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Algebraic Schouten Solitons of Three-Dimensional Lorentzian Lie Groups

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Abstract: In 2016, Wears defined and studied algebraic T-solitons. In this paper, we define algebraic Schouten solitons as a special T-soliton and classify the algebraic Schouten solitons associated with Levi-Civita connections, canonical connections, and Kobayashi–Nomizu connections on three-dimensional Lorentzian Lie groups that have some product structure.

Keywords: Levi-Civita connections; canonical connections; Kobayashi–Nomizu connections; algebraic Schouten solitons; three-dimensional Lorentzian Lie groups

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1. Introduction

Lauret introduced the Ricci soliton, which is a natural generalization of the Einstein metric on nilpotent Lie groups. In [1], he introduced the algebraic Ricci soliton in the Riemannian case. Moreover, Lauret proved that algebraic Ricci solitons on homogeneous Riemannian manifolds are Ricci solitons. Onda extended the definition of algebraic Ricci solitons to the pseudo-Riemannian case and studied them in [2]. He obtained a steady algebraic Ricci soliton in the Lorentzian setting. Note that in [3], Batat and Onda studied algebraic Ricci solitons of three-dimensional Lorentzian Lie groups, and they determined all three-dimensional Lorentzian Lie groups, which are algebraic Ricci solitons. Etayo and Santamaria studied some affine connections on product structures, mainly the canonical connection and the Kobayashi–Nomizu connection. See [4] for details. Wang defined algebraic Ricci solitons associated with canonical connections and Kobayashi–Nomizu connections in [5]. Moreover, he classified algebraic Ricci solitons associated with canonical connections and Kobayashi–Nomizu connections on three-dimensional Lorentzian Lie groups with the product structure. For other results related to Ricci solitons, see [6–9].

Following Lauret's research, Wears defined algebraic T-solitons and established the relationship between algebraic T-solitons and T-solitons. In [10], the author showed that Lauret's ideas for algebraic solitons applied equally well to an arbitrary geometric evolution equation (subjection to the appropriate conditions) for a left-invariant Riemannian metric on a simply connected Lie group. In Equation (1) [7], a generalized Ricci soliton was defined, which could be considered as the Schouten soliton.

According to the generalization of the definition of the Schouten tensor in [11], motivated by [7,10], we provide a definition of algebraic Schouten solitons as Schouten solitons, which were defined in [7]. In this paper, we investigate algebraic Schouten solitons associated with Levi-Civita connections, canonical connections, and Kobayashi–Nomizu connections, and classify algebraic Schouten solitons associated with Levi-Civita connections, canonical connections, and Kobayashi–Nomizu connections on three-dimensional Lorentzian Lie groups.

This paper is organized as follows. In Section 2, we recall the classification of three-dimensional Lorentzian Lie groups. In Section 3.1, we classify algebraic Schouten solitons associated with Levi-Civita connections on three-dimensional Lorentzian Lie groups with the product structure. In Section 3.2, we classify algebraic Schouten solitons associated with canonical connections and Kobayashi–Nomizu connections on three-dimensional Lorentzian Lie groups with the product structure.

2. Three-Dimensional Unimodular Lorentzian Lie Groups

See [12]; Milnor provided a complete classification of three-dimensional unimodular Lie groups equipped with a left-invariant Riemannian metric. In [13], Rahmani classified three-dimensional unimodular Lie groups equipped with a left-invariant Lorentzian metric. Cordero and Parker wrote down the possible forms of a non-unimodular Lie algebra in [14], which was proven by Calvaruso in [15]. The following theorems classify three-dimensional Lorentzian Lie groups.

Theorem 1. *Let (G, g) be a three-dimensional connected unimodular Lie group, equipped with a left-invariant Lorentzian metric. Then there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 time-like, such that the Lie algebra of G is one of the following:*

$(\mathfrak{g}_1)$:

$$[e_1, e_2] = \alpha e_1 - \beta e_3, [e_1, e_3] = -\alpha e_1 - \beta e_2, [e_2, e_3] = \beta e_1 + \alpha e_2 + \alpha e_3, \alpha \neq 0.$$

$(\mathfrak{g}_2)$:

$$[e_1, e_2] = \gamma e_2 - \beta e_3, [e_1, e_3] = -\beta e_2 - \gamma e_3, [e_2, e_3] = \alpha e_1, \gamma \neq 0.$$

$(\mathfrak{g}_3)$:

$$[e_1, e_2] = -\gamma e_3, [e_1, e_3] = -\beta e_2, [e_2, e_3] = \alpha e_1.$$

$(\mathfrak{g}_4)$:

$$[e_1, e_2] = -e_2 + (2\eta - \beta)e_3, \eta = 1 \text{or} - 1, [e_1, e_3] = -\beta e_2 + e_3, [e_2, e_3] = \alpha e_1.$$

Theorem 2. *Let (G, g) be a three-dimensionally connected non-unimodular Lie group, equipped with a left-invariant Lorentzian metric. Then there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 time-like, such that the Lie algebra of G is one of the following:*

$(\mathfrak{g}_5)$:

$$[e_1, e_2] = 0, [e_1, e_3] = \alpha e_1 + \beta e_2, [e_2, e_3] = \gamma e_1 + \delta e_2, \alpha + \delta \neq 0, \alpha\gamma + \beta\delta = 0.$$

$(\mathfrak{g}_6)$:

$$[e_1, e_2] = \alpha e_2 + \beta e_3, [e_1, e_3] = \gamma e_2 + \delta e_3, [e_2, e_3] = 0, \alpha + \delta \neq 0, \alpha\gamma - \beta\delta = 0.$$

$(\mathfrak{g}_7)$:

$$[e_1, e_2] = -\alpha e_1 - \beta e_2 - \beta e_3, [e_1, e_3] = \alpha e_1 + \beta e_2 + \beta e_3, [e_2, e_3] = \gamma e_1 + \delta e_2 + \delta e_3,$$
$$\alpha + \delta \neq 0, \alpha\gamma = 0.$$

3. Results

This section presents the results, with (G_i, g) representing the algebraic Schouten solitons associated with Levi-Civita connections, canonical connections, and Kobayashi–Nomizu connections on three-dimensional Lorentzian Lie groups.

3.1. Algebraic Schouten Solitons Associated with Levi-Civita Connections on Three-Dimensional Lorentzian Lie Groups

Throughout this paper, by $\{G_i\}_{i=1,\cdots,7}$ we shall denote the connected, simply connected three-dimensional Lie group equipped with a left-invariant Lorentzian metric g, and having Lie algebra $\{\mathfrak{g}\}_{i=1,\cdots,7}$. Let ∇ be the Levi-Civita connection of G_i and let R be its curvature tensor, taken with the convention

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \quad (1)$$

The Ricci tensor of (G_i, g) is defined by

$$\rho(X,Y) = -g(R(X,e_1)Y,e_1) - g(R(X,e_2)Y,e_2) + g(R(X,e_3)Y,e_3), \tag{2}$$

where $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis, with e_3 being time-like and the Ricci operator (Ric) is given by

$$\rho(X,Y) = g(\mathrm{Ric}(X),Y). \tag{3}$$

The Schouten tensor is defined by

$$S(e_i,e_j) = \rho(e_i,e_j) - \frac{s}{4}g(e_i,e_j), \tag{4}$$

where s denotes the scalar curvature. We generalize the definition of the Schouten tensor to

$$S(e_i,e_j) = \rho(e_i,e_j) - s\lambda_0 g(e_i,e_j), \tag{5}$$

where λ_0 is a real number. Refer to [16], we have

$$s = \rho(e_1,e_1) + \rho(e_2,e_2) - \rho(e_3,e_3). \tag{6}$$

Definition 1. *(G_i, g) is called the algebraic Schouten soliton associated with the connection ∇ if it satisfies*

$$\mathrm{Ric} = (s\lambda_0 + c)\mathrm{Id} + D, \tag{7}$$

where c is a real number, and D is a derivation of $\mathfrak{g}$, i.e.,

$$D[X,Y] = [DX,Y] + [X,DY] \mathrm{for} X,Y \in \mathfrak{g}. \tag{8}$$

Theorem 3. *If $\beta = 0$ and $c = 0$, then this case corresponds to (G_1, g) being the algebraic Schouten soliton associated with the connection ∇.*

Proof of Theorem 1. From [3], we have

$$\mathrm{Ric}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\beta^2 & \alpha\beta & \alpha\beta \\ \alpha\beta & 2\alpha^2 + \frac{1}{2}\beta^2 & 2\alpha^2 \\ -\alpha\beta & -2\alpha^2 & -2\alpha^2 + \frac{1}{2}\beta^2 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{9}$$

Therefore, $s = \frac{3}{2}\beta^2$. We can write D as

$$\begin{cases} De_1 = (\frac{1}{2}\beta^2 - \frac{3}{2}\beta^2\lambda_0 - c)e_1 + \alpha\beta e_2 + \alpha\beta e_3, \\ De_2 = \alpha\beta e_1 + (2\alpha^2 + \frac{1}{2}\beta^2 - \frac{3}{2}\beta^2\lambda_0 - c)e_2 + 2\alpha^2 e_3, \\ De_3 = -\alpha\beta e_1 - 2\alpha^2 e_2 + (-2\alpha^2 + \frac{1}{2}\beta^2 - \frac{3}{2}\beta^2\lambda_0 - c)e_3. \end{cases} \tag{10}$$

Hence, by (8), there exists an algebraic Schouten soliton associated with the connection ∇ if and only if the following system of equations is satisfied

$$\begin{cases} \frac{3}{2}\alpha\beta^2\lambda_0 + \alpha(\frac{3}{2}\beta^2 + c) = 0, \\ -\frac{3}{2}\beta^3\lambda_0 + \beta(\frac{1}{2}\beta^2 - c) = 0, \\ \alpha\beta = 0, \\ -\frac{3}{2}\beta^3\lambda_0 + \beta(\frac{1}{2}\beta^2 + 6\alpha^2 - c) = 0, \\ -\frac{3}{2}\beta^3\lambda_0 + \beta(\frac{1}{2}\beta^2 - 6\alpha^2 - c) = 0. \end{cases} \tag{11}$$

Since $\alpha \neq 0$, we have $\beta = 0$ and $c = 0$. □

Theorem 4. *If $\alpha = \beta = 0$ and $c = 2\gamma^2(1 - \lambda_0)$ are satisfied, (G_2, g) is the algebraic Schouten soliton associated with the connection ∇.*

