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On (p,q)–Fibonacci and (p,q)–Lucas Polynomials Associated with Changhee Numbers and Their Properties

Chuanjun Zhang [1], Waseem Ahmad Khan [2,*] and Can Kızılateş [3]

[1] School of Mathematics and Big Data, Guizhou Normal College, Guiyang 550018, China; zhangchuanjun@gznc.edu.cn
[2] Department of Mathematics and Natural Sciences, Prince Mohammad Bin Fahd University, P.O. Box 1664, Al Khobar 31952, Saudi Arabia
[3] Department of Mathematics, Zonguldak Bülent Ecevit University, Zonguldak 67100, Turkey; cankizilates@gmail.com
* Correspondence: wkhan1@pmu.edu.sa

Abstract: Many properties of special polynomials, such as recurrence relations, sum formulas, and symmetric properties have been studied in the literature with the help of generating functions and their functional equations. In this paper, using the (p,q)–Fibonacci polynomials, (p,q)–Lucas polynomials, and Changhee numbers, we define the (p,q)–Fibonacci–Changhee polynomials and (p,q)–Lucas–Changhee polynomials, respectively. We obtain some important identities and relations of these newly established polynomials by using their generating functions and functional equations. Then, we generalize the (p,q)–Fibonacci–Changhee polynomials and the (p,q)–Lucas–Changhee polynomials called generalized (p,q)–Fibonacci–Lucas–Changhee polynomials. We derive a determinantal representation for the generalized (p,q)–Fibonacci–Lucas–Changhee polynomials in terms of the special Hessenberg determinant. Finally, we give a new recurrent relation of the (p,q)–Fibonacci–Lucas–Changhee polynomials.

Keywords: (p,q)–Fibonacci polynomials; (p,q)–Lucas polynomials; Changhee numbers; generating function; Hessenberg determinant

MSC: 05A19; 11B37; 11B39; 11B83; 11C20; 11Y55

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1. Introduction

Special polynomials and numbers with special cases of these polynomials have been studied by many mathematicians. In particular, with the help of the generating functions of these polynomials, some identities, sum formulas, and symmetric identities containing these polynomials have been obtained. Special functions and numbers are frequently used in many branches of mathematics, especially in areas such as mathematical physics, mathematical modeling, and analytical number theory. Moreover, a large number of studies on families of generalized polynomials and their various applications in the solution of differential equations and approximation theory have appeared in the literature. For more details on special polynomials and some of their applications, please see [1–9].

Horadam [10] defined the sequence $\{\mathcal{W}_n(r,s;u,v)\}_{n\geq 0}$ or, in brief, $\{\mathcal{W}_n\}_{n\geq 0}$ by the recurrence relation

$$\mathcal{W}_n = u\mathcal{W}_{n-1} + v\mathcal{W}_{n-2}, \quad n \geq 2$$

where $r,s,u,v \in \mathbb{Z}$ with the initial values $\mathcal{W}_0 = r$ and $\mathcal{W}_1 = s$. For different values r,s,u and v, some special cases of the Horadam numbers $\mathcal{W}_n(r,s;u,v)$ are as shown in Table 1:

Table 1. Special cases of the Horadam sequence.

r	s	u	v	Sequence
0	1	1	1	Fibonacci; F_n
2	1	1	1	Lucas; L_n
0	1	2	1	Pell; P_n
2	2	2	1	Pell-Lucas; PL_n
0	1	k	1	k-Fibonacci; $F_{k,n}$
2	k	k	1	k-Lucas; $L_{k,n}$
0	1	1	2	Jacobsthal; J_n
2	1	1	2	Jacobsthal-Lucas; j_n

In addition to these numbers, polynomials containing these numbers and which have the same names as these numbers have also attracted the attention of mathematicians from the past to the present. For c_0, c_1, c_2, c_3 are constants and $d = 0$ or 1. In [11], Horadam defined the following polynomial sequence $\mathcal{W}_n(\xi)$ as

$$\mathcal{W}_n(\xi) = p(\xi)\mathcal{W}_{n-1}(\xi) + q(\xi)\mathcal{W}_{n-2}(\xi), \quad n \geq 2$$

where

$$\mathcal{W}_0(\xi) = c_0,\ \mathcal{W}_1(\xi) = c_1\xi^d,\ p(\xi) = c_2\xi^d,\ q(\xi) = c_3\xi^d.$$

With special choices of $p(\xi)$, $q(\xi)$, $\mathcal{W}_0(\xi)$, and $\mathcal{W}_1(\xi)$, the $\mathcal{W}_n(\xi)$ polynomials become important polynomials mentioned in the following Table 2:

Table 2. Special cases of the $W_n(\xi)$ polynomials.

$p(\xi)$	$q(\xi)$	$\mathcal{W}_0(\xi)$	$\mathcal{W}_1(\xi)$	Polynomial
ξ	1	0	1	Fibonacci; $f_n(\xi)$
ξ	1	2	ξ	Lucas; $l_n(\xi)$
2ξ	1	0	1	Pell; $P_n(\xi)$
2ξ	1	2	ξ	Pell–Lucas; $PL_n(\xi)$
1	2ξ	0	1	Jacobsthal; $J_n(\xi)$
1	2ξ	2	ξ	Jacobsthal–Lucas; $j_n(\xi)$
3ξ	-2	0	1	Fermat; $\mathcal{F}_n(\xi)$
3ξ	-2	2	ξ	Fermat Lucas; $\mathcal{FL}_n(\xi)$
2ξ	-1	0	1	Chebyshev polynomials of the second kind; $U_n(\xi)$
2ξ	-1	2	ξ	Chebyshev polynomials of the first kind; $w_n(\xi)$

