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Application of the Double Fuzzy Sawi Transform for Solving a Telegraph Equation

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Abstract: The main purpose of this study is to introduce a new double fuzzy transform called the double fuzzy Sawi transform. A proof of some basic properties of the single fuzzy Sawi transform and the double fuzzy Sawi transform are provided. These new results are implemented to obtain the exact solution of a non-homogeneous linear fuzzy telegraph equation under a generalized Hukuhara partial differentiability. In addition, by using the symmetric triangular fuzzy numbers, numerical examples are given to demonstrate the validity and superiority of the double fuzzy Sawi transform in solving the fuzzy linear telegraph equation.

Keywords: double fuzzy Sawi transform; linear fuzzy telegraph equation; generalized Hukuhara partial differentiability

MSC: 44A30; 35A22; 35N05

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1. Introduction

A lot of scientific fields, such as engineering, biology, and physics, use fuzzy differential equations for modeling uncertainty in various mathematical models [1–4]. Furthermore, fuzzy derivative definitions and concepts provide additional support in methods for solving these equations. In [5], Feuring and Buckley introduce fuzzy partial differential equations. Allahviranloo [6], who found solution for these equations based on a difference method via Taylor series, continues this work to a greater extent. Subsequently, Pownuk [7] uses algorithm-based sensitivity analysis applied to a finite element method model to obtain a solution of fuzzy partial differential equations. Another technique that was of high consideration in finding an approximate solution of the fuzzy heat equation in [8] pertains to the Adomian decomposition method. That method, along with the fuzzy variational iteration method, was applied by Osman, Gong, and Mustafa [9] to solve fuzzy heat-like and wave-like equations with variable coefficients in the sense of gH-differentiability. In later work [10], Gouyandeh and others introduced a fuzzy Fourier transformation. Based on this transformation, they examined analytical solutions of a fuzzy heat equation under a generalized Hukuhara partial differentiability. Many authors have presented various analytical and computational methods that led to solving these equations [11–15].

The Sawi transform was proposed by Mahgub [16] and is applied to ordinary linear differential equations with constant coefficients. Aggarwal and Gupta [17] provided supportive evidence for the relationship between the Sawi transformation and other fundamental transformations. Aggarwal discussed the application of the Sawi transform in finding solutions to biological problems such as growth and decay in [18]. Higary et al. demonstrated another example of how the Sawi transform can be used in finding a solution to a system of ordinary differential equations by its application in determining the concentration of chemical reactants of a chemical reaction in a series [19]. In [20], one-dimensional shallow water wave equations are solved using a new hybrid technique that involves homotopy perturbation based on the Sawi transformation.

In recent years, some scholars [21–25] applied different double fuzzy integral transforms (Laplace; Natural; Elzaki) to find the solution of a fuzzy partial differential equation. These transforms are very useful in solving linear fuzzy partial differential equations. They convert the fuzzy partial equations to algebraic equations about the unknown function.

The objective of the present paper is to obtain the solution of the linear fuzzy telegraph equation under generalized Hukuhara partial derivatives using a double fuzzy Sawi transform. To begin with, we define a single fuzzy Sawi transform. Then, we provide sufficient conditions for its existence and prove some key properties of this transformation. The fuzzy Sawi transform for a one-variable fuzzy function is used to define a new double fuzzy integral transformation for multivariate fuzzy functions, known as the double fuzzy Sawi transform (DFST). We introduce basic theorems and properties of the DFST and present some results related to the generalized Hukuhara partial derivatives. Based on the full DFST, we obtain a simple formula for the solution of the non-homogeneous linear fuzzy telegraph equation without converting it to two crisp equations. Finally, by using the symmetric triangular fuzzy numbers, we give numerical examples of homogeneous and non-homogeneous linear fuzzy telegraph equations with symmetric characteristics to demonstrate the usefulness of the provided double fuzzy integral transform in solving fuzzy partial differential equations.

The rest of this paper can be summarized as follows: in Section 2, we introduce theorems and some of the principal definitions that will be used. In Section 3, we illustrate fundamental facts and properties of the single fuzzy Sawi transformation. Section 4 presents the new double fuzzy Sawi transform: its basic properties; theorems; and several relationships related to its existence and gH-partial derivatives. In Section 5, we apply a double fuzzy Sawi transform to the linear fuzzy telegraph equation under partial gH-differentiability and obtain a formula for its solution. In Section 6, we provide numerical examples to be solved with this new double fuzzy transform. Finally, in Section 7, we summarize our results.

2. Basic Preliminaries

This section consists of some definitions and notations, which will be used in the paper. Let $\mathbb{R}$ be the set of all real numbers.

Definition 1 ([26])**.** *A fuzzy number is a mapping $u : \mathbb{R} \to [0,1]$ that possesses the following properties:*

(i) *u is normal, i.e., there exists $x \in \mathbb{R}$ with $u(x) = 1$;*
(ii) *u is upper semi-continuous on $\mathbb{R}$;*
(iii) *u is a convex fuzzy set, i.e., $u(rx + (1-r)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in \mathbb{R}$, $r \in [0,1]$;*
(iv) *$supp\ u = \{x \in \mathbb{R} : \ u(x) > 0\}$ is the support of u, and its closure $cl(supp u)$ is compact.*

Denote E^1 as the set of all fuzzy numbers. Any real number $a \in \mathbb{R}$ can be interpreted as a fuzzy number $\tilde{a} = \chi(a)$, and therefore, $\mathbb{R} \subset E^1$.

Definition 2 ([27])**.** *Let $u \in E^1$. The r-level set of u is denoted by*

$$[u]^r = \begin{cases} \{x \in \mathbb{R} : u(x) \geq r\}, & 0 < r \leq 1, \\ cl(supp\ u), \ r = 0. \end{cases}$$

Denote $[u]^r = [\underline{u}(r), \overline{u}(r)]$, so the r-level set $[u]^r$ is a bounded and closed interval for all $r \in [0,1]$.

Definition 3 ([27]). *A parametric form of the fuzzy number u is an ordered pair $u(r) = (\underline{u}(r), \overline{u}(r))$ of functions $\underline{u}(r)$ and $\overline{u}(r)$ for any $r \in [0,1]$, which satisfies the following conditions:*

(i) *the function $\underline{u}(r)$ is a bounded left continuous monotonic increasing in $[0,1]$;*
(ii) *the function $\overline{u}(r)$ is a bounded left continuous monotonic decreasing in $[0,1]$;*
(iii) *$\underline{u}(r) \leq \overline{u}(r)$.*

Definition 4 ([27]). *A triangular fuzzy number u is said to be an ordered triple $u = (a,b,c)$ with the conditions $a \leq b \leq c$ and $a,\ b,\ c \in \mathbb{R}$. The r-cuts associated with triangular fuzzy number u are $[a + (b-a)r, c - (c-b)r]$.*

Definition 5 ([28]). *A trapezoidal fuzzy number u is said to be an ordered four $u = (a,b,c,d)$ with the conditions $a \leq b \leq c \leq d$ and $a,\ b,\ c\ d \in \mathbb{R}$. The r-cuts associated with trapezoidal fuzzy number u are $[a + (b-a)r, d - (d-c)r]$.*

For fuzzy numbers $[u]^r = [\underline{u}(r), \overline{u}(r)]$, $[v]^r = [\underline{v}(r), \overline{v}(r)]$, and $\lambda \in \mathbb{R}$, the addition and scalar multiplication are defined by

$$[u \oplus v]^r = [u]^r + [v]^r = [\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)]$$

and

$$[\lambda \odot u]^r = k.[u]^r = \begin{cases} [\lambda \underline{u}(r), \lambda \overline{u}(r)], & \lambda \geq 0 \\ [\lambda \overline{u}(r), \lambda \underline{u}(r)], & \lambda < 0. \end{cases}$$

Denote $\tilde{0} = \chi_{\{0\}}$ as the neutral element with respect to $\oplus$ in E^1. In [26], the basic algebraic properties of fuzzy numbers are given.

