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Asymptotic Approximations of Higher-Order Apostol–Frobenius–Genocchi Polynomials with Enlarged Region of Validity

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Abstract: In this paper, the uniform approximations of the Apostol–Frobenius–Genocchi polynomials of order α in terms of the hyperbolic functions are derived through the saddle-point method. Moreover, motivated by the works of Corcino et al., an approximation with enlarged region of validity for these polynomials is also obtained. It is found out that the methods are also applicable for the case of the higher order generalized Apostol-type Frobenius–Genocchi polynomials and Apostol–Frobenius-type poly-Genocchi polynomials with parameters a, b, and c. These methods demonstrate the techniques of computing the symmetries of the defining equation of these polynomials. Graphs are illustrated to show the accuracy of the exact values and corresponding approximations of these polynomials with respect to some specific values of its parameters.

Keywords: Apostol–Frobenius–Genocchi polynomials; generalized Apostol-type Frobenius–Genocchi polynomials; Apostol–Frobenius-type poly-Genocchi polynomials; Genocchi polynomials; asymptotic approximation

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1. Introduction

The development of special functions involving the generalizations, extensions, and combinations of other special functions has become a flourishing area in mathematics. The established properties and identities of these special functions have relevant applications arising in mathematics and other fields of knowledge. A particular kind of these special functions is the Apostol-type polynomials, which have been mixed or combined with other classical polynomials to define new special polynomials. An interesting result in the combination of these polynomials is the construction of Apostol–Frobenius–Genocchi polynomials of order a denoted by $\mathcal{G}_n^{\alpha}(z;\lambda;u)$, which are defined by the generating function (see [1,2]),

$$\left(\frac{(1-u)w}{\lambda e^{w}-u}\right)^{\alpha} e^{zw} = \sum_{n=0}^{\infty} \mathcal{G}_n^{\alpha}(z;u;\lambda)\frac{w^n}{n!}, \quad |w| < \left|\log\left(\frac{\lambda}{u}\right)\right| \tag{1}$$

where $\lambda, u \in \mathbb{C}$ with $\lambda \neq 0, u \neq 1$ and $\alpha \in \mathbb{Z}$.

On setting $\lambda = 1$, (1) gives the Frobenius–Genocchi polynomials of order α, which were introduced by Yasar and Özarslan by means of following generating function [3]:

$$\left(\frac{(1-u)w}{e^{w}-u}\right)^{\alpha} e^{zw} = \sum_{n=0}^{\infty} \mathcal{G}_n^{\alpha}(z;u)\frac{w^n}{n!}, \quad |w| < |\log(u)|. \tag{2}$$

When $u = -1$, (1) gives the Apostol–Genocchi polynomials of order α defined by this generating function [4]

$$\left(\frac{2w}{\lambda e^{w}+1}\right)^{\alpha} e^{zw} = \sum_{n=0}^{\infty} \mathcal{G}_n^{\alpha}(z;\lambda)\frac{w^n}{n!}, \quad |w| < |\pi i - \log(\lambda)|. \tag{3}$$

When $\alpha = 1$ in (1) and (2), $\mathcal{G}_n^1(z;u;\lambda) = \mathcal{G}_n(z;u;\lambda)$ and $\mathcal{G}_n^1(z;u) = \mathcal{G}_n(z;u)$, where $\mathcal{G}_n(z;u;\lambda)$ and $\mathcal{G}_n(z;u)$ are called the Apostol–Frobenius–Genocchi polynomials and Frobenius–Genocchi polynomials, respectively [5].

The Apostol–Frobenius–Genocchi polynomials of order α are λ extensions of the Frobenius–Genocchi polynomials. The Frobenius–Genocchi polynomials are formed by mixing the definitions of the two classical polynomials, namely the Frobenius polynomials and the Genocchi polynomials. The Frobenius polynomials and numbers can be traced back to the works of the great German mathematician Ferdinand Georg Frobenius, who studied the context of these polynomials in number theory and the relation of its divisibility properties with the Stirling numbers of the second kind [6]. On the other hand, the Genocchi polynomials, which were named after Angelo Genocchi, have been studied extensively because of their relevant combinatorial relations and properties in number theory, complex analytic number theory, homotopy theory, quantum physics, etc. [7].

The Apostol-type polynomials have various forms which are generalizations of the Appell family [8]. In sciences and engineering, special polynomials associated with the Appell polynomial sequences are essential to solve problems involving differential equations. The solutions satisfying these equations may be expressed using special functions. Genocchi polynomials, Bernoulli polynomials and Euler polynomials are typical examples of Appell polynomial sequences. The applications of these sequences are important in other branches of mathematics to obtain polynomial expansions and approximation formulas both in analytic number theory and numerical analysis [9]. This motivates the present authors to investigate the further generalization of these polynomials and their asymptotic approximations.

Corcino et al. produced related studies on asymptotic approximations of some special polynomials in terms of hyperbolic functions (see [10–12]). It is observed that there is a resemblance in the generating function of the Apostol-tangent polynomials in [10] and the Apostol–Frobenius–Euler polynomials. However, the approximation of the Apostol–Frobenius–Genocchi polynomials parallel to the results of Corcino et al. remains to be unexplored in other related studies. In this study, the uniform approximation of the Apostol–Frobenius–Genocchi polynomials of order α for large n valid in some unbounded region of the complex variable z are derived using the saddle-point method. Using the technique of the contour integration in [10], an approximation with an enlarged region of validity is also obtained. Moreover, the same methods are applied to obtain the approximations for the case of the generalized Apostol-type Frobenius–Genocchi polynomials of order α with parameters a, b, and c denoted by $\mathcal{G}_n^{(\alpha)}(z;u;a,b,c,\lambda)$ and Apostol–Frobenius-type poly-Genocchi polynomials of order α with parameters a, b, and c denoted by $\mathcal{G}_n^{(\mu,\alpha)}(z;\lambda,u,a,b,c,)$. Corresponding asymptotic formulas of other special polynomials are given as corollaries. It is worth mentioning that the methods used in deriving the asymptotic formulas demonstrate the techniques of computing the symmetries of the defining equations of Apostol–Frobenius–Genocchi polynomials of order α as well as their generalizations $\mathcal{G}_n^{(\alpha)}(z;u;a,b,c,\lambda)$ and $\mathcal{G}_n^{(\mu,\alpha)}(z;\lambda,u,a,b,c,)$.

2. Uniform Approximation

In this section, the uniform approximation of the Apostol–Frobenius–Genocchi polynomials of order α is explored using the saddle-point method. The following theorem gives the said approximation.

Theorem 1. *For $n, \alpha \in \mathbb{Z}^+, u, \lambda \in \mathbb{C} \setminus \{0,1\}$, and $z \in \mathbb{C} \setminus \{0\}$ such that $|Im\, z^{-1}| < 2\pi - Arg\left(\frac{\lambda}{u}\right)$ or $|z^{-1}| < |z^{-1} - (2\pi i - \delta)|$, the Apostol–Frobenius–Genocchi polynomials of order α satisfy*

$$\mathcal{G}_n^{\alpha}\left(nz + \frac{\alpha}{2}; u; \lambda\right) = \frac{(nz)^n (1-u)^{\alpha} \operatorname{csch}^{\alpha}\left(\frac{\delta z+1}{2z}\right)}{(2z\sqrt{\lambda u})^{\alpha}} \left[1 - \frac{\alpha}{(2nz^2)}\left\{z^2(\alpha-1) - \alpha z \coth\left(\frac{\delta z+1}{2z}\right) + \frac{(\alpha+1)\coth^2\left(\frac{\delta z+1}{2z}\right) - 1}{4}\right\} + O\left(\frac{1}{n^2}\right)\right]. \tag{4}$$

where $\delta = \log\left(\frac{\lambda}{u}\right)$ and the logarithm is taken to be the principal branch.

Proof. Applying the Cauchy Integral formula to (1),

$$\mathcal{G}_n^{\alpha}(z; u; \lambda) = \frac{n!}{2\pi i} \int_C \frac{(1-u)^{\alpha}}{u^{\alpha}} \frac{w^{\alpha} e^{zw}}{(e^{\delta+w} - 1)^{\alpha}} \frac{dw}{w^{n+1}}, \tag{5}$$

where C is a circle about 0 with radius lesser than $\left|2\pi - \log\left(\frac{\lambda}{u}\right)\right|$ and $\delta = \log\left(\frac{\lambda}{u}\right)$ is a logarithm taken to be the principal branch.

With $\left(e^{\delta+w} - 1\right)^{\alpha} = \left(2e^{\frac{\delta+w}{2}} \sinh\left(\frac{\delta+w}{2}\right)\right)^{\alpha}$, it follows from shifting $z \to z + \alpha/2$ that

$$\mathcal{G}_n^{\alpha}\left(z + \frac{\alpha}{2}; u; \lambda\right) = \frac{n!}{2\pi i} \frac{(1-u)^{\alpha}}{(2\sqrt{\lambda u})^{\alpha}} \int_C f(w) e^{zw} \frac{dw}{w^{n+1}}, \tag{6}$$

where $f(w) = \frac{w^{\alpha}}{\sinh^{\alpha}\left(\frac{\delta+w}{2}\right)}$. Note that $f(w)$ is a meromorphic function with simple poles of order α at the zeros of $\sinh^{\alpha}\left(\frac{\delta+w}{2}\right)$, which are given by $w_j = 2j\pi - \delta, j = \pm 1, \pm 2, \cdots$.

By taking $z \to nz$ and letting $nz \to \infty$ with fixed z in (6),

$$\mathcal{G}_n^{\alpha}\left(nz + \frac{\alpha}{2}; u; \lambda\right) = \frac{n!}{2\pi i} \frac{(1-u)^{\alpha}}{(2\sqrt{\lambda u})^{\alpha}} \int_C f(w) e^{n(zw - \log w)} \frac{dw}{w}. \tag{7}$$

Note that on the saddle-point method, the main contribution of the integrand to the integral in (7) originates at the saddle-point of the argument of the exponential [13]. Thus, if the point $w = z^{-1}$ is not a pole, then the approximations of $\mathcal{G}_n^{\alpha}\left(nz + \frac{\alpha}{2}; u; \lambda\right)$ can be obtained by expanding $f(w)$ around the point. From Lemmas 1 and 2, and Theorem 1 of [14], it follows that

$$\mathcal{G}_n^{\alpha}\left(nz + \frac{\alpha}{2}; u; \lambda\right) = (nz)^n \frac{(1-u)^{\alpha}}{(2\sqrt{\lambda u})^{\alpha}} \sum_{k=0}^{\infty} \frac{f^{(k)}(z^{-1})}{k!} \frac{p_k(n)}{(nz)^k}, \tag{8}$$

where $p_k(n)$ are the polynomials

$$\begin{aligned} &p_0(n) = 1, p_1(n) = 0, p_2(n) = -n, p(3) = 2n, \\ &p_k(n) = (1-k)p_{k-1}(n) + np_{k-2}(n), k \geq 3. \end{aligned} \tag{9}$$

Computing the derivatives $f^{(k)}(z^{-1})$ for $k = 0, 1, 2$ gives

$$f^{(0)}(z^{-1}) = \frac{\operatorname{csch}^{\alpha}\left(\frac{\delta z+1}{2z}\right)}{z^{\alpha}},$$

$$f^{(1)}(z^{-1}) = \frac{\alpha \operatorname{csch}^{\alpha}\left(\frac{\delta z+1}{2z}\right)}{z^{\alpha}} \left\{ z - \frac{\coth\left(\frac{\delta z+1}{2z}\right)}{2} \right\}, \text{ and}$$

$$f^{(2)}(z^{-1}) = \frac{\alpha \operatorname{csch}^{\alpha}\left(\frac{\delta z+1}{2z}\right)}{z^{\alpha}} \left\{ z^2(\alpha-1) - \alpha z \coth\left(\frac{\delta z+1}{2z}\right) + \frac{(\alpha+1)\coth^2\left(\frac{\delta z+1}{2z}\right) - 1}{4} \right\}.$$

Expanding the sum in (8) and keeping only the first three terms gives

$$\begin{aligned}
\mathcal{G}_n^{\alpha}\left(nz + \frac{\alpha}{2}; u; \lambda\right) &= \frac{(nz)^n (1-u)^{\alpha}}{(2\sqrt{\lambda u})^{\alpha}} \left[\frac{f^{(0)}(z^{-1})}{0!} + \frac{f^{(1)}(z^{-1})}{1!} \frac{p_1(n)}{nz} + \frac{f^{(2)}(z^{-1})}{2!} \frac{p_2(n)}{(nz)^2} + O\left(\frac{1}{n^2}\right) \right] \\
&= \frac{(nz)^n (1-u)^{\alpha}}{(2\sqrt{\lambda u})^{\alpha}} \left[\frac{\operatorname{csch}^{\alpha}\left(\frac{\delta z+1}{2z}\right)}{z^{\alpha}} - \frac{\alpha \operatorname{csch}^{\alpha}\left(\frac{\delta z+1}{2z}\right)}{z^{\alpha}(2nz^2)} \left\{ z^2(\alpha-1) - \alpha z \coth\left(\frac{\delta z+1}{2z}\right) \right.\right. \\
&\left.\left. + \frac{\alpha \coth^2\left(\frac{\delta z+1}{2z}\right) + \operatorname{csch}^2\left(\frac{\delta z+1}{2z}\right)}{4} \right\} + O\left(\frac{1}{n^2}\right) \right] \\
&= \frac{(nz)^n (1-u)^{\alpha} \operatorname{csch}^{\alpha}\left(\frac{\delta z+1}{2z}\right)}{(2z\sqrt{\lambda u})^{\alpha}} \left[1 - \frac{\alpha}{(2nz^2)} \left\{ z^2(\alpha-1) - \alpha z \coth\left(\frac{\delta z+1}{2z}\right) \right.\right. \\
&\left.\left. + \frac{\alpha \coth^2\left(\frac{\delta z+1}{2z}\right) + \operatorname{csch}^2\left(\frac{\delta z+1}{2z}\right)}{4} \right\} + O\left(\frac{1}{n^2}\right) \right].
\end{aligned}$$

□

Figure 1 shows the accuracy of the approximation obtained in Theorem 1. The following corollaries give the uniform approximations of the Frobenius–Genocchi polynomials of order α, Apostol–Genocchi polynomials of order α and Genocchi polynomials of order α.