Next, Nalli and Haukkanen [12] defined the $h(\xi)$–Fibonacci polynomials $h(\xi)$–Lucas polynomials including the Fibonacci polynomials, Pell polynomials, Lucas polynomials, and Pell–Lucas polynomials given in Table 2. In [13] Lee and Asci considered the (p,q)–Fibonacci polynomials and (p,q)–Lucas polynomials, as follows:

$$F_{p,q,n}(\xi) = p(\xi)F_{p,q,n-1}(\xi) + q(\xi)F_{p,q,n-2}(\xi),$$

and

$$L_{p,q,n}(\xi) = p(\xi)L_{p,q,n-1}(\xi) + q(\xi)L_{p,q,n-2}(\xi),$$

with the initial values $F_{p,q,0}(\xi) = 0$, $F_{p,q,1}(\xi) = 1$, $L_{p,q,0}(\xi) = 2$ and $L_{p,q,1}(\xi) = p(\xi)$, respectively. Here, $p(\xi)$ and $q(\xi)$ are polynomials with real coefficients. They derived the generating functions of the (p,q)–Fibonacci polynomials and (p,q)–Lucas polynomials as

$$\sum_{n=0}^{\infty} F_{p,q,n}(\xi)w^n = \frac{w}{1 - p(\xi)w - q(\xi)w^2},$$

and

$$\sum_{n=0}^{\infty} L_{p,q,n}(\xi)w^n = \frac{2 - p(\xi)w}{1 - p(\xi)w - q(\xi)w^2}.$$

Very recently, Simsek [14] defined the general forms of ordinary generating functions for special numbers and polynomials involving Fibonacci-type numbers and polynomials, Lucas numbers, and polynomials. The new classes of multiple variables polynomials are defined by means of the following generating functions as

$$\sum_{n=0}^{\infty} \mathbb{Y}_n\left(P\left(\vec{\xi}_m\right)\right)w^n = \frac{1}{1 + \sum_{j=1}^{m} P_j(\xi_j)w^j},$$

and

$$\sum_{n=0}^{\infty} \mathbb{S}_n\left(P\left(\vec{\xi}_m\right); Q\left(\vec{\xi}_k\right)\right)w^n = \frac{\sum_{j=0}^{k} Q_j(\xi_j)w^j}{1 + \sum_{j=1}^{m} P_j(\xi_j)w^j},$$

where $P\left(\vec{\xi}_m\right) = (P_1(\xi_1), P_2(\xi_2), \ldots, P_m(\xi_m))$, $Q\left(\vec{\xi}_k\right) = (Q_1(\xi_1), Q_2(\xi_2), \ldots, Q_k(\xi_k))$,

$$P_j(\xi_j) = \sum_{v=0}^{d} a_v \xi_j^v, \; Q_l(\xi_l) = \sum_{v=0}^{c} b_v \xi_l^v,$$

and $c, d, k, m \in \mathbb{N}_0$, $0 \leq l \leq k$ and $0 \leq j \leq m$. For more details related to the above numbers and polynomials, please see [14–18]. Another well-known polynomial is the Euler polynomial. The classical Euler polynomials $E_n(\xi)$ are defined with the help of the following generating function as [19]

$$\frac{2}{e^w + 1}e^{\xi w} = \sum_{n=0}^{\infty} E_n(\xi)\frac{w^n}{n!}, \quad |w| < \pi. \tag{1}$$

Note that for $\xi = 0$, $E_n(0) = E_n$ are called the Euler numbers. In [20], Pathan and Khan defined the $h(\xi)$–Fibonacci–Euler and $h(\xi)$–Lucas–Euler polynomials and numbers and derived some important identities for these types of polynomials. For $n \geq 0$, the Stirling numbers of the second kind are defined by (see, [19])

$$\xi^n = \sum_{q=0}^{j} S_2(n, q)(\xi)_q, \tag{2}$$

where $(\xi)_q = \xi(\xi - 1)\ldots(\xi - q + 1)$. By virtue of (2), the Stirling numbers of the second kind can be expressed as follows:

$$\frac{1}{r!}(e^w - 1)^r = \sum_{n=r}^{\infty} S_2(n, r)\frac{w^n}{n!}.$$

The Changhee polynomials are defined with the help of the following generating function

$$\frac{2}{2 + w}(1 + w)^{\xi} = \sum_{n=0}^{\infty} Ch_n(\xi)\frac{w^n}{n!}. \tag{3}$$

Substituting $\xi = 0$ into (3), $Ch_n(0) = Ch_n$, Changhee numbers are obtained. Replacing w by $e^w - 1$ in (3), we can write (see [21])

$$\begin{aligned} \frac{2}{e^w+1}e^{\xi w} &= \sum_{m=0}^{\infty} Ch_m(\xi)\frac{(e^w-1)^m}{m!} \\ &= \sum_{m=0}^{\infty} Ch_m(\xi)\sum_{n=m}^{\infty} S_2(n,m)\frac{w^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} Ch_m(\xi)S_2(n,m)\frac{w^n}{n!}. \end{aligned} \tag{4}$$

Using (1) and (4), we can get (see [22])

$$E_n(\xi) = \sum_{m=0}^{n} Ch_m(\xi)S_2(n,m).$$

In [23] Kim et al. defined the modified Changhee–Genocchi polynomials defined by

$$\frac{2w}{2+w}(1+w)^{\xi} = \sum_{n=0}^{\infty} CG_n(\xi)\frac{w^n}{n!}. \tag{5}$$

Setting $\xi = 0$ into (5), $CG_n(0) = CG_n$ are called the modified Changhee–Genocchi numbers. For more details related to Stirling numbers of the second kind, the Changhee polynomials, and the modified Changhee–Genocchi polynomials, please see [23–25].