In this study, we suppose $\mathbb{R}_+ = [0, +\infty)$.

Definition 6 ([26]). *The Hausdorff distance between fuzzy numbers is given by $D : E^1 \times E^1 \to \mathbb{R}_+$ as*

$$D(u,v) = \sup_{r \in [0,1]} d([u]^r, [v]^r) = \sup_{r \in [0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\},$$

where d is the Hausdorff metric.

Some basic properties of the metric D are given in [26].

(i) (E^1, D) is a complete metric space;
(ii) $D(u \oplus w, v \oplus w) = D(u,v)$ for all $u,\ v,\ w \in E^1$;
(ii) $D(\lambda u, \lambda v) = |\lambda| D(u,v)$ for all $u,\ v \in E^1$ and $\lambda \in \mathbb{R}$;
(iii) $D(u \oplus v, w \oplus z) \leq D(u,w) + D(v,z)$ for all $u,\ v,\ w,\ z \in E^1$;
(iv) $D(u \ominus v, w \ominus z) \leq D(u,w) + D(v,z)$ for all $u,\ v,\ w,\ z \in E^1$, assuming that $u \ominus v$ and $w \ominus z$ exist.

The Hukuhara difference (H-difference) $\ominus$ between fuzzy numbers u and v is defined by $u \ominus v = w$ if and only if $v \oplus w = u$. For the r-level, we have

$$[u \ominus v]^r = [\underline{u}(r) - \underline{v}(r), \overline{u}(r) - \overline{v}(r)].$$

The H-difference does not always exist, but it is unique. A necessary condition is that u contains a translate $\{c\} \oplus v$ of v.

Definition 7 ([28]). *The generalized Hukuhara difference (gH-difference) $\ominus_{gH}$ between two fuzzy numbers $u, v \in E^1$ is defined by:*

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} (i) & u = v \oplus w, \\ or\ (ii) & v = u \oplus (-1) \odot w. \end{cases}$$

For the r-level, we obtain

$$[u \ominus_{gH} v]^r = [\min\{\underline{u}(r) - \underline{v}(r), \overline{u}(r) - \overline{v}(r)\}, \max\{\underline{u}(r) - \underline{v}(r), \overline{u}(r) - \overline{v}(r)\}].$$

If the H-difference exists, then $u \ominus v = u \ominus_{gH} v$. The sufficient conditions for the existence of $u \ominus_{gH} v$ are

(i) $\begin{cases} \underline{u}(r) - \underline{v}(r) = \underline{w}(r) \text{ and } \overline{u}(r) - \overline{v}(r) = \overline{w}(r), \text{ for all } 0 \leq r \leq 1, \\ \text{with } \underline{w}(r) \text{ increasing}, \overline{w}(r) \text{ decreasing}, \underline{w}(r) \leq \overline{w}(r). \end{cases}$

(ii) $\begin{cases} \overline{u}(r) - \overline{v}(r) = \underline{w}(r) \text{ and } \underline{u}(r) - \underline{v}(r) = \overline{w}(r), \text{ for all } 0 \leq r \leq 1, \\ \text{with } \underline{w}(r) \text{ increasing}, \overline{w}(r) \text{ decreasing}, \underline{w}(r) \leq \overline{w}(r). \end{cases}$

It is easy to show that both (i) and (ii) are true if and only if w is a crisp number.

In the fuzzy case, it is possible that the gH-difference of two fuzzy numbers does not exist.

Example 1. *Let $u = (0, 2, 2, 4)$ and $v = (0, 1, 2, 3)$ be a triangular and trapezoidal fuzzy number. In this case, $u(r) = (2r, 4 - 2r)$ and $v(r) = (r, 3 - r)$. For the level-wise, the gH-differences exist, but the gH-difference $u \ominus_{gH} v$ does not exist. If we suppose that it exists, then either case (i) or (ii) of Definition 2.7 should hold for any $r \in [0, 1]$. However,*

$$\underline{w}(0) = \underline{u}(0) - \underline{v}(0) = 0 < \overline{w}(0) = \overline{u}(0) - \overline{v}(0),$$

while

$$\underline{w}(1) = \underline{u}(1) - \underline{v}(1) = 1 > \overline{w}(1) = \overline{u}(1) - \overline{v}(1) = 0,$$

so neither case (i) or (ii) is true from Definition 2.7.

Proposition 1 ([28])**.** *Let $u, v \in E^1$ be two fuzzy numbers. Then*

(i) $(-1)[u]^r = [-\overline{u}(r), -\underline{u}(r)]$;

(ii) $\ominus[u]^r = [-\underline{u}(r), -\overline{u}(r)]$ *if a Hukuhara difference exists;*

(iii) $\tilde{0} \ominus_{gH} [u]^r = [-\overline{u}(r), -\underline{u}(r)] = (-1)[u]^r$;

(iv) $\tilde{0} \ominus_{gH} (-1)[u]^r = [\underline{u}(r), \overline{u}(r)] = [u]^r$;

(v) $[u]^r \ominus_{gH} [v]^r = [u]^r \ominus [v]^r$ *or* $[u]^r \ominus_{gH} [v]^r = -([v]^r \ominus [u]^r)$ *whenever the expressions on the right exist;*

(vi) $[v]^r \ominus_{gH} (-1)[u]^r \neq [v]^r + [u]^r$.

2.1. Fuzzy Function of One-Variable

Now, we present some definitions, theorems, and notations for a fuzzy-valued function of one-variable.

Consider the fuzzy-valued function $f : D \to E^1$. The parametric representation of this function is the ordered pair

$$f(x, r) = \left(\underline{f}(x, r), \overline{f}(x, r)\right) \text{ for any } 0 \leq r \leq 1.$$

Definition 8 ([29])**.** *A fuzzy-valued function $f : (a, b) \to E^1$ is said to be continuous at $x_0 \in (a, b)$ if for each $\varepsilon > 0$, there is $\delta > 0$ such that $D(f(x), f(x_0)) < \varepsilon$ whenever $|x - x_0| < \delta$.*

If the function f is continuous for each $x \in (a, b)$, then we say that f is continuous on $[a, b]$.

Definition 9 ([28])**.** *The generalized Hukuhara derivative (gH-derivative) of a fuzzy-valued function $f : (a, b) \to E^1$ at $x_0 \in (a, b)$ is defined as*

$$f'(x_0) = \lim_{h \to 0} \frac{1}{h} \left[f(x_0 + h) \ominus_{gH} f(x_0) \right]. \tag{1}$$

If $f'(x_0) \in E^1$ satisfying (1) exists, we say that f is a generalized Hukuhara differentiable at x_0.

Example 2. *Let $f(x) = g(x) \odot u$, where $a(x)$ is a crips differentiable function and $u \in E^1$. Then, it follows that the gH-derivative exists, and it is $f'(x) = g'(x) \odot u$.*

Definition 10 ([28])**.** *Let $f : (a,b) \to E^1$ and $x_0 \in (a,b)$, with $\underline{f}(.,r)$ and $\overline{f}(.,r)$ both differentiable at x_0. We say that*

(i) *the function f is (i)-gH-differentiable at x_0 if*

$$f'_{(i)}(x_0, r) = \left(\underline{f}'(x_0, r), \overline{f}'(x_0, r)\right) \quad for\ each\ r \in [0,1], \tag{2}$$

(ii) *the function f is (ii)-gH-differentiable at x_0 if*

$$f'_{(ii)}(x_0, r) = \left(\overline{f}'(x_0, r), \underline{f}'(x_0, r)\right) \quad for\ each\ r \in [0,1]. \tag{3}$$

We will use the following theorem and the formula for integration by parts to obtain the formulas of the fuzzy Sawi transform from derivatives of the function $f : (a,b) \to E'$.

