P}_2, \cdot))] \\
&\supseteq \lambda[\mathfrak{H}(\mathfrak{P}_1\mathfrak{r} + (1-\mathfrak{r})\mathfrak{P}_2, \cdot)\mathfrak{C}(\mathfrak{P}_1\mathfrak{r} + (1-\mathfrak{r})\mathfrak{P}_2, \cdot) + \mathfrak{H}(f(1-\mathfrak{r}) + \mathfrak{r}\mathfrak{P}_2, \cdot)\mathfrak{C}(f(1-\mathfrak{r}) + \mathfrak{r}\mathfrak{P}_2, \cdot)] \\
&+ \lambda[(h_1(\mathfrak{r})h_2(1-\mathfrak{r}) + h_1(1-\mathfrak{r})h_2(\mathfrak{r}))C(\mathfrak{P}_1, \mathfrak{P}_2) + (h_1(\mathfrak{r})h_2(\mathfrak{r}) + h_1(1-\mathfrak{r})h_2(1-\mathfrak{r}))D(\mathfrak{P}_1, \mathfrak{P}_2)].
\end{aligned}$$

Integration over $(0,1)$, we have

$$\int_0^1 \mathfrak{H}\left(\frac{\mathfrak{P}_1+\mathfrak{P}_2}{2},\cdot\right)\mathfrak{C}\left(\frac{\mathfrak{P}_1+\mathfrak{P}_2}{2},\cdot\right)d\mathfrak{f} = \left[\int_0^1 \underline{\mathfrak{H}}\left(\frac{\mathfrak{P}_1+\mathfrak{P}_2}{2},\cdot\right)\underline{\mathfrak{C}}\left(\frac{\mathfrak{P}_1+\mathfrak{P}_2}{2},\cdot\right)d\mathfrak{f},\right.$$
$$\left.\int_0^1 \overline{\mathfrak{H}}\left(\frac{\mathfrak{P}_1+\mathfrak{P}_2}{2},\cdot\right)\overline{\mathfrak{C}}\left(\frac{\mathfrak{P}_1+\mathfrak{P}_2}{2},\cdot\right)d\mathfrak{f}\right]$$
$$= \mathfrak{H}\left(\frac{\mathfrak{P}_1+\mathfrak{P}_2}{2},\cdot\right)\mathfrak{C}\left(\frac{\mathfrak{P}_1+\mathfrak{P}_2}{2},\cdot\right)d\mathfrak{f} \supseteq 2\lambda\left[\frac{1}{\mathfrak{P}_2-\mathfrak{P}_1}\int_{\mathfrak{P}_1}^{\mathfrak{P}_2}\mathfrak{H}(\mathfrak{f},\cdot)\mathfrak{C}(\mathfrak{f},\cdot)d\mathfrak{f}\right]$$
$$+ 2\lambda\left[C(\mathfrak{P}_1,\mathfrak{P}_2)\int_0^1 h_1(\mathfrak{r})h_2(1-\mathfrak{r})d + D(\mathfrak{P}_1,\mathfrak{P}_2)\int_0^1 h_1(\mathfrak{r})h_2(\mathfrak{r})d\mathfrak{f}\right].$$

By dividing $\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}$, we obtain the desired result:

$$\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\mathfrak{H}\left(\frac{\mathfrak{P}_1+\mathfrak{P}_2}{2},\cdot\right)\mathfrak{C}\left(\frac{\mathfrak{P}_1+\mathfrak{P}_2}{2},\cdot\right)$$
$$\supseteq \frac{1}{\mathfrak{P}_2-\mathfrak{P}_1}\int_{\mathfrak{P}_1}^{\mathfrak{P}_2}\mathfrak{H}(\mathfrak{f},\cdot)\mathfrak{C}(\mathfrak{f},\cdot)d\mathfrak{f} + C(\mathfrak{P}_1,\mathfrak{P}_2)\int_0^1 h_1(\mathfrak{r})h_2(1-\mathfrak{r})d\mathfrak{r}$$
$$+ D(\mathfrak{P}_1,\mathfrak{P}_2)\int_0^1 h_1(\mathfrak{r})h_2(\mathfrak{r})d\mathfrak{r}.$$

Hence, it is proved.
□

Example 4. *Let* $[\mathfrak{P}_1,\mathfrak{P}_2] = [0,1], h_1(\mathfrak{r}) = \mathfrak{r}, h_2(\mathfrak{r}) = 2\ \forall\, \mathfrak{r} \in (0,1)$. *If* $\mathfrak{H},\mathfrak{C} : [\mathfrak{P}_1,\mathfrak{P}_2] \subseteq \mathfrak{I} \to \mathbb{R}_{\mathfrak{I}}{}^{+}$ *are defined as*

$$\mathfrak{H}(\mathfrak{f},\cdot) = [\mathfrak{f}^2, 8-e^{\mathfrak{f}}] \text{ and } \mathfrak{C}(\mathfrak{f},\cdot) = [\mathfrak{f}, 7-\mathfrak{f}^2].$$