Our main purpose in this article is to define a new polynomial family with the help of (p,q)–Fibonacci polynomials, (p,q)–Lucas polynomials, and the Changhee number. In the second and third parts of our article, we define (p,q)–Fibonacci–Changhee polynomials and (p,q)–Lucas–Changhee polynomials with the help of the generating functions of these polynomials and their functional equations, and we derive many new and interesting identities and relations related to these classes of these numbers and polynomials. In the last section, we define a new polynomial containing the (p,q)–Fibonacci–Changhee and (p,q)–Lucas–Changhee polynomials. With the help of the generating function of this polynomial, we give the determinant expression for this polynomial. Based on this determinant, we give a new recurrence relation for this polynomial using the lower Hessenbeg matrix.

2. The (p,q)–Fibonacci–Changhee Numbers and Polynomials

In this part of the paper, we introduce the (p,q)–Fibonacci–Changhee numbers and polynomials, denoted by ${}_{Ch}F_{p,q,n}(\xi)$, and we derive some properties of these polynomials using their generating functions.

Definition 1. *Let $p(\xi)$ and $q(\xi)$ be polynomials with real coefficients. The (p,q)–Fibonacci–Changhee polynomials ${}_{Ch}F_{p,q,n}(\xi)$ are defined by the generating function as follows:*

$$\frac{w}{1-p(\xi)w-q(\xi)w^2}\,\frac{2}{2+w} = \sum_{n=0}^{\infty} {}_{Ch}F_{p,q,n}(\xi)\frac{w^n}{n!}. \tag{6}$$

Remark 1. *The (p,q)–Fibonacci–Changhee polynomials can be expressed as*

$${}_{Ch}F_{p,q,n}(\xi) = \mathbb{S}_n(1/2-p(\xi), -(q(\xi)+p(\xi)/2), -q(\xi)/2; 0, 1).$$

Using (6), we find that

$$\begin{aligned}\frac{2w}{(1-p(\xi)w-q(\xi)w^2)(2+w)} &= \sum_{n=0}^{\infty} {}_{Ch}F_{p,q,n}(\xi)\frac{w^n}{n!} \\ &= \sum_{n=0}^{\infty} F_{p,q,n}(\xi)w^n \sum_{m=0}^{\infty} Ch_m \frac{w^m}{m!}.\end{aligned}$$

Comparing the coefficients of w^n on both sides of the above equation, we obtain

$$ {}_{Ch}F_{p,q,n}(\xi) = n! \sum_{m=0}^{n} F_{p,q,n-m}(\xi)\frac{Ch_m}{m!}.$$

Theorem 1. *For $n \geq 1$, we have*

$$CG_n = {}_{Ch}F_{p,q,n}(\xi) - p(\xi)n\, {}_{Ch}F_{p,q,n-1}(\xi) - q(\xi)(n)_2\, {}_{Ch}F_{p,q,n-2}(\xi), \tag{7}$$

where $(n)_2 = n(n-1)$.

Proof. Using (5) and (6), we have

$$\begin{aligned}\frac{2w}{2+w} &= (1-p(\xi)w-q(\xi)w^2)\sum_{n=0}^{\infty} {}_{Ch}F_{p,q,n}(\xi)w^n \\ \sum_{n=0}^{\infty} CG_n\frac{w^n}{n!} &= \sum_{n=0}^{\infty} {}_{Ch}F_{p,q,n}(\xi)\frac{w^n}{n!} - p(\xi)\sum_{n=0}^{\infty} {}_{Ch}F_{p,q,n}(\xi)\frac{w^{n+1}}{n!} - q(\xi)\sum_{n=0}^{\infty} {}_{Ch}F_{p,q,n}(\xi)\frac{w^{n+2}}{n!} \\ \sum_{n=0}^{\infty} CG_n\frac{w^n}{n!} &= \sum_{n=0}^{\infty} {}_{Ch}F_{p,q,n}(\xi)\frac{w^n}{n!} - \sum_{n=0}^{\infty} p(\xi){}_{Ch}F_{p,q,n-1}(\xi)\frac{w^n}{(n-1)!} - \sum_{n=0}^{\infty} q(\xi){}_{Ch}F_{p,q,n-2}(\xi)\frac{w^n}{(n-2)!}.\end{aligned}$$

Comparing the coefficients of w^n on both sides of the above equation, we get the desired result (7). □

Theorem 2. *For $n \geq 1$, we get*

$$F_{p,q,n}(\xi) = \frac{1}{n!}\Big({}_{Ch}F_{p,q,n}(\xi) + \frac{n}{2}\,{}_{Ch}F_{p,q,n-1}(\xi)\Big).$$

Proof. Equation (6) can be written as

$$\begin{aligned}\frac{2w}{1-p(\xi)w-q(\xi)w^2} &= (2+w)\sum_{n=0}^{\infty} {}_{Ch}F_{p,q,n}(\xi)\frac{w^n}{n!} \\ 2\sum_{n=0}^{\infty} F_{p,q,n}(\xi)w^n &= 2\sum_{n=0}^{\infty} {}_{Ch}F_{p,q,n}(\xi)\frac{w^n}{n!} + \sum_{n=0}^{\infty} {}_{Ch}F_{p,q,n}(\xi)\frac{w^{n+1}}{n!} \\ 2\sum_{n=0}^{\infty} F_{p,q,n}(\xi)w^n &= 2\sum_{n=0}^{\infty} {}_{Ch}F_{p,q,n}(\xi)\frac{w^n}{n!} + \sum_{n=0}^{\infty} {}_{Ch}F_{p,q,n-1}(\xi)\frac{w^n}{(n-1)!}.\end{aligned}$$

Comparing the coefficients of w^n on both sides of the above equation, we get the desired results. □

Theorem 3. *For $n \geq 1$, we have*

$$ {}_{Ch}F_{p,q,n}(\xi) = n!\sum_{m=0}^{n}\sum_{i=0}^{[\frac{m-1}{2}]}\binom{m-i-1}{i}\frac{Ch_{n-m}}{(n-m)!}p^{m-2i-1}(\xi)q^i(\xi)$$

where $\left|p(\xi)w + q(\xi)w^2\right| < 1$.