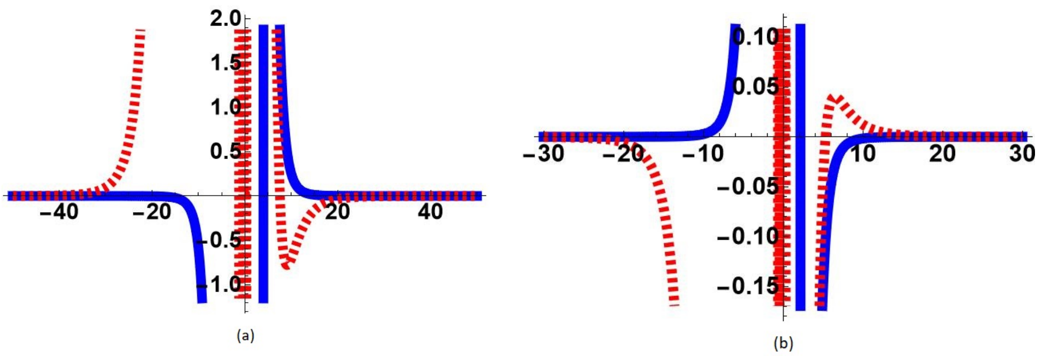

Figure 1. (**a**) $n = 9, \alpha = 8, u = 4$, and $\lambda = 5$. (**b**) $n = 8, \alpha = 7, u = 4$, and $\lambda = 6$. Solid lines represent the Apostol–Frobenius–Genocchi polynomials of order α $G_n^{\alpha}(nz + \frac{\alpha}{2}; u; \lambda)$ for several values of n, whereas dashed lines represent the right hand side of (4) with $z \equiv x$, both normalized by the factor $(1 + |\frac{x}{\sigma}|^n)^{-1}$ where we choose $\sigma = 0.5$.

Corollary 1. *For $n, \alpha \in \mathbb{Z}^+, u \in \mathbb{C} \setminus \{0, 1\}$, and $z \in \mathbb{C} \setminus \{0\}$ such that $|Im\ z^{-1}| < 2\pi - Arg\left(\frac{1}{u}\right)$ or $|z^{-1}| < |z^{-1} - (2\pi i + \nu)|$, the Frobenius–Genocchi polynomials of order α satisfy*

$$\begin{aligned}
G_n^{\alpha}\left(nz + \frac{\alpha}{2}; u\right) &= \frac{(nz)^n (1-u)^{\alpha} \operatorname{csch}^{\alpha}\left(\frac{1-z\nu}{2z}\right)}{(2z\sqrt{u})^{\alpha}} \left[1 - \frac{\alpha}{(2nz^2)} \left\{ z^2(\alpha-1) - \alpha z \coth\left(\frac{1-z\nu}{2z}\right) \right.\right. \\
&\left.\left. + \frac{(\alpha+1)\coth^2\left(\frac{1-z\nu}{2z}\right) - 1}{4} \right\} + O\left(\frac{1}{n^2}\right) \right] \qquad (10)
\end{aligned}$$

where $v = \log(u)$ and the logarithm is taken to be the principal branch.

Proof. This follows from Theorem 1 by taking $\lambda = 1$. $\square$

Corollary 2. *For $n, \alpha \in \mathbb{Z}^+, \lambda \in \mathbb{C} \setminus \{0,1\}$, and $z \in \mathbb{C} \setminus \{0\}$ such that $|Im\ z^{-1}| < \pi - Arg(\lambda)$ or $|z^{-1}| < |z^{-1} - (\pi i - \tau)|$, the Apostol–Genocchi polynomials of order α satisfy*

$$\mathcal{G}_n^{\alpha}\left(nz + \frac{\alpha}{2}; \lambda\right) = \frac{(nz)^n (1-u)^{\alpha} \operatorname{sech}^{\alpha}\left(\frac{1+z\tau}{2z}\right)}{(z\sqrt{u})^{\alpha}} \left[1 - \frac{\alpha}{(2nz^2)} \left\{z^2(\alpha - 1) - \alpha z \tanh\left(\frac{1+z\tau}{2z}\right) + \frac{(\alpha+1)\tanh^2\left(\frac{1+z\tau}{2z}\right) - 1}{4}\right\} + O\left(\frac{1}{n^2}\right)\right] \tag{11}$$

where $\tau = \log(\lambda)$ and the logarithm is taken to be the principal lunch.

Proof. This follows from Theorem 1 by taking $u = -1$. $\square$

Corollary 3. *For $n, \alpha \in \mathbb{Z}^+$ and $z \in \mathbb{C} \setminus \{0\}$ such that $|Im\ z^{-1}| < \pi$ or $|z^{-1}| < |z^{-1} - \pi|$, the Genocchi polynomials of order α satisfy*

$$G_n^{\alpha}\left(nz + \frac{\alpha}{2}\right) = \frac{(nz)^n \operatorname{sech}^{\alpha}\left(\frac{1}{2z}\right)}{z^{\alpha}} \left[1 - \frac{\alpha}{(2nz^2)} \left\{z^2(\alpha - 1) - \alpha z \tanh\left(\frac{1}{2z}\right) + \frac{(\alpha+1)\tanh^2\left(\frac{1}{2z}\right) - 1}{4}\right\} + O\left(\frac{1}{n^2}\right)\right] \tag{12}$$

Proof. This follows from Theorem 1 by taking $\lambda = 1$ and $u = -1$. $\square$

Remark 1. *Taking $\alpha = 1$ in (12), approximation for the classical Genocchi polynomials $G_n\left(nz + \frac{1}{2}\right)$ is obtained similar with that of Corcino et al. (see Theorem 2.5, [11]).*

The graphs in Figure 2, Figure 3, and Figure 4 show the accuracy of the asymptotic formulae obtained in Corollary 1, Corollary 2, Corollary 3, respectively.

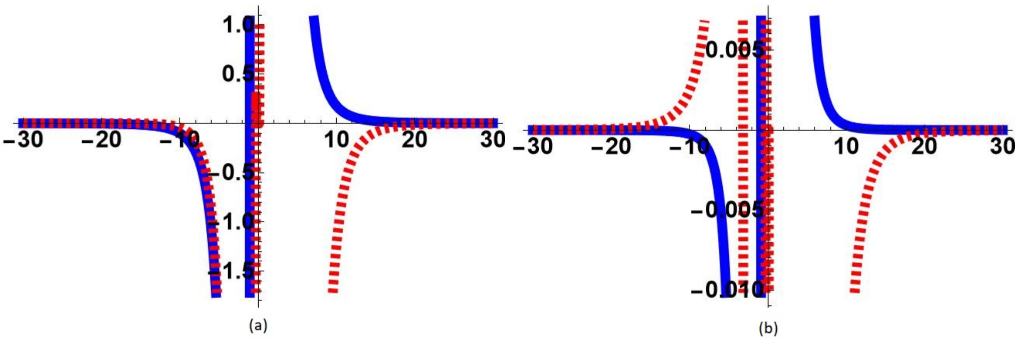

Figure 2. (**a**) $n = 8, \alpha = 5$, and $u = 2$. (**b**) $n = 7, \alpha = 6$, and $u = 3$. Solid lines represent the Frobenius–Genocchi polynomials of order α $G_n^{\alpha}(nz + \frac{\alpha}{2}; u)$ for several values of n, whereas dashed lines represent the right hand side of (10) with $z \equiv x$, both normalized by the factor $(1 + |\frac{x}{\sigma}|^n)^{-1}$, where we choose $\sigma = 0.5$.

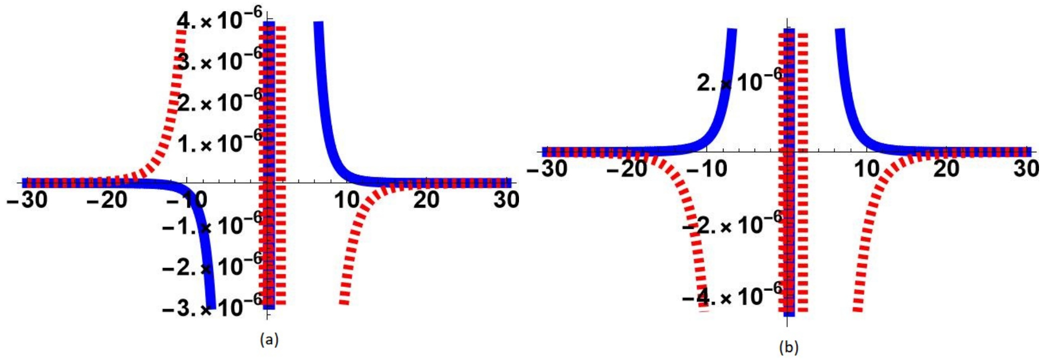

Figure 3. (**a**) $n = 10, \alpha = 7$, and $\lambda = 11$. (**b**) $n = 8, \alpha = 6$, and $\lambda = 9$. Solid lines represent the Apostol–Genocchi polynomials of order α $\mathcal{G}_n^{\alpha}(nz + \frac{\alpha}{2}; \lambda)$ for several values of n, whereas dashed lines represent the right-hand side of (11) with $z \equiv x$, both normalized by the factor $(1 + |\frac{x}{\sigma}|^n)^{-1}$ where we choose $\sigma = 0.5$.

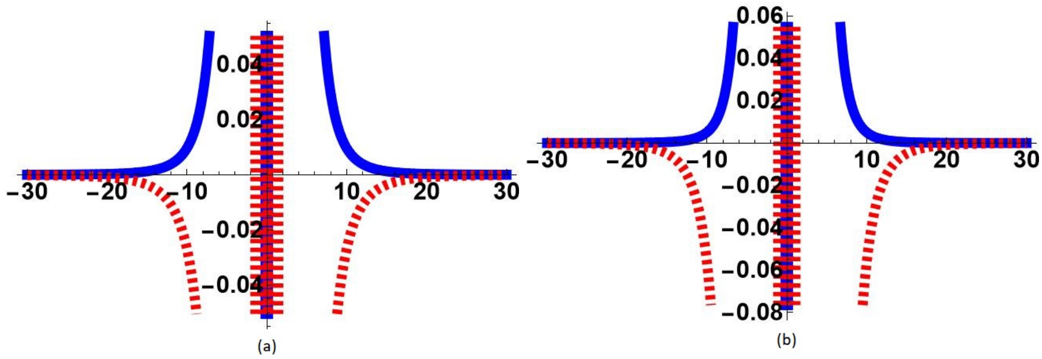

Figure 4. (**a**) $n = 7$ and $\alpha = 5$. (**b**) $n = 8$ and $\alpha = 6$. Solid lines represent the Genocchi polynomials of order α $G_n^{\alpha}(nz + \frac{\alpha}{2})$ for several values of n, whereas dashed lines represent the right hand side of (12) with $z \equiv x$, both normalized by the factor $(1 + |\frac{x}{\sigma}|^n)^{-1}$, where we choose $\sigma = 0.5$.

3. Enlarged Region of Validity

The validity of the approximations obtained from the previous section using the saddle-point method is valid in the region $|z^{-1}| < |z^{-1} - w_j|$ with poles $w_j = 2j\pi - \delta$, $j = \pm 1$, $\pm 2, \cdots$. In this section, approximation with an enlarged region of validity is derived by isolating the contribution of the poles. Motivated by the study of Corcino et al. [10], the approximation uses the method of contour integration, which introduces the incomplete gamma function in the formula. The following theorem describes the said approximation.