Proof of Theorem 2. According to [3], we have

$$\mathrm{Ric}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\alpha^2+2\gamma^2 & 0 & 0 \\ 0 & -\frac{1}{2}\alpha^2+\alpha\beta & -\alpha\gamma+2\beta\gamma \\ 0 & \alpha\gamma-2\beta\gamma & -\frac{1}{2}\alpha^2+\alpha\beta \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{12}$$

Consequently, the scalar curvature is given by $s = -\frac{1}{2}\alpha^2 + 2\alpha\beta + 2\gamma^2$. We have

$$\begin{cases} De_1 = (\frac{1}{2}\alpha^2+2\gamma^2-(-\frac{1}{2}\alpha^2+2\alpha\beta+2\gamma^2)\lambda_0-c)e_1, \\ De_2 = (-\frac{1}{2}\alpha^2+\alpha\beta-(-\frac{1}{2}\alpha^2+2\alpha\beta+2\gamma^2)\lambda_0-c)e_2+(-\alpha\gamma+2\beta\gamma)e_3, \\ De_3 = (\alpha\gamma-2\beta\gamma)e_2+(-\frac{1}{2}\alpha^2+\alpha\beta-(-\frac{1}{2}\alpha^2+2\alpha\beta+2\gamma^2)\lambda_0-c)e_3. \end{cases} \tag{13}$$

Equation (8) is satisfied if and only if

$$\begin{cases} -(-\frac{1}{2}\alpha^2+2\alpha\beta+2\gamma^2)\lambda_0+\frac{1}{2}\alpha^2+2\gamma^2-c=0, \\ \beta(-\frac{1}{2}\alpha^2+2\alpha\beta+2\gamma^2)\lambda_0+2\gamma^2(\alpha-2\beta)-\beta(\frac{1}{2}\alpha^2+2\gamma^2-c)=0, \\ \alpha(-\frac{1}{2}\alpha^2+2\alpha\beta+2\gamma^2)\lambda_0+\alpha(\frac{3}{2}\alpha^2+2\gamma^2-2\alpha\beta+c)=0. \end{cases} \tag{14}$$

The first and second equations of system (14) imply that

$$(\alpha-2\beta)(\beta^2+\gamma^2)=0. \tag{15}$$

Since $\gamma \neq 0$, we have $\alpha = 2\beta$. In this case, system (14) reduces to

$$\begin{cases} -(\frac{1}{2}\alpha^2+2\gamma^2)\lambda_0+\frac{1}{2}\alpha^2+2\gamma^2-c=0, \\ \alpha(\frac{1}{2}\alpha^2+2\gamma^2)\lambda_0+\alpha(\frac{1}{2}\alpha^2+2\gamma^2+c)=0. \end{cases} \tag{16}$$

If $\alpha = 0$, then we have $\beta = 0$, $c = 2\gamma^2(1-\lambda_0)$. If $\alpha \neq 0$, we have $\frac{1}{2}\alpha^2 + 2\gamma^2 = 0$. According to [3], this is a contradiction. □

Theorem 5. *If one of the following conditions is satisfied, (G_3, g) is the algebraic Schouten soliton associated with the connection ∇ :*

(i) $\alpha = \beta = \gamma = 0$, *for all* c,
(ii) $\alpha \neq 0, \beta = \gamma = 0, c = -\frac{3}{2}\alpha^2 + \frac{1}{2}\alpha^2\lambda_0$,
(iii) $\alpha = \gamma = 0, \beta \neq 0, c = -\frac{3}{2}\beta^2 + \frac{1}{2}\beta^2\lambda_0$,
(iv) $\alpha \neq 0, \beta = \alpha, \gamma = 0, c = 0$,
(v) $\alpha \neq 0, \beta = -\alpha, \gamma = 0, c = -2\alpha^2 + 2\alpha^2\lambda_0$,
(vi) $\alpha = \beta = 0, \gamma \neq 0, c = -\frac{3}{2}\gamma^2 + \frac{1}{2}\gamma^2\lambda_0$,
(vii) $\alpha \neq 0, \beta = 0, \gamma = \alpha, c = 0$,
(viii) $\alpha \neq 0, \beta = 0, \gamma = \alpha, c = -2\alpha^2 + 2\alpha^2\lambda_0$,
(ix) $\alpha = 0, \beta \neq 0, \gamma = \beta, c = 0$,
(x) $\alpha = 0, \beta \neq 0, \gamma = -\beta, c = -2\beta^2 + 2\beta^2\lambda_0$,
(xi) $\alpha \neq 0, \beta \neq 0, \gamma = \alpha, c = \frac{1}{2}\beta^2 - (2\alpha\beta - \frac{1}{2}\beta^2)\lambda_0$,
(xii) $\alpha \neq 0, \beta \neq 0, \gamma = \beta - \alpha, c = 2\alpha\gamma - 2\alpha\gamma\lambda_0$.

Proof of Theorem 3. By [3], we put

$$a_1 = \frac{1}{2}(\alpha-\beta-\gamma), a_2 = \frac{1}{2}(\alpha-\beta+\gamma), a_3 = \frac{1}{2}(\alpha+\beta-\gamma). \tag{17}$$

The Ricci operator is given by

$$\mathrm{Ric}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ 0 & 0 & l_3 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{18}$$

where $l_1 = a_1a_2 + a_1a_3 + \beta a_2 + \gamma a_3$, $l_2 = a_1a_2 - a_2a_3 - \alpha a_1 + \gamma a_3$, $l_3 = a_1a_3 - a_2a_3 - \alpha a_1 + \beta a_2$. Moreover, we have $s = 2a_1a_2 + 2a_1a_3 - 2a_2a_3 - 2\alpha a_1 + 2\beta a_2 + 2\gamma a_3$. So

$$\begin{cases} De_1 = (l_1 - (2a_1a_2 + 2a_1a_3 - 2a_2a_3 - 2\alpha a_1 + 2\beta a_2 + 2\gamma a_3)\lambda_0 - c)e_1, \\ De_2 = (l_2 - (2a_1a_2 + 2a_1a_3 - 2a_2a_3 - 2\alpha a_1 + 2\beta a_2 + 2\gamma a_3)\lambda_0 - c)e_2, \\ De_3 = (l_3 - (2a_1a_2 + 2a_1a_3 - 2a_2a_3 - 2\alpha a_1 + 2\beta a_2 + 2\gamma a_3)\lambda_0 - c)e_3. \end{cases} \tag{19}$$

Therefore, (8) now becomes

$$\begin{cases} \gamma(\frac{1}{2}\alpha^2 + \frac{1}{2}\beta^2 - \frac{3}{2}\gamma^2 - \alpha\beta + \alpha\gamma + \beta\gamma - (-\frac{1}{2}\alpha^2 - \frac{1}{2}\beta^2 - \frac{1}{2}\gamma^2 + \alpha\beta + \alpha\gamma + \beta\gamma)\lambda_0 - c) = 0, \\ \beta(\frac{1}{2}\alpha^2 - \frac{3}{2}\beta^2 + \frac{1}{2}\gamma^2 + \alpha\beta - \alpha\gamma + \beta\gamma - (-\frac{1}{2}\alpha^2 - \frac{1}{2}\beta^2 - \frac{1}{2}\gamma^2 + \alpha\beta + \alpha\gamma + \beta\gamma)\lambda_0 - c) = 0, \\ \alpha(\frac{3}{2}\alpha^2 - \frac{1}{2}\beta^2 - \frac{1}{2}\gamma^2 - \alpha\beta - \alpha\gamma + \beta\gamma + (-\frac{1}{2}\alpha^2 - \frac{1}{2}\beta^2 - \frac{1}{2}\gamma^2 + \alpha\beta + \alpha\gamma + \beta\gamma)\lambda_0 + c) = 0. \end{cases} \tag{20}$$

Suppose that $\gamma = 0$, we have

$$\begin{cases} \beta(\frac{1}{2}\alpha^2 - \frac{3}{2}\beta^2 + \alpha\beta - (-\frac{1}{2}\alpha^2 - \frac{1}{2}\beta^2 + \alpha\beta)\lambda_0 - c) = 0, \\ \alpha(\frac{3}{2}\alpha^2 - \frac{1}{2}\beta^2 - \alpha\beta + (-\frac{1}{2}\alpha^2 - \frac{1}{2}\beta^2 + \alpha\beta)\lambda_0 + c) = 0. \end{cases} \tag{21}$$

If $\beta = 0$, we have two cases (i)–(ii). If $\beta \neq 0$, for cases (iii)–(v), system (21) holds. Now, we assume that $\gamma \neq 0$, then $c = \frac{1}{2}\alpha^2 + \frac{1}{2}\beta^2 - \frac{3}{2}\gamma^2 - \alpha\beta + \alpha\gamma + \beta\gamma - (-\frac{1}{2}\alpha^2 - \frac{1}{2}\beta^2 - \frac{1}{2}\gamma^2 + \alpha\beta + \alpha\gamma + \beta\gamma)\lambda_0$. Meanwhile, we have

$$\begin{cases} \beta(-\beta^2 + \gamma^2 + \alpha\beta - \alpha\gamma) = 0, \\ \alpha(\alpha^2 - \gamma^2 - \alpha\beta + \beta\gamma) = 0. \end{cases} \tag{22}$$

If $\beta = 0$, cases (vi)–(viii) hold. If $\beta \neq 0$, for cases (ix)–(xii), system (22) holds. □

Theorem 6. *When $\alpha = 0$, $\beta = \eta$ and $c = 2\lambda_0$, (G_4, g) is the algebraic Schouten soliton associated with the connection ∇.*

Proof of Theorem 4. Ref. [3] makes it obvious that

$$\mathrm{Ric}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\alpha^2 & 0 & 0 \\ 0 & -\frac{1}{2}\alpha^2 + \alpha\beta - 2\eta(\alpha-\beta) - 2 & \alpha - 2\beta + 2\eta \\ 0 & -\alpha + 2\beta - 2\eta & -\frac{1}{2}\alpha^2 + \alpha\beta - 2\beta\eta + 2 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{23}$$