Proof. From (6), we obtain

$$\begin{aligned}\frac{w}{1-p(\xi)w-q(\xi)w^2}\frac{2}{2+w} &= w\frac{2}{2+w}\sum_{n=0}^{\infty}(p(\xi)w+q(\xi)w^2)^n \\ &= w\frac{2}{2+w}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}(p(\xi)w)^{n-i}(q(\xi)w^2)^i \\ &= \frac{2}{2+w}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}(p(\xi)w)^{n-i}(q(\xi))^i w^{2i+1}.\end{aligned}$$

On writing $n+i+1=m$ in the right-hand side of the above equation, we get

$$\begin{aligned}\frac{w}{1-p(\xi)w-q(\xi)w^2}\frac{2}{2+w} &= \frac{2}{2+w}\sum_{m=0}^{\infty}\left[\sum_{i=0}^{[\frac{m-1}{2}]}\binom{m-i-1}{i}p^{m-2i-1}(\xi)q^i(\xi)\right]w^m \\ \sum_{n=0}^{\infty}{}_{Ch}F_{p,q,n}(\xi)\frac{w^n}{n!} &= \sum_{n=0}^{\infty}Ch_n\frac{w^n}{n!}\sum_{m=0}^{\infty}\left[\sum_{i=0}^{[\frac{m-1}{2}]}\binom{m-i-1}{i}p^{m-2i-1}(\xi)q^i(\xi)\right]w^m.\end{aligned}$$

Next, we n by $n-m$ and compare the coefficients of w^n on both sides of the above equation, and thus obtain our assertion. □

Theorem 4. *The Representation of Changhee numbers in terms of (p,q)–Fibonacci–Changhee polynomials is*

$$Ch_n = \frac{{}_{Ch}F_{p,q,n+1}(\xi)}{n} - p(\xi)\,{}_{Ch}F_{p,q,n}(\xi) - q(\xi)n\,{}_{Ch}F_{p,q,n-1}(\xi), \quad n\geq 1. \tag{8}$$

Proof. Using (6), we find that

$$\begin{aligned}\frac{2}{2+w} &= (1-p(\xi)w-q(\xi)w^2)\sum_{n=0}^{\infty}{}_{Ch}F_{p,q,n}(\xi)\frac{w^{n-1}}{n!} \\ \sum_{n=0}^{\infty}Ch_n\frac{w^n}{n!} &= (1-p(\xi)w-q(\xi)w^2)\sum_{n=0}^{\infty}{}_{Ch}F_{p,q,n}(\xi)(\xi)\frac{w^{n-1}}{n!}.\end{aligned}$$

Comparing the coefficients of w^n on both sides of the above equation, we obtain the result (8). □

Theorem 5. *For $n\geq 0$, we have*

$$\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}F_{m,p,q}(\xi)m!S_2(k,m)E_{n-k} = \sum_{m=0}^{n}{}_{Ch}F_{p,q,m}(\xi)S_2(n,m). \tag{9}$$

Proof. Replacing w by e^w-1 in (6), we find that

$$\begin{aligned}\frac{e^w-1}{(1-p(\xi)(e^w-1)-q(\xi)(e^w-1)^2)}\frac{2}{e^w+1} &= \sum_{m=0}^{\infty}{}_{Ch}F_{p,q,m}(\xi)\frac{(e^w-1)^m}{m!} \\ &= \sum_{m=0}^{\infty}{}_{Ch}F_{p,q,m}(\xi)\sum_{n=m}^{\infty}S_2(n,m)\frac{w^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}{}_{Ch}F_{p,q,m}(\xi)S_2(n,m)\right)\frac{w^n}{n!}.\end{aligned} \tag{10}$$

On the other hand, we have

$$\begin{aligned}\frac{e^w-1}{(1-p(\xi)(e^w-1)-q(\xi)(e^w-1)^2)}\frac{2}{e^w+1} &= \sum_{m=0}^{\infty} F_{m,p,q}(\xi)m!\frac{(e^w-1)^m}{m!}\sum_{n=0}^{\infty} E_n\frac{w^n}{n!} \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{k} F_{m,p,q}(\xi)m!S_2(k,m)\frac{w^k}{k!}\sum_{n=0}^{\infty} E_n\frac{w^n}{n!}.\end{aligned}$$

So, we find that right hand-side of the above equation as

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}F_{m,p,q}(\xi)m!S_2(k,m)E_{n-k}\right)\frac{w^n}{n!}. \quad (11)$$

Therefore, according to (10) and (11), we obtain the desired result (9). □

Theorem 6. *For $n \geq 0$, we have*

$$ {}_{Ch}F_{2\xi,-1,n}(\xi) = \sum_{m=0}^{n}\binom{n}{m}U_m(\xi)m!Ch_{n-m}, \quad (12)$$

where $U_n(\xi)$ are the Chebyshev polynomials of the second kind.

