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Studies on Special Polynomials Involving Degenerate Appell Polynomials and Fractional Derivative

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Abstract: The focus of the research presented in this paper is on a new generalized family of degenerate three-variable Hermite–Appell polynomials defined here using a fractional derivative. The research was motivated by the investigations on the degenerate three-variable Hermite-based Appell polynomials introduced by R. Alyosuf. We show in the paper that, for certain values, the well-known degenerate Hermite–Appell polynomials, three-variable Hermite–Appell polynomials and Appell polynomials are seen as particular cases for this new family. As new results of the investigation, the operational rule for this new generalized family is introduced and the explicit summation formula is established. Furthermore, using the determinant formulation of the Appell polynomials, the determinant form for the new generalized family is obtained and the recurrence relations are also determined considering the generating expression of the polynomials contained in the new generalized family. Certain applications of the generalized three-variable Hermite–Appell polynomials are also presented showing the connection with the equivalent results for the degenerate Hermite–Bernoulli and Hermite–Euler polynomials with three variables.

Keywords: Hermite polynomials; Appell polynomials; three-variable Hermite-based Appell polynomials; fractional derivative; integral transforms; operational rule

MSC: 26A33; 33B10; 33C45

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1. Introduction and Preliminaries

Fractional calculus, a branch of mathematical analysis, examines the possibility of using the differentiation operators of real or complex number powers. Theoretical studies successfully employ fractional calculus operators, which are also applicable in a variety of science and engineering domains. A comprehensive overview of the theory and applications of the fractional-calculus operators can be seen in recent review papers [1,2].

A powerful method for dealing with fractional derivatives is the combination of integral transforms and special polynomials; see, for instance, [3].

For $\min\{\mathrm{Re}(\nu), \mathrm{Re}(b)\} > 0$, the integral of the form [4] (p. 218),

$$\int_0^\infty e^{-bt} t^{\nu-1} dt = \Gamma(\nu)\, b^{-\nu}, \tag{1}$$

is called Euler's integral of the second kind. Consequently, the following consequences are obtained in [3]:

$$\Gamma(\nu)\left(\alpha-\frac{\partial}{\partial u}\right)^{-\nu}f(u)=\int_0^\infty e^{-\alpha t}t^{\nu-1}\,e^{t\frac{\partial}{\partial u}}f(u)dt=\int_0^\infty e^{-\alpha t}t^{\nu-1}\,f(u+t)dt, \tag{2}$$

and

$$\Gamma(\nu)\left(\alpha-\frac{\partial^2}{\partial u^2}\right)^{-\nu}f(u)=\int_0^\infty e^{-\alpha t}t^{\nu-1}\,e^{t\frac{\partial^2}{\partial u^2}}f(u)dt. \tag{3}$$

Particularly in recent years, a number of generalizations of special functions in mathematical physics have seen a significant evolution. Many mathematical physics issues can be solved analytically thanks to the recent developments in special functions theory, which have various wide-range applications. Multi-variable and multi-index special functions represent a substantial improvement in the theory of generalized special functions. Both in the realm of pure mathematics and in real-world applications, special functions have been recognized for their importance. To address the problems appearing in the theory of abstract algebra and partial differential equations, the necessity for multi-variable and multi-index special functions is acknowledged. In physics, the Hermite polynomials are used to produce the quantum harmonic oscillator's eigenstates and to solve the Schrodinger equation for the harmonic oscillator. They are also employed as Gaussian quadrature in numerical analysis and the notion of multiple-index, multiple-variate Hermite polynomials were given by Hermite in [5]. Degenerate q-Hermite polynomials are defined by means of generating function in [6], and significant properties have been determined.

Recently, additional extensions of special polynomials have been built on the foundation of Euler's integral. When establishing operational definitions and generating relations for the generalized and innovative forms of special polynomials in [3], Dattoli em et al. employed Euler's integral. Thus, using (1), a generalization of a number of special polynomials including hybrid special polynomials was introduced by several authors. Extended Laguerre–Appell polynomials are considered for research in [7]. A new class of q-Sheffer–Appell polynomials was introduced and studied in [8] and certain positive linear operators together with the Sheffer–Appell polynomial sequences were investigated in [9]. Fractional calculus aspects were connected to special polynomials involving Appell sequences in the study presented in [10]. Complex Appell polynomials and their degenerate-type polynomials were studied in [11] and it iwas shown that the results can be applied to complex Bernoulli polynomials and complex Euler polynomials. Further studies involve Gould–Hopper-based Frobenius–Genocchi polynomials, Lagrange–Hermite polynomials [12] and generalized Legendre–Laguerre–Appell polynomials are investigated through fractional calculus.