Theorem 1 ([28])**.** *Let $f : [a,b] \to E^1$ be gH-differentiable with no switching point in the interval $[a,b]$. Then,*

$$(FR)\int_a^b f'(x)dx = f(b) \ominus_{gH} f(a).$$

Lemma 1 ([25])**.** *Let $f,\ g : \mathbb{R}_+ \to E^1$ be fuzzy-valued functions, which are improper fuzzy Riemann-integrable on $\mathbb{R}_+$. Then,*

(i) $(FR)\int_0^\infty [a \odot f(x) \ominus_{gH} b \odot g(x)]dx = a \odot (FR)\int_0^\infty f(x)dx \ominus_{gH} b \odot (FR)\int_0^\infty g(x)dx;$

(ii) $(FR)\int_0^\infty [a \odot f(x) \oplus b \odot g(x)]dx = a \odot (FR)\int_0^\infty f(x)dx \oplus b \odot (FR)\int_0^\infty g(x)dx,$

where a and b are nonzero constants.

Theorem 2 ([10], Integration by part)**.** *Let $f : [a,b] \to E^1$ be gH-differentiable with no switching point in the interval $[a,b]$ and $g : [a,b] \to \mathbb{R}$ be differentiable functions. Then,*

$$(FR)\int_a^b f'(x) \odot g(x)dx = (f(b) \odot g(b)) \ominus_{gH} (f(a) \odot g(a)) \ominus_{gH} (FR)\int_a^b f(x) \odot g'(x)dx.$$

2.2. Fuzzy Function of a Two-Variable

In this section, we present some definitions and theorems for the fuzzy-valued function of a two-variable. Furthermore, we introduce the generalized Hukuhara partial derivative.

Consider $f : D \subset \mathbb{R} \times \mathbb{R} \to E^1$ as a fuzzy-valued function of a two-variable. The parametric representation of this function is the ordered pair

$$f(x,y,r) = \left(\underline{f}(x,y,r), \overline{f}(x,y,r)\right) \ for\ any\ 0 \le r \le 1.$$

Definition 11 ([22])**.** *A fuzzy-valued function $f : D \to E^1$ is said to be continuous at $(x_0, y_0) \in D$ if for each $\varepsilon > 0$, there is $\delta > 0$ such that $D(f(x,y), f(x_0,y_0)) < \varepsilon$ whenever $|x - x_0| + |y - y_0| < \delta$. If f is continuous for each $(x,y) \in D$, then we say that f is continuous on D.*

Definition 12 ([12]). *The first generalized Hukuhara partial derivative (gH-p-derivative) of a fuzzy-valued function* $f : D \to E^1$ *at* $(x_0, y_0) \in D$ *with respect to variables x and y are the functions* $f'_x(x_0, y_0)$ *and* $f'_y(x_0, y_0)$*, defined by*

$$f'_x(x_0, y_0) = \lim_{h \to 0} \frac{1}{h} [f(x_0 + h, y_0) \ominus_{gH} f(x_0, y_0)],$$

$$f'_y(x_0, y_0) = \lim_{k \to 0} \frac{1}{k} [f(x_0, y_0 + k) \ominus_{gH} f(x_0, y_0)]$$

provided that $f'_x(x_0, y_0), \ \ f'_y(x_0, y_0) \in E^1$.

Definition 13 ([12]). *Let* $f : D \to E^1$*, and* $\underline{f}(x, y, r)$ *and* $\overline{f}(x, y, r)$ *are partial differentiable at* $(x_0, y_0) \in D$ *with respect to variable x. We say that*

(i) *the function* $f(x, y)$ *is* (i)*-gH-p-differentiable at* (x_0, y_0) *with respect to variable x if*

$$f'_{x_{(i)}}(x_0, y_0, r) = \left(\underline{f}'_x(x_0, y_0, r), \overline{f}'_x(x_0, y_0, r) \right) \ \text{for each } r \in [0, 1], \tag{4}$$

(ii) *the function* $f(x, y)$ *is* (ii)*-gH-p -differentiable at* (x_0, y_0) *with respect to variable x if*

$$f'_{x_{(ii)}}(x_0, y_0, r) = \left(\overline{f}'_x(x_0, y_0, r), \underline{f}'_x(x_0, y_0, r) \right) \ \text{for each } r \in [0, 1]. \tag{5}$$

Theorem 3 ([10]). *Let* $f : [a, \infty) \times [c, \infty) \to E^1$ *be a fuzzy-valued function. Suppose that* $(FR) \int_c^\infty f(x, y) dy$ *is convergent for each* $[a, \infty)$*, and* $(FR) \int_a^\infty f(x, y) dx$ *as a function y is convergent on* $[c, \infty)$*. Then,*

$$(FR) \int_a^\infty (FR) \int_c^\infty f(x, y) dy dx = (FR) \int_c^\infty (FR) \int_a^\infty f(x, y) dx dy,$$

for all $a, \ c \in \mathbb{R}$.

3. Fuzzy Sawi Transform

In this section, we introduce the fuzzy Sawi transform (FST) [30] and prove some basic properties for it.

Definition 14. *Let* $f : \mathbb{R}_+ \to E^1$ *be a fuzzy-valued function, and for* $\sigma > 0$*, the function* $e^{-\frac{x}{\sigma}} \odot f(x)$ *is an improper fuzzy Riemann-integrable on* $\mathbb{R}_+$*. Then, the FST of a function* $f(x)$ *is defined as*

$$F(\sigma) = S_x[f(x)] = (FR) \int_0^\infty \frac{1}{\sigma^2} e^{-\frac{x}{\sigma}} \odot f(x) dx, \tag{6}$$

where the variables σ *are used to factor the variable x in the argument of the fuzzy-valued function.*

Definition 15. *Let* $f : \mathbb{R}_+ \to E^1$ *and* $F(\sigma) = S_2[f(x)]$*. The fuzzy inverse Sawi transform of the function* $F(\sigma)$ *is defined as*

$$S_\sigma^{-1}[F(\sigma)] = f(x). \tag{7}$$

Definition 16. *A fuzzy-number-valued function* $f : \mathbb{R}_+ \to E^1$ *is said to be of exponential order* $\alpha > 0$ *if there exists a positive constant M such that for all* $x > X$

$$D(f(x), \tilde{0}) \leq M e^{\alpha x}.$$

The below theorem shows the sufficient existence condition for a fuzzy Sawi transform.

Theorem 4. *Let $f:\mathbb{R}_+ \to E^1$ be a continuous or piece-wise continuous fuzzy-valued function in every finite interval $(0,X)$ and of exponential order $\alpha > 0$. Then, the FST of $f(x)$ exists for all σ, provided $Re(\frac{1}{\sigma}) > \alpha$.*

Proof. Let $S_x[f(x)] = F(\sigma)$. From the definition of FST, we have

$$\begin{aligned} D(F(\sigma),\tilde{0}) &= D\left((FR)\int_0^\infty \tfrac{1}{\sigma^2}e^{-\frac{x}{\sigma}} \odot f(x)dx, \tilde{0}\right) \leq \int_0^\infty \tfrac{1}{\sigma^2}e^{-\frac{x}{\sigma}} D(f(x),\tilde{0})dx \\ &\leq \tfrac{M}{\sigma^2}\int_0^\infty e^{-(\frac{1}{\sigma}-\alpha)x}dx = \tfrac{M}{\sigma^2(\frac{1}{\sigma}-\alpha)} = \tfrac{M}{\sigma(1-\sigma\alpha)}, \end{aligned}$$

$for\ Re(\frac{1}{\sigma}) > \alpha$. □

In [19], a classical Sawi transform is applied on some special functions.