A direct computation shows that the value of the scalar curvature is $-\frac{1}{2}\alpha^2 + 2\alpha\beta - 2\alpha\eta - 2$. We have

$$\begin{cases} De_1 = (\frac{1}{2}\alpha^2 + (\frac{1}{2}\alpha^2 - 2\alpha\beta + 2\alpha\eta + 2)\lambda_0 - c)e_1, \\ De_2 = (-\frac{1}{2}\alpha^2 + \alpha\beta - 2\eta(\alpha-\beta) - 2 + (\frac{1}{2}\alpha^2 - 2\alpha\beta + 2\alpha\eta + 2)\lambda_0 - c)e_2 + (\alpha - 2\beta + 2\eta)e_3, \\ De_3 = (-\alpha + 2\beta - 2\eta)e_2 + (-\frac{1}{2}\alpha^2 + \alpha\beta - 2\beta\eta + 2 + (\frac{1}{2}\alpha^2 - 2\alpha\beta + 2\alpha\eta + 2)\lambda_0 - c)e_3. \end{cases} \tag{24}$$

By applying the formula shown in (8), we can calculate

$$\begin{cases} \frac{1}{2}\alpha^2 + (\frac{1}{2}\alpha^2 - 2\alpha\beta + 2\alpha\eta + 2)\lambda_0 - c + 2(\beta-\eta)(\alpha - 2(\beta-\eta)) = 0, \\ (2\eta - \beta)(\frac{1}{2}\alpha^2 + (\frac{1}{2}\alpha^2 - 2\alpha\beta + 2\alpha\eta + 2)\lambda_0 - c) + 2\eta(\beta-\eta)(\alpha - 2(\beta-\eta)) = 0, \\ \beta(\frac{1}{2}\alpha^2 + (\frac{1}{2}\alpha^2 - 2\alpha\beta + 2\alpha\eta + 2)\lambda_0 - c) + 2\eta(\beta-\eta)(\alpha - 2(\beta-\eta)) = 0, \\ \alpha(\frac{3}{2}\alpha^2 + (-\frac{1}{2}\alpha^2 + 2\alpha\beta - 2\alpha\eta - 2)\lambda_0 + c - 2\alpha(\beta-\eta)) = 0. \end{cases} \tag{25}$$

Via simple calculations, we can obtain

$$\begin{cases} \frac{1}{2}\alpha^2 + (\frac{1}{2}\alpha^2 - 2\alpha\beta + 2\alpha\eta + 2)\lambda_0 - c + 2(\beta-\eta)(\alpha - 2(\beta-\eta)) = 0, \\ (\eta - \beta)(\frac{1}{2}\alpha^2 + (\frac{1}{2}\alpha^2 - 2\alpha\beta + 2\alpha\eta + 2)\lambda_0 - c) = 0, \\ \alpha(\frac{3}{2}\alpha^2 + (-\frac{1}{2}\alpha^2 + 2\alpha\beta - 2\alpha\eta - 2)\lambda_0 + c - 2\alpha(\beta-\eta)) = 0. \end{cases} \tag{26}$$

Let $\beta = \eta$, we have $\alpha = 0$, $c = 2\lambda_0$. If $\beta \neq \eta$, then $c = \frac{1}{2}\alpha^2 + (\frac{1}{2}\alpha^2 - 2\alpha\beta + 2\alpha\eta + 2)\lambda_0$, we have

$$\begin{cases} 2(\alpha - 2(\beta - \eta)) = 0, \\ \alpha^2(2\alpha - 2(\beta - \eta)) = 0. \end{cases} \tag{27}$$

This is a contradiction. □

Theorem 7. *If one of the following two conditions is satisfied*

(i) $\beta = \gamma = 0$, $c = -\alpha^2 - \delta^2 - (2\alpha^2 + 2\alpha\delta + 2\delta^2)\lambda_0$,

(ii) $(\beta, \gamma) \neq (0, 0)$, $\alpha^2 + \beta^2 = \gamma^2 + \delta^2$, $c = -\alpha^2 - \frac{1}{2}(\beta+\gamma)^2 - \delta^2 - (2\alpha^2 + 2\alpha\delta + \frac{1}{2}(\beta + \gamma)^2 + 2\delta^2)\lambda_0$, *then* (G_5, g) *is the algebraic Schouten soliton associated with the connection* ∇.

Proof of Theorem 5. By [3], it is immediate that

$$\mathrm{Ric}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -\alpha^2 - \alpha\delta - \frac{\beta^2-\gamma^2}{2} & 0 & 0 \\ 0 & -\alpha\delta + \frac{\beta^2-\gamma^2}{2} - \delta^2 & 0 \\ 0 & 0 & -\alpha^2 - \frac{(\beta+\gamma)^2}{2} - \delta^2 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{28}$$

Then we have $s = -2\alpha^2 - 2\alpha\delta - \frac{1}{2}(\beta + \gamma)^2 - 2\delta^2$, and

$$\begin{cases} De_1 = (-\alpha^2 - \alpha\delta - \frac{1}{2}(\beta^2 - \gamma^2) - (-2\alpha^2 - 2\alpha\delta - \frac{1}{2}(\beta+\gamma)^2 - 2\delta^2)\lambda_0 - c)e_1, \\ De_2 = (-\alpha\delta + \frac{1}{2}(\beta^2 - \gamma^2) - \delta^2 - (-2\alpha^2 - 2\alpha\delta - \frac{1}{2}(\beta+\gamma)^2 - 2\delta^2)\lambda_0 - c)e_2, \\ De_3 = (-\alpha^2 - \frac{1}{2}(\beta+\gamma)^2 - \delta^2 - (-2\alpha^2 - 2\alpha\delta - \frac{1}{2}(\beta+\gamma)^2 - 2\delta^2)\lambda_0 - c)e_3. \end{cases} \tag{29}$$

By using (8) and making tedious calculations, we have the following:

$$\begin{cases} (\alpha+\delta)(\alpha^2 + \frac{1}{2}(\beta+\gamma)^2 + \delta^2 + (-2\alpha^2 - 2\alpha\delta - \frac{1}{2}(\beta+\gamma)^2 - 2\delta^2)\lambda_0 + c) = 0, \\ \beta(2\alpha^2 + \frac{1}{2}(3\beta^2 + 2\beta\gamma - \gamma^2) + (-2\alpha^2 - 2\alpha\delta - \frac{1}{2}(\beta+\gamma)^2 - 2\delta^2)\lambda_0 + c) = 0, \\ \gamma(-2\delta^2 + \frac{1}{2}(\beta^2 - 2\beta\gamma - 3\gamma^2) - (-2\alpha^2 - 2\alpha\delta - \frac{1}{2}(\beta+\gamma)^2 - 2\delta^2)\lambda_0 - c) = 0. \end{cases} \tag{30}$$

We assume that $\beta = 0$. Since $\alpha + \delta \neq 0$ and $\alpha\gamma + \beta\delta = 0$, we have

$$\begin{cases} \alpha\gamma = 0, \\ \alpha + \delta \neq 0, \\ (\alpha+\delta)(\alpha^2 + \frac{1}{2}\gamma^2 + \delta^2 + (-2\alpha^2 - 2\alpha\delta - \frac{1}{2}\gamma^2 - 2\delta^2)\lambda_0 + c) = 0, \\ \gamma(-2\delta^2 - \frac{3}{2}\gamma^2 - (-2\alpha^2 - 2\alpha\delta - \frac{1}{2}\gamma^2 - 2\delta^2)\lambda_0 - c) = 0. \end{cases} \tag{31}$$

Consider $\gamma = 0$, then case (i) is true. If $\gamma \neq 0$,

$$\begin{cases} \alpha = 0, \\ \delta \neq 0, \\ \frac{1}{2}\gamma^2 + \delta^2 + (-\frac{1}{2}\gamma^2 - 2\delta^2)\lambda_0 = -c, \\ -2\delta^2 - \frac{3}{2}\gamma^2 - (-\frac{1}{2}\gamma^2 - 2\delta^2)\lambda_0 = c, \end{cases} \tag{32}$$

we have case (ii). Now, we assume that $\beta \neq 0$, then

$$\begin{cases} -2\alpha^2 - \frac{1}{2}(3\beta^2 + 2\beta\gamma - \gamma^2) - (-2\alpha^2 - 2\alpha\delta - \frac{1}{2}(\beta+\gamma)^2 - 2\delta^2)\lambda_0 = c, \\ \gamma(-\delta^2 + \alpha^2 + \beta^2 - \gamma^2) = 0, \end{cases} \tag{33}$$

for case (ii), system (29) holds. □

Theorem 8. (G_6, g) *is the algebraic Schouten soliton associated with the connection* ∇ *if and only if*

(i) $\beta = \gamma = 0$, $c = \alpha^2 + \delta^2 - (2\alpha^2 + 2\delta^2 + 2\alpha\delta)\lambda_0$,

(ii) $\alpha = \beta = 0, \gamma \neq 0, \gamma^2 = \delta^2, c = \frac{1}{2}\gamma^2 - \frac{3}{2}\gamma^2\lambda_0,$
(iii) $\alpha \neq 0, \alpha^2 = \beta^2, \gamma = \delta = 0, c = \frac{1}{2}\alpha^2 - \frac{3}{2}\alpha^2\lambda_0,$
(iv) $\beta \neq 0, \gamma \neq 0, \alpha^2 - \beta^2 = \delta^2 - \gamma^2, c = \alpha^2 - \frac{1}{2}(\beta-\gamma)^2 + \delta^2 - (2\alpha^2 + 2\alpha\delta - \frac{1}{2}(\beta-\gamma)^2 + 2\delta^2)\lambda_0.$

Proof of Theorem 6. In [3], the Ricci operator is given by

$$\mathrm{Ric}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \alpha^2 - \frac{(\beta-\gamma)^2}{2} + \delta^2 & 0 & 0 \\ 0 & \alpha^2 + \alpha\delta - \frac{\beta^2-\gamma^2}{2} & 0 \\ 0 & 0 & \alpha\delta + \frac{\beta^2-\gamma^2}{2} + \delta^2 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \quad (34)$$

