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Investigating the Impact of Fractional Non-Linearity in the Klein–Fock–Gordon Equation on Quantum Dynamics

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Abstract: In this paper, we investigate the fractional-order Klein–Fock–Gordon equations on quantum dynamics using a new iterative method and residual power series method based on the Caputo operator. The fractional-order Klein–Fock–Gordon equation is a generalization of the traditional Klein–Fock–Gordon equation that allows for non-integer orders of differentiation. This equation has been used in the study of quantum dynamics to model the behavior of particles with fractional spin. The Laplace transform is employed to transform the equations into a simpler form, and the resulting equations are then solved using the proposed methods. The accuracy and efficiency of the method are demonstrated through numerical simulations, which show that the method is superior to existing numerical methods in terms of accuracy and computational time. The proposed method is applicable to a wide range of fractional-order differential equations, and it is expected to find applications in various areas of science and engineering.

Keywords: residual power series; Laplace transform; fractional-order Klein–Fock–Gordon equations; new iterative method; Caputo operator

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1. Introduction

Fractional calculus (FC), which has existed since classical calculus, has recently received much interest due to its connections to basic ideas. Leibniz and L'Hospital were the first to present fractional calculus, but it has since gained popularity among academics due to its wide range of applications. Following that, it was widely used to examine a variety of occurrences. However, several types of research emphasized the disadvantages of using this operator, specifically the physical importance of the starting condition and the derivative of a non-zero constant. Then, Caputo introduced a novel fractional operator that overcame the earlier limitations. Most models explored and analyzed under the FC framework use the Caputo operator. Momani and Shawagfeh provide several basic works of fractional calculus on various aspects [1]: Podlubny [2], Jafari and Seifi [3,4], Kiryakova [5], Oldham and Spanier [6], Miller and Ross [7], Diethelm et al. [8], Trujillo [9], Kilbas and Kemple and Beyer [10] and so on [11–13].

The Klein–Fock–Gordon equation is related to quantum dynamics because it describes the time evolution of the wave function of a relativistic particle. The wave function contains all the information about the particle's position, momentum, and other physical properties. The solutions of the Klein–Fock–Gordon equation are used to calculate the probabilities of different outcomes of measurements on the particle [14–18]. These probabilities are related to the behavior of the wave function over time, as governed by the equation. The Klein–Fock–Gordon equation is a fundamental equation in quantum dynamics that describes the behavior of spin-zero particles in the context of special relativity. Its solutions provide information about the probabilities of different outcomes of measurements on these particles [19–22].

The connection between symmetry and the Klein–Gordon equation is also of great interest. The Klein–Gordon equation is invariant under the Lorentz transformation, a symmetry transformation that preserves the speed of light and the space–time interval. This symmetry is related to the special theory of relativity and has important consequences for the behavior of particles in high-energy physics [23–26]. In addition, the Klein–Gordon equation can exhibit other symmetries, such as gauge symmetry and super symmetry. Gauge symmetry is a local symmetry related to particle behavior under electromagnetic and other gauge interactions. Super symmetry is a symmetry that relates particles with different spins and has important implications for studying fundamental particles. In summary, investigating the impact of fractional non-linearity in the Klein–Gordon equation on quantum dynamics and the connection between symmetry is an active area of research that has important implications for theoretical and experimental physics [27–31].

The Klein–Fock–Gordon (KFG) equation is a fundamental quantum mechanics equation that describes spinless particles' behavior in relativistic settings. It is named after the physicists Klein, Fock, and Gordon, who developed the equation. The KFG equation, known as the relativistic wave equation, is a quantized form of the relationship between relativistic energy and momentum. It is closely related to the Schrodinger equation and is used to describe relativistic electrons. However, the classical Klein–Fock–Gordon equation has limitations in describing the dynamics of some physical systems, such as viscoelastic materials and biological systems, which exhibit non-local and memory effects. To overcome these limitations, researchers have proposed using fractional calculus to generalize the Klein–Fock–Gordon equation, resulting in the fractional-order Klein–Fock–Gordon equation.

The fractional-order Klein–Fock–Gordon equation is a partial differential equation that incorporates fractional derivatives, which are non-local operators that describe a system's memory and hereditary effects. These derivatives offer a more precise depiction of the actions of viscoelastic materials, biological systems, and other physical systems. This equation has attracted significant attention from researchers due to its potential applications in many areas, including nanotechnology, condensed-matter physics, and medical imaging. Moreover, studying the fractional-order Klein–Fock–Gordon equation has also led to the development of new mathematical tools and techniques, which can be used to solve other problems in fractional calculus and quantum mechanics.