Let $F(\sigma) = S_x[f(x)]$. Then,

(i) if $f(x) = 1$ for $x > 0$, then $F(\sigma) = \frac{1}{\sigma}$;
(ii) if $f(x) = x^n$, where n are positive integers, then $F(\sigma) = (n!)\sigma^{n-1}$;
(iii) if $f(x) = e^{ax}$, then $F(\sigma) = \frac{1}{\sigma(1-a\sigma)}$;
(iv) if $f(x) = \sin ax$, then $F(\sigma) = \frac{a}{1+a^2\sigma^2}$;
(v) if $f(x) = \cos ax$, then $F(\sigma) = \frac{1}{\sigma(1+a^2\sigma^2)}$.

Now, we present some basic properties of FST. To start with, we will prove the linearity property of the fuzzy Sawi transform.

Theorem 5. *Let $f,\ g:\mathbb{R}_+ \to E^1$ be fuzzy-valued functions for which the FST exists. Then, the FST of a $a\odot f(x) \oplus b\odot g(x)$ and $a\odot f(x) \ominus_{gH} b\odot g(x)$ exists, and*

(i) $a\odot S_x[f(x)] \ominus_{gH} b\odot S_x[g(x)] = S_x[a\odot f(x) \ominus_{gH} b\odot g(x)]$;
(ii) $a\odot S_x[f(x)] \oplus b\odot S_x[g(x)] = S_x[a\odot f(x) \oplus b\odot g(x)]$,

where a and b are nonzero constants.

Proof. Using Definition 15 and Lemma 1, we obtain

$$\begin{aligned} a\odot S_x[f(x)] \ominus_{gH} b\odot S_x[g(x)] &= (FR)\int_0^\infty \tfrac{a}{\sigma^2}e^{-\frac{x}{\sigma}} \odot f(x)dx \ominus_{gH} (FR)\int_0^\infty \tfrac{b}{\sigma^2}e^{-\frac{x}{\sigma}} \odot g(x)dx \\ &= (FR)\int_0^\infty (\tfrac{a}{\sigma^2}e^{-\frac{x}{\sigma}} \odot f(x) \ominus_{gH} \tfrac{b}{\sigma^2}e^{-\frac{x}{\sigma}} \odot g(x))dx \\ &= (FR)\int_0^\infty \tfrac{1}{\sigma^2}e^{-\frac{x}{\sigma}}(a\odot f(x) \ominus_{gH} b\odot g(x))dx \\ &= S_x[a\odot f(x) \ominus_{gH} b\odot g(x)]. \end{aligned}$$

In a similar manner, we obtain the proof for part (ii). □

The fuzzy Sawi transform for a derivative of a continuous fuzzy-valued function is given in the following theorem.

Theorem 6. *Let $f:\mathbb{R}_+ \to E^1$ be a continuous fuzzy-valued function of exponential order $\alpha > 0$. Moreover, let $f'(x)$ be piece-wise continuous in every finite closed interval $0 \leq x \leq b$. Then, for $Re(\frac{1}{\sigma}) > \alpha$, we have*

(i) $S_x[f'(x)] = (-1)\frac{1}{\sigma^2} \odot f(0) \ominus_{gH} (-1)\frac{1}{\sigma} \odot F(\sigma)$;
(ii) $S_x[f''(x)] = (-1)\frac{1}{\sigma^2} \odot f'(0) \ominus_{gH} \left(\frac{1}{\sigma^3} \odot f(0) \ominus_{gH} \frac{1}{\sigma^2} \odot F(\sigma)\right)$,

where $S_x[f(x)] = F(\sigma)$.

Proof. From the definition of the improper fuzzy Riemann-integral, we have

$$S_x[f'(x)] = (FR)\int_0^{\infty} \frac{1}{\sigma^2} e^{-\frac{x}{\sigma}} \odot f'(x)dx = \lim_{R\to\infty}(FR)\int_0^{R} \frac{1}{\sigma^2} e^{-\frac{x}{\sigma}} \odot f'(x)dx$$

provided this limit exist, which is, Theorem 2

$$\begin{aligned}&(FR)\int_0^{R} \tfrac{1}{\sigma^2} e^{-\frac{x}{\sigma}} \odot f'(x)dx = \tfrac{1}{\sigma^2} e^{-\frac{x}{\sigma}} \odot f(x)|_0^R \ominus_{gH} (-1)(FR)\int_0^{R} \tfrac{1}{\sigma^3} e^{-\frac{x}{\sigma}} \odot f(x)dx \\ &= \tfrac{1}{\sigma^2} e^{-\frac{R}{\sigma}} \odot f(R) \ominus_{gH} \tfrac{1}{\sigma^2} \odot f(0) \ominus_{gH} (-1)(FR)\int_0^{R} \tfrac{1}{\sigma^3} e^{-\frac{x}{\sigma}} \odot f(x)dx.\end{aligned}$$

The fuzzy-valued function f is of exponential order $\alpha > 0$. Thus, there exists $M > 0$ and $x_0 > 0$ such that $D(f(x), \tilde{0}) \leq Me^{\alpha x}$ for $x > x_0$. Thus, if $Re(\frac{1}{\sigma}) > \alpha$, we have

$$\lim_{R\to\infty} D(e^{-\frac{R}{\sigma}} \odot f(R), \tilde{0}) \leq \lim_{R\to\infty} Me^{-(\frac{1}{\sigma}-\alpha)x} = 0$$

and

$$\lim_{R\to\infty}(FR)\int_0^{R} \frac{1}{\sigma^3} e^{-\frac{x}{\sigma}} \odot f(x)dx = \frac{1}{\sigma} \odot S_x[f(x)].$$

Hence, by the above equations and Proposition 1, we obtain

$$S_x[f'(x)] = (-1)\frac{1}{\sigma^2} \odot f(0) \ominus_{gH} (-1)\frac{1}{\sigma} \odot S_x[f(x)].$$

Using Definition 14, Proposition 1, and equation (i) of this theorem, we obtain

$$\begin{aligned} S_x[f''(x)] &= (-1)\tfrac{1}{\sigma^2} \odot f'(0) \ominus_{gH} (-1)\tfrac{1}{\sigma} \odot S_x[f'(x)] \\ &= (-1)\tfrac{1}{\sigma^2} \odot f'(0) \ominus_{gH} (-1)\tfrac{1}{\sigma}\Big((-1)\tfrac{1}{\sigma^2} \odot f(0) \ominus_{gH} (-1)\tfrac{1}{\sigma} \odot S_x[f(x)]\Big) \\ &= (-1)\tfrac{1}{\sigma^2} \odot f'(0) \ominus_{gH} \Big(\tfrac{1}{\sigma^3} \odot f(0) \ominus_{gH} \tfrac{1}{\sigma^2} \odot S_x[f(x)]\Big). \end{aligned}$$

Hence, the proof of (ii) is established. □

Corollary 1. *Let $f : \mathbb{R}_+ \times \mathbb{R}_+ \to E^1$ be continuous fuzzy-valued functions and of exponential order $\alpha > 0$. Moreover, let $f'_x(x, y)$ be piece-wise continuous in every finite closed interval $0 \leq x \leq b$. Then, for $Re(\frac{1}{\sigma}) > \alpha$, we have*

(i) $S_x[f'_x(x, y)] = (-1)\frac{1}{\sigma^2} \odot f(0, y) \ominus_{gH} (-1)\frac{1}{\sigma} \odot F(\sigma, y);$

(ii) $S_x[f''_{xx}(x, y)] = (-1)\frac{1}{\sigma^2} \odot f'_x(0, y) \ominus_{gH} \Big(\frac{1}{\sigma^3} \odot f(0, y) \ominus_{gH} \frac{1}{\sigma^2} \odot F(\sigma, y)\Big),$

where $S_x[f(x, y)] = F(\sigma, y)$.