In a recent study, R. Alyosuf [13] introduced degenerate three-variable Hermite-based Appell polynomials (D-3VHAP) listed by the generating relation:

$$\sum_{m=0}^{\infty}{}_{\mathcal{J}}R_m(u,v,w;\chi)\frac{t^m}{m!}=\mathcal{Y}(t,u,v,w;\chi)=R(t)(1+\chi)^{\frac{ut}{\chi}}(1+\chi)^{\frac{vt^2}{\chi}}(1+\chi)^{\frac{wt^3}{\chi}}, \tag{4}$$

which possess the series definition:

$$\sum_{k=0}^{m}\binom{m}{k}\,\mathcal{J}_{m-k}(u,v,w;\chi)\,R_k={}_{\mathcal{J}}R_m(u,v,w;\chi), \tag{5}$$

and are represented by operational rule:

$$\exp\left(\frac{v\,\chi}{log(1+\chi)}\,D_u^2+w\Big(\frac{\chi}{log(1+\chi)}\Big)^2\,D_u^3\right)\{R_m(u)\}={}_{\mathcal{J}}R_m(u,v,w;\chi), \tag{6}$$

where, $R_m(u)$ are Appell polynomials [14]given by generating relation:

$$\sum_{k=0}^{\infty} R_k(u)\frac{t^k}{k!} = R(t)\exp(ut), \tag{7}$$

with $R(t)$ being the convergent power series given by:

$$\sum_{k=0}^{\infty} R_k\frac{t^k}{k!} = R(t), \qquad R_0 \neq 0. \tag{8}$$

Fractional operators offer a more accurate representation of complex systems that cannot be modeled using integer-order derivatives. Hence, they have significant applications in various fields, including numerous branches of mathematics, physics [15], engineering [16], and finance [17]. For example, the behavior of viscoelastic materials, biological systems and electrical networks can be described using fractional operators [18]. Additionally, fractional operators have applications in electromagnetics, where those operators are used to describe the behavior of electromagnetic waves in media with fractional-order dielectric and magnetic properties [19]. Other applications of fractional calculus can be seen in [20].

The work of Datolli and colleagues [3] and that of R. Alyusof [13] served as a source of inspiration and motivation for the investigation reported in this paper due to the tremendous relevance of fractional operators. The generalized form of a convoluted degenerate hybrid special polynomial family is constructed here by using the fractional operator called Eulers' integral given by (1). A generalized degenerate Hermite-based Appell polynomial family denoted by ${}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)$ is introduced using the generating expression:

$$\frac{R(z)(1+\chi)^{\frac{uz}{\chi}}}{\left[\beta - \frac{(vz^2+wz^3)}{\chi}\, log(1+\chi)\right]^{\nu}} = \sum_{m=0}^{\infty} {}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)\frac{z^m}{m!}. \tag{9}$$

These hybrid special polynomials could be useful in image processing and computer vision to enhance image quality and extract features. Further, they have applications in financial mathematics, where they model the behavior of stock prices, interest rates, and other financial variables.

The focus of the present article is to present the study on the features of the generalized forms of the hybrid degenerate special polynomials connected to the Hermite polynomials through the extensive use of integral transforms and operational principles. The main contributions of the paper are contained in Sections 2 and 3, after a comprehensive introduction where all the necessary previously known results are listed. The novelty starts in Section 2, where fractional derivatives are used to introduce a generalized version of degenerate three-variable Hermite–Appell polynomials. These polynomials are further investigated and for them, summation formula, determinant form and recurrence relations are also deduced. Section 3 includes several applications of the new results involving generalized degenerate three-variable Hermite-Appell polynomials as well as equivalent results for the degenerate Hermite–Bernoulli and Hermite–Euler polynomials with three variables.

2. Generalized Forms of Mixed Special Polynomials

We first establish the following result before introducing the generalized version of the degenerate three-variable Hermite–Appell polynomials:

Theorem 1. *For the generalized degenerate three-variable Hermite–Appell polynomials* ${}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)$*, the following operational rule holds true:*

$$\left(\beta - \left(v\frac{\chi}{log(1+\chi)}\, D_u^2 + w\left(\frac{\chi}{log(1+\chi)}\right)^2 D_u^3\right)\right)^{-\nu} R_m(u) = {}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta). \tag{10}$$

Proof. Substituting b with $\alpha - \left(v\frac{\chi}{log(1+\chi)}\ D_u^2 + w\left(\frac{\chi}{log(1+\chi)}\right)^2\ D_u^3\right)$ in integral (1) and the resulting equation on $R_m(u)$, we discover

$$\left(\alpha - \left(v\frac{\chi}{log(1+\chi)}\ D_u^2 + w\left(\frac{\chi}{log(1+\chi)}\right)^2\ D_u^3\right)\right)^{-\nu} R_m(u)$$
$$= \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t} t^{\nu-1} \exp\left(vt\frac{\chi}{log(1+\chi)}\ D_u^2 + wt\left(\frac{\chi}{log(1+\chi)}\right)^2\ D_u^3\right) R_m(u)dt, \quad (11)$$

which in view of Equation (6) gives

$$\left(\alpha - \left(v\frac{\chi}{log(1+\chi)}\ D_u^2 + w\left(\frac{\chi}{log(1+\chi)}\right)^2\ D_u^3\right)\right)^{-\nu} R_m(u) =$$
$$\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t} t^{\nu-1}\, {}_{\mathcal{J}}R_m(u, vt, wt; \chi)dt. \quad (12)$$