In this context, this paper aims to provide an overview of the fractional-order Klein–Fock–Gordon equation and its application to quantum dynamics. We discuss the mathematical framework of the equation, its physical interpretation, and its properties. Furthermore, we review recent research on the numerical methods used to solve this equation and its applications to various physical systems. Overall, this paper comprehensively reviews the fractional-order Klein–Fock–Gordon equation and its role in quantum dynamics. The following fractional-order Klein–Fock–Gordon equation is taken into consideration in this article as:

$$D_\tau^\rho \Psi(\varsigma,\tau) = \Psi_{\varsigma\varsigma}(\varsigma,\tau) + a\Psi(\varsigma,\tau) + b\Psi^n(\varsigma,\tau), \quad 1 < \rho \leq 2,$$

along with initial conditions: $\Psi(\varsigma,0) = f(\varsigma)$ and $\Psi_\tau(\varsigma,0) = g(\varsigma)$. The Klein–Fock–Gordon equation, with n being a positive integer and a, b being real constants, has appeared in various physical phenomena, including non-linear optics, quantum field theory, the interaction of solitons in collisionless plasma, and condensed-matter physics. The KFG equation has been studied by different methods, such as the homotopy analysis method [32], variational iteration method [33], modified differential transform method [34], differential transform method [35], q-homotopy analysis transform method [36], and homotopy analysis transform method [37].

The residual power series method (RPSM), a powerful and simple approach for determining the coefficients of power series solutions for first- and second-order fuzzy differential equations, was developed by Jordanian mathematician Omar Abu Arqub in 2013. Unlike other techniques, the RPSM does not necessitate perturbation, linearization, or discretization and can provide effective solutions for both linear and non-linear equations [38–40]. In recent years, the method has been applied to solve a wide range of non-linear ordinary and partial differential equations of various orders and classes [41–45]. The RPSM has been applied in several areas, including the prediction of solitary patterns in non-linear fractional dispersive partial differential equations, the resolution of the highly non-linear singular differential equation known as the generalized Lane–Emden equation, and the approximation of the solution to fractional non-linear KdV–Burger equations [46,47].

Compared to other analytical and numerical approaches, the RPSM has some distinct advantages. Firstly, it does not require a recursion connection or the comparison of coefficients of related terms. Secondly, it offers a straightforward way to ensure the convergence of the series solution by reducing the associated residual error. Thirdly, it does not suffer from computational rounding errors and does not require much time or memory [48–50]. Lastly, the RPSM can be directly applied to a specific issue by selecting a suitable initial approximation and does not necessitate any transformations when shifting from low-order to high-order or from simple linearity to complex non-linearity [51–53].

The structure of this work is organized as follows: Section 2 covers fundamental aspects of calculus theory. Sections 3 and 4 present the RPSTM and NITM formulations used to derive the general solution. To demonstrate the effectiveness and feasibility of both approaches, Section 5 includes numerical examples and comparisons to the exact solution. The conclusion is provided in Section 6.

2. Fundamental Definitions

Definition 1. *The fractional Caputo derivative of the function $\psi(\xi,\tau)$ of order ρ is expressed as* [54]

$$^{C}D_{\tau}^{\rho}\Psi(\xi,\tau) = J_{\tau}^{m-\rho}\Psi^{m}(\xi,\tau),\ \ m-1<\rho\leq m,\ \ \tau>0. \tag{1}$$

where $m \in N$ and J_{τ}^{ρ} are the fractional integral Riemann–Liouville of $\psi(\xi,\tau)$ of the fractional-order ρ, which is defined as

$$J_{\tau}^{\rho}\Psi(\xi,\tau) = \frac{1}{\Gamma(\rho)}\int_{0}^{\tau}(\tau-\eta)^{\rho-1}\Psi(\xi,\eta)d\eta, \tag{2}$$

supposing that the provided integral exists.