4. Double Fuzzy Sawi Transform

In this section, we introduce a new double fuzzy integral transform that combines two fuzzy Sawi transforms of order one. We prove basic properties concerning the existence, linearity, and the double fuzzy inverse Sawi transform. Moreover, we provide the double fuzzy Sawi transform of some basic functions. We establish new results relative to gH-partial derivatives of the new transform.

Definition 17. *Let $f : \mathbb{R}_+ \times \mathbb{R}_+ \to E^1$ be a fuzzy-valued function, and for $\sigma > 0$, $\delta > 0$, the function $e^{-\frac{x}{\sigma}-\frac{y}{\delta}} \odot f(x,y)$ is improper fuzzy Riemann-integrable on $\mathbb{R}_+ \times \mathbb{R}_+$. Then, the double fuzzy Sawi transform of a function $f(x,y)$ is defined as*

$$\begin{aligned} F(\sigma,\delta) &= S_2[f(x,y)] = S_x[S_y[f(x,y)]] \\ &= (FR)\int_0^\infty (FR)\int_0^\infty \frac{1}{\sigma^2\delta^2} e^{-\frac{x}{\sigma}-\frac{y}{\delta}} \odot f(x,y)dxdy, \end{aligned} \tag{8}$$

where the variables σ and δ are used to factor the variable x and y in the arguments of the fuzzy-valued function.

Definition 18. *Let $f : \mathbb{R}_+ \times \mathbb{R}_+ \to E^1$ and $F(\sigma,\delta) = S_2[f(x,y)]$. The double fuzzy inverse Sawi transform of the function $F(\sigma,\delta)$ is defined as*

$$S_\sigma^{-1}[S_\delta^{-1}[F(\sigma,\delta)]] = S_\sigma^{-1}[F(\sigma,y)] = f(x,y). \tag{9}$$

Definition 19. *A fuzzy-number-valued function $f : \mathbb{R}_+ \times \mathbb{R}_+ \to E^1$ is said to be of exponential order $\alpha > 0$ and $\beta > 0$ if there exists a positive constant M such that for all $x > X$ and $y > Y$*

$$D(f(x,y),\tilde{0}) \leq Me^{\alpha x+\beta x}.$$

The following theorem gives the existence conditions for a double fuzzy Sawi transform.

Theorem 7. *Let $f : \mathbb{R}_+ \times \mathbb{R}_+ \to E^1$ be a continuous or piece-wise continuous fuzzy-valued function in every finite interval $(0,X)$ and $(0,Y)$ and of exponential order $\alpha > 0$ and $\beta > 0$. Then, the DFST of $f(x,y)$ exists for all σ an δ provided $Re(\frac{1}{\sigma}) > \alpha$ and $Re(\frac{1}{\delta}) > \beta$.*

Proof. Let $S_2[f(x,y)] = F(\sigma,\delta)$. From the definition of DFST, we have

$$\begin{aligned} D(F(\sigma,\delta),\tilde{0}) &= D\left((FR)\int_0^\infty (FR)\int_0^\infty \frac{1}{\sigma^2\delta^2} e^{-\frac{x}{\sigma}-\frac{y}{\delta}} \odot f(x,y)dxdy, \tilde{0}\right) \\ &\leq \int_0^\infty \frac{1}{\sigma^2\delta^2} e^{-\frac{x}{\sigma}-\frac{y}{\delta}} D(f(x,y),\tilde{0})dxdy \\ &\leq \frac{M}{\sigma^2}\int_0^\infty e^{-(\frac{1}{\sigma}-\alpha)x}\left(\frac{1}{\delta^2}\int_0^\infty e^{-(\frac{1}{\delta}-\beta)y}dy\right)dx \\ &= \frac{M}{\sigma^2\delta^2(\frac{1}{\sigma}-\alpha)(\frac{1}{\delta}-\beta)} = \frac{M}{\sigma\delta(1-\sigma\alpha)(1-\delta\beta)}, \end{aligned}$$

for $Re(\frac{1}{\sigma}) > \alpha$, $Re(\frac{1}{\delta}) > \beta$. □

Lemma 2. *Let $f(x,y) = a \in E^1$. Then, for $x > 0$ and $y > 0$, we have*

$$S_2[f(x,y)] = \frac{1}{\sigma\delta} \odot a.$$

Proof. By Definition 17, we can write

$$\begin{aligned} S_2[f(x,y)] &= (FR)\int_0^\infty (FR)\int_0^\infty \frac{1}{\sigma^2\delta^2} e^{-\frac{x}{\sigma}-\frac{y}{\delta}} \odot adxdy \\ &= a \odot \lim_{\tau\to\infty}\int_0^\tau \frac{1}{\sigma^2} e^{-\frac{x}{\sigma}}dx \lim_{\xi\to\infty}\int_0^\xi \frac{1}{\delta^2} e^{-\frac{y}{\delta}}dy \\ &= a \odot \lim_{\tau\to\infty}\left(-\frac{1}{\tau}e^{-\frac{\tau}{\sigma}} + \frac{1}{\sigma}\right)\lim_{\xi\to\infty}\left(-\frac{1}{\xi}e^{-\frac{\xi}{\delta}} + \frac{1}{\delta}\right) = \frac{1}{\sigma\delta} \odot a. \end{aligned}$$

Hence, the proof is obtained. □

A classical double Sawi transform is performed on some special functions. Let $F(\sigma,\delta) = S_2[f(x,y)]$. Then,

(i) if $f(x,y) = 1$ for $x > 0,\ y > 0$, then $F(\sigma,\delta) = \frac{1}{\sigma\delta}$;
(ii) if $g(x,y) = x^m y^n$, where $m,\ n$ are positive integers, then $F(\sigma,\delta) = (m!)(n!)\sigma^{m-1}\delta^{n-1}$;
(iii) if $f(x,y) = e^{ax+by}$, where $a,\ b$ are any constants, then $F(\sigma,\delta) = \frac{1}{\sigma\delta(1-a\sigma)(1-b\delta)}$;
(iv) if $f(x,y) = e^{i(ax+by)}$, where $a,\ b \in \mathbb{R}$, then $F(\sigma,\delta) = \frac{1}{\sigma\delta(1-ia\sigma)(1-ib\delta)} = \frac{(1-ia\sigma)(1-ib\delta)}{\sigma\delta(1+a^2\sigma^2)(1+b^2\delta^2)} = \frac{1-ab\sigma\delta+i(a\sigma+b\delta)}{\sigma\delta(1+a^2\sigma^2)(1+b^2\delta^2)}$.
Consequently,

$S_2[\cos(ax+by)] = \frac{1-ab\sigma\delta}{\sigma\delta(1+a^2\sigma^2)(1+b^2\delta^2)}$, $S_2[\sin(ax+by)] = \frac{a\sigma+b\delta}{\sigma\delta(1+a^2\sigma^2)(1+b^2\delta^2)}$.