A new family of polynomials is defined by the transform on the right-hand side of Equation (12). Using the symbol ${}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)$ to identify this unique family of polynomials, we may create the generalized degenerate three-variable Hermite Appell polynomials (D3VHAP) given by expression

$${}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta) = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t} t^{\nu-1}\, {}_{\mathcal{J}}R_m(u, vt, wt; \chi)dt. \quad (13)$$

In view of Equations (12) and (13), assertion (10) follows. □

Next, we prove the following result, which will be applied to construct the generating function of the generalized D3VHAP ${}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)$:

Theorem 2. *For the generalized D3VHAP ${}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)$, the following generating expression holds true:*

$$\frac{R(z)(1+\chi)^{\frac{uz}{\chi}}}{\left[\beta - \frac{(vz^2+wz^3)}{\chi}\ log(1+\chi)\right]^\nu} = \sum_{m=0}^{\infty} {}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)\frac{z^m}{m!} \quad (14)$$

Proof. When we multiply both sides of expression (13) by $\frac{z^m}{m!}$ and summing over m adding the results, we obtain

$$\sum_{m=0}^{\infty} {}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)\frac{z^m}{m!} = \sum_{m=0}^{\infty}\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\beta t} t^{\nu-1}\, {}_{\mathcal{J}}R_m(u,vt,wt;\chi)\frac{z^m}{m!}dt. \quad (15)$$

Using Equation (4) with t replaced by z in the right-hand side of Equation (15), it follows that

$$\sum_{m=0}^{\infty} {}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)\frac{z^m}{m!} = \sum_{m=0}^{\infty}\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\beta t} t^{\nu-1} R(z)(1+\chi)^{\frac{uz+vz^2t+wz^3t}{\chi}}dt, \quad (16)$$

which in view of expression (1) yields assertion (14). □

Corollary 1. *For $R(z) = 1$, the generalized D3VHAP ${}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)$ reduces to the degenerate three-variable Hermite polynomials $\mathcal{J}_{m,\nu}(u,v,w;\chi,\beta)$, therefore the corresponding operational rule and generating function for these polynomials are given by the expressions:*

$$\left(\beta - \left(v\frac{\chi}{log(1+\chi)}\ D_u^2 + w\left(\frac{\chi}{log(1+\chi)}\right)^2\ D_u^3\right)\right)^{-\nu} u^m = \mathcal{J}_{m,\nu}(u,v,w;\chi,\beta) \quad (17)$$

and

$$\frac{(1+\chi)^{\frac{uz}{\chi}}}{\left[\beta - \frac{(vz^2+wz^3)}{\chi}\, log(1+\chi)\right]^{\nu}} = \sum_{m=0}^{\infty} \mathcal{J}_{m,\nu}(u,v,w;\chi,\beta)\frac{z^m}{m!}, \tag{18}$$

respectively.

Remark 1. *For $\beta = \nu = 1$, the generalized D3VHAP ${}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)$ reduces to the degenerate Hermite–Appell polynomials ${}_{\mathcal{J}}R_m(u,v,w;\chi)$ [13].*

Remark 2. *For $\alpha = \nu = 1$ and $\chi \to 0$, the generalized D3VHAP ${}_{\mathcal{J}}R_m(u,v,w;\chi)$ becomes the 3VHAP [21].*

Remark 3. *For $\alpha = \nu = 1$, $v = w = 0$ and $\chi \to 0$, the generalized D3VHAP ${}_{\mathcal{J}}R_m(u,v,w;\chi)$ reduces to the Appell polynomials [14].*

The next step is to prove the explicit summation formula for the generalized D3VHAP ${}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)$:

Theorem 3. *For, ${}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)$ i-e, the generalized D3VHAP, the below listed explicit summation formula in terms of the generalized D3VHP $\mathcal{J}_{m,\nu}(u,v,w;\chi,\beta)$ holds true:*

$${}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta) = \sum_{r=0}^{m}\binom{m}{r} R_r\, \mathcal{J}_{m-r,\nu}(u,v,w;\chi,\beta) \tag{19}$$

Proof. By inserting Equations (18) and (8) into the left-hand side of the expression (14), assertion (19) is obtained. □