Then, we have

$$\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\mathfrak{H}\left(\frac{\mathfrak{P}_1+\mathfrak{P}_2}{2},\cdot\right)\mathfrak{C}\left(\frac{\mathfrak{P}_1+\mathfrak{P}_2}{2},\cdot\right) = \left[\frac{1}{4}, \frac{27\left(-\sqrt{e}+8\right)}{2}\right],$$
$$\frac{1}{\mathfrak{P}_2-\mathfrak{P}_1}\int_{\mathfrak{P}_1}^{\mathfrak{P}_2}\mathfrak{H}(\mathfrak{f},\cdot)\mathfrak{C}(\mathfrak{f},\cdot)d\mathfrak{f} = \left[\frac{1}{4}, -6e+\frac{175}{3}\right],$$
$$C(\mathfrak{P}_1,\mathfrak{P}_2)\int_0^1 h_1(\mathfrak{r})h_2(1-\mathfrak{r})d\mathfrak{r} = \left[\frac{1}{4}, \frac{17-2e}{4}\right]$$

and

$$D(\mathfrak{P}_1,\mathfrak{P}_2)\int_0^1 h_1(\mathfrak{r})h_2(\mathfrak{r})d\mathfrak{r} = \left[0, \frac{18-3e}{8}\right].$$

It follows that

$$\left[\frac{1}{4}, \frac{27\left(-\sqrt{e}+8\right)}{2}\right] \supseteq \left[\frac{1}{2}, \frac{13}{2} - \frac{5(33e-280)}{24}\right].$$

This proves the above theorem.

4.2. Stochastically Ostrowski-Type Inclusions

We can accomplish our goal using the following lemma [33].

Lemma 1. *Let $\mathfrak{H} : \Im \times \Omega \subseteq \mathbb{R} \to \mathbb{R}$ be a stochastic process that can be mean-square differentiated on $\Im^o$ such that $\mathfrak{H}' \in L[\mathfrak{P}_1, \mathfrak{P}_2]$. Then the equality stated below is true:*

$$\mathfrak{H}(\mathfrak{Z}, \cdot) - \frac{1}{\mathfrak{P}_2 - \mathfrak{P}_1} \int_{\mathfrak{P}_1}^{\mathfrak{P}_2} \mathfrak{H}(\mathfrak{T}, \cdot) d\mathfrak{T}$$
$$= \frac{(\mathfrak{Z} - \mathfrak{P}_1)^2}{\mathfrak{P}_2 - \mathfrak{P}_1} \int_0^1 \mathfrak{r}\mathfrak{H}'(\mathfrak{Z}\mathfrak{r} + (1-\mathfrak{r})\mathfrak{P}_1, \cdot) d\mathfrak{r} -$$
$$\frac{(\mathfrak{P}_2 - \mathfrak{Z})^2}{\mathfrak{P}_2 - \mathfrak{P}_1} \int_0^1 \mathfrak{r}\mathfrak{H}'(\mathfrak{Z}\mathfrak{r} + (1-\mathfrak{r})\mathfrak{P}_2, \cdot) d\mathfrak{r}, \forall \mathfrak{Z} \in [\mathfrak{P}_1, \mathfrak{P}_2].$$

Theorem 8. *Let $h : (0,1) \to \mathbb{R}$ be a super-multiplicative, non-negative function, and consider a differentiable map $\mathfrak{H} : \Im \times \Omega \subseteq \mathbb{R} \to \mathbb{R}_{\Im}$ on $\Im^o$ such that $\mathfrak{H}' \in L[\mathfrak{P}_1, \mathfrak{P}_2]$ and $\mathfrak{r} \le h(\mathfrak{r})$. If $|\mathfrak{H}'|$ is a h-convex stochastic process on $\Im$, with $|\mathfrak{H}'(\mathfrak{Z}, \cdot)| \le \Delta$ for each $\mathfrak{Z} \in [\mathfrak{P}_1, \mathfrak{P}_2]$, then*