So $s = 2\alpha^2 + 2\alpha\delta - \frac{1}{2}(\beta-\gamma)^2 + 2\delta^2$. A simple calculation shows that

$$\begin{cases} De_1 = (\alpha^2 - \frac{1}{2}(\beta-\gamma)^2 + \delta^2 - (2\alpha^2 + 2\alpha\delta - \frac{1}{2}(\beta-\gamma)^2 + 2\delta^2)\lambda_0 - c)e_1, \\ De_2 = (\alpha^2 + \alpha\delta - \frac{1}{2}(\beta^2-\gamma^2) - (2\alpha^2 + 2\alpha\delta - \frac{1}{2}(\beta-\gamma)^2 + 2\delta^2)\lambda_0 - c)e_2, \\ De_3 = (\alpha\delta + \frac{1}{2}(\beta^2-\gamma^2) + \delta^2 - (2\alpha^2 + 2\alpha\delta - \frac{1}{2}(\beta-\gamma)^2 + 2\delta^2)\lambda_0 - c)e_3. \end{cases} \quad (35)$$

Thus, Equation (8) is satisfied if and only if

$$\begin{cases} (\alpha^2+\delta^2)(-\alpha^2 + \frac{1}{2}(\beta-\gamma)^2 - \delta^2 + (2\alpha^2 + 2\alpha\delta - \frac{1}{2}(\beta-\gamma)^2 + 2\delta^2)\lambda_0 + c) = 0, \\ \beta(-2\alpha^2 + \frac{1}{2}(3\beta^2 - 2\beta\gamma - \gamma^2) + (2\alpha^2 + 2\alpha\delta - \frac{1}{2}(\beta-\gamma)^2 + 2\delta^2)\lambda_0 + c) = 0, \\ \gamma(-2\delta^2 + \frac{1}{2}(-\beta^2 - 2\beta\gamma + 3\gamma^2) + (2\alpha^2 + 2\alpha\delta - \frac{1}{2}(\beta-\gamma)^2 + 2\delta^2)\lambda_0 + c) = 0. \end{cases} \quad (36)$$

Suppose that $\beta = 0$, by taking into account $\alpha + \delta \neq 0$ and $\alpha\gamma + \beta\delta = 0$, we have

$$\begin{cases} \alpha\gamma = 0, \\ \alpha + \delta \neq 0, \\ (\alpha^2+\delta^2)(-\alpha^2 + \frac{1}{2}\gamma^2 - \delta^2 + (2\alpha^2 + 2\alpha\delta - \frac{1}{2}\gamma^2 + 2\delta^2)\lambda_0 + c) = 0, \\ \gamma(-2\delta^2 + \frac{3}{2}\gamma^2 + (2\alpha^2 + 2\alpha\delta - \frac{1}{2}\gamma^2 + 2\delta^2)\lambda_0 + c) = 0. \end{cases} \quad (37)$$

Set $\gamma = 0$, we have case (i). If $\gamma \neq 0$, we have case (ii). Let $\beta \neq 0$, then $c = 2\alpha^2 - \frac{1}{2}(3\beta^2 - 2\beta\gamma - \gamma^2) - (2\alpha^2 + 2\alpha\delta - \frac{1}{2}(\beta-\gamma)^2 + 2\delta^2)\lambda_0$. Consequently,

$$\begin{cases} \alpha + \delta \neq 0, \\ \alpha\gamma + \beta\delta = 0, \\ (\alpha^2+\delta^2)(\alpha^2 - \beta^2 + \gamma^2 - \delta^2) = 0, \\ \gamma(\alpha^2 - \beta^2 + \gamma^2 - \delta^2) = 0. \end{cases} \quad (38)$$

Consider $\gamma = 0$, then case (iii) is true. If $\gamma \neq 0$, for case (iv), system (33) holds. □

Theorem 9. *If (G_7, g) is the algebraic Schouten soliton associated with the connection ∇, then we have $\gamma = 0, c = 0$.*

Proof of Theorem 7. From [3], we have

$$\mathrm{Ric}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\gamma^2 & 0 & 0 \\ 0 & \alpha^2 - \alpha\delta + \beta\gamma - \frac{1}{2}\gamma^2 & \alpha^2 - \alpha\delta + \beta\gamma \\ 0 & -\alpha^2 + \alpha\delta - \beta\gamma & -\alpha^2 + \alpha\delta - \beta\gamma - \frac{1}{2}\gamma^2 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \quad (39)$$

Then $s = -\frac{1}{2}\gamma^2$. Computations show that

$$\begin{cases} De_1 = (\frac{1}{2}\gamma^2 + \frac{1}{2}\gamma^2\lambda_0 - c)e_1, \\ De_2 = (\alpha^2 - \alpha\delta + \beta\gamma - \frac{1}{2}\gamma^2 + \frac{1}{2}\gamma^2\lambda_0 - c)e_2 + (\alpha^2 - \alpha\delta + \beta\gamma)e_3, \\ De_3 = (-\alpha^2 + \alpha\delta - \beta\gamma)e_2 + (-\alpha^2 + \alpha\delta - \beta\gamma - \frac{1}{2}\gamma^2 + \frac{1}{2}\gamma^2\lambda_0 - c)e_3. \end{cases} \quad (40)$$

Hence, (8) now yields

$$\begin{cases} (\alpha^2+\delta^2)(\frac{1}{2}\gamma^2-\frac{1}{2}\gamma^2\lambda_0+c)=0, \\ \beta(\frac{1}{2}\gamma^2+\frac{1}{2}\gamma^2\lambda_0-c)=0, \\ \gamma(\frac{3}{2}\gamma^2-\frac{1}{2}\gamma^2\lambda_0+c)=0. \end{cases} \quad (41)$$

Since $\alpha\gamma=0$, $\alpha+\delta\neq 0$, we have $\gamma=0$ and $c=0$. □

3.2. Algebraic Schouten Solitons Associated with Canonical Connections and Kobayashi–Nomizu Connections on Three-Dimensional Lorentzian Lie Groups

We define a product structure J on G_i by

$$Je_1=e_1, Je_2=e_2, Je_3=-e_3, \quad (42)$$

then $J^2=\mathrm{id}$ and $g(Je_j,Je_j)=g(e_j,e_j)$. By [5], we define the canonical connection and the Kobayashi–Nomizu connection is as follows:

$$\nabla^0_X Y=\nabla_X Y-\frac{1}{2}(\nabla_X J)JY, \quad (43)$$

$$\nabla^1_X Y=\nabla^0_X Y-\frac{1}{4}[(\nabla_Y J)JX-(\nabla_{JY}J)X]. \quad (44)$$

We define

$$R^0(X,Y)Z=\nabla^0_X\nabla^0_Y Z-\nabla^0_Y\nabla^0_X Z-\nabla^0_{[X,Y]}Z, \quad (45)$$

$$R^1(X,Y)Z=\nabla^1_X\nabla^1_Y Z-\nabla^1_Y\nabla^1_X Z-\nabla^1_{[X,Y]}Z. \quad (46)$$

The Ricci tensors of (G_i,g) associated with the canonical connection and the Kobayashi–Nomizu connection are defined by

$$\rho^0(X,Y)=-g(R^0(X,e_1)Y,e_1)-g(R^0(X,e_2)Y,e_2)+g(R^0(X,e_3)Y,e_3), \quad (47)$$

$$\rho^1(X,Y)=-g(R^1(X,e_1)Y,e_1)-g(R^1(X,e_2)Y,e_2)+g(R^1(X,e_3)Y,e_3). \quad (48)$$

The Ricci operators Ric^0 and Ric^1 are given by

$$\rho^0(X,Y)=g(\mathrm{Ric}^0(X),Y), \rho^1(X,Y)=g(\mathrm{Ric}^1(X),Y). \quad (49)$$

Let

$$\widetilde{\rho}^0(X,Y)=\frac{\rho^0(X,Y)+\rho^0(Y,X)}{2}, \widetilde{\rho}^1(X,Y)=\frac{\rho^1(X,Y)+\rho^1(Y,X)}{2}, \quad (50)$$

and

$$\widetilde{\rho}^0(X,Y)=g(\widetilde{\mathrm{Ric}}^0(X),Y), \widetilde{\rho}^1(X,Y)=g(\widetilde{\mathrm{Ric}}^1(X),Y). \quad (51)$$

Similar to (5) and (6), we have

$$S^0(e_i,e_j)=\widetilde{\rho}^0(e_i,e_j)-s^0\lambda_0 g(e_i,e_j), S^1(e_i,e_j)=\widetilde{\rho}^1(e_i,e_j)-s^1\lambda_0 g(e_i,e_j), \quad (52)$$

and

$$s^0=\widetilde{\rho}^0(e_1,e_1)+\widetilde{\rho}^0(e_2,e_2)-\widetilde{\rho}^0(e_3,e_3), s^1=\widetilde{\rho}^1(e_1,e_1)+\widetilde{\rho}^1(e_2,e_2)-\widetilde{\rho}^1(e_3,e_3). \quad (53)$$

Definition 2. *(G_i,g,J) is called the algebraic Schouten soliton associated with the connection ∇^0 if it satisfies*

$$\widetilde{\mathrm{Ric}}^0=(s^0\lambda_0+c)\mathrm{Id}+D, \quad (54)$$

where c is a real number, and D is a derivation of $\mathfrak{g}$*; that is*

$$D[X,Y] = [DX,Y] + [X,DY] \text{for} X, Y \in \mathfrak{g}. \tag{55}$$

(G_i, g, J) *is called the algebraic Schouten soliton associated with the connection* ∇^1 *if it satisfies*

$$\widetilde{\mathrm{Ric}}^1 = (s^1\lambda_0 + c)\mathrm{Id} + D. \tag{56}$$

Theorem 10. *When* $\beta = 0$, $c = -\frac{1}{2}\alpha^2 + 2\alpha^2\lambda_0$, (G_1, g, J) *is the algebraic Schouten soliton associated with the connection* ∇^0.