Clearly, the double fuzzy Sawi transform is a linear transformation as shown below.

Theorem 8. *Let $f,\ g : \mathbb{R}_+ \times \mathbb{R}_+ \to E^1$ be fuzzy-valued functions for which the DFST exists. Then, the DFST of a $a \odot f(x,y) \oplus b \odot g(x,y)$ and $a \odot f(x,y) \ominus_{gH} b \odot g(x,y)$ exists, and*

(i) $a \odot S_2[f(x,y)] \ominus_{gH} b \odot S_2[g(x,y)] = S_2[a \odot f(x,y) \ominus_{gH} b \odot g(x,y)]$;
(ii) $a \odot S_2[f(x,y)] \oplus b \odot S_2[g(x,y)] = S_2[a \odot f(x,y) \oplus b \odot g(x,y)]$,

where a and b are nonzero constants.

Proof. The proof follows from Definition 17 and Theorem 5. □

Now, we present some fundamental properties of DFST.

Theorem 9. *(Shifting property) Let $f(x,y)$ be a continuous fuzzy-valued function. If $F(\sigma,\delta)$ is the double fuzzy Sawi transform of $f(x,y)$ and $a,\ b$ are arbitrary constants, then*

$$S_2[e^{ax+by} \odot f(x,y)] = \frac{1}{(1-a\sigma)^2(1-b\delta)^2} \odot F\left(\frac{\sigma}{1-a\sigma}, \frac{\delta}{1-b\delta}\right). \tag{10}$$

Proof. From Definition 17, we have

$$\begin{aligned} S_2[e^{ax+by} \odot f(x,y)] &= (FR)\int_0^\infty (FR)\int_0^\infty \frac{1}{\sigma^2\delta^2} e^{-x(\frac{1}{\sigma}-a)-y(\frac{1}{\delta}-b)} \odot f(x,y)dxdy \\ &= (FR)\int_0^\infty (FR)\int_0^\infty \frac{1}{\sigma^2\delta^2} e^{-x(\frac{1-a\sigma}{\sigma})-y(\frac{1-b\delta}{\delta})} \odot f(x,y)dxdy \\ &= (FR)\int_0^\infty (FR)\int_0^\infty \frac{1}{(1-a\sigma)^2(1-b\delta)^2} e^{-x(\frac{1-a\sigma}{\sigma})-y(\frac{1-b\delta}{\delta})} \odot f(x,y)dxdy \\ &= \frac{1}{(1-a\sigma)^2(1-b\delta)^2} \odot F\left(\frac{\sigma}{1-a\sigma}, \frac{\delta}{1-b\delta}\right). \end{aligned}$$

Hence, the proof is obtained. □

Theorem 10. *(Change of scale property) Let $f(x,y)$ be a continuous fuzzy-valued function. If $F(\sigma,\delta)$ is the double fuzzy Sawi transform of $f(x,y)$ and $a,\ b$ are arbitrary constants, then*

$$S_2[f(ax,by)] = ab \odot F(\sigma,\delta). \tag{11}$$

Proof. From Definition 17, we have

$$S_2[f(ax,by)] = (FR)\int\limits_0^\infty (FR)\int\limits_0^\infty \frac{1}{\sigma^2\delta^2}e^{-\frac{x}{\sigma}-\frac{y}{\delta}} \odot f(ax,by)dxdy.$$

Put $ax = x_1, adx = dx_1, by = y_1, bdy = dy_1$ in the above equation, and we have

$$S_2[f(ax,by)] = (FR)\int\limits_0^\infty (FR)\int\limits_0^\infty \frac{ab}{a^2\sigma^2 b^2\delta^2}e^{-\frac{x_1}{a\sigma}-\frac{y_1}{b\delta}} \odot f(x_1,y_1)dx_1dy_1 = ab \odot F(\sigma,\delta). \quad \square$$

Theorem 11. *(Derivatives' properties)*

If $S_2[f(x,y)] = F(\sigma,\delta)$, then

(i) $S_2[f'_x(x,y)] = (-1)\frac{1}{\sigma^2} \odot S_y[f(0,y)] \ominus_{gH} (-1)\frac{1}{\sigma} \odot F(\sigma,\delta)$;

(ii) $S_2[f'_y(x,y)] = (-1)\frac{1}{\delta^2} \odot S_x[f(x,0)] \ominus_{gH} (-1)\frac{1}{\delta} \odot F(\sigma,\delta)$;

(iii) $S[f''_{xx}(x,y)] = (-1)\frac{1}{\sigma^2} \odot S_y[f'_x(0,y)] \ominus_{gH} \left(\frac{1}{\sigma^3}S_y[f(0,y)] \ominus_{gH} \frac{1}{\sigma^2} \odot F(\sigma,\delta)\right)$;

(iv) $S_2[f''_{yy}(x,y)] = (-1)\frac{1}{\delta^2} \odot S_x[f'_y(x,0)] \ominus_{gH} \left(\frac{1}{\delta^3} \odot S_x[f(x,0)] \ominus \frac{1}{\delta^2} \odot F(\sigma,\delta)\right)$;

(v) $S_2[f''_{xy}(x,y)] = (-1)\frac{1}{\delta^2} \odot S_x[f'_x(x,0)] \ominus_{gH} \left(\frac{1}{\sigma^2\delta} \odot S_y[f(0,y)] \ominus \frac{1}{\sigma\delta} \odot F(\sigma,\delta)\right)$;

(vi) $S_2[f''_{yx}(x,y)] = (-1)\frac{1}{\sigma^2} \odot S_y[f'_y(0,y)] \ominus_{gH} \left(\frac{1}{\sigma\delta^2} \odot S_x[f(x,0)] \ominus_{gH} \frac{1}{\sigma\delta} \odot F(\sigma,\delta)\right)$.

Proof. From Definition 17, Theorem 5, and Corollary 1, we obtain

$$\begin{aligned}
&S_2[f'_x(x,y)] = S_y[S_x[f'_x(x,y)]] \\
&= S_y\left[(-1)\frac{1}{\sigma^2} \odot f(0,y) \ominus_{gH} (-1)\frac{1}{\sigma} \odot S_x[f(x,y)]\right] \\
&= (-1)\frac{1}{\sigma^2} \odot S_y[f(0,y)] \ominus_{gH} (-1)\frac{1}{\sigma} \odot S_y[S_x[f(x,y)]] \\
&= (-1)\frac{1}{\sigma^2} \odot S_y[f(0,y)] \ominus_{gH} (-1)\frac{1}{\sigma} \odot F(\sigma,\delta).
\end{aligned}$$

A similar procedure can be used to prove the case (ii).

$$\begin{aligned}
&S_2[f''_{xx}(x,y)] = S_y[S_x[f''_{xx}(x,y)]] \\
&= S_y\left[(-1)\frac{1}{\sigma^2} \odot f'_x(0,y) \ominus_{gH} \left(\frac{1}{\sigma^3} \odot f(0,y) \ominus_{gH} \frac{1}{\sigma^2} \odot S_x[f(x,y)]\right)\right] \\
&= (-1)\frac{1}{\sigma^2} \odot S_y[f'_x(0,y)] \ominus_{gH} \left(\frac{1}{\sigma^3} \odot S_y[f(0,y)] \ominus_{gH} \frac{1}{\sigma^2} \odot S_y[S_x[f(x,y)]]\right) \\
&= (-1)\frac{1}{\sigma^2} \odot S_y[f'_x(0,y)] \ominus_{gH} \left(\frac{1}{\sigma^3} \odot S_y[f(0,y)] \ominus_{gH} \frac{1}{\sigma^2} \odot F(\sigma,\delta)\right).
\end{aligned}$$