$$H\bigg(\big[\underline{\mathfrak{H}}(\mathfrak{Z}, \cdot), \overline{\mathfrak{H}}(\mathfrak{Z}, \cdot)\big], \bigg[\frac{1}{\mathfrak{P}_2 - \mathfrak{P}_1} \int_{\mathfrak{P}_1}^{\mathfrak{P}_2} \underline{\mathfrak{H}}(\mathfrak{T}, \cdot) d\mathfrak{T}, \frac{1}{\mathfrak{P}_2 - \mathfrak{P}_1} \int_{\mathfrak{P}_1}^{\mathfrak{P}_2} \overline{\mathfrak{H}}(\mathfrak{T}, \cdot) d\mathfrak{T}\bigg]\bigg)$$
$$= \max\bigg\{ \bigg|\underline{\mathfrak{H}}(\mathfrak{Z}, \cdot) - \frac{1}{\mathfrak{P}_2 - \mathfrak{P}_1} \int_{\mathfrak{P}_1}^{\mathfrak{P}_2} \underline{\mathfrak{H}}(\mathfrak{T}, \cdot) d\mathfrak{T}\bigg|, \bigg|\overline{\mathfrak{H}}(\mathfrak{Z}, \cdot) - \frac{1}{\mathfrak{P}_2 - \mathfrak{P}_1} \int_{\mathfrak{P}_1}^{\mathfrak{P}_2} \overline{\mathfrak{H}}(\mathfrak{T}, \cdot) d\mathfrak{T}\bigg|\bigg\}$$
$$\supseteq \frac{\Delta\big[(\mathfrak{Z} - \mathfrak{P}_1)^2 + (\mathfrak{P}_2 - \mathfrak{Z})^2\big]}{\mathfrak{P}_2 - \mathfrak{P}_1} \int_0^1 \Big[h(\mathfrak{r}^2) + h(\mathfrak{r} - \mathfrak{r}^2)\Big] d\mathfrak{r}$$

$\forall\ \Im \in [\mathfrak{P}_1, \mathfrak{P}_2]$.

Proof. From the above Lemma 1, one has $|\mathfrak{H}'|$ a h-convex stochastic process for $\mathbb{IVFS}$, and then

$$\max\bigg\{ \bigg|\underline{\mathfrak{H}}(\mathfrak{Z}, \cdot) - \frac{1}{\mathfrak{P}_2 - \mathfrak{P}_1} \int_{\mathfrak{P}_1}^{\mathfrak{P}_2} \underline{\mathfrak{H}}(\mathfrak{T}, \cdot) d\mathfrak{T}\bigg|, \bigg|\overline{\mathfrak{H}}(\mathfrak{Z}, \cdot) - \frac{1}{\mathfrak{P}_2 - \mathfrak{P}_1} \int_{\mathfrak{P}_1}^{\mathfrak{P}_2} \overline{\mathfrak{H}}(\mathfrak{T}, \cdot) d\mathfrak{T}\bigg|\bigg\}$$
$$\supseteq \frac{(\mathfrak{Z} - \mathfrak{P}_1)^2}{\mathfrak{P}_2 - \mathfrak{P}_1} \int_0^1 \mathfrak{r}\big|\mathfrak{H}'(\mathfrak{Z}\mathfrak{r} + (1-\mathfrak{r})\mathfrak{P}_1, \cdot)\big| d\mathfrak{r} + \frac{(\mathfrak{P}_2 - \mathfrak{Z})^2}{\mathfrak{P}_2 - \mathfrak{P}_1} \int_0^1 \mathfrak{r}\big|\mathfrak{H}'(\mathfrak{Z}\mathfrak{r} + (1-\mathfrak{r})\mathfrak{P}_2, \cdot)\big| d\mathfrak{r}$$
$$\supseteq \frac{(\mathfrak{Z} - \mathfrak{P}_1)^2}{\mathfrak{P}_2 - \mathfrak{P}_1} \int_0^1 \mathfrak{r}\big[h(\mathfrak{r})|\mathfrak{H}'(\mathfrak{Z}, \cdot)| + h(1-\mathfrak{r})|\mathfrak{H}'(\mathfrak{P}_1, \cdot), \cdot|\big] d\mathfrak{r}$$
$$+ \frac{(\mathfrak{P}_2 - \mathfrak{Z})^2}{\mathfrak{P}_2 - \mathfrak{P}_1} \int_0^1 \mathfrak{r}[h(\mathfrak{r})|\mathfrak{H}'(\mathfrak{Z}, \cdot)| + h(1-\mathfrak{r})|\mathfrak{H}'(\mathfrak{P}_2, \cdot), \cdot|] d\mathfrak{r}$$
$$\supseteq \frac{\Delta(\mathfrak{Z} - \mathfrak{P}_1)^2}{\mathfrak{P}_2 - \mathfrak{P}_1} \int_0^1 \Big[h^2(\mathfrak{r}) + h(\mathfrak{r})h(1-\mathfrak{r})\Big] d\mathfrak{r} + \frac{(\mathfrak{P}_2 - \mathfrak{Z})^2}{\mathfrak{P}_2 - \mathfrak{P}_1} \int_0^1 \Big[h^2(\mathfrak{r}) + h(\mathfrak{r})h(1-\mathfrak{r})\Big] d\mathfrak{r}$$
$$\supseteq \frac{\Delta\big[(\mathfrak{Z} - \mathfrak{P}_1)^2 + (\mathfrak{P}_2 - \mathfrak{Z})^2\big]}{\mathfrak{P}_2 - \mathfrak{P}_1} \int_0^1 \Big[h^2(\mathfrak{r}) + h(\mathfrak{r})h(1-\mathfrak{r})\Big] d\mathfrak{r}.$$