Corollary 2. *The determinant formulation listed in [22] (p. 1533) of the Appell polynomials is used to obtain the determinant form of the generalized D3VHAP:*

$$R_0(u) = \frac{1}{\gamma_0}, \quad \gamma_0 = \frac{1}{R_0}, \tag{20}$$

$$R_m(u) = \frac{(-1)^m}{(\gamma_0)^{n+1}}\begin{vmatrix} 1 & u & u^2 & \cdots & u^{m-1} & u^m \\ \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{m-1} & \gamma_m \\ 0 & \gamma_0 & \binom{2}{1}\gamma_1 & \cdots & \binom{m-1}{1}\gamma_{m-2} & \binom{m}{1}\gamma_{m-1} \\ 0 & 0 & \gamma_0 & \cdots & \binom{m-1}{2}\gamma_{m-3} & \binom{m}{2}\gamma_{m-2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \gamma_0 & \binom{m}{m-1}\gamma_1 \end{vmatrix}, \tag{21}$$

$$\gamma_m = -\frac{1}{R_0}\left(\sum_{k=1}^{m}\binom{m}{k} R_k\, \gamma_{m-k}\right), m = 1,2,3,\cdots,$$

where $\gamma_0,\ \gamma_1, \cdots, \gamma_m \in \mathbb{R},\ \gamma_0 \neq 0.$

Theorem 4. *For the generalized D3VHAP ${}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)$, the following determinant form holds true:*

$${}_{\mathcal{J}}R_{0,\nu}(u,v,w;\chi,\beta) = \frac{1}{\gamma_0}\,\mathcal{J}_{m,\nu}(u,v,w;\chi,\beta), \qquad \gamma_0 = \frac{1}{R_0}, \tag{22}$$

$$
{}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)
$$

$$
= \frac{(-1)^m}{(\gamma_0)^{m+1}} \begin{vmatrix} \mathcal{J}_{0,\nu}(u,v,w;\chi,\beta) & \mathcal{J}_{1,\nu}(u,v,w;\chi,\beta) & \cdots & \mathcal{J}_{m-1,\nu}(u,v,w;\chi,\beta) & \mathcal{J}_{m,\nu}(u,v,w;\chi,\beta) \\ \gamma_0 & \gamma_1 & \cdots & \gamma_{m-1} & \beta_n \\ 0 & \gamma_0 & \cdots & \binom{m-1}{1}\gamma_{m-2} & \binom{m}{1}\gamma_{m-1} \\ 0 & 0 & \cdots & \binom{m-1}{2}\gamma_{m-3} & \binom{m}{2}\gamma_{m-2} \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & \gamma_0 & \binom{m}{m-1}\gamma_1 \end{vmatrix}, \tag{23}
$$

$$
\gamma_m = -\frac{1}{R_0}\Big(\sum_{k=1}^{m}\binom{m}{k}R_k\gamma_{m-k}\Big), \qquad m = 1,2,3,\cdots,
$$

where $\gamma_0,\ \gamma_1,\cdots,\gamma_m \in \mathbb{R},\ \gamma_0 \neq 0$ *and* $\mathcal{J}_{m,\nu}(u,v,w;\chi,\beta)$ $(m = 0,1,\cdots)$ *are the generalized D3VHP defined by Equation* (18).

Proof. Taking $m = 0$ in Equation (19) and then using Equation (17) in the resultant equation, it follows that:

$$
{}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta) = \frac{1}{\gamma_0}\ \mathcal{J}_{0,\nu}(u,v,w;\chi,\beta),\ \ \gamma_0 = \frac{1}{R_0}. \tag{24}
$$

Expansion of the determinant in Equation (20) with respect to the first row gives

$$
\begin{aligned}
R_m(u) = \frac{(-1)^m}{(\gamma_0)^{m+1}} &\begin{vmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_{m-1} & \gamma_m \\ \gamma_0 & \binom{2}{1}\gamma_1 & \cdots & \binom{m-1}{1}\gamma_{m-2} & \binom{m}{1}\gamma_{m-1} \\ 0 & \gamma_0 & \cdots & \binom{m-1}{2}\gamma_{m-3} & \binom{m}{2}\gamma_{m-2} \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & \gamma_0 & \binom{m}{m-1}\gamma_1 \end{vmatrix} \\
-\frac{(-1)^m u}{(\gamma_0)^{m+1}} &\begin{vmatrix} \gamma_0 & \gamma_2 & \cdots & \gamma_{m-1} & \gamma_m \\ 0 & \binom{2}{1}\gamma_1 & \cdots & \binom{m-1}{1}\gamma_{m-2} & \binom{m}{1}\gamma_{m-1} \\ 0 & \gamma_0 & \cdots & \binom{m-1}{2}\gamma_{m-3} & \binom{m}{2}\gamma_{m-2} \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & \gamma_0 & \binom{m}{m-1}\gamma_1 \end{vmatrix} \\
+\frac{(-1)^m u^2}{(\gamma_0)^{m+1}} &\begin{vmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{m-1} & \gamma_m \\ 0 & \gamma_0 & \cdots & \binom{m-1}{1}\gamma_{m-2} & \binom{m}{1}\gamma_{m-1} \\ 0 & 0 & \cdots & \binom{m-1}{2}\gamma_{m-3} & \binom{m}{2}\gamma_{m-2} \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & \gamma_0 & \binom{m}{m-1}\gamma_1 \end{vmatrix}
\end{aligned} \tag{25}
$$