The proof of case (iv) is analogous to the proof of case (iii).

$$\begin{aligned}
&S_2[f''_{xy}(x,y)] = S_x[S_y[f''_{xy}(x,y)]] \\
&= S_x\left[(-1)\frac{1}{\delta^2} \odot f'_x(x,0) \ominus_{gH} \frac{1}{\delta} \odot S_y[f'_x(x,y)]\right] \\
&= (-1)\frac{1}{\delta^2} \odot S_x[f'_x(x,0)] \ominus_{gH} \frac{1}{\delta} \odot S_y[S_x[f'_x(x,y)]] \\
&= (-1)\frac{1}{\delta^2} \odot S_x[f'_x(x,0)] \ominus_{gH} \frac{1}{\delta} \odot S_y\left[(-1)\frac{1}{\sigma^2} \odot f(0,y) \ominus_{gH} (-1)\frac{1}{\sigma} \odot S_x[f(x,y)]\right] \\
&= (-1)\frac{1}{\delta^2} \odot S_x[f'_x(x,0)] \ominus_{gH} \left(\frac{1}{\delta\sigma^2} \odot S_y[f(0,y)] \ominus_{gH} \frac{1}{\delta\sigma} \odot S_y[S_x[f(x,y)]]\right) \\
&= (-1)\frac{1}{\delta^2} \odot S_x[f'_x(x,0)] \ominus_{gH} \left(\frac{1}{\delta\sigma^2} \odot S_y[f(0,y)] \ominus_{gH} \frac{1}{\delta\sigma} \odot F(\sigma,\delta)\right).
\end{aligned}$$

The proof of case (vi) is analogous to the proof of case (v). $\square$

5. Double Fuzzy Sawi Transform to Solve Fuzzy Telegraph Equation

In this section, we demonstrate the application of the DFST method for solving of the linear fuzzy telegraph equation. This equation is defined as

$$w''_{xx}(x,t) = a \odot w''_{tt}(x,t) \oplus b \odot w'_t(x,t) \oplus c \odot w(x,t) \oplus g(x,t), \tag{12}$$

where $a,\ b,\ c \in \mathbb{R}_+$, with initial conditions

$$w(x,0) = \psi_0(x),\ \ w'_t(x,0) = \psi_1(x) \tag{13}$$

and boundary conditions

$$w(0,t) = \varphi_0(t),\ \ w'_x(0,t) = \varphi_1(t). \tag{14}$$

We apply DFST for Definition 17 on both side of the Equation (12). By using Theorem 8, we obtain the following

$$S_2[w''_{xx}(x,t)] = a \odot S_2[w''_{tt}(x,t)] \oplus b \odot S_2[w'_t(x,t)] \oplus c \odot S_2[w(x,t)] \oplus S_2[g(x,t)].$$

Applying Theorem 11, we have

$$\begin{aligned}
&(-1)\tfrac{1}{\sigma^2} \odot S_t[w'_x(0,t)] \ominus_{gH} \left(\tfrac{1}{\sigma^3} S_t[w(0,t)] \ominus_{gH} \tfrac{1}{\sigma^2} \odot W(\sigma,\delta)\right) \\
&= a \odot \left[(-1)\tfrac{1}{\delta^2} \odot S_x[w'_t(x,0)] \ominus_{gH} \left(\tfrac{1}{\delta^3} \odot S_x[w(x,0)] \ominus_{gH} \tfrac{1}{\delta^2} \odot W(\sigma,\delta)\right)\right] \\
&\oplus b \odot \left[(-1)\tfrac{1}{\delta^2} \odot S_x[w(x,0)] \ominus_{gH} (-1)\tfrac{1}{\delta} \odot W(\sigma,\delta)\right] \oplus c \odot W(\sigma,\delta) \oplus G(\sigma,\delta),
\end{aligned}$$

where $W(\sigma,\delta) = S_2[w(x,t)]$ and $G(\sigma,\delta) = S_2[g(x,t)]$. Then, from initial conditions (13) and boundary conditions (14), we have

$$\begin{aligned}
&(-1)\tfrac{1}{\sigma^2} \odot S_t[\varphi_1(t)] \ominus_{gH} \left(\tfrac{1}{\sigma^3} S_t[\varphi_0(t)] \ominus_{gH} \tfrac{1}{\sigma^2} \odot W(\sigma,\delta)\right) \\
&= a \odot \left[(-1)\tfrac{1}{\delta^2} \odot S_x[\psi_1(x)] \ominus_{gH} \left(\tfrac{1}{\delta^3} \odot S_x[\psi_0(x)] \ominus_{gH} \tfrac{1}{\delta^2} \odot W(\sigma,\delta)\right)\right] \\
&\oplus b \odot \left[(-1)\tfrac{1}{\delta^2} \odot S_x[\psi_0(x)] \ominus_{gH} (-1)\tfrac{1}{\delta} \odot W(\sigma,\delta)\right] \oplus c \odot W(\sigma,\delta) \oplus G(\sigma,\delta).
\end{aligned}$$

Denote $\Psi_0(\sigma) = S_x[\psi_0(x)]$, $\Psi_1(\sigma) = S_x[\psi_1(x)]$, $\Phi_0(\delta) = S_t[\varphi_0(t)]$ and $\Phi_1(\delta) = S_t[\varphi_1(t)]$. Then, we obtain

$$\begin{aligned}
&(-1)\tfrac{1}{\sigma^2} \odot \Phi_1(\delta) \ominus_{gH} \left(\tfrac{1}{\sigma^3} \Phi_0(\delta) \ominus_{gH} \tfrac{1}{\sigma^2} \odot W(\sigma,\delta)\right) \\
&= a \odot \left[(-1)\tfrac{1}{\delta^2} \odot \Psi_1(\sigma) \ominus_{gH} \left(\tfrac{1}{\delta^3} \odot \Psi_0(\sigma) \ominus_{gH} \tfrac{1}{\delta^2} \odot W(\sigma,\delta)\right)\right] \\
&\oplus b \odot \left[(-1)\tfrac{1}{\delta^2} \odot \Psi_0(\sigma) \ominus_{gH} (-1)\tfrac{1}{\delta} \odot W(\sigma,\delta)\right] \oplus c \odot W(\sigma,\delta) \oplus G(\sigma,\delta).
\end{aligned}$$

Using Proposition 1, we have

$$\begin{aligned}
&\left(\tfrac{1}{\sigma^2} - \tfrac{a}{\delta^2} - \tfrac{b}{\delta} - c\right) \odot W(\sigma,\delta) \\
&= \tfrac{1}{\sigma^2} \odot \Phi_1(\delta) \oplus \tfrac{1}{\sigma^3} \odot \Phi_0(\delta) \oplus \tfrac{-a}{\delta^2} \odot \Psi_1(\sigma) \oplus \left(\tfrac{-a}{\delta^3} - \tfrac{b}{\delta^2}\right) \odot \Psi_0(\sigma) \oplus G(\sigma,\delta).
\end{aligned}$$

Hence,

$$\begin{aligned}
W(\sigma,\delta) &= \tfrac{A}{\sigma^2} \odot \Phi_1(\delta) \oplus \tfrac{A}{\sigma^3} \odot \Phi_0(\delta) \\
&\oplus \tfrac{-a}{\delta^2} A \odot \Psi_1(\sigma) \oplus A\left(\tfrac{-a}{\delta^3} - \tfrac{b}{\delta^2}\right) \odot \Psi_0(\sigma) \oplus A \odot G(\sigma,\delta),
\end{aligned} \tag{15}$$

where

$$A = \frac{\sigma^2\delta^2}{\delta^2 - a\sigma^2 - b\sigma^2\delta - c\sigma^2\delta^2}. \tag{16}$$

Finally, by using the inverse of DFST, we obtain $w(x,t)$.