Definition 2. *The Laplace transformation of the term $u(\varphi,\tau)$ is defined as* [54]

$$\Psi(\xi,s) = \mathcal{L}_{\tau}[\Psi(\xi,\tau)] = \int_{0}^{\infty}e^{-s\tau}\Psi(\xi,\tau)d\tau,\ \ s>\rho, \tag{3}$$

where the inverse Laplace transform is defined as

$$\Psi(\xi,\tau) = \mathcal{L}_{\tau}^{-1}[\Psi(\xi,s)] = \int_{l-i\infty}^{l+i\infty}e^{s\tau}\Psi(\xi,s)ds,\ \ l=Re(s)>l_0, \tag{4}$$

Lemma 1. *Suppose that $\Psi(\varsigma,\tau)$ is piecewise continuous term and of exponential-order ζ and $\Psi(\varsigma,s)=\mathcal{L}_{\tau}[\Psi(\varsigma,\tau)]$, we obtain*

1. $\mathcal{L}_{\tau}[J_{\tau}^{\rho}\Psi(\varsigma,\tau)] = \frac{\Psi(\varsigma,s)}{s^{\rho}},\ \ \rho>0.$
2. $\mathcal{L}_{\tau}[D_{\tau}^{\rho}\Psi(\varsigma,\tau)] = s^{\rho}\Psi(\varsigma,s) - \sum_{k=0}^{m-1}s^{\rho-k-1}\Psi^{k}(\varsigma,0),\ \ m-1<\rho\leq m.$
3. $\mathcal{L}_{\tau}[D_{\tau}^{n\rho}\Psi(\varsigma,\tau)] = s^{n\rho}\Psi(\varsigma,s) - \sum_{k=0}^{n-1}s^{(n-k)\rho-1}D_{\tau}^{k\rho}\Psi(\varsigma,0),\ \ 0<\rho\leq 1.$

Proof. For proof, see Ref. [54]. □

Theorem 1. *Let $\Psi(\varsigma,\tau)$ be a function that is piecewise continuous on the interval $I \times [0,\infty)$ and has an exponential order of ζ. Assuming that the function $\psi(\xi,s) = \mathcal{L}_\tau[\Psi(\varsigma,\tau)]$ has a fractional expansion, we have:*

$$\Psi(\varsigma,s) = \sum_{n=0}^{\infty} \frac{f_n(\varsigma)}{s^{1+n\rho}}, \ \ 0 < \rho \le 1, \ \ \xi \in I, \ \ s > \zeta. \tag{5}$$

Then, $f_n(\varsigma) = D_\tau^{n\rho}\Psi(\varsigma,0)$.

Proof. For proof, see Ref. [54]. □

3. Road Map of RPSTM

In this section, we show the general methodology LRPS method for the fractional-order partial differential equations

$$D_\tau^\rho \Psi(\varsigma,\tau) = \frac{\partial^2 \Psi(\varsigma,\tau)}{\partial \varsigma^2} + a\Psi(\varsigma,\tau) + b\Psi^n(\varsigma,\tau) = 0, \quad \text{where} \quad 1 < \rho \le 2, \tag{6}$$

and consider the following IC's:

$$\Psi(\varsigma,0) = f_0(x), \ \ \frac{\partial \Psi(\varsigma,0)}{\partial \tau} = g_0(x). \tag{7}$$

Applying the LT of Equation (6) and making use of (7), we obtain

$$\Psi(\varsigma,s) - \frac{f_0(x)}{s^\rho} - \frac{1}{s^\rho}\left(\frac{\partial^2 \Psi(\varsigma,s)}{\partial \varsigma^2}\right) - a\frac{1}{s^\rho}(\Psi(\varsigma,s)) - b\frac{1}{s^\rho}\left(\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}[\Psi(\varsigma,s)]\right]^n\right) = 0, \tag{8}$$

Suppose that the result of Equation (8) has the following

$$\Psi(\varsigma,s) = \sum_{n=0}^{\infty} \frac{f_n(\varsigma,s)}{s^{n\rho+1}}. \tag{9}$$

The k^{th}-truncated term series are

$$\Psi(\varsigma,s) = \frac{f_0(\varsigma,s)}{s} + \sum_{n=1}^{k} \frac{f_n(\varsigma,s)}{s^{n\rho+1}}, \ \ n = 1,2,3,4\cdots. \tag{10}$$

The Laplace residual functions are

$$\mathcal{L}_\tau Res(\varsigma,s) = \Psi(\varsigma,s) - \frac{f_0(\varsigma,s)}{s} - \frac{1}{s^\rho}\left(\frac{\partial^2 \Psi(\varsigma,s)}{\partial \varsigma^2}\right) - a\frac{1}{s^\rho}(\Psi(\varsigma,s)) - b\frac{1}{s^\rho}\left(\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}[\Psi(\varsigma,s)]\right]^n\right). \tag{11}$$