Proof. On setting $p(\xi) = 2\xi$ and $q(\xi) = -1$ in (6), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} {}_{Ch}F_{2\xi,-1,n}(\xi)\frac{w^n}{n!} &= \frac{w}{1-2\xi w+w^2}\frac{2}{2+w} \\ &= \sum_{m=0}^{\infty} U_m(\xi)w^m\sum_{n=0}^{\infty} Ch_n\frac{w^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}U_m(\xi)m!Ch_{n-m}\right)\frac{w^n}{n!}. \quad (13)\end{aligned}$$

Comparing the coefficients of w^n on both sides of the above equation, we obtain (12). □

3. The (p,q)–Lucas–Changhee Numbers and Polynomials

Now, in this part of the our paper, the (p,q)–Lucas–Changhee numbers and polynomials, denoted by ${}_{Ch}L_{p,q,n}(\xi)$, are defined, and certain properties of these polynomials are obtained.

Definition 2. *Let $p(\xi)$ and $q(\xi)$ be a polynomial with real coefficients. We define (p,q)–Lucas–Changhee polynomials ${}_{Ch}L_{p,q,n}(\xi)$ by the generating function*

$$\frac{2-p(\xi)w}{1-p(\xi)w-q(\xi)w^2}\frac{2}{2+w} = \sum_{n=0}^{\infty} {}_{Ch}L_{p,q,n}(\xi)\frac{w^n}{n!}. \quad (14)$$

Remark 2. *The (p,q)–Lucas–Changhee polynomials can be expressed as*

$$ {}_{Ch}L_{p,q,n}(\xi) = \mathbb{S}_n(1/2-p(\xi), -(q(\xi)+p(\xi)/2), -q(\xi)/2; 2, -p(\xi)).$$

We may now rewrite (14) as

$$\sum_{n=0}^{\infty} {}_{Ch}L_{p,q,n}(\xi)\frac{w^n}{n!} = \sum_{n=0}^{\infty} L_{p,q,n}(\xi)w^n\sum_{m=0}^{\infty} Ch_m\frac{w^m}{m!}.$$

We replace n by $n-m$ in the right hand-side and compare the coefficients of w^n to obtain the following representation for (p,q)–Lucas–Changhee polynomials

$$ {}_{Ch}L_{p,q,n}(\xi) = n! \sum_{m=0}^{n} L_{p,q,n-m}(\xi) \frac{Ch_m}{m!}. $$

Theorem 7. *For $n \geq 1$, we have*

$$ \begin{aligned} \frac{2Ch_n}{n!} &= p(\xi)_{Ch}F_{p,q,n}(\xi)\frac{1}{n!} + {}_{Ch}L_{p,q,n}(\xi)\frac{1}{n!} \\ &-p(\xi)\left[p(\xi)_{Ch}F_{p,q,n-1}(\xi)\frac{1}{(n-1)!} + {}_{Ch}L_{p,q,n-1}(\xi)\frac{1}{(n-1)!}\right] \\ &-q(\xi)\left[p(\xi)_{Ch}F_{p,q,n-2}(\xi)\frac{1}{(n-2)!} + {}_{Ch}L_{p,q,n-2}(\xi)\frac{1}{(n-2)!}\right], \end{aligned} \tag{15} $$

and

$$ L_{p,q,n}(\xi) = {}_{Ch}L_{p,q,n}(\xi)\frac{1}{n!} + \frac{1}{2}{}_{Ch}L_{p,q,n-1}(\xi)\frac{1}{(n-1)!}. \tag{16} $$

Proof. Using Equation (14), we can write

$$ \begin{aligned} \frac{2}{1-p(\xi)w-q(\xi)w^2}\frac{2}{2+w} &= p(\xi)\sum_{n=0}^{\infty} {}_{Ch}F_{p,q,n}(\xi)\frac{w^n}{n!} + \sum_{n=0}^{\infty} {}_{Ch}L_{p,q,n}(\xi)\frac{w^n}{n!} \\ \frac{2}{1-p(\xi)w-q(\xi)w^2}\sum_{n=0}^{\infty} Ch_n\frac{w^n}{n!} &= p(\xi)\sum_{n=0}^{\infty} {}_{Ch}F_{p,q,n}(\xi)\frac{w^n}{n!} + \sum_{n=0}^{\infty} {}_{Ch}L_{p,q,n}(\xi)\frac{w^n}{n!}. \end{aligned} $$

Thus, we have

$$ \begin{aligned} 2\sum_{n=0}^{\infty} Ch_n\frac{w^n}{n!} &= (1-p(\xi)w-q(\xi)w^2)\left[p(\xi)\sum_{n=0}^{\infty} {}_{Ch}F_{p,q,n}(\xi)\frac{w^n}{n!} + \sum_{n=0}^{\infty} {}_{Ch}L_{p,q,n}(\xi)\frac{w^n}{n!}\right] \\ 2\sum_{n=0}^{\infty} Ch_n\frac{w^n}{n!} &= p(\xi)\sum_{n=0}^{\infty} {}_{Ch}F_{p,q,n}(\xi)\frac{w^n}{n!} + \sum_{n=0}^{\infty} {}_{Ch}L_{p,q,n}(\xi)\frac{w^n}{n!} \\ &-p(\xi)w\left[p(\xi)\sum_{n=0}^{\infty} {}_{Ch}F_{h,n}(\xi)\frac{w^n}{n!} + \sum_{n=0}^{\infty} {}_{Ch}L_{p,q,n}(\xi)\frac{w^n}{n!}\right] \\ &-q(\xi)w^2\left[p(\xi)\sum_{n=0}^{\infty} {}_{Ch}F_{p,q,n}(\xi)\frac{w^n}{n!} + \sum_{n=0}^{\infty} {}_{Ch}L_{p,q,n}(\xi)\frac{w^n}{n!}\right]. \end{aligned} $$