Theorem 2. *For $\lambda, u \in \mathbb{C} \setminus \{0, 1\}$, $\alpha \in \mathbb{Z}^+$ and $z \in \mathbb{C}$ such that $|z^{-1}| < |z^{-1} - w_k|$ for all $k = l+1, l+2, \cdots$, the Apostol–Frobenius–Euler polynomials of order α satisfy*

$$\begin{aligned} & G_n^{\alpha}\left(nz + \frac{\alpha}{2}; u; \lambda\right) \\ & = \frac{(1-u)^{\alpha}}{(2\sqrt{\lambda u})^{\alpha}} \left\{ \sum_{k=1}^{l} \sum_{j=1}^{\alpha} e^{w_k nz} r_{k_j} \left[\sum_{s=0}^{n} \binom{n}{s} (-1)^{(j-1)} \langle j-1 \rangle_s (w_k)^{-(j-1+s)} \left(\frac{(n-s)!}{w_k^{n-s+1}} \right.\right.\right. \\ & \left.\left.\left. - \frac{\Gamma(n-s+1, w_k nz)}{w_k^{n-s+1}} \right) + \frac{(-1)^j \langle j \rangle_n}{w_k^{j+n}} \right] + (nz)^n \sum_{k=0}^{\infty} \frac{f^{(k)}(z^{-1}) - h_l^{(k)}(z^{-1})}{k!} \frac{p_k(n)}{(nz)^k} \right\} \end{aligned} \tag{13}$$

where the polynomials $p_k(n)$ are given in (9), h_l^k is the kth derivative of the function $h_l(w)$ given by (21) and

$$\sum_{j=1}^{m} \frac{r_{k_j}}{(w - w_k)^j}$$

are the given principal parts of the Laurent series corresponding to the poles $w_k = 2k\pi - \delta$*, where* $\delta = \log\left(\frac{\lambda}{u}\right)$ *is a logarithm taken to be the principal branch. The entire function* $h_l(w)$ *is determined by* $f(w) = w^\alpha \operatorname{csch}^\alpha\left(\frac{\delta+w}{2}\right)$.

Proof. Using the Mittag–Leffler theorem ([15,16]), write $f(w) = w^\alpha / \sinh^\alpha\left(\frac{\delta+w}{2}\right)$ as

$$f(w) = \sum_{k=1}^{l}\left[\sum_{j=1}^{m}\frac{r_{k_j}}{(w-w_k)^j} + q_k(w)\right] + g(w) = \sum_{k=1}^{l}\sum_{j=1}^{\alpha}\frac{r_{k_j}}{(w-w_k)^j} + f_l(w), \tag{14}$$

where

$$f_l(w) = \sum_{k=1}^{l} q_k(w) + g(w), \tag{15}$$

$q_k(w)$ is a polynomial of w, r_{k_j} are residues at $w_k, k = 1, 2, \cdots, l$. With this, $f_l(w)$ has no poles inside the disk $|w| < |w_{m+1}|$. Substituting (14) to (7) gives

$$\begin{aligned} G_n^\alpha\left(nz + \frac{\alpha}{2}; u; \lambda\right) &= \frac{n!}{2\pi i}\frac{(1-u)^\alpha}{(2\sqrt{\lambda u})^\alpha}\int_C\left(\sum_{k=1}^{l}\sum_{j=1}^{\alpha}\frac{r_{k_j}}{(w-w_k)^j} + f_l(w)\right)e^{wnz}\frac{dw}{w^{n+1}} &(16)\\ &= X_l^{n,\alpha}(z) + Y_l^{n,\alpha}(z) &(17)\end{aligned}$$

where

$$X_l^{n,\alpha}(z) = \frac{n!}{2\pi i}\frac{(1-u)^\alpha}{(2\sqrt{\lambda u})^\alpha}\int_C f_l(w)e^{wnz}\frac{dw}{w^{n+1}}, \tag{18}$$

$$Y_l^{n,\alpha}(z) = \frac{(1-u)^\alpha}{(2\sqrt{\lambda u})^\alpha}\int_C \frac{n!}{2\pi i}\sum_{k=1}^{l}\sum_{j=1}^{\alpha}\frac{r_{k_j}}{(w-w_k)^j}e^{wnz}\frac{dw}{w^{n+1}}. \tag{19}$$

To evaluate (18), repeat the process from the last section using the saddle-point method to expand $f_l(w)$ around the saddle-point $w = z^{-1}$ instead of $f(w)$. It follows from Lemmas 1 and 2 and Theorem 1 of [14] that (18) may be expanded as the infinite sum

$$X_l^{n,\alpha}(z) = (nz)^n\frac{(1-u)^\alpha}{(2\sqrt{\lambda u})^\alpha}\sum_{k=0}^{\infty}\frac{f_l^{(k)}(z^{-1})}{k!}\frac{p_k(n)}{(nz)^k}, \tag{20}$$

where $p_k(n)$ are the polynomials in (9). Note that (20) is valid for $\alpha \in \mathbb{Z}^+, z \in \mathbb{C}\setminus\{0\}$ such that $|z^{-1}| < |z^{-1} - w_j|$ for $j = l+1, l+2, \cdots, \ldots$ given the first $2l$ poles of $f(w)$. From (14), the kth derivative of $f_l(w)$ is

$$f_l^{(k)}(w) = f^{(k)}(w) - h_l^{(k)}(w),$$

where

$$h_l(w) = -\sum_{k=1}^{l}\sum_{j=1}^{\alpha}\frac{r_{k_j}}{(w-w_k)^j}. \tag{21}$$

Thus, the expansion of (18) is

$$X_l^{n,\alpha}(z) = (nz)^n\frac{(1-u)^\alpha}{(2\sqrt{\lambda u})^\alpha}\sum_{k=0}^{\infty}\frac{f_l^{(k)}(z^{-1}) - h_l^{(k)}(z^{-1})}{k!}\frac{p_k(n)}{(nz)^k} \tag{22}$$

valid for $|z^{-1}| < |z^{-1} - w_j|, j = l+1, l+2, \cdots$ and $z \neq 0$. The expansion's range of validity is larger than that of the expansion in Theorem 1.

On the other hand, similar computations are employed from the methods of Corcino et al. (see [10]) to derive an expansion for $Y_l^{n,\alpha}(z)$. The technique involves shifting of the integration contour by $w = w_k + t$ in each integral in (19). Consequently, $dw = dt$ and

$$Y_l^{n,\alpha}(z) = \frac{(1-u)^\alpha}{(2\sqrt{\lambda u})^\alpha} \sum_{k=1}^{l} \sum_{j=1}^{\alpha} e^{w_k nz} r_{k_j} \frac{n!}{2\pi i} \int_{C'} \frac{e^{tnz}}{t^j} \frac{dt}{(w_k + t)^{n+1}}, \tag{23}$$

where $C' : t = -w_k + Re^{i\theta}, -\pi < \theta \leq \pi$ is a circle with radius R and center at $-w_k$. Note that 0 is not on the $w_k's$. This C' is the image of $C : w = Re^{i\theta}$ through the shift $w = w_k + t$. To proceed, observe that

$$\frac{e^{tnz}}{t^j} = \int_0^{nz} \frac{e^{tx}}{t^{j-1}} + \frac{1}{t^j}. \tag{24}$$

Substituting (24) to (23) gives

$$Y_l^{n,\alpha}(z) = \frac{(1-u)^\alpha}{(2\sqrt{\lambda u})^\alpha} \sum_{k=1}^{l} \sum_{j=1}^{\alpha} e^{w_k nz} r_{k_j} \frac{n!}{2\pi i} \int_{C'} \left(\int_0^{nz} \frac{e^{tx}}{t^{j-1}} + \frac{1}{t^j} \right) \frac{dw}{(w_k + t)^{n+1}}, \tag{25}$$

The proceeding part of the computations is expositions from [10]. Note that when $j = 1$,

$$\frac{n!}{2\pi i} \int_{C'} \frac{e^{tx}}{t^{j-1}} \frac{dw}{(w_k + t)^{n+1}} = \frac{d^n}{dt^n} e^{tx} t^{-(j-1)} \bigg|_{t=-w_k}. \tag{26}$$

is x^n. For $j \geq 1$, using the Leibniz rule for differentiation, it becomes

$$\begin{aligned} \frac{d^n}{dt^n} e^{tx} t^{-(j-1)} &= \sum_{s=0}^{n} \binom{n}{s} x^{n-s} e^{tx} \frac{d^s}{dt^s} t^{-(j-1)} \bigg|_{t=-w_k} \\ &= \sum_{s=0}^{n} \binom{n}{s} x^{n-s} e^{-w_k x} (-1)^{(j-1)} \langle j-1 \rangle_s (w_k)^{-(j-1+s)} \end{aligned} \tag{27}$$

where $\langle j-1 \rangle_s$ denote the rising factorial of $j-1$ with increment s. Thus, (26) can be written as

$$\frac{n!}{2\pi i} \int_{C'} \frac{e^{tx}}{t^{j-1}} \frac{dw}{(w_k + t)^{n+1}} = \sum_{s=0}^{n} \binom{n}{s} x^{n-s} e^{-w_k x} (-1)^{(j-1)} \langle j-1 \rangle_s (w_k)^{-(j-1+s)}. \tag{28}$$

It can also be evaluated that

$$\frac{n!}{2\pi i} \int_{C'} t^{-j} \frac{dw}{(w_k + t)^{n+1}} = \frac{d^n}{dt^n} (t^{-j}) \bigg|_{t=w_k} = \frac{(-1)^j \langle j \rangle_n}{w_k^{j+n}}. \tag{29}$$

Now, consider the incomplete gamma function

$$\Gamma(n-s+1, w_k z) = \int_{w_k z}^{\infty} e^{-t} t^{n-s} dt. \tag{30}$$

Let $\eta = \frac{t}{w_k}$. Then $t = \eta w_k$ and $w_k d\eta = dt$. Moreover, $t = \infty \iff \eta = \infty; t = w_k z \iff \eta = z$. Thus, (30) becomes

$$\Gamma(n-s+1, w_k z) = \int_{z}^{\infty} e^{-w_k \eta} (w_k \eta)^{n-s} w_k d\eta. \tag{31}$$

It can be shown that taking $z \mapsto nz$,

$$\int_0^{nz} e^{-w_k\eta}\eta^{n-s}d\eta = \int_0^{\infty} e^{-w_k\eta}\eta^{n-s}d\eta - \frac{\Gamma(n-s+1, w_k nz)}{w_k^{n-s+1}}. \tag{32}$$

Substituting (28) and (29) to (25) gives

$$Y_l^{n,\alpha}(z) = \frac{(1-u)^\alpha}{(2\sqrt{\lambda u})^\alpha}\sum_{k=1}^{l}\sum_{j=1}^{\alpha} e^{w_k nz} r_{k_j}\left[\sum_{s=0}^{n}\binom{n}{s}(-1)^{(j-1)}\langle j-1\rangle_s (w_k)^{-(j-1+s)}\left(\int_0^{nz} x^{n-s}e^{-w_k x}\right)dx\right] + \frac{(-1)^j\langle j\rangle_n}{w_k^{j+n}}\Bigg]. \tag{33}$$

Moreover, substituting (32) into (33), and noting that for $n \geq s$,

$$\int_0^{\infty} t^{n-s}e^{-w_k t}dt = \frac{(n-s)!}{w_k^{n-s+1}}, \tag{34}$$

results in

$$Y_l^{n,\alpha}(z) = \frac{(1-u)^\alpha}{(2\sqrt{\lambda u})^\alpha}\sum_{k=1}^{l}\sum_{j=1}^{\alpha} e^{w_k nz} r_{k_j}\left[\sum_{s=0}^{n}\binom{n}{s}(-1)^{(j-1)}\langle j-1\rangle_s (w_k)^{-(j-1+s)}\left(\frac{(n-s)!}{w_k^{n-s+1}} - \frac{\Gamma(n-s+1, w_k nz)}{w_k^{n-s+1}}\right) + \frac{(-1)^j\langle j\rangle_n}{w_k^{j+n}}\right]. \tag{35}$$

Using the values of (22) and (35) in (17) gives

$$G_n^\alpha\left(nz + \frac{\alpha}{2}; u; \lambda\right) = \frac{(1-u)^\alpha}{(2\sqrt{\lambda u})^\alpha}\left\{\sum_{k=1}^{l}\sum_{j=1}^{\alpha} e^{w_k nz} r_{k_j}\left[\sum_{s=0}^{n}\binom{n}{s}(-1)^{(j-1)}\langle j-1\rangle_s (w_k)^{-(j-1+s)}\left(\frac{(n-s)!}{w_k^{n-s+1}} - \frac{\Gamma(n-s+1, w_k nz)}{w_k^{n-s+1}}\right) + \frac{(-1)^j\langle j\rangle_n}{w_k^{j+n}}\right] + (nz)^n\sum_{k=0}^{\infty}\frac{f^{(k)}(z^{-1}) - h_l^{(k)}(z^{-1})}{k!}\frac{p_k(n)}{(nz)^k}\right\}$$

valid for $\alpha \in \mathbb{Z}^+$, $z \in \mathbb{C} \setminus \{0\}$ such that $|z^{-1}| < |z^{-1} - w_k|$ for all $k = l+1, l+2, \cdot$, where where the polynomials $p_k(n)$ are given in (9) and $h_l^{(k)}$ is the kth derivative of $h_l(w)$ given by (21). □

The accuracy of the asymptotic formula obtained in (4) and (13) is shown in Figure 5.

The following corollary gives the approximation with enlarged region of validity for the Frobenius–Genocchi polynomials.