Furthermore, the k^{th}-LRFs are:

$$\mathcal{L}_\tau Res_k(\varsigma,s) = \Psi_k(\varsigma,s) - \frac{f_0(\varsigma,s)}{s} - \frac{1}{s^\rho}\left(\frac{\partial^2 \Psi_k(\varsigma,s)}{\partial \varsigma^2}\right) - a\frac{1}{s^\rho}(\Psi_k(\varsigma,s)) - b\frac{1}{s^\rho}\left(\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}[\Psi_k(\varsigma,s)]\right]^n\right). \tag{12}$$

Some characteristics arising in the RPSTM are given as:

- $\mathcal{L}_\tau Res(\varsigma,s) = 0$ and $\lim_{j\to\infty} \mathcal{L}_\tau Res_k(\varsigma,s) = \mathcal{L}_\tau Res_\Psi(\varsigma,s)$ for each $s > 0$.
- $\lim_{s\to\infty} s\mathcal{L}_\tau Res_\Psi(\varsigma,s) = 0 \Rightarrow \lim_{s\to\infty} s\mathcal{L}_\tau Res_{\Psi,k}(\varsigma,s) = 0$.
- $\lim_{s\to\infty} s^{k\rho+1}\mathcal{L}_\tau Res_{\Psi,k}(\varsigma,s) = \lim_{s\to\infty} s^{k\rho+1}\mathcal{L}_\tau Res_{\Psi,k}(\varsigma,s) = 0, \ \ 0 < \rho \le 1,$ $k = 1,2,3,\cdots$.

To find the coefficients $f_n(\varsigma, s)$, we recursively solve the following system

$$\lim_{s\to\infty} s^{k\rho+1}\mathcal{L}_\tau Res_{\Psi,k}(\varsigma, s) = 0, \;\; k = 1, 2, \cdots. \tag{13}$$

Finally, we apply inverse Laplace transform to Equation (11) to achieve the k^{th} analytic solution of $\Psi_k(\varsigma, \tau)$.

4. Basic Idea of New Iterative Method

To explain the fundamental concept of the new iterative method, we examine the general functional equation:

$$\Psi(\varsigma) = f(\varsigma) + N(\Psi(\varsigma)), \tag{14}$$

Here, N is a non-linear operator from a Banach space B to B, and f is an unknown function. We seek a solution to Equation (14) in the form of a series:

$$\Psi(\varsigma) = \sum_{i=0}^{\infty} \Psi_i(\varsigma). \tag{15}$$

The non-linear term can be decomposed as

$$N(\sum_{i=0}^{\infty} \Psi_i(\varsigma)) = N(\omega_0) + \sum_{i=0}^{\infty}\left[N(\sum_{j=0}^{i}\Psi_j(\varsigma)) - N(\sum_{j=0}^{i-1}\Psi_j(\varsigma))\right]. \tag{16}$$

From Equations (15) and (16), Equation (14) is equivalent to

$$\sum_{i=0}^{\infty} \Psi_i(\varsigma) = f + N(\Psi_0) + \sum_{i=0}^{\infty}\left[N(\sum_{j=0}^{i}\Psi_j(\varsigma)) - N(\sum_{j=0}^{i-1}\Psi_j(\varsigma))\right]. \tag{17}$$

We define the following recurrence relation:

$$\begin{aligned}
&\Psi_0 = f,\\
&\Psi_1 = N(\Psi_0),\\
&\Psi_2 = N(\Psi_0 + \Psi_1) - N(\Psi_0),\\
&\Psi_{n+1} = N(\Psi_0 + \Psi_1 + \cdots \Psi_n) - N(\Psi_0 + \Psi_1 + \cdots \Psi_{n-1}), \;\; n = 1, 2, 3\cdots.
\end{aligned} \tag{18}$$

Then,

$$\begin{aligned}
&(\Psi_0 + \Psi_1 + \cdots \Psi_n) = N(\Psi_0 + \Psi_1 + \cdots \Psi_n), n = 1, 2, 3\cdots,\\
&\Psi = \sum_{i=0}^{\infty} \Psi_i(\varsigma) = f + N(\sum_{i=0}^{\infty} \Psi_i(\varsigma)).
\end{aligned} \tag{19}$$