$$+\cdots+\frac{(-1)^{2m-1}u^{m-1}}{(\gamma_0)^{m+1}}\begin{vmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_m \\ 0 & \gamma_0 & \binom{2}{1}\gamma_1 & \cdots & \binom{m}{1}\gamma_{m-1} \\ 0 & 0 & \gamma_0 & \cdots & \binom{m}{2}\gamma_{m-2} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & \binom{m}{m-1}\gamma_1 \end{vmatrix}$$

$$+\frac{u^m}{(\gamma_0)^{m+1}}\begin{vmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{m-1} \\ 0 & \gamma_0 & \binom{2}{1}\gamma_1 & \cdots & \binom{m-1}{1}\gamma_{m-2} \\ 0 & 0 & \gamma_0 & \cdots & \binom{m-1}{2}\gamma_{m-3} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & \gamma_0 \end{vmatrix}. \tag{26}$$

Since each minor in Equation (26) is independent of u, operating $\left(\beta - \left(v\frac{\chi}{log(1+\chi)}\, D_u^2 + w\left(\frac{\chi}{log(1+\chi)}\right)^2 D_u^3\right)\right)^{-\nu}$ on both sides of Equation (26) and then using Equations (10) and (17), we find

$$_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta) = \frac{(-1)^m \mathcal{J}_{0,\nu}(u,v,w;\chi,\beta)}{(\gamma_0)^{m+1}}\begin{vmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_{m-1} & \gamma_m \\ \gamma_0 & \binom{2}{1}\gamma_1 & \cdots & \binom{m-1}{1}\gamma_{m-2} & \binom{m}{1}\gamma_{m-1} \\ 0 & \gamma_0 & \cdots & \binom{m-1}{2}\gamma_{m-3} & \binom{m}{2}\gamma_{m-2} \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & \gamma_0 & \binom{m}{m-1}\gamma_1 \end{vmatrix}$$

$$-\frac{(-1)^m \mathcal{J}_{1,\nu}(u,v,w;\chi,\beta)}{(\gamma_0)^{m+1}}\begin{vmatrix} \gamma_0 & \gamma_2 & \cdots & \gamma_{m-1} & \gamma_m \\ 0 & \binom{2}{1}\gamma_1 & \cdots & \binom{m-1}{1}\gamma_{m-2} & \binom{m}{1}\gamma_{m-1} \\ 0 & \gamma_0 & \cdots & \binom{m-1}{2}\gamma_{m-3} & \binom{m}{2}\gamma_{m-2} \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & \gamma_0 & \binom{m}{m-1}\gamma_1 \end{vmatrix}$$

$$+\frac{(-1)^m \mathcal{J}_{2,\nu}(u,v,w;\chi,\beta)}{(\gamma_0)^{m+1}}\begin{vmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{m-1} & \gamma_m \\ 0 & \gamma_0 & \cdots & \binom{m-1}{1}\gamma_{m-2} & \binom{m}{1}\gamma_{m-1} \\ 0 & 0 & \cdots & \binom{m-1}{2}\gamma_{m-3} & \binom{m}{2}\gamma_{m-2} \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & \gamma_0 & \binom{m}{m-1}\gamma_1 \end{vmatrix} + \cdots$$

$$+\frac{(-1)^{2m-1}\mathcal{J}_{m-1,\nu}(u,v,w;\chi,\beta)}{(\gamma_0)^{m+1}}\begin{vmatrix} \beta_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_m \\ 0 & \gamma_0 & \binom{2}{1}\gamma_1 & \cdots & \binom{m}{1}\gamma_{m-1} \\ 0 & 0 & \gamma_0 & \cdots & \binom{m}{2}\gamma_{m-2} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & \binom{m}{m-1}\gamma_1 \end{vmatrix}$$

$$+\frac{\mathcal{J}_{m,\nu}(u,v,w;\chi,\beta)}{(\gamma_0)^{m+1}}\begin{vmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{m-1} \\ 0 & \gamma_0 & \binom{2}{1}\gamma_1 & \cdots & \binom{m-1}{1}\gamma_{m-2} \\ 0 & 0 & \gamma_0 & \cdots & \binom{m-1}{2}\gamma_{m-3} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & \binom{m}{m-1}\gamma_1 \end{vmatrix}. \tag{27}$$