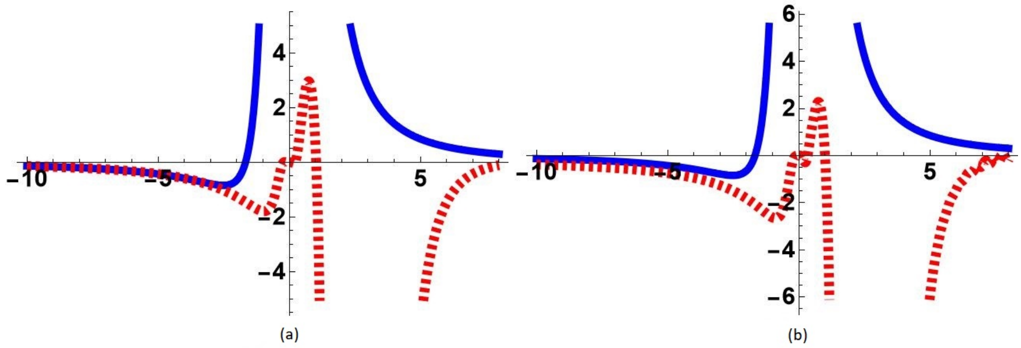

Figure 5. Solid lines in (**a**,**b**) represent $G_n^\alpha(nz+\frac{\alpha}{2};u;\lambda)$ whereas dashed lines in (**a**,**b**) represent the right hand side of (4) and (13) for $n=3, \alpha=2, u=9$ and $\lambda=6$, respectively, with $z\equiv x$, both normalized by the factor $(1+|\frac{x}{\sigma}|^n)^{-1}$ where we choose $\sigma=0.5$. When the solid and dashed lines in the subfigures (**a**,**b**) did not coincide, it indicates that the corresponding normalized values of z are located outside the specified range of validity.

Corollary 4. *For $u\in\mathbb{C}\setminus\{0,1\}$, $\alpha, n\in\mathbb{Z}^+$ and $z\in\mathbb{C}$ such that $|z^{-1}|<|z^{-1}-w_k|$ for all $k=l+1, l+2,\cdots$, the Frobenius–Genocchi polynomials of order α satisfy*

$$G_n^\alpha\Big(nz+\frac{\alpha}{2};u\Big)=\frac{(1-u)^\alpha}{(2\sqrt{u})^\alpha}\Bigg\{\sum_{k=1}^{l}\sum_{j=1}^{\alpha}e^{w_k nz}r_{k_j}\Bigg[\sum_{s=0}^{n}\binom{n}{s}(-1)^{(j-1)}\langle j-1\rangle_s(w_k)^{-(j-1+s)}$$
$$\left(\frac{(n-s)!}{w_k^{n-s+1}}-\frac{\Gamma(n-s+1,w_k nz)}{w_k^{n-s+1}}\right)+\frac{(-1)^j\langle j\rangle_n}{w_k^{j+n}}\Bigg]$$
$$+(nz)^n\sum_{k=0}^{\infty}\frac{f^{(k)}(z^{-1})-h_l^{(k)}(z^{-1})}{k!}\frac{p_k(n)}{(nz)^k}\Bigg\}$$

where the polynomials $p_k(n)$ are given in (9), h_l^k is the kth derivative of the function $h_l(w)$ given by (21), and

$$\sum_{j=1}^{m}\frac{r_{k_j}}{(w-w_k)^j}$$

are the given principal parts of the Laurent series corresponding to the poles $w_k=2\pi i+\nu$, where $\nu=\log(u)$ is a logarithm taken to be the principal branch. The entire function $h_l(w)$ is determined by $f(w)=w^\alpha\operatorname{csch}^\alpha\big(\frac{w-\nu}{2}\big)$.

Proof. This follows from Theorem 2 by taking $\lambda=1$. □

4. Generalized Apostol-Type Frobenius–Genocchi Polynomials

Khan and Srivastasa [17] introduced the generalized Apostol-type Frobenius–Genocchi polynomials of order a, denoted by $\mathcal{G}_n^{(\alpha)}(z;u;a,b,c,\lambda)$ by means of the following generating function

$$\left(\frac{(a^w-u)w}{\lambda b^w-u}\right)^\alpha c^{zw}=\sum_{n=0}^{\infty}\mathcal{G}_n^\alpha(z;u;a,b,c,\lambda)\frac{w^n}{n!},\quad |w|<\left|\frac{\log\left(\frac{\lambda}{u}\right)}{\ln(b)}\right|, \tag{36}$$

for parameters $\lambda, u\in\mathbb{C}, u\neq\lambda$ and $a,b,$ and $c\in\mathbb{R}^+$ with $a\neq b$.

Setting $a=1$ and $b=c=e$ immediately reduces (36) to the Apostol–Frobenius–Genocchi polynomials defined in (1).

When $a = 1, b = e^m, c = e$, and $\lambda = 1$ in (36), this results in the generalized Frobenius–Genocchi polynomials of order α with parameter m, denoted by $G_n^\alpha(z;u;m)$, defined by Belbachir and Souddi [18] by means of the following generating function:

$$\left(\frac{(1-u)w}{e^{mw}-u}\right)^\alpha e^{zw} = \sum_{n=0}^\infty G_n^\alpha(z;u;m)\frac{w^n}{n!}, \quad |w| < \left|\frac{\log(u)}{m}\right|. \tag{37}$$

When $a = 1, b = e^m, c = e, u = -1$, and $\lambda = 1$ in (36), this results in the generalized Genocchi polynomials of order α with parameter m, denoted by $G_n^\alpha(z;\alpha)$, defined by the generating function [18]

$$\left(\frac{2w}{e^{mw}+1}\right)^\alpha e^{zw} = \sum_{n=0}^\infty G_n^\alpha(z;m)\frac{w^n}{n!}, \quad |w| < \frac{\pi}{m}. \tag{38}$$

In this section, the approximations of the generalized Apostol-type Frobenius–Genocchi polynomials of order α with parameters a, b, and c are obtained using the methods applied in Theorems 1 and 2. Approximations for the generalized Frobenius–Genocchi polynomials and generalized Genocchi polynomials of order α with parameter m are given as corollaries.

4.1. Uniform Approximations

Using the saddle-point method, uniform approximations for the generalized Apostol-type Frobenius–Genocchi polynomials are derived. The following theorem describes the said approximation:

Theorem 3. *For $n, \alpha \in \mathbb{Z}^+, a, b, c \in \mathbb{R}^+, u, \lambda \in \mathbb{C} \setminus \{0,1\}$, and $z \in \mathbb{C} \setminus \{0\}$ such that $|Im\ z^{-1}| < \frac{2\pi - Arg\left(\frac{\lambda}{u}\right)}{Arg(b)}$ or $|z^{-1}| < \left|z^{-1} - \left(\frac{2\pi i - \delta}{\beta}\right)\right|$, the generalized Apostol-type Frobenius–Genocchi polynomials of order α satisfy*

$$\begin{aligned}
\mathcal{G}_n^\alpha\left(\frac{nz}{\gamma} + \frac{\alpha}{2}; u; a, b, c, \lambda\right) = \frac{(nz)^n \left(\frac{c}{b}\right)^{\frac{\alpha}{2z}} (a^{1/z} - u)^\alpha}{\left(2z\sqrt{\lambda u}\right)^\alpha \sinh^\alpha\left(\frac{z\delta+\beta}{2z}\right)} \Bigg[1 - \frac{\alpha}{2nz^2}\Bigg\{\alpha\Bigg(\frac{\log\left(\frac{c}{b}\right)}{2} \\
+ \frac{a^{1/z}\log(a)}{a^{1/z} - u} + z - \frac{\beta}{2}\coth\left(\frac{z\delta+\beta}{2z}\right)\Bigg)^2 \\
+ \frac{(a^{1/z} - u)a^{1/z}\log^2(a) - (a^{1/z}\log(a))^2}{(a^{1/z} - u)^2} - z^2 \\
+ \frac{\beta^2}{4}\operatorname{csch}^2\left(\frac{z\delta+\beta}{2z}\right)\Bigg\} + O\left(\frac{1}{n^2}\right)\Bigg]
\end{aligned} \tag{39}$$

where $\delta = \log\left(\frac{\lambda}{u}\right)$ and the logarithm is taken to be the principal branch.

Proof. Applying the Cauchy integral formula to (36),

$$\mathcal{G}_n^\alpha(z; u; a, b, c, \lambda) = \frac{n!}{2\pi i}\int_C \frac{1}{u^\alpha}\frac{((a^w - u)w)^\alpha}{(e^{\delta + w\beta} - 1)^\alpha} e^{zw\gamma}\frac{dw}{w^{n+1}}, \tag{40}$$

where C is a circle 0 with radius lesser than $\left|\frac{2\pi i - \delta}{\beta}\right|$ and $\delta = \log\left(\frac{\lambda}{u}\right), \beta = \log(b)$, and $\gamma = \log(c)$ are logarithms taken to be the principal branch.

With $\left(2\sqrt{\frac{\lambda}{u}}\right)^\alpha (\sqrt{b})^{\alpha w}\sinh^\alpha\left(\frac{\delta + w\beta}{2}\right) = \left(e^{\delta + w\beta} - 1\right)^\alpha$, (40) becomes

$$\mathcal{G}_n^\alpha(z; u; a, b, c, \lambda) = \frac{n!}{2\pi i}\int_C \frac{1}{\left(2\sqrt{\lambda u}\right)^\alpha}\frac{((a^w - u)w)^\alpha}{(\sqrt{b})^{\alpha w}\sinh^\alpha\left(\frac{\delta + w\beta}{2}\right)} e^{zw\gamma}\frac{dw}{w^{n+1}}. \tag{41}$$

Shifting the variable $z \to \frac{z}{\gamma} + \frac{\alpha}{2}$ in (41) gives

$$\mathcal{G}_n^{\alpha}\left(\frac{z}{\gamma}+\frac{\alpha}{2};u;a,b,c,\lambda\right) = \frac{n!}{2\pi i}\frac{1}{\left(2\sqrt{\lambda u}\right)^{\alpha}}\int_C f(w)e^{zw}\frac{dw}{w^{n+1}} \tag{42}$$

where

$$f(w) = \left(\frac{c}{b}\right)^{\frac{\alpha w}{2}}\frac{((a^w-u)w)^{\alpha}}{\sinh^{\alpha}\left(\frac{\delta+w\beta}{2}\right)} \tag{43}$$

is a meromorphic function with simple poles of order α at the zeros of $\sinh^{\alpha}\left(\frac{\delta+w\beta}{2}\right)$ which are given by $w_j = \frac{2j\pi i-\delta}{\beta}, j = \pm1, \pm2, \cdots$. Now taking $z \to nz$ and letting $nz \to \infty$ with fixed z,

$$\mathcal{G}_n^{\alpha}\left(\frac{nz}{\gamma}+\frac{\alpha}{2};u;a,b,c,\lambda\right) = \frac{n!}{2\pi i}\frac{1}{\left(2\sqrt{\lambda u}\right)^{\alpha}}\int_C f(w)e^{n(zw-\log w)}\frac{dw}{w}. \tag{44}$$

With the used of the saddle-point method from Section 2, it can be noted that the main value of the integrand to the integral in (44) originates from the saddle-point of the argument of the exponent. Thus, the approximations $\mathcal{G}_n^{\alpha}\left(\frac{nz}{\gamma}+\frac{\alpha}{2};u;a,b,c,\lambda\right)$ are obtained by expanding $f(w)$ around the saddle-point $w = z^{-1}$. It follows from Lemmas 1 and 2 and Theorem 1 of [14] that

$$\mathcal{G}_n^{\alpha}\left(\frac{nz}{\gamma}+\frac{\alpha}{2};u;a,b,c,\lambda\right) = \frac{(nz)^n}{\left(2\sqrt{\lambda u}\right)^{\alpha}}\sum_{k=0}^{\infty}\frac{f^{(k)}(z^{-1})}{k!}\frac{p_k(n)}{(nz)^k}, \tag{45}$$

where $p_k(n)$ are the polynomials in (9). Computing the derivatives $f^{(k)}(z^{-1})$ for $k = 0,1,2$ gives

$$f^{(0)}(z^{-1}) = f(z^{-1}) = \left(\frac{c}{b}\right)^{\frac{\alpha}{2z}}\frac{(a^{1/z}-u)^{\alpha}}{z^{\alpha}\sinh^{\alpha}\left(\frac{z\delta+\beta}{2z}\right)},$$

$$f^{(1)}(z^{-1}) = \frac{\alpha\left(\frac{c}{b}\right)^{\frac{\alpha}{2z}}\left(a^{1/z}-u\right)^{\alpha}}{z^{\alpha}\sinh^{\alpha}\left(\frac{z\delta+\beta}{2z}\right)}\left\{\frac{\log\left(\frac{c}{b}\right)}{2}+\frac{a^{1/z}\log(a)}{a^{1/z}-u}+z-\frac{\beta}{2}\coth\left(\frac{z\delta+\beta}{2z}\right)\right\}, \text{ and}$$

$$f^{(2)}(z^{-1}) = \frac{\alpha\left(\frac{c}{b}\right)^{\frac{\alpha}{2z}}\left(a^{1/z}-u\right)^{\alpha}}{z^{\alpha}\sinh^{\alpha}\left(\frac{z\delta+\beta}{2z}\right)}\left\{\alpha\left(\frac{\log\left(\frac{c}{b}\right)}{2}+\frac{a^{1/z}\log(a)}{a^{1/z}-u}+z-\frac{\beta}{2}\coth\left(\frac{z\delta+\beta}{2z}\right)\right)^2\right.$$
$$\left.+\frac{(a^{1/z}-u)a^{1/z}\log^2(a)-(a^{1/z}\log(a))^2}{(a^{1/z}-u)^2}-z^2+\frac{\beta^2}{4}\operatorname{csch}^2\left(\frac{z\delta+\beta}{2z}\right)\right\}.$$