5. Numerical Problem

In Figure 1, show that the comparison of NIM and LRPSM solution of Problem 1. Figure 2, two-dimensional comparison of NIM and LRPSM solution for $\psi(\varsigma, \tau)$ at different values of $\rho = 0.25$, $\rho = 0.55$, $\rho = 0.75$ and $\rho = 1$, and $\tau = 0.4$. In Figure 3, show that the three dimensional graph of NIM and LRPSM solution of Problem 1. In Table 1, compare the solutions obtained using the proposed technique and the exact method for various fractional orders ρ with $\tau = 0.04$ of Problem 1.

Problem 1. *Consider that the non-linear FKFG equation is given as*

$$D_\tau^\rho \Psi(\varsigma, \tau) = \frac{\partial^2 \Psi(\varsigma, \tau)}{\partial \varsigma^2} - \Psi^2(\varsigma, \tau) = 0, \quad \textit{where} \quad 1 < \rho \leq 2, \tag{20}$$

along with the initial conditions:

$$\Psi(\varsigma,0) = 1+\sin(\varsigma), \quad \frac{\partial\Psi(\varsigma,0)}{\partial\tau} = 0. \tag{21}$$

Solution by LRPSM

Applying LT to Equation (20) and making use of Equation (21), we obtain

$$\Psi(\varsigma,s) - \frac{1+\sin(k\varsigma)}{s} - \frac{1}{s^{\rho}}\left(\frac{\partial^2\Psi(\varsigma,s)}{\partial\varsigma^2}\right) + \frac{1}{s^{\rho}}\left(\mathcal{L}_{\tau}\left[\mathcal{L}_{\tau}^{-1}[\Psi(\varsigma,s)]\right]^2\right) = 0, \tag{22}$$

and so the k^{th}-truncated term series for Equation (33)

$$\Psi(\eta,s) = \frac{(1+\sin(\varsigma))}{s} + \sum_{n=1}^{k}\frac{f_n(\varsigma,s)}{s^{n\rho+1}}, \quad n = 1,2,3,4\cdots. \tag{23}$$

and the k^{th}-LRFs is provided as:

$$\mathcal{L}_{\tau}Res_k(\varsigma,s) = \Psi_k(\varsigma,s) - \frac{1+\sin(k\varsigma)}{s} - \frac{1}{s^{\rho}}\left(\frac{\partial^2\Psi_k(\varsigma,s)}{\partial\varsigma^2}\right) + \frac{1}{s^{\rho}}\left(\mathcal{L}_{\tau}\left[\mathcal{L}_{\tau}^{-1}[\Psi_k(\varsigma,s)]\right]^2\right), \tag{24}$$

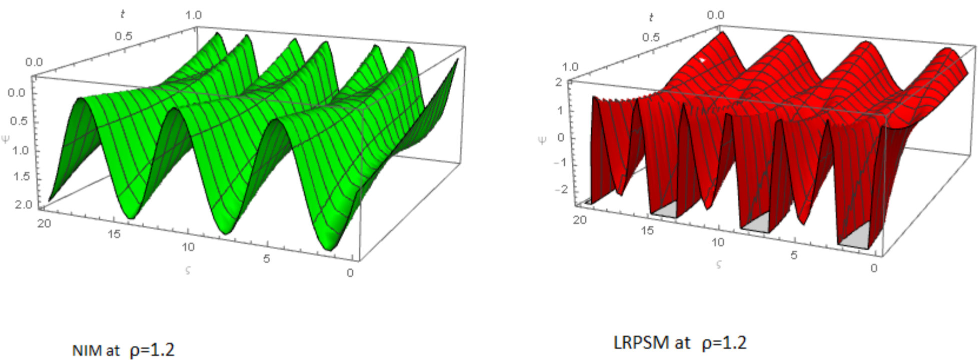

Figure 1. Comparison of NIM and LRPSM solution graph.