Combining the components in Equation (27), the right-hand side leads to the theorem's proof (12). □

Next, we derive the recurrence relations of the generalized D3VHAP ${}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)$ by considering their generating expression. A recurrence relation is an equation that iteratively creates a sequence or multidimensional array of values after one or more initial terms are given. The definition of each subsequent term in the series or array depends on the preceding terms. The listed recurrence relations of the generalized D3VHAP ${}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)$ are discovered by differentiating generating function (14) with respect to u, v, w, and β:

$$\begin{aligned} \frac{\chi}{log(1+\chi)}\frac{\partial}{\partial u}\Big({}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)\Big) &= m\ {}_{\mathcal{J}}R_{m-1,\nu}(u,v,w;\chi,\beta), \\ \frac{\chi}{log(1+\chi)}\frac{\partial}{\partial v}\Big({}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)\Big) &= \nu\ m(m-1)\ {}_{\mathcal{J}}R_{m-2,\nu+1}(u,v,w;\chi,\beta), \\ \frac{\chi}{log(1+\chi)}\frac{\partial}{\partial w}\Big({}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)\Big) &= \nu\ m(m-1)(m-2)\ {}_{\mathcal{J}}R_{m-3,\nu+1}(u,v,w;\chi,\beta), \\ \frac{\partial}{\partial \beta}\Big({}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)\Big) &= -\nu\ {}_{\mathcal{J}}R_{m,\nu+1}(u,v,w;\chi,\beta). \end{aligned}$$

Given the aforementioned relationships, we have

$$\frac{\chi}{log(1+\chi)}\frac{\partial}{\partial v}\Big({}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)\Big) = -\frac{\partial^3}{\partial u^2\partial\beta}\ {}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta),$$

$$\left(\frac{n}{log(1+\chi)}\right)^2\frac{\partial}{\partial w}\Big({}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)\Big) = -\frac{\partial^4}{\partial u^3\partial\beta}\ {}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta).$$

3. Applications

A variety of members of the Appell polynomial family can be obtained depending on the proper choice for the function $\mathcal{R}(t)$. Several applications in number theory, combinatorics, numerical analysis, and other areas of practical mathematics make use of these polynomials and numbers of Bernoulli, Euler, and Genocchi. The Taylor expansion, the trigonometric and hyperbolic tangent and cotangent functions, and the sums of powers of natural numbers are only a few examples of mathematical formulas where the Bernoulli numbers can be found. In close proximity to the trigonometric and hyperbolic secant function origins, the Euler numbers enter the Taylor expansion. In graph theory, automata theory, and calculating the number of up–down ascending sequences, the Genocchi numbers are useful.

Thus, for suitable selection of $R(z)$ in (14), the following generating expressions for degenerate 3VH-Bernoulli, Euler and Genocchi polynomials hold:

$$\frac{\frac{z}{e^z-1}(1+\chi)^{\frac{uz}{\chi}}}{\left[\beta-\frac{(vz^2+wz^3)}{\chi}\,log(1+\chi)\right]^{\nu}}=\sum_{m=0}^{\infty}{}_{\mathcal{J}}\mathfrak{B}_{m,\nu}(u,v,w;\chi,\beta)\frac{z^m}{m!},$$

$$\frac{\frac{z}{e^z+1}(1+\chi)^{\frac{uz}{\chi}}}{\left[\beta-\frac{(vz^2+wz^3)}{\chi}\,log(1+\chi)\right]^{\nu}}=\sum_{m=0}^{\infty}{}_{\mathcal{J}}\mathfrak{E}_{m,\nu}(u,v,w;\chi,\beta)\frac{z^m}{m!}$$

and

$$\frac{\frac{2z}{e^z+1}(1+\chi)^{\frac{uz}{\chi}}}{\left[\beta-\frac{(vz^2+wz^3)}{\chi}\,log(1+\chi)\right]^{\nu}}=\sum_{m=0}^{\infty}{}_{\mathcal{J}}G_{m,\nu}(u,v,w;\chi,\beta)\frac{z^m}{m!}.$$

The generalized D3VH-Bernoulli polynomials ${}_{\mathcal{J}}\mathfrak{B}_{m,\nu}(u,v,w;\chi,\beta)$ and the generalized D3VH-Euler polynomials ${}_{\mathcal{J}}\mathfrak{E}_{m,\nu}(u,v,w;\chi,\beta)$ in view of (10) are defined using the following operational rules:

$$\left(\beta-\left(v\frac{\chi}{log(1+\chi)}\,D_u^2+w\left(\frac{\chi}{log(1+\chi)}\right)^2\,D_u^3\right)\right)^{-\nu}\mathfrak{B}_m(u)={}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)$$

and

$$\left(\beta-\left(v\frac{\chi}{log(1+\chi)}\,D_u^2+w\left(\frac{\chi}{log(1+\chi)}\right)^2\,D_u^3\right)\right)^{-\nu}\mathfrak{E}_m(u)={}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta),$$

respectively.