Expanding the sum in (45) and keeping only the first three terms give

$$\begin{aligned}
&\mathcal{G}_n^{(\alpha)}\left(\frac{nz}{\gamma}+\frac{\alpha}{2};u;a,b,c,\lambda\right)\\
&= \frac{(nz)^n}{\left(2\sqrt{\lambda u}\right)^{\alpha}}\left[\frac{f^{(0)}(z^{-1})}{0!}+\frac{f^{(1)}(z^{-1})}{1!}\frac{p_1(n)}{nz}+\frac{f^{(2)}(z^{-1})}{2!}\frac{p_2(n)}{(nz)^2}+O\left(\frac{1}{n^2}\right)\right]\\
&= \frac{(nz)^n}{\left(2\sqrt{\lambda u}\right)^{\alpha}}\left[\left(\frac{c}{b}\right)^{\frac{\alpha}{2z}}\frac{(a^{1/z}-u)^{\alpha}}{z^{\alpha}\sinh^{\alpha}\left(\frac{z\delta+\beta}{2z}\right)}-\frac{\alpha\left(\frac{c}{b}\right)^{\frac{\alpha}{2z}}\left(a^{1/z}-u\right)^{\alpha}}{2nz^{\alpha+2}\sinh^{\alpha}\left(\frac{z\delta+\beta}{2z}\right)}\times\right.
\end{aligned}$$

$$\left\{\alpha\left(\frac{\log\left(\frac{c}{b}\right)}{2}+\frac{a^{1/z}\log(a)}{a^{1/z}-u}+z-\frac{\beta}{2}\coth\left(\frac{z\delta+\beta}{2z}\right)\right)^2\right.$$
$$+\frac{(a^{1/z}-u)a^{1/z}\log^2(a)-(a^{1/z}\log(a))^2}{(a^{1/z}-u)^2}-z^2$$
$$\left.\left.+\frac{\beta^2}{4}\operatorname{csch}^2\left(\frac{z\delta+\beta}{2z}\right)\right\}+O\left(\frac{1}{n^2}\right)\right]$$
$$=\frac{(nz)^n\left(\frac{c}{b}\right)^{\frac{\alpha}{2z}}(a^{1/z}-u)^\alpha}{\left(2z\sqrt{\lambda u}\right)^\alpha\sinh^\alpha\left(\frac{z\delta+\beta}{2z}\right)}\left[1-\frac{\alpha}{2nz^2}\left\{\alpha\left(\frac{\log\left(\frac{c}{b}\right)}{2}+\frac{a^{1/z}\log(a)}{a^{1/z}-u}\right.\right.\right.$$
$$\left.+z-\frac{\beta}{2}\coth\left(\frac{z\delta+\beta}{2z}\right)\right)^2+\frac{(a^{1/z}-u)a^{1/z}\log^2(a)-(a^{1/z}\log(a))^2}{(a^{1/z}-u)^2}-z^2$$
$$\left.\left.+\frac{\beta^2}{4}\operatorname{csch}^2\left(\frac{z\delta+\beta}{2z}\right)\right\}+O\left(\frac{1}{n^2}\right)\right].$$

□

Remark 2. *Taking $a=1, b=e$ and $c=e$, Theorem 3 gives a uniform approximation, which is similar with that obtained in Theorem 1 for the Apostol–Frobenius–Genocchi polynomials of order α.*

The graphs in Figure 6 show the accuracy of the asymptotic formula obtained in Theorem 3.

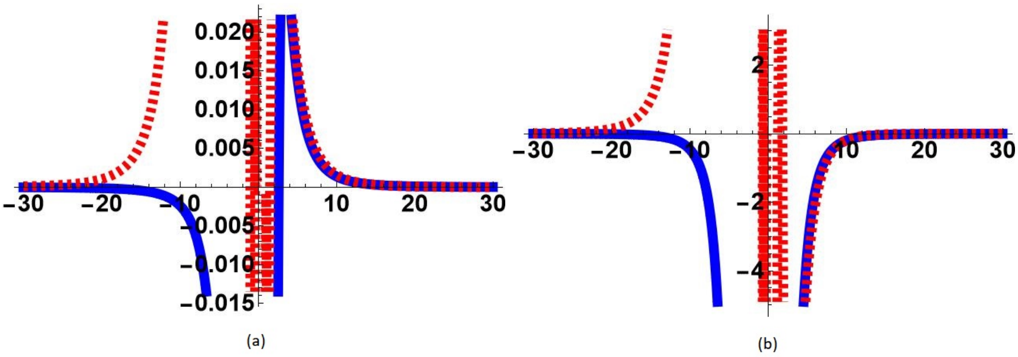

Figure 6. (**a**) $n=5, \alpha=4, u=2, a=2, b=3, c=4$ and $\lambda=3$. (**b**) $n=7, \alpha=5, u=3, a=4, b=2, c=3$ and $\lambda=4$. Solid lines represent the generalized Apostol-type Frobenius–Genocchi polynomials of order α $\mathcal{G}_n^\alpha(\frac{nz}{\gamma}+\frac{\alpha}{2};u;a,b,c,\lambda)$ for several values of n, whereas dashed lines represent the right hand side of (3) with $z\equiv x$, both normalized by the factor $(1+|\frac{x}{\sigma}|^n)^{-1}$, where we choose $\sigma=0.5$.

The following corollaries give the uniform approximations for the generalized Frobenius–Genocchi polynomials and generalized Genocchi polynomials of order α with parameter m.

Corollary 5. *For $n,\alpha\in\mathbb{Z}^+$, $m\in\mathbb{R}^+$, $u\in\mathbb{C}\setminus\{0,1\}$, and $z\in\mathbb{C}\setminus\{0\}$ such that $|Im\, z^{-1}|<\frac{2\pi+Arg(u)}{m}$ or $|z^{-1}|<|z^{-1}-\frac{(2\pi i+\nu)}{m}|$, the generalized Frobenius–Genocchi polynomials of order α with parameter m satisfy*

$$\mathcal{G}_n^\alpha\left(nz+\frac{\alpha}{2};u;m\right)$$
$$=\frac{(nz)^n\left(e^{1-m}\right)^{\frac{\alpha}{2z}}(1-u)^\alpha}{\left(2z\sqrt{u}\right)^\alpha\sinh^\alpha\left(\frac{m-z\nu}{2z}\right)}\left[1-\frac{\alpha}{2nz^2}\left\{\alpha\left(\frac{(1-m)}{2}+z-\frac{m}{2}\coth\left(\frac{m-z\nu}{2z}\right)\right)^2\right.\right.$$

$$-z^2+\frac{m^2}{4}\operatorname{csch}^2\left(\frac{m-zv}{2z}\right)\Bigg\}+O\left(\frac{1}{n^2}\right)\Bigg] \tag{46}$$

where $v=\log(u)$ and the logarithm is taken to be the principal branch.

Proof. This follows from Theorem 3 by taking $a=1, b=e^m, c=e$ and $\lambda=1$. □

Corollary 6. *For $n,\alpha\ \in\mathbb{Z}^+, m\in\mathbb{R}^+$ and $z\in C\setminus 0$ such that $|Im\ z^{-1}|<\frac{\pi}{m}$ or $|z^{-1}|<|z^{-1}-\frac{\pi}{m}|$, the generalized Genocchi polynomials of order α with parameter m satisfy*

$$\begin{aligned}&\mathcal{G}_n^{\alpha}\left(nz+\frac{\alpha}{2};m\right)\\&=\frac{(nz)^n\left(e^{1-m}\right)^{\frac{\alpha}{2z}}}{z^{\alpha}}\operatorname{sech}\left(\frac{m}{2z}\right)\Bigg\{1-\frac{\alpha}{2nz^2}\Bigg\{\alpha\left(\frac{(1-m)}{2}+z-\frac{m}{2}\tanh\left(\frac{m}{2z}\right)\right)^2\\&-z^2-\frac{m^2}{4}\operatorname{sech}^2\left(\frac{m}{2z}\right)\Bigg\}+O\left(\frac{1}{n^2}\right)\Bigg]\end{aligned} \tag{47}$$

Proof. This follows from Theorem 3 by taking $a=1, b=e^m, c=e, u=-1$ and $\lambda=1$. □

The graphs in Figures 7 and 8 show the approximations of Corollaries 5 and 6, respectively.

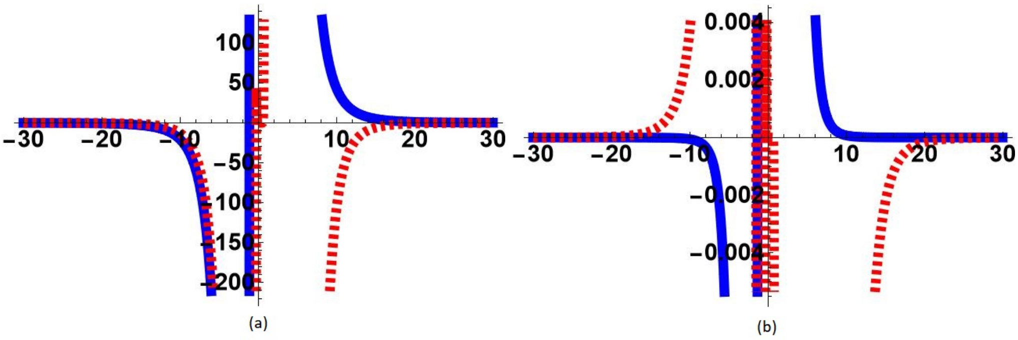

Figure 7. (**a**) $n=8, \alpha=5$ and $u=2$. (**b**) $n=7, \alpha=6$ and $u=3$. Solid lines represent the generalized Frobenius–Genocchi polynomials of order α with parameter m $\mathcal{G}_n^{\alpha}(nz+\frac{\alpha}{2};u;m)$ for several values of n, whereas dashed lines represent the right hand side of (46) with $z\equiv x$, both normalized by the factor $(1+|\frac{x}{\sigma}|^n)^{-1}$ where we choose $\sigma=0.5$.

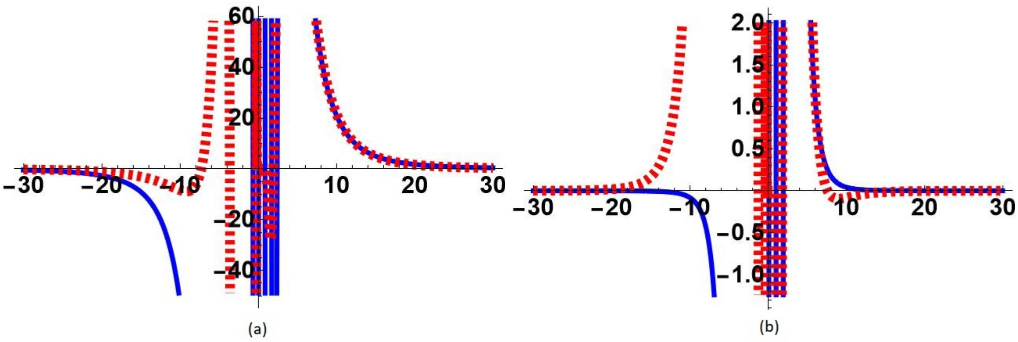

Figure 8. (**a**) $n=10, \alpha=7$ and $\lambda=11$. (**b**) $n=8, \alpha=6$ and $\lambda=9$. Solid lines represent the generalized Genocchi polynomials of order α with parameter m $\mathcal{G}_n^{\alpha}(nz+\frac{\alpha}{2};m)$ for several values of n, whereas dashed lines represent the right hand side of (6) with $z\equiv x$, both normalized by the factor $(1+|\frac{x}{\sigma}|^n)^{-1}$ where we choose $\sigma=0.5$.

4.2. Enlarged Region of Validity

In Section 4.1, the use of the saddle-point method resulted in an approximation valid in the region $|z^{-1}| < |z^{-1} - w_j|$ with poles $w_j = \frac{2j\pi-\delta}{\beta}, j = \pm 1, \pm 2, \cdots$. In this subsection, an approximation with enlarged region of validity for the generalized Apostol-type Frobenius–Genocchi polynomials of order α with parameters a, b and c is obtained following the process used in Section 3. The following theorem contains the said approximation.

Theorem 4. *For $n, \alpha \in \mathbb{Z}^+, a, b,$ and $c \in \mathbb{R}^+$, $\lambda, u \in \mathbb{C} \setminus \{0, 1\}$, $\alpha \in \mathbb{Z}^+$ and $z \in \mathbb{C}$ such that $|z^{-1}| < |z^{-1} - w_k|$ for all $k = l+1, l+2, \cdots$, the generalized Apostol-type Frobenius–Genocchi polynomials of order α with parameters $a, b,$ and c satisfy*

$$\begin{aligned}\mathcal{G}_n^{\alpha}\left(\frac{nz}{\gamma} + \frac{\alpha}{2}; u; a, b, c, \lambda\right) = \frac{1}{(2\sqrt{\lambda u})^{\alpha}} \Bigg\{ & \sum_{k=1}^{l} \sum_{j=1}^{\alpha} e^{w_k nz} r_{k_j} \Bigg[\sum_{s=0}^{n} \binom{n}{s} (-1)^{(j-1)} \langle j-1 \rangle_s (w_k)^{-(j-1+s)} \\ & \left(\frac{(n-s)!}{w_k^{n-s+1}} - \frac{\Gamma(n-s+1, w_k nz)}{w_k^{n-s+1}} \right) + \frac{(-1)^j \langle j \rangle_n}{w_k^{j+n}} \Bigg] \\ & + (nz)^n \sum_{k=0}^{\infty} \frac{f^{(k)}(z^{-1}) - h_l^{(k)}(z^{-1})}{k!} \frac{p_k(n)}{(nz)^k} \Bigg\} \end{aligned} \tag{48}$$

where the polynomials $p_k(n)$ are given in (9), h_l^k is the kth derivative of the function $h_l(w)$given by (21) and

$$\sum_{j=1}^{m} \frac{r_{k_j}}{(w - w_k)^j}$$

are the given principal parts of the Laurent series corresponding to the poles $w_k = \frac{2k\pi-\delta}{\beta}$ where $\delta = \log\left(\frac{\lambda}{u}\right), \beta = \log(b)$ and $\log(c)$ are logarithms taken to be the principal branch. The entire function $h_l(w)$ is determined by $f(w) = \left(\frac{c}{b}\right)^{\frac{\alpha w}{2}} ((a^w - u)w)^{\alpha} \operatorname{csch}^{\alpha}\left(\frac{\delta + w\beta}{2}\right)$.