Comparing the coefficients of w^n, we have the result (15).
Again, we rewrite Equation (14) as

$$ \begin{aligned} 2\frac{2-p(\xi)w}{1-p(\xi)w-q(\xi)w^2} &= (2+w)\sum_{n=0}^{\infty} {}_{Ch}L_{p,q,n}(\xi)\frac{w^n}{n!} \\ 2\sum_{n=0}^{\infty} L_{p,q,n}(\xi)w^n &= 2\sum_{n=0}^{\infty} {}_{Ch}L_{p,q,n}(\xi)\frac{w^n}{n!} + \sum_{n=0}^{\infty} {}_{Ch}L_{p,q,n}(\xi)\frac{w^{n+1}}{n!}. \end{aligned} $$

Comparing the coefficients of w^n, we get the result (16). □

Theorem 8. *For $n \geq 0$, we have*

$$ \sum_{k=0}^{n}\sum_{k=0}^{m}\binom{n}{k} L_{p,q,m}(\xi)S_2(k,m)E_{n-k}m! = \sum_{m=0}^{n} {}_{Ch}L_{p,q,m}(\xi)S_2(n,m). \tag{17} $$

Proof. Replacing w with $e^w - 1$ in (14), we get

$$\frac{2-p(\xi)(e^{w}-1)}{1-p(\xi)(e^{w}-1)-q(\xi)(e^{w}-1)^{2}}\frac{2}{e^{w}+1} = \sum_{m=0}^{\infty}{}_{Ch}L_{p,q,m}(\xi)\frac{(e^{w}-1)^{m}}{m!}$$
$$= \sum_{m=0}^{\infty}{}_{Ch}L_{p,q,m}(\xi)\sum_{n=m}^{\infty}S_{2}(n,m)\frac{w^{n}}{n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}{}_{Ch}L_{p,q,m}(\xi)S_{2}(n,m)\right)\frac{w^{n}}{n!}. \tag{18}$$

On the other hand, we have

$$\frac{2-p(\xi)(e^{w}-1)}{1-p(\xi)(e^{w}-1)-q(\xi)(e^{w}-1)^{2}}\frac{2}{e^{w}+1} = \sum_{m=0}^{\infty}L_{p,q,m}(\xi)m!\frac{(e^{w}-1)^{m}}{m!}\sum_{n=0}^{\infty}E_{n}\frac{w^{n}}{n!}$$
$$= \sum_{k=0}^{m}L_{p,q,m}(\xi)\sum_{k=m}^{\infty}S_{2}(k,m)m!\frac{w^{k}}{k!}\sum_{n=0}^{\infty}E_{n}\frac{w^{n}}{n!}.$$

From the above equation, we get

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{k=0}^{m}\binom{n}{k}L_{p,q,m}(\xi)S_{2}(k,m)E_{n-k}m!\right)\frac{w^{n}}{n!}. \tag{19}$$

In view of (18) and (19), we obtain the result (17). □

Theorem 9. *For $n \geq 0$, we have*

$$_{Ch}L_{2\xi,-1,n}(\xi) = 2\sum_{m=0}^{n}\binom{n}{m}w_{m}(\xi)m!Ch_{n-m}, \tag{20}$$

where $w_m(\xi)$ are the Chebyshev polynomials of the first kind.

Proof. On taking $p(\xi) = 2\xi$ and $q(\xi) = -1$ in (14), we get

$$\sum_{n=0}^{\infty}{}_{Ch}L_{2\xi,-1,n}(\xi)\frac{w^{n}}{n!} = 2\frac{1-\xi w}{1-2\xi w+w^{2}}\frac{2}{2+w}$$
$$= 2\sum_{m=0}^{\infty}w_{m}(\xi)m!\frac{w^{m}}{m!}\sum_{n=0}^{\infty}Ch_{n}\frac{w^{n}}{n!}$$
$$= \sum_{n=0}^{\infty}\left(2\sum_{m=0}^{n}\binom{n}{m}w_{m}(\xi)m!Ch_{n-m}\right)\frac{w^{n}}{n!}.$$

Comparing the coefficients of w^n, we get the result (20). □

4. Some Applications of the Generalized (p,q)–Fibonacci–Lucas–Changhee Polynomials in Matrices

In this section, firstly, we define a polynomial including the (p,q)–Fibonacci–Changhee polynomials and the (p,q)–Lucas–Changhee polynomials that we defined in Sections 2 and 3. We define this newly established polynomial with the help of the following generating function as follows.

Let (p,q) be a polynomial with real coefficients. The generalized (p,q)–Fibonacci–Lucas–Changhee polynomials $_{Ch}\mathcal{P}^{a,b}_{p,q,n}(\xi)$ are defined by the following generating function

$$\sum_{n=0}^{\infty}{}_{Ch}\mathcal{P}^{a,b}_{p,q,n}(\xi)\frac{w^{n}}{n!} = \frac{aw+b(2-p(\xi)w)}{(1-p(\xi)w-q(\xi)w^{2})}\frac{2}{2+w}. \tag{21}$$

Remark 3. *The (p,q)–Fibonacci–Lucas–Changhee polynomials can be expressed as*

$$ {}_{Ch}\mathcal{P}^{a,b}_{p,q,n}(\xi) = \mathbb{S}_n(1/2 - p(\xi), -(q(\xi) + p(\xi)/2), -q(\xi)/2; 2b, a - p(\xi)). $$

Setting $a = 1$ and $b = 0$ into (21), we obtain the (p,q)–Fibonacci–Changhee polynomials defined by (6). Setting $a = 0$ and $b = 1$ into (21), we obtain the (p,q)–Lucas–Changhee polynomials defined by (14).