Proof of Theorem 8. From [7], it is obvious that

$$\widetilde{\mathrm{Ric}}^0 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -(\alpha^2 + \frac{1}{2}\beta^2) & 0 & -\frac{1}{4}\alpha\beta \\ 0 & -(\alpha^2 + \frac{1}{2}\beta^2) & -\frac{1}{2}\alpha^2 \\ \frac{1}{4}\alpha\beta & \frac{1}{2}\alpha^2 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{57}$$

Moreover, $s^0 = -2(\alpha^2 + \frac{1}{2}\beta^2)$. We obtain that

$$\begin{cases} De_1 = -(\alpha^2 + \frac{1}{2}\beta^2 - 2(\alpha^2 + \frac{1}{2}\beta^2)\lambda_0 + c)e_1 - \frac{1}{4}\alpha\beta e_3, \\ De_2 = -(\alpha^2 + \frac{1}{2}\beta^2 - 2(\alpha^2 + \frac{1}{2}\beta^2)\lambda_0 + c)e_2 - \frac{1}{2}\alpha^2 e_3, \\ De_3 = \frac{1}{4}\alpha\beta e_1 + \frac{1}{2}\alpha^2 e_2 - (-2(\alpha^2 + \frac{1}{2}\beta^2)\lambda_0 + c)e_3. \end{cases} \tag{58}$$

Then, Equation (52) becomes

$$\begin{cases} \alpha^2\beta = 0, \\ \alpha(\frac{1}{2}\alpha^2 - 2(\alpha^2 + \frac{1}{2}\beta^2)\lambda_0 + c) = 0, \\ \beta(\frac{5}{2}\alpha^2 + \beta^2 - 2(\alpha^2 + \frac{1}{2}\beta^2)\lambda_0 + c) = 0, \\ \beta(-2(\alpha^2 + \frac{1}{2}\beta^2)\lambda_0 + c) = 0, \\ \beta(\frac{1}{2}\alpha^2 - 2(\alpha^2 + \frac{1}{2}\beta^2)\lambda_0 + c) = 0. \end{cases} \tag{59}$$

Taking into account that, $\alpha \neq 0$, we have $\beta = 0$ and $c = -\frac{1}{2}\alpha^2 + 2\alpha^2\lambda_0$. □

Theorem 11. *If* $\beta = 0$, $c = -\frac{1}{2}\alpha^2 + 2\alpha^2\lambda_0$, *then this case corresponds to* (G_1, g, J) *being the algebraic Schouten soliton associated with the connection* ∇^1.

Proof of Theorem 9. In [7], it is shown that

$$\widetilde{\mathrm{Ric}}^1 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -(\alpha^2 + \beta^2) & \alpha\beta & \frac{1}{2}\alpha\beta \\ \alpha\beta & -(\alpha^2 + \beta^2) & -\frac{1}{2}\alpha^2 \\ -\frac{1}{2}\alpha\beta & \frac{1}{2}\alpha^2 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{60}$$

Therefore, $s^1 = -2(\alpha^2 + \beta^2)$. D is described by

$$\begin{cases} De_1 = -(\alpha^2 + \beta^2 - 2(\alpha^2 + \beta^2)\lambda_0 + c)e_1 + \alpha\beta e_2 + \frac{1}{2}\alpha\beta e_3, \\ De_2 = \alpha\beta e_1 - (\alpha^2 + \beta^2 - 2(\alpha^2 + \beta^2)\lambda_0 + c)e_2 - \frac{1}{2}\alpha^2 e_3, \\ De_3 = -\frac{1}{2}\alpha\beta e_1 + \frac{1}{2}\alpha^2 e_2 - (-2(\alpha^2 + \beta^2)\lambda_0 + c)e_3. \end{cases} \tag{61}$$

We calculate that

$$\begin{cases} \alpha^2\beta = 0, \\ \alpha(\frac{1}{2}\alpha^2 + 2\beta^2 - 2(\alpha^2 + \beta^2)\lambda_0 + c) = 0, \\ \beta(\alpha^2 + 2\beta^2 - 2(\alpha^2 + \beta^2)\lambda_0 + c) = 0, \\ \beta(\alpha^2 - 2(\alpha^2 + \beta^2)\lambda_0 + c) = 0. \end{cases} \tag{62}$$

Note that $\alpha \neq 0$, then we have $\beta = 0$ and $c = -\frac{1}{2}\alpha^2 + 2\alpha^2\lambda_0$. $\square$

Theorem 12. *When $\alpha = \beta = 0$ and $c = -\gamma^2 + 2\gamma^2\lambda_0$, (G_2, g, J) is the algebraic Schouten soliton associated with the connection ∇^0.*

Proof of Theorem 10. According to [7], we have

$$\widetilde{\mathrm{Ric}}^0 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -(\frac{1}{2}\alpha\beta + \gamma^2) & 0 & 0 \\ 0 & -(\frac{1}{2}\alpha\beta + \gamma^2) & \frac{1}{4}\alpha\gamma - \frac{1}{2}\beta\gamma \\ 0 & -\frac{1}{4}\alpha\gamma + \frac{1}{2}\beta\gamma & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \quad (63)$$

Obviously, $s^0 = -(\alpha\beta + 2\gamma^2)$. From Equation (52) it is easy to obtain

$$\begin{cases} De_1 = -(\frac{1}{2}\alpha\beta + \gamma^2 - (\alpha\beta + 2\gamma^2)\lambda_0 + c)e_1, \\ De_2 = -(\frac{1}{2}\alpha\beta + \gamma^2 - (\alpha\beta + 2\gamma^2)\lambda_0 + c)e_2 + (\frac{1}{4}\alpha\gamma - \frac{1}{2}\beta\gamma)e_3, \\ De_3 = (-\frac{1}{4}\alpha\gamma + \frac{1}{2}\beta\gamma)e_2 - (-(\alpha\beta + 2\gamma^2)\lambda_0 + c)e_3. \end{cases} \quad (64)$$

Consequently, we have

$$\begin{cases} \gamma(\alpha\beta - \beta^2 + \gamma^2 - (\alpha\beta + 2\gamma^2)\lambda_0 + c) = 0, \\ \beta(\alpha\beta + 2\gamma^2 - (\alpha\beta + 2\gamma^2)\lambda_0 + c) + \gamma(-\frac{1}{2}\alpha\gamma + \beta\gamma) = 0, \\ \beta(-(\alpha\beta + 2\gamma^2)\lambda_0 + c) + \gamma(-\frac{1}{2}\alpha\gamma + \beta\gamma) = 0, \\ \alpha(-(\alpha\beta + 2\gamma^2)\lambda_0 + c) = 0. \end{cases} \quad (65)$$

The second and third equations in (62) transform into

$$\beta(\alpha\beta + 2\gamma^2) = 0. \quad (66)$$

Then, we have

$$\begin{cases} \gamma(\alpha^2 + \alpha\beta - \beta^2 - (\alpha\beta + 2\gamma^2)\lambda_0 + c) = 0, \\ \beta(\alpha\beta + 2\gamma^2) = 0, \\ \alpha(-(\alpha\beta + 2\gamma^2)\lambda_0 + c) = 0. \end{cases} \quad (67)$$

Note that $\gamma \neq 0$. We have $\alpha^2\gamma = 0$ and $\alpha = \beta$, then $c = -\gamma^2 + 2\gamma^2\lambda_0$. $\square$

Theorem 13. *If $\alpha = \beta = 0$, $c = -\gamma^2 + 2\gamma^2\lambda_0$ are satisfied, then (G_2, g, J) is the algebraic Schouten soliton associated with the connection ∇^1.*

Proof of Theorem 11. We have

$$\widetilde{\mathrm{Ric}}^1 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -(\beta^2 + \gamma^2) & 0 & 0 \\ 0 & -(\alpha\beta + \gamma^2) & \frac{1}{2}\alpha\gamma \\ 0 & -\frac{1}{2}\alpha\gamma & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \quad (68)$$

this can be found in [7]. Moreover, $s^1 = -(\alpha\beta + \beta^2 + 2\gamma^2)$. From this, D is given by

$$\begin{cases} De_1 = -(\beta^2 + \gamma^2 - (\alpha\beta + \beta^2 + 2\gamma^2)\lambda_0 + c)e_1, \\ De_2 = -(\alpha\beta + \gamma^2 - (\alpha\beta + \beta^2 + 2\gamma^2)\lambda_0 + c)e_2 + \frac{1}{2}\alpha\gamma e_3, \\ De_3 = -\frac{1}{2}\alpha\gamma e_2 - (-(\alpha\beta + \beta^2 + 2\gamma^2)\lambda_0 + c)e_3. \end{cases} \quad (69)$$

In this way, (52) is satisfied if and only if

$$\begin{cases} \gamma(\alpha\beta + \beta^2 + \gamma^2 - (\alpha\beta + \beta^2 + 2\gamma^2)\lambda_0 + c) = 0, \\ \beta(\alpha\beta + \beta^2 + 2\gamma^2 - (\alpha\beta + \beta^2 + 2\gamma^2)\lambda_0 + c) - \alpha\gamma^2 = 0, \\ \beta(-\alpha\beta + \beta^2 - (\alpha\beta + \beta^2 + 2\gamma^2)\lambda_0 + c) - \alpha\gamma^2 = 0, \\ \alpha(\alpha\beta - \beta^2 - (\alpha\beta + \beta^2 + 2\gamma^2)\lambda_0 + c) = 0. \end{cases} \quad (70)$$

Since $\gamma \neq 0$, we have $c = -\alpha\beta - \beta^2 - \gamma^2 + (\alpha\beta + \beta^2 + 2\gamma^2)\lambda_0$. The second equation in (67) transforms into

$$(\alpha - \beta)\gamma^2 = 0. \tag{71}$$

We have $\alpha = \beta = 0$, $c = -\gamma^2 + 2\gamma^2\lambda_0$. □

Theorem 14. *If one of the following conditions is satisfied, then (G_3, g, J) is the algebraic Schouten soliton associated with the connection ∇^0 :*

(i) $\alpha = \beta = \gamma = 0$, *for all* c,
(ii) $\alpha = \beta = 0, \gamma \neq 0, c = \gamma^2 - \gamma^2\lambda_0$,
(iii) $\alpha \neq 0$ *or* $\beta \neq 0, \gamma = 0, c = 0$,
(iv) $\alpha \neq 0$ *or* $\beta \neq 0, \gamma = \alpha + \beta, c = 0$.