Proof. Recall the generalized Apostol-type Frobenius–Genocchi polynomials of order α with parameters $a, b,$ and c in (44)

$$\mathcal{G}_n^{\alpha}\left(\frac{nz}{\gamma} + \frac{\alpha}{2}; u; a, b, c, \lambda\right) = \frac{n!}{2\pi i} \frac{1}{\left(2\sqrt{\lambda u}\right)^{\alpha}} \int_C f(w) e^{n(zw - \log w)} \frac{dw}{w}. \tag{49}$$

where

$$f(w) = \left(\frac{c}{b}\right)^{\frac{\alpha w}{2}} \frac{((a^w - u)w)^{\alpha}}{\sinh^{\alpha}\left(\frac{\delta + w\beta}{2}\right)} \tag{50}$$

and $\delta = \log\left(\frac{\lambda}{u}\right), \gamma = \log(c)$, and $\beta = \log(b)$ are logarithms taken to be the principal branch. Substituting the Mittag–Leffler expansion of $f(w)$ in (14) to (49) gives

$$\mathcal{G}_n^{\alpha}\left(\frac{nz}{\gamma} + \frac{\alpha}{2}; u; a, b, c, \lambda\right) = S_l^{n,\alpha}(z) + T_l^{n,\alpha}(z). \tag{51}$$

where

$$S_l^{n,\alpha}(z) = \frac{n!}{2\pi i} \frac{1}{(2\sqrt{\lambda u})^{\alpha}} \int_C f_l(w) e^{wnz} \frac{dw}{w^{n+1}}, \tag{52}$$

$$T_l^{n,\alpha}(z) = \frac{1}{(2\sqrt{\lambda u})^{\alpha}} \int_C \frac{n!}{2\pi i} \sum_{k=1}^{l} \sum_{j=1}^{\alpha} \frac{r_{k_j}}{(w - w_k)^j} e^{wnz} \frac{dw}{w^{n+1}}. \tag{53}$$

To determine (52), repeat the process of the saddle-point method in Section 3 to expand $f_l(w)$ around the saddle-point z^{-1}. The expansion is given as

$$S_l^{n,\alpha}(z) = (nz)^n \frac{1}{(2\sqrt{\lambda u})^\alpha} \sum_{k=0}^{\infty} \frac{f_l^{(k)}(z^{-1}) - h_l^{(k)}(z^{-1})}{k!} \frac{p_k(n)}{(nz)^k} \tag{54}$$

valid for $|z^{-1}| < |z^{-1} - w_j|, j = l+1, l+2, \cdots$ and $z \neq 0$. The expansion's range of validity is larger than that of the expansion in Theorem 3.

On the other hand, an expansion for $T_l^{n,\alpha}(z)$ can be derived by performing the method of contour integration used in Section 3 to evaluate the integral $Y_l^{n,\alpha}(z)$ in. Thus, the expansion is given as

$$\begin{aligned} T_l^{n,\alpha}(z) = \frac{1}{(2\sqrt{\lambda u})^\alpha} \sum_{k=1}^{l} \sum_{j=1}^{\alpha} e^{w_k nz} r_{k_j} \Bigg[\sum_{s=0}^{n} \binom{n}{s} (-1)^{(j-1)} \langle j-1 \rangle_s (w_k)^{-(j-1+s)} \Bigg(\frac{(n-s)!}{w_k^{n-s+1}} \\ - \frac{\Gamma(n-s+1, w_k nz)}{w_k^{n-s+1}} \Bigg) + \frac{(-1)^j \langle j \rangle_n}{w_k^{j+n}} \Bigg]. \end{aligned} \tag{55}$$

Substituting the values of (54) and (55) to (51) gives

$$\begin{aligned} \mathcal{G}_n^\alpha\left(\frac{nz}{\gamma} + \frac{\alpha}{2}; u; a, b, c, \lambda\right) = \frac{1}{(2\sqrt{\lambda u})^\alpha} \Bigg\{ \sum_{k=1}^{l} \sum_{j=1}^{\alpha} e^{w_k nz} r_{k_j} \Bigg[\sum_{s=0}^{n} \binom{n}{s} (-1)^{(j-1)} \langle j-1 \rangle_s (w_k)^{-(j-1+s)} \\ \Bigg(\frac{(n-s)!}{w_k^{n-s+1}} - \frac{\Gamma(n-s+1, w_k nz)}{w_k^{n-s+1}} \Bigg) + \frac{(-1)^j \langle j \rangle_n}{w_k^{j+n}} \Bigg] \\ + (nz)^n \sum_{k=0}^{\infty} \frac{f^{(k)}(z^{-1}) - h_l^{(k)}(z^{-1})}{k!} \frac{p_k(n)}{(nz)^k} \Bigg\} \end{aligned}$$

valid for $\alpha \in \mathbb{Z}^+$, $z \in \mathbb{C} \setminus \{0\}$ such that $|z^{-1}| < |z^{-1} - w_k|$ for all $k = l+1, l+2, \cdot$, where the polynomials $p_k(n)$ are given in (9) and $h_l^{(k)}$ is the kth derivative of $h_l(w)$ given by (21). □

The accuracy of the asymptotic formula obtained in (39) and (48) is shown in Figure 9.

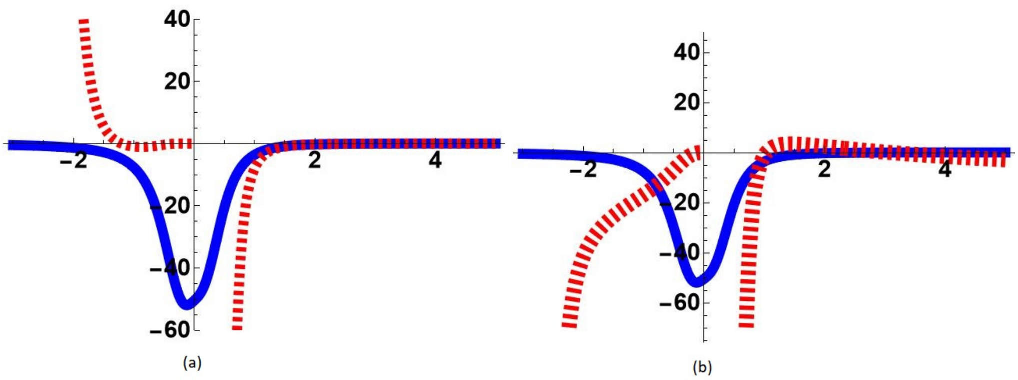

Figure 9. Solid lines in (**a**,**b**) represent $\mathcal{G}_n^\alpha\left(\frac{nz}{\gamma} + \frac{\alpha}{2}; u; a, b, c, \lambda\right)$ whereas dashed lines in (**a**,**b**) represent the right-hand side of (39) and (48), respectively, for $n = 3, \alpha = 2, u = 2, a = 5, b = 3, c = 4$ and $\lambda = 3$ with $z \equiv x$, both normalized by the factor $(1 + |\frac{x}{\sigma}|^n)^{-1}$, where we choose $\sigma = 0.5$. When the solid and dashed lines in the subfigures (**a**,**b**) did not coincide, it indicates that the corresponding normalized values of z are located outside the specified range of validity.

5. Apostol–Frobenius-Type Poly-Genocchi Polynomials of Order α with Parameters a, b, and c

Kim et al. [19] introduced the poly-Genocchi polynomials, which are a generalization of the Genocchi numbers formed by mixing these numbers with the following definition of polylogarithm $Li_\mu(z)$

$$Li_\mu(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^\mu}, \ \mu \in \mathbb{Z}. \tag{56}$$

In the study of Corcino et al. [20], the definitions of Apostol and Frobenius polynomials were mixed, leading to the construction of a new variation of poly-Genocchi polynomials called the Apostol–Frobenius-type poly-Genocchi polynomials of order α with parameters a, b, and c, denoted by $\mathcal{G}_n^{(\mu,\alpha)}(z;\lambda,u,a,b,c)$, defined as follows:

$$\begin{aligned}&\sum_{n=0}^{\infty} \mathcal{G}_n^{(\mu,\alpha)}(z;\lambda,u,a,b,c)\frac{w^n}{n!}\\ &= \left(\frac{Li_\mu(1-(ab)^{-(1-u)w})}{\lambda b^w - ua^{-w}}\right)^{\alpha} c^{zw}, \ \ |t| < \frac{\sqrt{\left(\log\left(\frac{\lambda}{u}\right)\right)^2 + 4\pi^2}}{|\log(a)+\log(b)|}.\end{aligned} \tag{57}$$

Using the fact that

$$Li_\mu(z) = -\log(1-z),$$

when $\mu = 1$, (57) gives

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(z;\lambda,u,a,b,c)\frac{w^n}{n!} = \left(\frac{(1-u)w\log(ab)}{\lambda b^w - ua^{-w}}\right)^{\alpha} c^{zw} \tag{58}$$

where the polynomials $\mathcal{G}_n^{(\alpha)}(z;\lambda,u,a,b,c) = \mathcal{G}_n^{(1,\alpha)}(z;\lambda,u,a,b,c)$ are called the Apostol–Frobenius-type Genocchi polynomials of order α with parameters a, b, and c.

Setting $a = 1, b = c = e$, (57) yields

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(\mu,\alpha)}(z;\lambda,u,1,e,e)\frac{w^n}{n!} = \left(\frac{Li_\mu(1-e^{-(1-u)w})}{\lambda e^w - u}\right)^{\alpha} e^{zw} \tag{59}$$

where the polynomials $\mathcal{G}_n^{(\mu,\alpha)}(z;\lambda,u,1,e,e) = \mathcal{G}_n^{(\mu,\alpha)}(z;\lambda,u)$ are called the Apostol–Frobenius-type poly-Genocchi polynomials of order α. When $\mu = 1$, (59) gives the Apostol–Frobenius–Genocchi polynomials of order α equation defined in (1).

In this section, approximations for the Apostol–Frobenius-type poly-Genocchi polynomials of order α with parameters a, b, and c are obtained using the methods in Theorems 1 and 2.

5.1. Uniform Approximations

Using the saddle-point method, uniform approximations for the Apostol–Frobenius-Type poly-Genocchi polynomials of order a with parameters a, b, and c are derived. The following theorem satisfies the said approximation.

Theorem 5. *For $n, \alpha \in \mathbb{Z}^+, \mu \in \mathbb{Z}, a, b, c \in \mathbb{R}^+, u, \lambda \in \mathbb{C}\setminus\{0,1\}$, and $z \in \mathbb{C}\setminus\{0\}$ such that $|Im\ z^{-1}| < \frac{2\pi - Arg\left(\frac{\lambda}{u}\right)}{Arg(ab)}$ or $|z^{-1}| < \left|z^{-1} - \left(\frac{2\pi i - \delta}{\psi}\right)\right|$, the Apostol-type Frobenius poly-Genocchi polynomials of order α with parameters a, b, and c satisfy*

$$\begin{aligned}&\mathcal{G}_n^{(\mu,\alpha)}\left(\frac{nz}{\gamma} + \frac{\alpha}{2};\lambda,u,a,b,c\right)\\ &= \frac{(nz)^n\left(\frac{ac}{b}\right)^{\frac{\alpha}{2z}}\left(Li_\mu(1-(ab)^{-(1-u)/z})\right)^{\alpha}}{\left(2\sqrt{\lambda u}\right)^{\alpha}\sinh^{\alpha}\left(\frac{z\delta+\psi}{2z}\right)}\left[1 - \frac{1}{2nz^2}\left\{\left(\frac{\alpha\log\left(\frac{ac}{b}\right)}{2}\right.\right.\right.\end{aligned}$$

$$+\frac{\alpha(1-u)\log(ab)(ab)^{-(1-u)/z}Li_{\mu-1}(1-(ab)^{-(1-u)/z})}{(1-(ab)^{-(1-u)/z})Li_{\mu}(1-(ab)^{-(1-u)/z})}$$
$$-\frac{\beta\psi}{2}\coth\left(\frac{z\delta+\psi}{2z}\right)\Bigg)^2+\frac{1}{((1-(ab)^{-(1-u)/z})Li_{\mu}(1-(ab)^{-(1-u)/z}))^2}\times$$
$$\Bigg\{\alpha(1-u)^2(\log(ab))^2(ab)^{-(1-u)/z}\Bigg(-(ab)^{-(1-u)/z}\Big(Li_{\mu-1}(1-(ab)^{-(1-u)/z})\Big)^2$$
$$+Li_{\mu}(1-(ab)^{-(1-u)/z})\Bigg((ab)^{-(1-u)/z}Li_{\mu-2}(1-(ab)^{-(1-u)/z})$$
$$-Li_{\mu-1}(1-(ab)^{-(1-u)/z})\Bigg)\Bigg)\Bigg\}+\frac{\alpha\psi^2}{4}\operatorname{csch}^2\left(\frac{z\delta+\psi}{2z}\right)\Bigg\}+O\left(\frac{1}{n^2}\right)\Bigg] \tag{60}$$

where $\delta=\log\left(\frac{\lambda}{u}\right)$, $\gamma=\log(c)$, *and* $\psi=\log(ab)$ *are logarithms taken to be the principal branch.*

Proof. Applying the Cauchy integral formula to (57),

$$\mathcal{G}_n^{(\mu,\alpha)}(z;\lambda,u,a,b,c)=\frac{n!}{2\pi i}\int_C\frac{1}{u^\alpha}\frac{(a^w)^\alpha(Li_\mu(1-(ab)^{-(1-u)w}))^\alpha}{(e^{\delta+\psi w}-1)^\alpha}e^{zw\gamma}\frac{dw}{w^{n+1}} \tag{61}$$

where C is a circle 0 with radius lesser than $\left|\frac{2\pi i-\delta}{\psi}\right|$ and $\delta=\log\left(\frac{\lambda}{u}\right)$, $\psi=\log(ab)$ and $\gamma=\log(c)$ are logarithms taken to be the principal branch.