Secondly, we present a closed formula for the generalized (p,q)–Fibonacci–Lucas–Changhee polynomials ${}_{Ch}\mathcal{P}^{a,b}_{p,q,n}(\xi)$, in terms of the following determinant. For more details related to determinantal expressions for special polynomials and numbers, please see [26–31].

Theorem 10. *The generalized (p,q)–Fibonacci–Lucas–Changhee polynomials ${}_{Ch}\mathcal{P}^{a,b}_{p,q,n}(\xi)$ for $n \geq 0$ can be expressed as following determinant*

$$ {}_{Ch}\mathcal{P}^{a,b}_{p,q,n}(\xi) = \frac{1}{2^{n+1}} \begin{vmatrix} 4b & -2 & \cdots & 0 & 0 \\ 2a - 2bp(\xi) & (2p(\xi)-1)\binom{1}{0} & \cdots & 0 & 0 \\ 0 & (4q(\xi)+2p(\xi))\binom{2}{0} & \ddots & 0 & 0 \\ 0 & 6q(\xi)\binom{3}{0} & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -2 & 0 \\ 0 & 0 & \cdots & (2p(\xi)-1)\binom{n-1}{n-2} & -2 \\ 0 & 0 & \cdots & (4q(\xi)+2p(\xi))\binom{n}{n-2} & (2p(\xi)-1)\binom{n}{n-1} \end{vmatrix}. \quad (22) $$

Proof. We recall that a general and fundamental formula for derivatives of a ratio of two differential functions ([32], page 40, Exercise 5). Let $\gamma(w)$ and $\delta(w) \neq 0$ be differentiable functions, let $\mathcal{U}_{(n+1)\times 1}(w)$ be an $(n+1) \times 1$ matrix whose elements $\gamma_{k,1}(w) = \gamma^{(k-1)}(w)$ for $1 \leq k \leq n+1$, and let $\mathcal{V}_{(n+1)\times n}(w)$ be an $(n+1) \times n$ matrix whose elements

$$ \delta_{i,j}(w) = \begin{cases} \binom{i-1}{j-1}\delta^{(i-j)}(w), & i-j \geq 0; \\ 0, & i-j < 0 \end{cases} $$

for $1 \leq i \leq n+1$ and $1 \leq j \leq n$, and let $\mathbf{W}_{(n+1)\times(n+1)}(\xi)$ denote the lower Hessenberg determinant of the $(n+1) \times (n+1)$ lower Hessenberg matrix

$$ \mathbf{W}_{(n+1)\times(n+1)}(w) = \begin{bmatrix} \mathcal{U}_{(n+1)\times 1}(w) & \mathcal{V}_{(n+1)\times n}(w) \end{bmatrix}. $$

Then, the nth derivative of the ratio $\frac{\gamma(w)}{\delta(w)}$ can be computed by

$$ \frac{d^n}{d\xi^n}\left[\frac{\gamma(w)}{\delta(w)}\right] = (-1)^n \frac{\left|\mathbf{W}_{(n+1)\times(n+1)}(w)\right|}{\delta^{n+1}(w)}. \quad (23) $$

Using (21) and (23), we find that

$$
\begin{aligned}
&\frac{d^n}{dw^n}\left[\frac{aw+b(2-p(\xi)w)}{(1-p(\xi)w-q(\xi)w^2)}\frac{2}{2+w}\right] \\
=\ &\frac{(-1)^n}{[(1-p(\xi)w-q(\xi)w^2)(2+w)]^{n+1}} \\
&\times\begin{vmatrix}
\gamma(w) & \delta(w) & 0 & \cdots & 0 & 0 \\
\gamma'(w) & \delta'(w) & \delta(w) & \cdots & 0 & 0 \\
\gamma''(w) & \delta''(w) & \binom{2}{1}\delta'(w) & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\gamma^{(n-2)}(w) & \delta^{(n-2)}(w) & \binom{n-2}{1}\delta^{(n-3)}(w) & \cdots & \delta(w) & 0 \\
\gamma^{(n-1)}(w) & \delta^{(n-1)}(w) & \binom{n-1}{1}\delta^{(n-2)}(w) & \cdots & \binom{n-1}{n-2}\delta'(w) & \delta(w) \\
\gamma^{(n)}(w) & \delta^{(n)}(w) & \binom{n}{1}\delta^{(n-1)}(w) & \cdots & \binom{n}{n-2}\delta''(w) & \binom{n}{n-1}\delta'(w)
\end{vmatrix}.
\end{aligned}
$$

So, we find the right hand-side above equation

$$
\to \frac{1}{2^{n+1}}\times\begin{vmatrix}
4b & -2 & \cdots & 0 & 0 \\
2a-2bp(\xi) & (2p(\xi)-1)\binom{1}{0} & \cdots & 0 & 0 \\
0 & (2p(\xi)+4q(\xi))\binom{2}{0} & \cdots & 0 & 0 \\
0 & 6q(\xi)\binom{3}{0} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -2 & 0 \\
0 & 0 & \cdots & (2p(\xi)-1)\binom{n-1}{n-2} & -2 \\
0 & 0 & \cdots & (2p(\xi)+4q(\xi))\binom{n}{n-2} & (2p(\xi)-1)\binom{n}{n-1}
\end{vmatrix},
$$