Appell polynomials are involved in various identities. The operational formalism outlined in the preceding section can be used to acquire the appropriate identification using the generalized Hermite–Appell polynomials. To do this, we take the following course of action:

The operator $\left(\beta-\left(v\frac{\chi}{log(1+\chi)}\,D_u^2+w\left(\frac{\chi}{log(1+\chi)}\right)^2\,D_u^3\right)\right)^{-\nu}$, referred to as $(\mathcal{O})$, is applied on both sides of a given relation.

We have the four applications listed below.

1. Consider first the following connections involving Bernoulli polynomials [23] (pp. 29–30):

$$\mathfrak{B}_m(u+1)-\mathfrak{B}_m(u)=m\,u^{m-1},\quad m=0,1,2,\dots$$

$$\sum_{k=0}^{m-1}\binom{m}{k}\mathfrak{B}_k(u)=mu^{m-1},\quad m=2,3,4,\dots$$

$$\mathfrak{B}_m(ku)=k^{m-1}\sum_{k=0}^{m-1}\mathfrak{B}_m\left(u+\frac{k}{m}\right),\quad m=0,1.2,\dots;\quad k=1,2,3,\dots$$

The identities that contain the generalized D3VH-Bernoulli polynomials ${}_{\mathcal{J}}\mathfrak{B}_{m,\nu}(u,v,w;\chi,\beta)$ are obtained by applying the operator $(\mathcal{O})$ to earlier expressions and taking into account operational rules (14) and (17) on the resulting expressions. They are listed as follows:

$$ {}_{\mathcal{J}}\mathfrak{B}_{m,\nu}(u+1,v,w;\chi,\beta) - {}_{\mathcal{J}}\mathfrak{B}_{m,\nu}(u,v,w;\chi,\beta) = m\ {}_{\mathcal{J}}\mathfrak{B}_{m-1,\nu}(u,v,w;\chi,\beta), \quad m = 0,1,2..., $$

$$ \sum_{k=0}^{m-1} \binom{m}{k} {}_{\mathcal{J}}\mathfrak{B}_{m,\nu}(u,v,w;\chi,\beta) = m\ {}_{\mathcal{J}}\mathfrak{B}_{m-1,\nu}(u,v,w;\chi,\beta), \quad m = 2,3,4..., $$

$$ {}_{\mathcal{J}}\mathfrak{B}_{m,\nu}(ku,k^2v,k^3w;\chi,\beta) = k^{m-1} \sum_{k=0}^{m-1} {}_{\mathcal{J}}\mathfrak{B}_{m-1,\nu}(u+k/m,v,w;\chi,\beta), \quad m = 0,1,2,...; \ k = 1,2,\cdots. $$

2. We now use the the following relationships involving Euler polynomials [23] (pp. 29–30):

$$ \mathfrak{E}_m(u+1) + \mathfrak{E}_m(u) = 2u^m. $$

$$ \mathfrak{E}_m(kx) = k^m \sum_{k=0}^{m-1} (-1)^k \mathfrak{E}_m\left(u + \frac{k}{m}\right) \quad m = 0,1,2...; \ k \ odd, $$

The following identities involving the generalized D3VH-Euler polynomials ${}_{\mathcal{J}}\mathfrak{E}_{m,\nu}(u,v,w;\chi,\beta)$ are obtained:

$$ {}_{\mathcal{J}}\mathfrak{E}_{m,\nu}(u+1,v,w;\chi,\beta) + {}_{\mathcal{J}}\mathfrak{E}_{m,\nu}(u,v,w;\chi,\beta) = 2\ \mathcal{J}_{m,\nu}(u,v,w;\chi,\beta). $$

$$ {}_{\mathcal{J}}\mathfrak{E}_{m,\nu}(ku,k^2v,k^3w;\chi,\beta) = k^m \sum_{k=0}^{m-1} (-1)^k {}_{\mathcal{J}}\mathfrak{E}_{m,\nu}(u+k/m,v,w;\chi,\beta), \quad m = 0,1.2,...; \ k \ odd. $$

3. Next, we review the relationships between Bernoulli and Euler polynomials [23] (pp. 29–30), which are listed below:

$$ \mathfrak{B}_m(u) = 2^{-m} \sum_{k=0}^{m} \binom{m}{k} \mathfrak{B}_{m-k} \mathfrak{E}_k(2u), \quad m = 0,1,2..., $$

$$ \mathfrak{E}_m(u) = \frac{2^{m+1}}{m+1}\left[\mathfrak{B}_{m+1}\left(\frac{u+1}{2}\right) - \mathfrak{B}_{m+1}\left(\frac{u}{2}\right)\right], \quad m = 0,1,2..., $$

$$ \mathfrak{E}_m(ku) = -\frac{2k^m}{m+1} \sum_{k=0}^{m-1} (-1)^k \mathfrak{B}_{m+1}\left(\frac{u+k}{m}\right), \quad m = 0,1,2...; k \ even. $$