With $\left(2\sqrt{\frac{\lambda}{u}}\right)^\alpha(\sqrt{ab})^{\alpha w}\sinh^\alpha\left(\frac{\delta+w\psi}{2}\right)=\left(e^{\delta+\psi w}-1\right)^\alpha$, (61) becomes

$$\mathcal{G}_n^{(\mu,\alpha)}(z;\lambda,u,a,b,c)=\frac{n!}{2\pi i}\int_C\frac{1}{(2\sqrt{\lambda u})^\alpha}\frac{\left(\frac{a}{b}\right)^{\frac{w\alpha}{2}}(Li_\mu(1-(ab)^{-(1-u)w}))^\alpha}{\sinh^\alpha\left(\frac{\delta+w\psi}{2}\right)}e^{zw\gamma}\frac{dw}{w^{n+1}} \tag{62}$$

From (62), the shifting of $z\to\frac{z}{\gamma}+\frac{\alpha}{2}$ results in

$$\mathcal{G}_n^{(\mu,\alpha)}\left(\frac{z}{\gamma}+\frac{\alpha}{2};\lambda,u,a,b,c\right)=\frac{n!}{2\pi i}\frac{1}{\left(2\sqrt{\lambda u}\right)^\alpha}\int_C f(w)e^w\frac{dw}{w^{n+1}} \tag{63}$$

where

$$f(w)=\frac{\left(\frac{ac}{b}\right)^{\frac{w\alpha}{2}}(Li_\mu(1-(ab)^{-(1-u)w}))^\alpha}{\sinh^\alpha\left(\frac{\delta+w\psi}{2}\right)} \tag{64}$$

is a meromorphic function with simple poles of order α at the zeros of $\sinh^\alpha\left(\frac{\delta+w\psi}{2}\right)$ which are given by $w_j=\frac{2j\pi i-\delta}{\psi}$, $j=\pm1,\pm2,\cdots$.

Note that taking $z\to nz$ and letting $nz\to\infty$ with fixed z,

$$\mathcal{G}_n^{(\mu,\alpha)}\left(\frac{nz}{\gamma}+\frac{\alpha}{2};\lambda,u,a,b,c\right)=\frac{n!}{2\pi i}\frac{1}{\left(2\sqrt{\lambda u}\right)^\alpha}\int_C f(w)e^{n(zw-\log w)}\frac{dw}{w}. \tag{65}$$

Using the saddle-point method, it can be similarly observed that the approximations of (65) can be derived by expanding $f(w)$ around the saddle- point $w=z^{-1}$. Thus, it follows from Lemmas 1 and 2 and Theorem 1 of [14] that

$$\mathcal{G}_n^{(\mu,\alpha)}\left(\frac{nz}{\gamma}+\frac{\alpha}{2};u;a,b,c,\lambda\right)=\frac{(nz)^n}{\left(2\sqrt{\lambda u}\right)^\alpha}\sum_{k=0}^{\infty}\frac{f^{(k)}(z^{-1})}{k!}\frac{p_k(n)}{(nz)^k}, \tag{66}$$

where $p_k(n)$ are the polynomials in (9). Solving the derivatives $f^{(k)}(z^{-1})$ for $k = 0,1,2$ gives

$$f^{(0)}(z^{-1}) = f(z^{-1}) = \left(\frac{ac}{b}\right)^{\frac{\alpha}{2z}} \frac{(Li_\mu(1-(ab)^{-(1-u)/z}))^\alpha}{\sinh^\alpha\left(\frac{z\delta+\psi}{2z}\right)},$$

$$f^{(1)}(z^{-1}) = \frac{\left(\frac{ac}{b}\right)^{\frac{\alpha}{2z}}(Li_\mu(1-(ab)^{-(1-u)w}))^\alpha}{\sinh^\alpha\left(\frac{z\delta+\psi}{2z}\right)}\Bigg\{\frac{\alpha\log\left(\frac{ac}{b}\right)}{2}$$
$$+\frac{\alpha(1-u)\log(ab)(ab)^{-(1-u)/z}Li_{\mu-1}(1-(ab)^{-(1-u)/z})}{(1-(ab)^{-(1-u)/z})Li_\mu(1-(ab)^{-(1-u)/z})} - \frac{\alpha\psi}{2}\coth\left(\frac{z\delta+\psi}{2z}\right)\Bigg\},$$

$$f^{(2)}(z^{-1}) = \frac{\left(\frac{ac}{b}\right)^{\frac{\alpha}{2z}}(Li_\mu(1-(ab)^{-(1-u)w}))^\alpha}{\sinh^\alpha\left(\frac{z\delta+\psi}{2z}\right)}\Bigg\{\Bigg(\frac{\alpha\log\left(\frac{ac}{b}\right)}{2}$$
$$+\frac{\alpha(1-u)\log(ab)(ab)^{-(1-u)/z}Li_{\mu-1}(1-(ab)^{-(1-u)/z})}{(1-(ab)^{-(1-u)/z})Li_\mu(1-(ab)^{-(1-u)/z})} - \frac{\alpha\psi}{2}\coth\left(\frac{z\delta+\psi}{2z}\right)\Bigg)^2$$
$$+\frac{1}{((1-(ab)^{-(1-u)/z})Li_\mu(1-(ab)^{-(1-u)/z}))^2}\times$$
$$\Bigg\{\alpha(1-u)^2(\log(ab))^2(ab)^{-(1-u)/z}\Bigg(-(ab)^{-(1-u)/z}\left(Li_{\mu-1}(1-(ab)^{-(1-u)/z})\right)^2$$
$$+(ab)^{-(1-u)/z}Li_\mu(1-(ab)^{-(1-u)/z})Li_{\mu-2}(1-(ab)^{-(1-u)/z})$$
$$-Li_\mu(1-(ab)^{-(1-u)/z})Li_{\mu-1}(1-(ab)^{-(1-u)/z})\Bigg)\Bigg\} + \frac{\alpha\psi^2}{4}\operatorname{csch}^2\left(\frac{z\delta+\psi}{2z}\right)\Bigg\}.$$

Expanding the sum in (66) and keeping only the first three terms give

$$\mathcal{G}_n^{(\mu,\alpha)}\left(\frac{nz}{\gamma}+\frac{\alpha}{2};\lambda,u,a,b,c\right)$$
$$=\frac{(nz)^n}{\left(2\sqrt{\lambda u}\right)^\alpha}\left[\frac{f^{(0)}(z^{-1})}{0!}+\frac{f^{(1)}(z^{-1})}{1!}\frac{p_1(n)}{nz}+\frac{f^{(2)}(z^{-1})}{2!}\frac{p_2(n)}{(nz)^2}+O\left(\frac{1}{n^2}\right)\right]$$
$$=\frac{(nz)^n\left(\frac{ac}{b}\right)^{\frac{\alpha}{2z}}(Li_\mu(1-(ab)^{-(1-u)/z}))^\alpha}{\left(2\sqrt{\lambda u}\right)^\alpha\sinh^\alpha\left(\frac{z\delta+\psi}{2z}\right)}\Bigg[1-\frac{1}{2nz^2}\Bigg\{\Bigg(\frac{\alpha\log\left(\frac{ac}{b}\right)}{2}$$
$$+\frac{\alpha(1-u)\log(ab)(ab)^{-(1-u)/z}Li_{\mu-1}(1-(ab)^{-(1-u)/z})}{(1-(ab)^{-(1-u)/z})Li_\mu(1-(ab)^{-(1-u)/z})}$$
$$-\frac{\alpha\psi}{2}\coth\left(\frac{z\delta+\psi}{2z}\right)\Bigg)^2+\frac{1}{((1-(ab)^{-(1-u)/z})Li_\mu(1-(ab)^{-(1-u)/z}))^2}\times$$
$$\Bigg\{\alpha(1-u)^2(\log(ab))^2(ab)^{-(1-u)/z}\Bigg(-(ab)^{-(1-u)/z}\left(Li_{\mu-1}(1-(ab)^{-(1-u)/z})\right)^2$$
$$+(ab)^{-(1-u)/z}Li_\mu(1-(ab)^{-(1-u)/z})Li_{\mu-2}(1-(ab)^{-(1-u)/z})$$
$$-Li_\mu(1-(ab)^{-(1-u)/z})Li_{\mu-1}(1-(ab)^{-(1-u)/z})\Bigg)\Bigg\}$$
$$+\frac{\alpha\psi^2}{4}\operatorname{csch}^2\left(\frac{z\delta+\psi}{2z}\right)\Bigg\}+O\left(\frac{1}{n^2}\right)\Bigg].$$

□

The following corollary gives the uniform approximations for the Apostol–Frobenius-type poly-Genocchi polynomials of order α.

Corollary 7. *For $n, \alpha \in \mathbb{Z}^+, \mu \in \mathbb{Z}, u, \lambda \in \mathbb{C} \setminus \{0,1\}$, and $z \in \mathbb{C} \setminus \{0\}$ such that $|Imz^{-1}| < 2\pi - Arg\left(\frac{\lambda}{u}\right)$ or $|z^{-1}| < \left|z^{-1} - (2\pi i - \delta)\right|$, the Apostol–Frobenius-type poly-Genocchi polynomials of order α satisfy*

$$
\begin{aligned}
&\mathcal{G}_n^{(\mu,\alpha)}\left(nz + \frac{\alpha}{2}; \lambda, u\right) \\
&= \frac{(nz)^n (Li_\mu(1 - e^{-(1-u)/z}))^\alpha}{\left(2\sqrt{\lambda u}\right)^\alpha \sinh^\alpha\left(\frac{z\delta+1}{2z}\right)} \Bigg[1 \\
&- \frac{1}{2nz^2}\Bigg\{\left(\frac{\alpha(1-u)e^{-(1-u)/z} Li_{\mu-1}(1 - e^{-(1-u)/z})}{(1 - e^{-(1-u)/z}) Li_\mu(1 - e^{-(1-u)/z})} - \frac{\alpha}{2}\coth\left(\frac{z\delta+1}{2z}\right)\right)^2 \\
&+ \frac{1}{((1 - e^{-(1-u)/z}) Li_\mu(1 - e^{-(1-u)/z}))^2}\Bigg\{\alpha(1-u)^2 e^{-(1-u)/z} \\
&\Bigg(- e^{-(1-u)/z}\left(Li_{\mu-1}(1 - e^{-(1-u)/z})\right)^2 \\
&+ Li_\mu(1 - e^{-(1-u)/z})\left(e^{-(1-u)/z} Li_{\mu-2}(1 - e^{-(1-u)/z}) - Li_{\mu-1}(1 - e^{-(1-u)/z})\right)\Bigg)\Bigg\} \\
&+ \frac{\alpha}{4}\operatorname{csch}^2\left(\frac{z\delta+1}{2z}\right)\Bigg\} + O\left(\frac{1}{n^2}\right)\Bigg] \qquad (67)
\end{aligned}
$$

where $\delta = \log\left(\frac{\lambda}{u}\right)$ and the logarithm is taken to be the principal branch.

The graphs in Figures 10 and 11 show the approximations of Theorem 5 and Corollary 7, respectively.