as $w \to 0$ for $n \geq 0$. Therefore, we have,

$$
\begin{aligned}
{}_{Ch}\mathcal{P}^{a,b}_{p,q,n}(\xi) &= \lim_{\xi\to 0}\frac{d^n}{dw^n}\left[\frac{aw+b(2-p(\xi)w)}{(1-p(\xi)w-q(\xi)w^2)}\frac{2}{2+w}\right] \\
&= \frac{1}{2^{n+1}}\begin{vmatrix}
4b & -2 & \cdots & 0 & 0 \\
2a-2bp(\xi) & (2p(\xi)-1)\binom{1}{0} & \cdots & 0 & 0 \\
0 & (2p(\xi)+4q(\xi))\binom{2}{0} & \ddots & 0 & 0 \\
0 & 6q(\xi)\binom{3}{0} & \ddots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -2 & 0 \\
0 & 0 & \cdots & (2p(\xi)-1)\binom{n-1}{n-2} & -2 \\
0 & 0 & \cdots & (2p(\xi)+4q(\xi))\binom{n}{n-2} & (2p(\xi)-1)\binom{n}{n-1}
\end{vmatrix}.
\end{aligned}
$$

Thus, the proof is completed. □

Remark 4. *By choosing $a = 1$ and $b = 0$ in Theorem 10, we can find the determinantal expression for the (p,q)–Fibonacci–Changhee polynomials. Similarly, taking $a = 0$ and $b = 1$ in Theorem 10, we can get the determinantal expression for the (p,q)–Lucas–Changhee polynomials.*

Remark 5. *Equality (22) can be obtained using a different method. For nice and short proofs, please see [33–36].*

Now, in the following theorem, we give a new recurrent relation for the generalized (p,q)–Fibonacci–Lucas–Changhee polynomials ${}_{Ch}\mathcal{P}^{a,b}_{p,q,n}(\xi)$.

Theorem 11. *For $n \geq 3$, we have the recurrent relation*

$$ {}_{Ch}\mathcal{P}^{a,b}_{p,q,n}(\xi) = \frac{1}{2}\left(\begin{array}{c} (n)_3 q(\xi)\ {}_{Ch}\mathcal{P}^{a,b}_{p,q,n-3}(\xi) + (n)_2(p(\xi) + 2q(\xi))\ {}_{Ch}\mathcal{P}^{a,b}_{p,q,n-2}(\xi) \\ +(n)_1(2p(\xi) - 1)\ {}_{Ch}\mathcal{P}^{a,b}_{p,q,n-1}(\xi) \end{array} \right). \tag{24} $$

Proof. Let $D_0 = 1$ and

$$ \mathcal{D}_n = \begin{vmatrix} u_{1,1} & u_{1,2} & 0 & \cdots & 0 & 0 \\ u_{2,1} & u_{2,2} & u_{2,3} & \cdots & 0 & 0 \\ u_{3,1} & u_{3,2} & u_{3,3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{n-2,1} & u_{n-2,2} & u_{n-2,3} & \cdots & u_{n-2,n-1} & 0 \\ u_{n-1,1} & u_{n-1,2} & u_{n-1,3} & \cdots & u_{n-1,n-1} & u_{n-1,n} \\ u_{n,1} & u_{n,2} & u_{n,3} & \cdots & u_{n,n-1} & u_{n,n} \end{vmatrix}, $$

for $n \in \mathbb{N}$. In ([37], Page 222, Theorem), the authors proved that the sequence $\mathcal{D}_n$ for $n \geq 0$ satisfies $D_1 = e_{1,1}$ and

$$ \mathcal{D}_n = \sum_{r=1}^{n} (-1)^{n-r} u_{n,r} \left(\prod_{j=r}^{n-1} u_{j,j+1} \right) \mathcal{D}_{r-1}, \tag{25} $$

for $n \geq 2$, where the empty product is understood to be 1. If we apply the recurrent relation (25) to Theorem 10, we have

$$ \begin{aligned} 2^n\ {}_{Ch}\mathcal{P}^{a,b}_{p,q,n-1}(\xi) &= 6q(\xi)\binom{n-1}{n-4} 2^{n-1}\ {}_{Ch}\mathcal{P}^{a,b}_{p,q,n-4}(\xi) \\ &\quad + (2p(\xi) + 4q(\xi))\binom{n-1}{n-3} 2^{n-1}\ {}_{Ch}\mathcal{P}^{a,b}_{p,q,n-3}(\xi) \\ &\quad + (2p(\xi) - 1)\binom{n-1}{n-2} 2^{n-1}\ {}_{Ch}\mathcal{P}^{a,b}_{p,q,n-2}(\xi) \end{aligned} $$

for $n \geq 3$, which can be simplified as (24). □

5. Conclusions

In our present investigation, we defined the (p, q)–Fibonacci–Changhee polynomials and (p, q)–Lucas–Changhee polynomials, respectively. Then, we derived several fundamental properties and relations of these types of polynomials. Next, we generalized the (p, q)–Fibonacci–Changhee polynomials and the (p, q)–Lucas–Changhee polynomials called generalized (p, q)–Fibonacci–Lucas–Changhee polynomials. We derived a determinantal representation for the generalized (p, q)–Fibonacci–Lucas–Changhee polynomials. Finally, we provided a new recurrent relation of the generalized (p, q)–Fibonacci–Lucas–Changhee polynomials. Our results can be derived not only for the generalized (p, q)–Fibonacci–Lucas–Changhee polynomials defined by Equation (21), but also for many polynomials and numbers (please see Tables 1 and 2), according to the special cases of $p(\xi)$, $q(\xi)$. Thus, our main results are more general. For the interested reader, the results presented here could motivate further research such as symmetric identities, sum formulas, and recurrence relations on the subject of other mixed-type polynomials.

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