When we apply the operator $(\mathcal{O})$ to the prior listed equations, we obtain:

$$ {}_{\mathcal{J}}\mathfrak{B}_{m,\nu}(u,v,w;\chi,\beta) = 2^{-m} \sum_{k=0}^{m} \binom{m}{k} \mathfrak{B}_{m-k} {}_{\mathcal{J}}\mathfrak{E}_{m,\nu}(2u,4v,8w;\chi,\beta), \quad m = 0,1,2..., $$

$$ {}_{\mathcal{J}}\mathfrak{E}_{m,\nu}(u,v,w;\chi,\beta) = \frac{2^{m+1}}{m+1}\left[{}_{\mathcal{J}}R_{m+1,\nu}(\frac{u+1}{2},\frac{v}{4},\frac{w}{8};\chi,\beta) - {}_{\mathcal{J}}\mathfrak{B}_{m+1,\nu}(\frac{u}{2},\frac{v}{4},\frac{w}{8};\chi,\beta)\right], \ m = 0,1,2,... $$

$$ {}_{\mathcal{J}}\mathfrak{E}_{m,\nu}(ku,k^2v,k^3w;\chi,\beta) = -\frac{2k^m}{m+1} \sum_{k=0}^{m-1} (-1)^k {}_{\mathcal{J}}\mathfrak{B}_{m+1,\nu}(\frac{u+k}{m},v,w;\chi,\beta), \quad m = 0,1.2,\cdots; \ k \ \ even. $$

4. Further, the determinant definition of the generalized D3VH-Bernoulli polynomials ${}_{\mathcal{J}}\mathfrak{B}_{m,\nu}(u,v,w;\chi,\beta)$ is derived by assuming $\gamma_0 = 1$ and $\gamma_i = \frac{1}{i+1}$ $(i = 1,2,\cdots,n)$ in (22) and (23) and the determinant formulation of the generalized D3VH-Euler polynomials $\mathcal{J}Em,\nu(u,v,w;\chi,\beta)$ is derived by taking $\gamma 0 = 1$ and $\gamma i = \frac{1}{2}$ $(i = 1,2,\cdots,n)$ in expressions (22) and (23).

The examples above show how the operational connection between the Appell and generalized D3VHAP polynomials may be used to find solutions for the generalized D3VHAP polynomials.

4. Conclusions

Inspired by the study conducted in [13], where three-variable degenerate Hermite-based Appell polynomials have been introduced and studied, the new generalized family of degenerate three-variable Hermite–Appell polynomials ${}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)$ is introduced in Section 2 of this paper. For these polynomials ${}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)$, Theorem 1 provides the operational rule. Theorem 2 gives the generating expression for the function ${}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)$ and the connection of this family to the degenerate Hermite–Appell polynomials, three-variable Hermite–Appell polynomials and Appell polynomials. The explicit summation formula for polynomials ${}_{\mathcal{J}}R_{m,\nu}(u,v,w;\chi,\beta)$ is proved in Theorem 3 and the determinant form for the generalized family is obtained in Theorem 4. The recurrence relations of the generalized three-variable degenerate Hermite-based Appell polynomials are also derived. In Section 3, certain applications of the results obtained in Section 2 are presented giving the equivalent results for the degenerate Hermite–Bernoulli and Hermite–Euler polynomials with three variables. These generalized degenerate hybrid special polynomials associated with Hermite polynomials have a wide range of applications in mathematics and physics. These polynomials may arise naturally in the study of quantum mechanics, in probability theory, where these polynomials may be related to the normal distribution, which is one of the most important distributions in probability theory. In approximation theory, these polynomials can be used as a basis for approximating functions and serve as a powerful tool for numerical analysis. Further, in statistical mechanics, Hermite polynomials are used to calculate the partition function and thermodynamic properties of ideal gases and can be used in Fourier analysis to decompose functions into a sum of orthogonal functions.

By using operational approaches, the development of new functional families is facilitated as well as the derivation of the characteristics of those functional families linked to regular and generalized special functions. Dattoli and his colleagues recognized the significance of the use of operational techniques in the study of special functions that are intended to provide explicit solutions for families of partial differential equations, including those of the Heat and D'Alembert type, and their applications; see, for example [3,24,25] when applied to multi-variable generalized special functions in conjunction with the monomiality principle. This article's method can be utilized as a helpful tool in novel analytical techniques for the solutions of a large class of partial differential equations that are regularly encountered in physical issues.

Further, future research can be conducted in order to find the symmetric identities and determinant forms for these polynomials. Additionally, implicit summation formulae can be taken as future observations.

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