6. Examples

Example 3. *Consider the homogeneous linear fuzzy telegraphic equation given by*

$$w''_{xx}(x,t) = w''_{tt}(x,t) \oplus w'_t(x,t) \oplus w(x,t), \quad x \geq 0, \; t \geq 0 \tag{17}$$

with initial conditions

$$w(x,0,r) = e^x \odot (1,2,3), \quad w'_t(x,0,r) = -e^x \odot (1,2,3) \tag{18}$$

and boundary conditions

$$w(0,t,r) = e^{-t} \odot (1,2,3), \quad w'_x(0,t,r) = e^{-t} \odot (1,2,3). \tag{19}$$

In this case, $a = b = c = 1$, the functions

$$\psi_0(x) = e^x \odot (1,2,3), \quad \psi_1(x) = -e^x \odot (1,2,3),$$
$$\varphi_0(t) = e^{-t} \odot (1,2,3), \quad \varphi_1(t) = e^{-t} \odot (1,2,3).$$

From Definition 14, we have

$$S_x[\psi_0(x)] = \Psi_0(\sigma) = \frac{1}{\sigma(1-\sigma)} \odot (1,2,3), \quad S_x[\psi_1(x)] = \Psi_1(\sigma) = -\frac{1}{\sigma(1-\sigma)} \odot (1,2,3),$$

$$S_t[\varphi_0(t)] = \Phi_0(\delta) = \frac{1}{\delta(1+\delta)} \odot (1,2,3), \quad S_t[\varphi_1(t)] = \Phi_1(\delta) = \frac{1}{\delta(1+\delta)} \odot (1,2,3).$$

By using the Equations (15) and (16), we obtain

$$\begin{aligned} W(\sigma,\delta) &= \tfrac{A}{\sigma^2} \tfrac{1}{\delta(1+\delta)} \odot (1,2,3) \oplus \tfrac{A}{\sigma^3} \tfrac{1}{\delta(1+\delta)} \odot (1,2,3) \\ &\oplus \tfrac{A}{\delta^2 \sigma(1-\sigma)} \odot (1,2,3) \oplus A\left(\tfrac{-1}{\delta^3} - \tfrac{1}{\delta^2}\right) \tfrac{1}{\sigma(1-\sigma)} \odot (1,2,3), \end{aligned}$$

where

$$A = \frac{\sigma^2 \delta^2}{\delta^2 - \sigma^2 - \sigma^2 \delta - \sigma^2 \delta^2}.$$

Hence,

$$\begin{aligned} W(\sigma,\delta) &= A\left(\tfrac{1}{\sigma^2 \delta(1+\delta)} + \tfrac{1}{\sigma^3 \delta(1+\delta)} - \tfrac{1}{\sigma \delta^3 (1-\sigma)}\right) \odot (1,2,3) \\ &= \tfrac{1}{\sigma(1-\sigma)\delta(1+\delta)} \odot (1,2,3). \end{aligned}$$

Applying the inverse double fuzzy Sawi transform, we find the solution of the Equations (17)–(19) is

$$w(x,t) = e^{x-t} \odot (1,2,3).$$

Example 4. *Suppose the non-homogeneous linear fuzzy telegraph equation that is given as*

$$w''_{xx}(x,t) = w''_{tt}(x,t) \oplus w'_t(x,t) \oplus w(x,t) \oplus g(x,t) \tag{20}$$

with initial conditions

$$w(x,0,r) = e^x \odot (1,2,3), \quad w'_t(x,0,r) = e^x \odot (1,2,3) \tag{21}$$

and boundary conditions

$$w(0,t,r) = e^t \odot (1,2,3), \quad w'_x(0,t,r) = e^t \odot (1,2,3), \tag{22}$$

where the function $g(x,t) = -2e^{x+t} \odot (1,2,3)$.

In this case, $a = b = c = 1$, the functions

$$\psi_0(x) = e^x \odot (1,2,3), \quad \psi_1(x) = e^x \odot (1,2,3),$$

$$\varphi_0(t) = e^t \odot (1,2,3), \quad \varphi_1(t) = e^t \odot (1,2,3).$$

From Definition 14, we have

$$S_x[\psi_0(x)] = \Psi_0(\sigma) = \frac{1}{\sigma(1-\sigma)} \odot (1,2,3), \quad S_x[\psi_1(x)] = \Psi_1(\sigma) = \frac{1}{\sigma(1-\sigma)} \odot (1,2,3),$$

$$S_t[\varphi_0(t)] = \Phi_0(\delta) = \frac{1}{\delta(1-\delta)} \odot (1,2,3), \quad S_t[\varphi_1(t)] = \Phi_1(\delta) = \frac{1}{\delta(1-\delta)} \odot (1,2,3),$$

$$G(\sigma,\delta) = \frac{-2}{\sigma\delta(1-\sigma)(1-\delta)} \odot (1,2,3).$$

By using the Equations (15) and (16), we obtain

$$\begin{aligned} &W(\sigma,\delta) = \tfrac{A}{\sigma^2}\tfrac{1}{\delta(1-\delta)} \odot (1,2,3) \oplus \tfrac{A}{\sigma^3}\tfrac{1}{\delta(1-\delta)} \odot (1,2,3) \\ &\oplus \tfrac{-A}{\delta^2}\tfrac{1}{\sigma(1-\sigma)} \odot (1,2,3) \oplus A\left(\tfrac{-1}{\delta^3} - \tfrac{1}{\delta^2}\right)\tfrac{1}{\sigma(1-\sigma)} \odot (1,2,3) \oplus \tfrac{-2A}{\sigma\delta(1-\sigma)(1-\delta)} \odot (1,2,3), \end{aligned}$$

where

$$A = \frac{\sigma^2\delta^2}{\delta^2 - \sigma^2 - \sigma^2\delta - \sigma^2\delta^2}.$$

Hence,

$$\begin{aligned} W(\sigma,\delta) \quad &= A\left(\tfrac{1}{\sigma^2\delta(1-\delta)} + \tfrac{1}{\sigma^3\delta(1-\delta)} - \tfrac{2}{\sigma\delta^2(1-\sigma)} - \tfrac{1-\delta+2\delta^2}{\sigma\delta^3(1-\sigma)(1-\delta)}\right) \odot (1,2,3) \\ &= \tfrac{1}{\sigma(1-\sigma)\delta(1-\delta)} \odot (1,2,3). \end{aligned}$$

Applying the inverse double fuzzy Sawi transform, we find the solution of the Equations (20)–(22) is

$$w(x,t) = e^{x+t} \odot (1,2,3).$$

7. Conclusions

The main idea of this paper is to develop the double fuzzy Sawi transform to solve a fuzzy telegraph equation under gH-differentiability. Fundamental properties of the new double fuzzy transformation are introduced and proven. New results on DFST for fuzzy partial gH-derivatives are obtained. Finally, we apply DFST to solve the linear non-homogeneous fuzzy telegraph equation. A simple formula for its solution is obtained. Some illustrative examples are given to show the validity and efficiency of DFST in solving the linear fuzzy telegraph equations with symmetric characteristics. The results obtained of this study show the strength and simplicity of DFST in solving linear fuzzy partial differential equations.

For future research, we intend to use DFST in solving some fuzzy integro-differential equations, as well as fuzzy partial differential equations from mathematical physics [31,32] due to the advantage of this transform in preserving the constants' values, which reduces the calculations in comparison to other double fuzzy integral transforms.

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