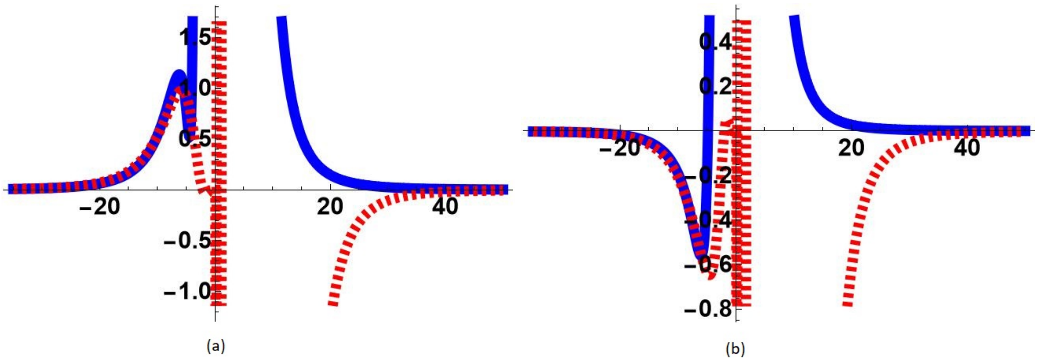

Figure 10. (**a**) $n = 6, \alpha = 4, \lambda = 2, u = 7, a = 3, b = 4, c = 3$, and $\mu = 2$. (**b**) $n = 5, \alpha = 4, \lambda = 2, u = 4, a = 3, b = 4, c = 3$, and $\mu = 2$. Solid lines represent the Apostol-Frobenius-type poly-Genocchi polynomials of order α with parameters a, b, and c $\mathcal{G}_n^{(\mu,\alpha)}\left(\frac{nz}{\gamma} + \frac{\alpha}{2}; \lambda, u, a, b, c\right)$ for several values of n, whereas dashed lines represent the right hand side of (60) with $z \equiv x$, both normalized by the factor $(1 + |\frac{x}{\sigma}|^n)^{-1}$ where we choose $\sigma = 0.5$.

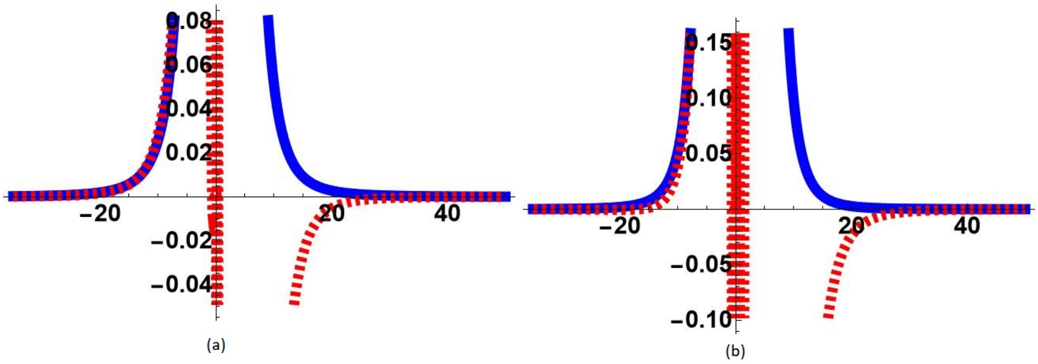

Figure 11. (**a**) $n = 6, \alpha = 4, \lambda = 2, u = 8$, and $\mu = 2$. (**b**) $n = 7, \alpha = 5, \lambda = 2, u = 4$, and $\mu = 2$. Solid lines represent the Apostol–Frobenius-type poly-Genocchi polynomials of order α $\mathcal{G}_n^{(\mu,\alpha)}(nz + \frac{\alpha}{2}; \lambda, u)$ for several values of n, whereas dashed lines represent the right hand side of (67) with $z \equiv x$, both normalized by the factor $(1 + |\frac{x}{\sigma}|^n)^{-1}$, where we choose $\sigma = 0.5$.

5.2. Enlarged Region of Validity

In this subsection, an approximation with enlarged region of validity for the Apostol–Frobenius-type poly-Genocchi polynomials of order α with parameters a, b, and c is obtained following the method of contour integration employed and discussed in Section 3. The following theorem contains the said approximation.

Theorem 6. *For $n, \alpha \in \mathbb{Z}^+, a, b$, and $c \in \mathbb{R}^+$, $\lambda, u \in \mathbb{C} \setminus \{0, 1\}$, $\alpha \in \mathbb{Z}^+$ and $z \in \mathbb{C}$ such that $|z^{-1}| < |z^{-1} - w_k|$ for all $k = l+1, l+2, \cdots$, the Apostol–Frobenius-type poly-Genocchi polynomials of order α with parameters a, b, and c satisfy*

$$
\begin{aligned}
&\mathcal{G}_n^{(\mu,\alpha)}\left(\frac{nz}{\gamma} + \frac{\alpha}{2}; \lambda, u, a, b, c\right) \\
&\quad = \frac{1}{(2\sqrt{\lambda u})^\alpha}\left\{\sum_{k=1}^{l}\sum_{j=1}^{\alpha} e^{w_k nz} r_{k_j}\left[\sum_{s=0}^{n}\binom{n}{s}(-1)^{(j-1)}\langle j-1\rangle_s (w_k)^{-(j-1+s)}\right.\right. \\
&\quad \left.\left(\frac{(n-s)!}{w_k^{n-s+1}} - \frac{\Gamma(n-s+1, w_k nz)}{w_k^{n-s+1}}\right) + \frac{(-1)^j \langle j\rangle_n}{w_k^{j+n}}\right] \\
&\quad \left. + (nz)^n \sum_{k=0}^{\infty} \frac{f^{(k)}(z^{-1}) - h_l^{(k)}(z^{-1})}{k!} \frac{p_k(n)}{(nz)^k}\right\}
\end{aligned} \tag{68}
$$

where the polynomials $p_k(n)$ are given in (9), h_l^k is the kth derivative of the function $h_l(w)$ given by (21), and

$$\sum_{j=1}^{m} \frac{r_{k_j}}{(w - w_k)^j}$$

are the given principal parts of the Laurent series corresponding to the poles $w_k = \frac{2k\pi - \delta}{\psi}$ where $\delta = \log\left(\frac{\lambda}{u}\right), \gamma = \log(c), \psi = \log(ab)$ are logarithms taken to be the principal branch. The entire function $h_l(w)$ is determined by $f(w) = \left(\frac{c}{b}\right)^{\frac{\alpha w}{2}} (Li_\mu(1 - (ab)^{-(1-u)w}))^\alpha \operatorname{csch}^\alpha\left(\frac{\delta + w\psi}{2}\right)$.

Proof. Recall the Apostol–Frobenius-type poly-Genocchi polynomials of order α with parameters a, b, and c in (65)

$$\mathcal{G}_n^{(\mu,\alpha)}\left(\frac{nz}{\gamma} + \frac{\alpha}{2}; \lambda, u, a, b, c\right) = \frac{n!}{2\pi i} \frac{1}{\left(2\sqrt{\lambda u}\right)^\alpha} \int_C f(w) e^{n(zw - \log w)} \frac{dw}{w}. \tag{69}$$

where

$$f(w) = \left(\frac{c}{b}\right)^{\frac{\alpha w}{2}} \frac{(Li_\mu(1-(ab)^{-(1-u)w}))^\alpha}{\sinh^\alpha\left(\frac{\delta+w\psi}{2}\right)} \tag{70}$$

and $\delta = \log\left(\frac{\lambda}{u}\right)$, $\gamma = \log(c)$, and $\psi = \log(ab)$. Substituting the expansion of $f(w)$ in (14) to (65) gives

$$\mathcal{G}_n^\alpha\left(\frac{nz}{\gamma} + \frac{\alpha}{2}; u; a, b, c, \lambda\right) = U_l^{n,\alpha}(z) + V_l^{n,\alpha}(z). \tag{71}$$

where

$$U_l^{n,\alpha}(z) = \frac{n!}{2\pi i} \frac{1}{(2\sqrt{\lambda u})^\alpha} \int_C f_l(w) e^{wnz} \frac{dw}{w^{n+1}}, \tag{72}$$

$$V_l^{n,\alpha}(z) = \frac{1}{(2\sqrt{\lambda u})^\alpha} \int_C \frac{n!}{2\pi i} \sum_{k=1}^{l} \sum_{j=1}^{\alpha} \frac{r_{k_j}}{(w-w_k)^j} e^{wnz} \frac{dw}{w^{n+1}}. \tag{73}$$

To evaluate (72), repeat the process of the saddle-point method in Section 3 to expand $f_l(w)$ around the saddle-point z^{-1}. This gives the expansion of

$$U_l^{n,\alpha}(z) = (nz)^n \frac{1}{(2\sqrt{\lambda u})^\alpha} \sum_{k=0}^{\infty} \frac{f_l^{(k)}(z^{-1}) - h_l^{(k)}(z^{-1})}{k!} \frac{p_k(n)}{(nz)^k}, \tag{74}$$

valid for $|z^{-1}| < |z^{-1} - w_j|, j = l+1, l+2, \cdots$ and $z \neq 0$. The expansion's range of validity is larger than that of the expansion in Theorem 5.

On the other hand, perform the technique of contour integration discussed in Section 3 to derive the expansion of $V_l^{n,\alpha}(z)$. Thus, the expansion is given as

$$\begin{aligned} V_l^{n,\alpha}(z) = \frac{1}{(2\sqrt{\lambda u})^\alpha} \sum_{k=1}^{l} \sum_{j=1}^{\alpha} e^{w_k nz} r_{k_j} \Bigg[\sum_{s=0}^{n} \binom{n}{s} (-1)^{(j-1)} \langle j-1 \rangle_s (w_k)^{-(j-1+s)} \Bigg(\frac{(n-s)!}{w_k^{n-s+1}} \\ - \frac{\Gamma(n-s+1, w_k nz)}{w_k^{n-s+1}} \Bigg) + \frac{(-1)^j \langle j \rangle_n}{w_k^{j+n}} \Bigg]. \end{aligned} \tag{75}$$

Substituting the values of (74) and (75) to (71) gives

$$\begin{aligned} &\mathcal{G}_n^{(\mu,\alpha)}\left(\frac{nz}{\gamma} + \frac{\alpha}{2}; \lambda, u, a, b, c\right) \\ &= \frac{1}{(2\sqrt{\lambda u})^\alpha} \Bigg\{ \sum_{k=1}^{l} \sum_{j=1}^{\alpha} e^{w_k nz} r_{k_j} \Bigg[\sum_{s=0}^{n} \binom{n}{s} (-1)^{(j-1)} \langle j-1 \rangle_s (w_k)^{-(j-1+s)} \\ &\left(\frac{(n-s)!}{w_k^{n-s+1}} - \frac{\Gamma(n-s+1, w_k nz)}{w_k^{n-s+1}} \right) + \frac{(-1)^j \langle j \rangle_n}{w_k^{j+n}} \Bigg] \\ &+ (nz)^n \sum_{k=0}^{\infty} \frac{f^{(k)}(z^{-1}) - h_l^{(k)}(z^{-1})}{k!} \frac{p_k(n)}{(nz)^k} \Bigg\} \end{aligned}$$

valid for $\alpha \in \mathbb{Z}^+$, $z \in \mathbb{C} \setminus \{0\}$ such that $|z^{-1}| < |z^{-1} - w_k|$ for all $k = l+1, l+2, \cdot$, where the polynomials $p_k(n)$ are given in (9) and $h_l^{(k)}$ is the kth derivative of $h_l(w)$ given by (21). □

The accuracy of the asymptotic formula obtained in (60) and (68) is shown in Figure 12.

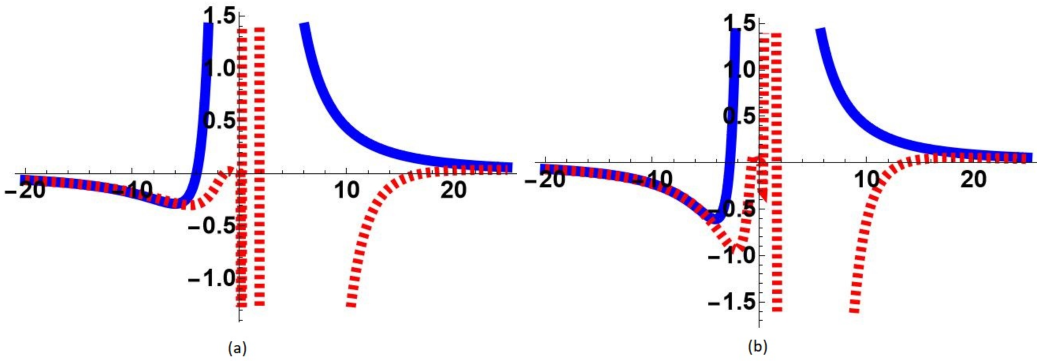

Figure 12. Solid lines in (**a**,**b**) represent $\mathcal{G}_n^{(\mu,\alpha)}\left(\frac{nz}{\gamma}+\frac{\alpha}{2};\lambda,u,a,b,c\right)$ whereas dashed lines in (**a**,**b**) represent the right hand side of (60) and (68), respectively, for $n=3,\alpha=2,u=4,a=3$, $b=4$, $c=3,\lambda=2$ and $\mu=2$ with $z\equiv x$, both normalized by the factor $(1+|\frac{x}{\sigma}|^n)^{-1}$ where we choose $\sigma=0.5$.

6. Conclusions

Uniform approximations for the Apostol–Frobenius–Genocchi polynomials of order α in terms of the hyperbolic functions are obtained using the saddle-point method of Lopez and Temme in [14]. In addition, another approximation with enlarged region of validity is also obtained for these polynomials using the technique of contour integration of Corcino et al. in [10]. Moreover, these methods have shown to provide approximations for the generalized Apostol-type Frobenius–Genocchi polynomials and Apostol–Frobenius-type poly-Genocchi polynomials of order α with parameters a, b, and c. By considering the different values of the parameters, corollaries are established as corresponding special cases of these polynomials. These corollaries can be used as check formulas of the general cases. It is interesting to further explore the applicability of these methods with other special polynomials arising from combinations and generalizations of other classical polynomials.

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