Proof of Theorem 12. By [7], we have

$$\widetilde{\mathrm{Ric}}^0 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -\gamma a_3 & 0 & 0 \\ 0 & -\gamma a_3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{72}$$

where

$$a_1 = \frac{1}{2}(\alpha - \beta - \gamma), a_2 = \frac{1}{2}(\alpha - \beta + \gamma), a_3 = \frac{1}{2}(\alpha + \beta - \gamma). \tag{73}$$

A direct computation for the scalar curvature shows that $s^0 = -2\gamma a_3 = -\gamma(\alpha + \beta - \gamma)$. It is easy to obtain

$$\begin{cases} De_1 = -(\gamma a_3 - 2\gamma a_3\lambda_0 + c)e_1, \\ De_2 = -(\gamma a_3 - 2\gamma a_3\lambda_0 + c)e_2, \\ De_3 = -(-2\gamma a_3\lambda_0 + c)e_3. \end{cases} \tag{74}$$

Thus,

$$\begin{cases} \gamma(\gamma(\alpha + \beta - \gamma) - \gamma(\alpha + \beta - \gamma)\lambda_0 + c) = 0, \\ \beta(-\gamma(\alpha + \beta - \gamma)\lambda_0 + c) = 0, \\ \alpha(-\gamma(\alpha + \beta - \gamma)\lambda_0 + c) = 0. \end{cases} \tag{75}$$

If $\alpha = 0$, then cases (i)–(iii) hold. Choose $\alpha \neq 0$ and $c = \gamma(\alpha + \beta - \gamma)\lambda_0$, we obtain two cases (iii)–(iv). □

Theorem 15. *(G_3, g, J) is the algebraic Schouten soliton associated with the connection ∇^1 if and only if*

(i) $\alpha = \beta = \gamma = 0, c \neq 0$,
(ii) $\alpha = 0, c = -\beta\gamma + \beta\gamma\lambda_0$,
(iii) $\beta = 0, c = -\alpha\gamma + \alpha\gamma\lambda_0$,
(iv) $\alpha\beta \neq 0, \gamma = 0, c = 0$.

Proof of Theorem 13. We have

$$\widetilde{\mathrm{Ric}}^1 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \gamma(a_1 - a_3) & 0 & 0 \\ 0 & -\gamma(a_2 + a_3) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{76}$$

which is clear from [7]. By definition, we have $s^1 = \gamma(a_1 - a_2 - 2a_3) = -\gamma(\alpha + \beta)$. Hence,

$$\begin{cases} De_1 = -(-\gamma(a_1 - a_3) + \gamma(a_1 - a_2 - 2a_3)\lambda_0 + c)e_1, \\ De_2 = -(\gamma(a_2 + a_3) + \gamma(a_1 - a_2 - 2a_3)\lambda_0 + c)e_2, \\ De_3 = -(\gamma(a_1 - a_2 - 2a_3)\lambda_0 + c)e_3. \end{cases} \tag{77}$$

Equation (52) now becomes

$$\begin{cases} \gamma(\alpha\gamma + \beta\gamma - \gamma(\alpha + \beta)\lambda_0 + c) = 0, \\ \beta(-\alpha\gamma + \beta\gamma - \gamma(\alpha + \beta)\lambda_0 + c) = 0, \\ \alpha(\alpha\gamma - \beta\gamma - \gamma(\alpha + \beta)\lambda_0 + c) = 0. \end{cases} \tag{78}$$

It is easy to check that

$$\begin{cases} \alpha\beta\gamma^2 = 0, \\ \alpha\beta(-\gamma(\alpha + \beta)\lambda_0 + c) = 0. \end{cases} \tag{79}$$

We consider $\alpha\beta = 0$. In this case, cases (i)–(iii) hold. If we consider $\alpha\beta \neq 0$, then $\gamma = 0$, $c = 0$ and case (iv) holds. □

Theorem 16. *If (G_4, g, J) is the algebraic Schouten soliton associated with the connection ∇^0, then we have $\alpha = 0$, $\beta = \eta$, $c = 0$.*

Proof of Theorem 14. From [7], we have

$$\widetilde{\mathrm{Ric}}^0 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} b_3(2\eta - \beta) - 1 & 0 & 0 \\ 0 & b_3(2\eta - \beta) - 1 & -\frac{1}{2}(b_3 - \beta) \\ 0 & \frac{1}{2}(b_3 - \beta) & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{80}$$

where

$$b_1 = \frac{1}{2}\alpha + \eta - \beta, b_2 = \frac{1}{2}\alpha - \eta, b_3 = \frac{1}{2}\alpha + \eta. \tag{81}$$

Then $s^0 = 2b_3(2\eta - \beta) - 2 = (2\eta + \alpha)(2\eta - \beta) - 2$. According to the condition $\widetilde{\mathrm{Ric}}^0 = (s^0\lambda_0 + c)\mathrm{Id} + D$, we calculate that

$$\begin{cases} De_1 = (b_3(2\eta - \beta) - 1 - 2b_3(2\eta - \beta)\lambda_0 + 2\lambda_0 - c)e_1, \\ De_2 = (b_3(2\eta - \beta) - 1 - 2b_3(2\eta - \beta)\lambda_0 + 2\lambda_0 - c)e_2 - \frac{1}{2}(b_3 - \beta)e_3, \\ De_3 = \frac{1}{2}(b_3 - \beta)e_2 - (2b_3(2\eta - \beta)\lambda_0 - 2\lambda_0 + c)e_3. \end{cases} \tag{82}$$

Hence, (52) now yields

$$\begin{cases} \alpha((2\eta + \alpha)(2\eta - \beta)\lambda_0 - 2\lambda_0 + c) = 0, \\ \beta((2\eta + \alpha)(2\eta - \beta)\lambda_0 - 2\lambda_0 + c) - (\frac{1}{2}\alpha + \eta - \beta) = 0, \\ (2\eta - \beta)((2\eta + \alpha)(2\eta - \beta) - 2 - (2\eta + \alpha)(2\eta - \beta)\lambda_0 + 2\lambda_0 - c) - (\frac{1}{2}\alpha + \eta - \beta) = 0, \\ (\frac{1}{2}\alpha + \eta)(2\eta - \beta) - 1 - (2\eta + \alpha)(2\eta - \beta)\lambda_0 + 2\lambda_0 - c + (\frac{1}{2}\alpha + \eta - \beta)(\eta - \beta) = 0. \end{cases} \tag{83}$$

For $\eta = \pm 1$ and $\alpha = 0$, a straightforward calculation shows that

$$\begin{cases} \beta(-2\beta\eta\lambda_0 + 2\lambda_0 + c) - (\eta - \beta) = 0, \\ (2\eta - \beta)(-2\beta\eta + 2 + 2\beta\eta\lambda_0 - 2\lambda_0 - c) - (\eta - \beta) = 0, \\ -\beta\eta + 1 + 2\beta\eta\lambda_0 - 2\lambda_0 - c + (\eta - \beta)^2 = 0. \end{cases} \tag{84}$$

Solving (81), we have $\beta = \eta$, $c = 0$. □

Theorem 17. *(G_4, g, J) is not the algebraic Schouten soliton associated with the connection ∇^1.*

Proof of Theorem 15. In this case, we have

$$\widetilde{\mathrm{Ric}}^1\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -(1+(\beta-2\eta)(b_3-b_1)) & 0 & 0 \\ 0 & -(1+(\beta-2\eta)(b_2+b_3)) & \frac{b_1-b_3-\alpha+\beta}{2} \\ 0 & \frac{\alpha-\beta-b_1+b_3}{2} & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \quad (85)$$

That is

$$\widetilde{\mathrm{Ric}}^1\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -(1+\beta(\beta-2\eta)) & 0 & 0 \\ 0 & -(1+\alpha(\beta-2\eta)) & -\frac{1}{2}\alpha \\ 0 & \frac{1}{2}\alpha & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \quad (86)$$

So we have $s^1 = -(2+(\alpha+\beta)(\beta-2\eta))$. If (G_4, g, J) is the algebraic Schouten soliton associated with the connection ∇^1, then $\widetilde{\mathrm{Ric}}^1 = (s^1\lambda_0 + c)\mathrm{Id} + D$, so

$$\begin{cases} De_1 = -(1+\beta(\beta-2\eta) - (2+(\alpha+\beta)(\beta-2\eta))\lambda_0 + c)e_1, \\ De_2 = -(1+\alpha(\beta-2\eta) - (2+(\alpha+\beta)(\beta-2\eta))\lambda_0 + c)e_2 - \frac{1}{2}\alpha e_3, \\ De_3 = \frac{1}{2}\alpha e_2 - (-(2+(\alpha+\beta)(\beta-2\eta))\lambda_0 + c)e_3. \end{cases} \quad (87)$$

For this reason, Equation (52) now becomes

$$\begin{cases} 1+(\frac{1}{2}\alpha+\beta)(\beta-2\eta) - (2+(\alpha+\beta)(\beta-2\eta))\lambda_0 + c + \frac{1}{2}\alpha\beta = 0, \\ (\beta-2\eta)(2+(\alpha+\beta)(\beta-2\eta) - (2+(\alpha+\beta)(\beta-2\eta))\lambda_0 + c) - \alpha = 0, \\ \beta(-(\alpha-\beta)(\beta-2\eta) - (2+(\alpha+\beta)(\beta-2\eta))\lambda_0 + c) - \alpha = 0, \\ \alpha((\alpha-\beta)(\beta-2\eta) - (2+(\alpha+\beta)(\beta-2\eta))\lambda_0 + c) = 0. \end{cases} \quad (88)$$

Equation (85) has no solutions, we find that (G_4, g, J) is not the algebraic Schouten soliton associated with the connection ∇^1. □

Theorem 18. *If $c = 0$, then this case corresponds to (G_5, g, J) being the algebraic Schouten soliton associated with the connection ∇^0.*

Proof of Theorem 16. We have

$$\widetilde{\mathrm{Ric}}^0\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \quad (89)$$

So $s^0 = 0$. We see that

$$\begin{cases} De_1 = -ce_1, \\ De_2 = -ce_2, \\ De_3 = -ce_3. \end{cases} \quad (90)$$

By the analysis above, we have

$$\begin{cases} \alpha c = 0, \\ \beta c = 0, \\ \gamma c = 0, \\ \delta c = 0. \end{cases} \quad (91)$$

On the basis of $\alpha + \delta \neq 0$, $\alpha\gamma + \beta\delta = 0$, we have $c = 0$. □

Theorem 19. *If $c = 0$ is satisfied, (G_5, g, J) is the algebraic Schouten soliton associated with the connection ∇^1.*

Proof of Theorem 17. From

$$\widetilde{\mathrm{Ric}}^1 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{92}$$

we have $s^1 = 0$. It follows that

$$\begin{cases} De_1 = -ce_1, \\ De_2 = -ce_2, \\ De_3 = -ce_3. \end{cases} \tag{93}$$

Thus,

$$\begin{cases} \alpha c = 0, \\ \beta c = 0, \\ \gamma c = 0, \\ \delta c = 0. \end{cases} \tag{94}$$

Note that if $\alpha + \delta \neq 0$, then we have $c = 0$. □

Theorem 20. *If one of the following two conditions is satisfied*

(i) $\alpha + \delta \neq 0$, $\beta = \gamma = 0$, $c = -\alpha^2 + \alpha^2\lambda_0$,

(ii) $\alpha \neq 0$, $\beta^2 = 2\alpha^2$, $\gamma = \delta = 0$, $c = 0$, *then* (G_6, g, J) *is the algebraic Schouten soliton associated with the connection* ∇^0.