Hence, it is proved. □

4.3. Stochastically Jensen-Type Inclusion

Theorem 9. *Let $e_j \in \mathbb{R}^+$. If h is a non-negative function, then $\mathfrak{H} : \Im \times \Omega \to \mathbb{R}_{\Im}$ is a non-negative h-convex stochastic process for $\mathbb{IVFS}$ with $g_j \in I$. Almost everywhere, the following inclusion is satisfied:*

$$\mathfrak{H}\left(\frac{1}{E_p} \sum_{j=1}^{p} e_j g_j, \cdot \right) \supseteq \sum_{j=1}^{p} \left[h\left(\frac{e_j}{E_p}\right) \mathfrak{H}(g_j, \cdot) \right], \tag{13}$$

where

$$E_p = \sum_{j=1}^{p} e_j.$$

Proof. By induction, if $p = 2$, then Equation (13) is true. Assume that inclusion (13) also holds for $p - 1$; then,

$$\begin{aligned}
\mathfrak{H}\left(\frac{1}{E_p}\sum_{j=1}^{p} e_j g_j, \cdot\right) &= \mathfrak{H}\left(\frac{e_p}{E_p} g_p + \sum_{j=1}^{p-1} \frac{e_j}{E_p} g_j, \cdot\right) \\
&= \mathfrak{H}\left(\frac{e_p}{E_p} g_p + \frac{E_{p-1}}{E_p}\sum_{j=1}^{p-1} \frac{e_j}{E_{p-1}} g_j, \cdot\right) \\
&\supseteq h\left(\frac{e_p}{E_p}\right)\mathfrak{H}(g_p, \cdot) + h\left(\frac{E_{p-1}}{E_p}\right)\mathfrak{H}\left(\sum_{j=1}^{p-1} \frac{e_j}{E_{p-1}} g_j, \cdot\right) \\
&\supseteq h\left(\frac{e_p}{E_p}\right)\mathfrak{H}(g_p, \cdot) + h\left(\frac{E_{p-1}}{E_p}\right)\sum_{j=1}^{p-1}\left[h\left(\frac{e_j}{E_{p-1}}\right)\mathfrak{H}(g_j, \cdot)\right] \\
&\supseteq h\left(\frac{e_p}{E_p}\right)\mathfrak{H}(g_p, \cdot) + \sum_{j=1}^{p-1}\left[h\left(\frac{e_j}{E_p}\right)\mathfrak{H}(g_j, \cdot)\right] \\
&\supseteq \sum_{j=1}^{p}\left[h\left(\frac{e_j}{E_p}\right)\mathfrak{H}(g_j, \cdot)\right].
\end{aligned}$$

Hence, it is proved. □

5. Conclusions

As a result of its numerous potential benefits, convex analysis is currently a very attractive and captivating field of research. Today's mathematical investigations rely heavily upon the concept of convexity, along with the perception of inequalities. The results of the *h*-convex stochastic process were extended from partial-order stochastic process to interval-order stochastic process using set inclusion. This allowed us to construct the Hermite–Hadamard, Ostrowski, and Jensen inequalities. Furthermore, some examples that are not trivial were provided to support our main findings. Our study's findings can be used in various contexts to produce a range of both novel and well-known results. In this paper, additional improvements and refinements to previously published findings were presented. Further exploration will focus on the fuzzy interval Katugampola integral operator, Riemann–Liouville, as well as other fractional integral operators. In addition, stochastic processes with variational and integral inequalities play an important role in a wide range of disciplines, including symmetric stochastic Markov processes, stochastic integrals, as well as finding a solution and assessing differential equation stability using symmetry analysis methods, which are very powerful tools for finding exact solutions. It will also be interesting to develop these inequalities using quantum calculus, a recently emerging field in various disciplines. Furthermore, we hope that these inequalities will play an important role in the development of various stochastic models. Eventually, we hope to be able to use the concept of this study in many different modes, such as time-scale calculus, coordinates, interval analysis, fuzzy fractional, fractional calculus, quantum calculus, and so forth. The current developments and style of this paper should pique readers' interest and encourage more research in this field.

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