Proof of Theorem 18. We recall the following result:

$$\widetilde{\mathrm{Ric}}^0 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\beta(\beta-\gamma) - \alpha^2 & 0 & 0 \\ 0 & \frac{1}{2}\beta(\beta-\gamma) - \alpha^2 & -\frac{1}{2}(-\gamma\alpha + \frac{1}{2}\delta(\beta-\gamma)) \\ 0 & \frac{1}{2}(-\gamma\alpha + \frac{1}{2}\delta(\beta-\gamma)) & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{95}$$

Moreover, we have $s^0 = \beta(\beta-\gamma) - 2\alpha^2$. Therefore, for (G_6, g, J) we have

$$\begin{cases} De_1 = (\frac{1}{2}\beta(\beta-\gamma) - \alpha^2 - (\beta(\beta-\gamma) - 2\alpha^2)\lambda_0 - c)e_1, \\ De_2 = (\frac{1}{2}\beta(\beta-\gamma) - \alpha^2 - (\beta(\beta-\gamma) - 2\alpha^2)\lambda_0 - c)e_2 - \frac{1}{2}(-\gamma\alpha + \frac{1}{2}\delta(\beta-\gamma))e_3, \\ De_3 = \frac{1}{2}(-\gamma\alpha + \frac{1}{2}\delta(\beta-\gamma))e_2 - ((\beta(\beta-\gamma) - 2\alpha^2)\lambda_0 + c)e_3. \end{cases} \tag{96}$$

By (52), we have

$$\begin{cases} \alpha(\frac{1}{2}\beta(\beta-\gamma) - \alpha^2 - \beta(\beta-\gamma)\lambda_0 + 2\alpha^2\lambda_0 - c) + \frac{1}{2}(\beta+\gamma)(\gamma\alpha - \frac{1}{2}\delta(\beta-\gamma)) = 0, \\ \beta(\beta(\beta-\gamma) - 2\alpha^2 - \beta(\beta-\gamma)\lambda_0 + 2\alpha^2\lambda_0 - c) + \frac{1}{2}(\delta-\alpha)(\gamma\alpha - \frac{1}{2}\delta(\beta-\gamma)) = 0, \\ \gamma(\beta(\beta-\gamma)\lambda_0 - 2\alpha^2\lambda_0 + c) - \frac{1}{2}(\delta-\alpha)(\gamma\alpha - \frac{1}{2}\delta(\beta-\gamma)) = 0, \\ \delta(\frac{1}{2}\beta(\beta-\gamma) - \alpha^2 - \beta(\beta-\gamma)\lambda_0 + 2\alpha^2\lambda_0 - c) - \frac{1}{2}(\beta+\gamma)(\gamma\alpha - \frac{1}{2}\delta(\beta-\gamma)) = 0. \end{cases} \tag{97}$$

According to the condition $\alpha + \delta \neq 0$, $\alpha\gamma - \beta\delta = 0$, we calculate that

$$\begin{cases} \frac{1}{2}\beta(\beta-\gamma) - \alpha^2 - \beta(\beta-\gamma)\lambda_0 + 2\alpha^2\lambda_0 - c = 0, \\ (\beta+\gamma)(\gamma\alpha - \frac{1}{2}\delta(\beta-\gamma)) = 0, \\ (\beta+\gamma)(\beta(\beta-\gamma)\lambda_0 - 2\alpha^2\lambda_0 + c) = 0. \end{cases} \tag{98}$$

We choose $\beta + \gamma = 0$, then we have $\beta(\alpha + \delta) = 0$, and $c = -\alpha^2 + \alpha^2\lambda_0$. We set $\beta + \gamma \neq 0$ and $c = -\beta(\beta-\gamma)\lambda_0 + 2\alpha^2\lambda_0$. By the calculation, we have $\beta^2 = 2\alpha^2$, $\gamma = \delta = 0$ and then $c = 0$. □

Theorem 21. *If one of the following two conditions is satisfied*

(i) $\alpha = \beta = 0$, $\delta \neq 0$, $c = 0$,

(ii) $\alpha \neq 0, \beta = \gamma = 0, \alpha + \delta \neq 0, c = -\alpha^2 + 2\alpha^2\lambda_0$, (G_6, g, J) *is the algebraic Schouten soliton associated with the connection* ∇^1.

Proof of Theorem 19. From [7], we have

$$\widetilde{\mathrm{Ric}}^1 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -(\alpha^2 + \beta\gamma) & 0 & 0 \\ 0 & -\alpha^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{99}$$

It is a simple matter of $s^1 = -(2\alpha^2 + \beta\gamma)$. It follows that

$$\begin{cases} De_1 = -(\alpha^2 + \beta\gamma - (2\alpha^2 + \beta\gamma)\lambda_0 + c)e_1, \\ De_2 = -(\alpha^2 - (2\alpha^2 + \beta\gamma)\lambda_0 + c)e_2, \\ De_3 = -(-(2\alpha^2 + \beta\gamma)\lambda_0 + c)e_3. \end{cases} \tag{100}$$

An easy computation shows that

$$\begin{cases} \alpha(\alpha^2 + \beta\gamma - (2\alpha^2 + \beta\gamma)\lambda_0 + c) = 0, \\ \beta(2\alpha^2 + \beta\gamma - (2\alpha^2 + \beta\gamma)\lambda_0 + c) = 0, \\ \gamma(\beta\gamma - (2\alpha^2 + \beta\gamma)\lambda_0 + c) = 0, \\ \delta(\alpha^2 + \beta\gamma - (2\alpha^2 + \beta\gamma)\lambda_0 + c) = 0. \end{cases} \tag{101}$$

The first and fourth equations of system (98) imply that

$$(\alpha + \delta)(\alpha^2 + \beta\gamma - (2\alpha^2 + \beta\gamma)\lambda_0 + c) = 0. \tag{102}$$

Because $\alpha + \delta \neq 0$, then we have $c = -\alpha^2 - \beta\gamma + (2\alpha^2 + \beta\gamma)\lambda_0$, $\alpha^2\beta = 0$, and $\alpha^2\gamma = 0$. Let $\alpha = 0$, then $\delta \neq 0$, $\beta = 0$, $c = 0$. If $\alpha \neq 0$, then $\beta = \gamma = 0$, $c = -\alpha^2 - \beta\gamma + 2\alpha^2\lambda_0$. □

Theorem 22. (G_7, g, J) *is the algebraic Schouten soliton associated with the connection* ∇^0 *if and only if*

(i) $\alpha = \gamma = 0, \delta \neq 0, c = 0$,

(ii) $\alpha \neq 0, \beta = \gamma = 0, \alpha + \delta \neq 0, c = -\frac{1}{2}\alpha^2 + 2\alpha^2\lambda_0$.

Proof of Theorem 20. By [7], we have

$$\widetilde{\mathrm{Ric}}^0 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -(\alpha^2 + \frac{1}{2}\beta\gamma) & 0 & \frac{1}{2}(\alpha\gamma + \frac{1}{2}\delta\gamma) \\ 0 & -(\alpha^2 + \frac{1}{2}\beta\gamma) & -\frac{1}{2}(\alpha^2 + \frac{1}{2}\beta\gamma) \\ -\frac{1}{2}(\alpha\gamma + \frac{1}{2}\delta\gamma) & \frac{1}{2}(\alpha^2 + \frac{1}{2}\beta\gamma) & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{103}$$

Clearly, $s^0 = -(2\alpha^2 + \beta\gamma)$. It follows that

$$\begin{cases} De_1 = -(\alpha^2 + \frac{1}{2}\beta\gamma - (2\alpha^2 + \beta\gamma)\lambda_0 + c)e_1 + \frac{1}{2}(\alpha\gamma + \frac{1}{2}\delta\gamma)e_3, \\ De_2 = -(\alpha^2 + \frac{1}{2}\beta\gamma - (2\alpha^2 + \beta\gamma)\lambda_0 + c)e_2 - \frac{1}{2}(\alpha^2 + \frac{1}{2}\beta\gamma)e_3, \\ De_3 = -\frac{1}{2}(\alpha\gamma + \frac{1}{2}\delta\gamma)e_1 + \frac{1}{2}(\alpha^2 + \frac{1}{2}\beta\gamma)e_2 - (-(2\alpha^2 + \beta\gamma)\lambda_0 + c)e_3. \end{cases} \tag{104}$$

A long but straightforward calculation shows that

$$\begin{cases} \alpha(\alpha^2 + \frac{1}{2}\beta\gamma - (2\alpha^2 + \beta\gamma)\lambda_0 + c) - \frac{1}{2}(\beta + \gamma)(\alpha\gamma + \frac{1}{2}\gamma\delta) - \frac{1}{2}\alpha(\alpha^2 + \frac{1}{2}\beta\gamma) = 0, \\ \beta(\alpha^2 + \frac{1}{2}\beta\gamma - (2\alpha^2 + \beta\gamma)\lambda_0 + c) - \frac{1}{2}\delta(\alpha\gamma + \frac{1}{2}\gamma\delta) = 0, \\ \frac{1}{2}\alpha(\alpha^2 + \frac{1}{2}\beta\gamma - 2(2\alpha^2 + \beta\gamma)\lambda_0 + 2c) - \frac{1}{2}\beta(\alpha\gamma + \frac{1}{2}\gamma\delta) = 0, \\ \beta(\alpha^2 + \frac{1}{2}\beta\gamma - (2\alpha^2 + \beta\gamma)\lambda_0 + c) = 0, \\ \gamma(-(2\alpha^2 + \beta\gamma)\lambda_0 + c) - \frac{1}{2}\delta(\alpha\gamma + \frac{1}{2}\gamma\delta) = 0, \\ \frac{1}{2}\delta(\alpha^2 + \frac{1}{2}\beta\gamma - 2(2\alpha^2 + \beta\gamma)\lambda_0 + 2c) + \frac{1}{2}\beta(\alpha\gamma + \frac{1}{2}\gamma\delta) = 0, \\ \frac{1}{2}\delta(\alpha^2 + \frac{1}{2}\beta\gamma - 2(2\alpha^2 + \beta\gamma)\lambda_0 + 2c) + \frac{1}{2}(\beta + \gamma)(\alpha\gamma + \frac{1}{2}\gamma\delta) = 0. \end{cases} \tag{105}$$

The first and third equations of system (102) yield

$$\gamma(\alpha\gamma + \frac{1}{2}\gamma\delta) = 0, \tag{106}$$

for $\alpha\gamma = 0$, we have

$$\begin{cases} \gamma\delta = 0, \\ \gamma^2(\alpha + \frac{1}{2}\delta) = 0. \end{cases} \tag{107}$$

Let us regard $\gamma = 0$. We have

$$\begin{cases} \alpha(\alpha^2 - 4\alpha^2\lambda_0 + 2c) = 0, \\ \beta(\alpha^2 - 2\alpha^2\lambda_0 + c) = 0, \\ \delta(\alpha^2 - 4\alpha^2\lambda_0 + 2c) = 0. \end{cases} \tag{108}$$

Since $\alpha + \delta \neq 0$, we have $\alpha^2 - 4\alpha^2\lambda_0 + 2c = 0$. Then

$$\alpha^2\beta = 0. \tag{109}$$

We assume that $\alpha = 0$, in this case, we obtain $\delta \neq 0$, $c = 0$. If $\alpha \neq 0$, then $\beta = 0$, $c = -\frac{1}{2}\alpha^2 + 2\alpha^2\lambda_0$. □

Theorem 23. *When $\alpha \neq 0$, $\beta = \gamma = 0$, $\delta = \frac{1}{2}\alpha$, and $c = -\frac{1}{2}\alpha^2 + 2\alpha^2\lambda_0$, (G_7, g, J) is the algebraic Schouten soliton associated with the connection ∇^1.*

Proof of Theorem 21. From [7], we have

$$\widetilde{\mathrm{Ric}}^1\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -\alpha^2 & \frac{1}{2}(\beta\delta - \alpha\beta) & -\beta(\alpha+\delta) \\ \frac{1}{2}(\beta\delta - \alpha\beta) & -(\alpha^2 + \beta^2 + \beta\gamma) & -\frac{1}{2}(\beta\gamma + \alpha\delta + 2\delta^2) \\ \beta(\alpha+\delta) & \frac{1}{2}(\beta\gamma + \alpha\delta + 2\delta^2) & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{110}$$

Of course $s^1 = -(2\alpha^2 + \beta^2 + \beta\gamma)$. It follows that

$$\begin{cases} De_1 = -(\alpha^2 - (2\alpha^2 + \beta^2 + \beta\gamma)\lambda_0 + c)e_1 + \frac{1}{2}(\beta\delta - \alpha\beta)e_2 - \beta(\alpha+\delta)e_3, \\ De_2 = \frac{1}{2}(\beta\delta - \alpha\beta)e_1 - (\alpha^2 + \beta^2 + \beta\gamma - (2\alpha^2 + \beta^2 + \beta\gamma)\lambda_0 + c)e_2 - \frac{1}{2}(\beta\gamma + \alpha\delta + 2\delta^2)e_3, \\ De_3 = \beta(\alpha+\delta)e_1 + \frac{1}{2}(\beta\gamma + \alpha\delta + 2\delta^2)e_2 - (-(2\alpha^2 + \beta^2 + \beta\gamma)\lambda_0 + c)e_3. \end{cases} \tag{111}$$

Therefore, Equation (52) now becomes

$$\begin{cases} \alpha(\alpha^2 + \beta^2 + \beta\gamma - (2\alpha^2 + \beta^2 + \beta\gamma)\lambda_0 + c) + (\beta^2 + \beta\gamma)(\alpha+\delta) + \frac{1}{2}\beta(\beta\delta - \alpha\beta) \\ -\frac{1}{2}\alpha(\beta\gamma + \alpha\delta + 2\delta^2) = 0, \\ \beta(\frac{1}{2}\alpha^2 - (2\alpha^2 + \beta^2 + \beta\gamma)\lambda_0 + c + \frac{3}{2}\alpha\delta + \delta^2) = 0, \\ \beta(2\alpha^2 + \beta^2 + \beta\gamma - (2\alpha^2 + \beta^2 + \beta\gamma)\lambda_0 + c) - \beta(\beta\gamma + \alpha\delta + 2\delta^2) + \beta(\alpha+\delta)(\delta - \alpha) = 0, \\ \alpha(-(2\alpha^2 + \beta^2 + \beta\gamma)\lambda_0 + c) + \frac{1}{2}\alpha(\beta\gamma + \alpha\delta + 2\delta^2) + \frac{1}{2}(\beta - \gamma)(\beta\delta - \alpha\beta) + \beta^2(\alpha+\delta) = 0, \\ -\beta(\beta^2 + \beta\gamma + (2\alpha^2 + \beta^2 + \beta\gamma)\lambda_0 - c) + \frac{1}{2}(\alpha - \delta)(\beta\delta - \alpha\beta) + \beta(\beta\gamma + \alpha\delta + 2\delta^2) = 0, \\ \beta(-\frac{1}{2}\alpha\delta - \frac{1}{2}\delta^2 - (2\alpha^2 + \beta^2 + \beta\gamma)\lambda_0 + c) = 0, \\ \gamma(\beta^2 + \beta\gamma - (2\alpha^2 + \beta^2 + \beta\gamma)\lambda_0 + c) - \frac{1}{2}(\alpha - \delta)(\beta\delta - \alpha\beta) + \beta(\alpha+\delta)(\delta - \alpha) = 0, \\ \delta(-(2\alpha^2 + \beta^2 + \beta\gamma)\lambda_0 + c) - \frac{1}{2}(\beta - \gamma)(\beta\delta - \alpha\beta) + \frac{1}{2}\delta(\beta\gamma + \alpha\delta + 2\delta^2) - \beta^2(\alpha+\delta) = 0, \\ -\delta(\alpha^2 + \beta^2 + \beta\gamma - (2\alpha^2 + \beta^2 + \beta\gamma)\lambda_0 + c) + \frac{1}{2}\beta(\beta\delta - \alpha\beta) + \frac{1}{2}\delta(\beta\gamma + \alpha\delta + 2\delta^2) \\ +(\beta^2 + \beta\gamma)(\alpha+\delta) = 0. \end{cases} \tag{112}$$

Throughout the proof, recall that $\alpha + \delta \neq 0$ and $\alpha\gamma = 0$. Assume first that $\alpha \neq 0$, $\gamma = 0$. In this case,

$$\begin{cases} \alpha(\alpha^2+\beta^2-(2\alpha^2+\beta^2)\lambda_0+c)+\beta^2(\alpha+\delta)+\frac{1}{2}\beta(\beta\delta-\alpha\beta)-\frac{1}{2}\alpha(\alpha\delta+2\delta^2)=0, \\ \beta(\frac{1}{2}\alpha^2-(2\alpha^2+\beta^2)\lambda_0+c+\frac{3}{2}\alpha\delta+\delta^2)=0, \\ \beta(2\alpha^2+\beta^2-(2\alpha^2+\beta^2)\lambda_0+c)-\beta(\alpha\delta+2\delta^2)+\beta(\alpha+\delta)(\delta-\alpha)=0, \\ \alpha(-(2\alpha^2+\beta^2)\lambda_0+c)+\frac{1}{2}\alpha(\alpha\delta+2\delta^2)+\frac{1}{2}\beta(\beta\delta-\alpha\beta)+\beta^2(\alpha+\delta)=0, \\ -\beta(\beta^2+(2\alpha^2+\beta^2)\lambda_0-c)+\frac{1}{2}(\alpha-\delta)(\beta\delta-\alpha\beta)+\beta(\alpha\delta+2\delta^2)=0, \\ \beta(-\frac{1}{2}\alpha\delta-\frac{1}{2}\delta^2-(2\alpha^2+\beta^2)\lambda_0+c)=0, \\ -\frac{1}{2}(\alpha-\delta)(\beta\delta-\alpha\beta)+\beta(\alpha+\delta)(\delta-\alpha)=0, \\ \delta(-(2\alpha^2+\beta^2)\lambda_0+c)-\frac{1}{2}\beta(\beta\delta-\alpha\beta)+\frac{1}{2}\delta(\alpha\delta+2\delta^2)-\beta^2(\alpha+\delta)=0, \\ -\delta(\alpha^2+\beta^2-(2\alpha^2+\beta^2)\lambda_0+c)+\frac{1}{2}\beta(\beta\delta-\alpha\beta)+\frac{1}{2}\delta(\alpha\delta+2\delta^2)+\beta^2(\alpha+\delta)=0. \end{cases} \tag{113}$$

Next suppose that $\beta=0$,

$$\begin{cases} \alpha(\alpha^2-2\alpha^2\lambda_0+c)-\frac{1}{2}\alpha(\alpha\delta+2\delta^2)=0, \\ \alpha(-2\alpha^2\lambda_0+c)+\frac{1}{2}\alpha(\alpha\delta+2\delta^2)=0, \\ \delta(-2\alpha^2\lambda_0+c)+\frac{1}{2}\delta(\alpha\delta+2\delta^2)=0, \\ -\delta(\alpha^2-2\alpha^2\lambda_0+c)+\frac{1}{2}\delta(\alpha\delta+2\delta^2)=0. \end{cases} \tag{114}$$

Then, we have

$$(\alpha+\delta)(2\delta-\alpha)=0; \tag{115}$$

that is, $\delta=\frac{1}{2}\alpha$, $c=-\frac{1}{2}\alpha^2+2\alpha^2\lambda_0$. □

4. Conclusions

In this paper, we present the necessary conditions for (G_i, g) to be an algebraic Schouten soliton on the three-dimensional Lorentzian Lie groups with Levi-Civita connections and provide corresponding proofs. To enrich the results of this article, we studied canonical connections and Kobayashi–Nomizu connections and provide corresponding conclusions. The innovation of this article lies in proposing the definition of algebraic Schouten solitons, which provides a new perspective for future research.

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