The first few terms are obtained by substituting the n^{th}-truncate series Equation (23) into the n^{th}-Laplace residual function Equation (24), multiplying the resulting equation by $s^{n\rho+1}$, and then recursively solving the relation $\lim_{s\to\infty}(s^{n\rho+1}\mathcal{L}\tau Res\psi, n(\varsigma,s)) = 0$, $n = 1,2,3,\cdots$ to determine $f_k(\varsigma,s)$ for $k = 1,2,3,\cdots$.

$$\begin{aligned}
f_1(\varsigma) &= -(1+3\sin(\varsigma)+\sin^2(\varsigma)),\\
f_2(\varsigma) &= 6-6\cos(2\varsigma)+\frac{25}{2}\sin(\varsigma)-\frac{1}{2}\sin(3\varsigma),\\
f_3(\varsigma) &= \frac{1}{4}\left(-\frac{\Gamma(1+2\rho)(-3+\cos(2\varsigma)-6\sin^2(\varsigma))^2}{\Gamma^2(1+\rho)}\right)\\
&-\frac{1}{2}(74-123\cos(4\varsigma)+\cos(4\varsigma)+111\sin(\varsigma)-23\sin(3\varsigma))
\end{aligned} \tag{25}$$

and so on.

Now, putting the values of $f_k(\varsigma)$, $k = 1,2,3,\cdots$, in Equation (23), we obtain

$$\Psi(\varsigma,s) = \frac{(1+\sin(\varsigma))}{s} - \frac{(1+3\sin(\varsigma)+\sin^2(\varsigma))}{s^{\rho+1}} + \frac{6-6\cos(2\varsigma)+\frac{25}{2}\sin(\varsigma)-\frac{1}{2}\sin(3\varsigma)}{s^{2\mu+1}} + \cdots. \tag{26}$$

Using the inverse Laplace transform, we obtain

$$\Psi(\varsigma,\tau) = (1+\sin(\varsigma)) - \frac{(1+3\sin(\varsigma)+\sin^2(\varsigma))\tau^{\rho}}{\Gamma(\rho+1)} + \frac{\left(6-6\cos(2\varsigma)+\frac{25}{2}\sin(\varsigma)-\frac{1}{2}\sin(3\varsigma)\right)\tau^{2\rho}}{\Gamma(2\mu+1)} + \cdots. \quad (27)$$

Solution by NIM

We obtain the equivalent integral equation of the initial-value Equations (20) and (21)

$$\Psi(\varsigma,\tau) = (1+\sin(\varsigma)) + I_{\tau}^{\rho}\left[\frac{\partial^2\Psi(\varsigma,\tau)}{\partial\varsigma^2} - \Psi(\varsigma,\tau)\right]. \quad (28)$$

where $N(\Psi) = \Psi^2$.

and using the algorithm (18) of Nim we obtain

$$\begin{aligned}
\Psi_0(\varsigma,\tau) &= (1+\sin(\varsigma)),\\
\Psi_1(\varsigma,\tau) &= -\frac{(1+\sin(\varsigma)(3+\sin(\varsigma)))\tau^{\rho}}{\Gamma(\rho+1)},\\
\Psi_2(\varsigma,\tau) &= -\frac{\tau^{3\rho}\Gamma(1+2\rho)(-3+\cos(2\varsigma)-6\sin(\varsigma))^2}{4\Gamma(1+\rho)^2\Gamma(1+3\rho)} - \frac{(-12+12\cos(2\varsigma)-25\sin(\varsigma)+\sin(3\varsigma))}{2\Gamma(1+2\rho)},
\end{aligned} \quad (29)$$

and the third-order solution using the new iterative method

$$\begin{aligned}
\Psi(\varsigma,\tau) =&(1+\sin(\varsigma)) - \frac{(1+\sin(\varsigma)(3+\sin(\varsigma)))\tau^{\rho}}{\Gamma(\rho+1)} - \frac{\tau^{3\rho}\Gamma(1+2\rho)(-3+\cos(2\varsigma)-6\sin(\varsigma))^2}{4\Gamma(1+\rho)^2\Gamma(1+3\rho)}\\
&-\frac{(-12+12\cos(2\varsigma)-25\sin(\varsigma)+\sin(3\varsigma))}{2\Gamma(1+2\rho)} + \cdots.
\end{aligned} \quad (30)$$

Table 1. For example, we compare the solutions obtained using the proposed technique and the exact method for various fractional orders ρ with $\tau = 0.04$.

ς	NIM	LRPSM	NIM Absolute Error	LRPSM Absolute Error
−1	0.161830	0.161830	-1.45015×10^{-6}	-4.84885×10^{-5}
−0.9	0.219649	0.219649	-9.26956×10^{-6}	-4.20187×10^{-5}
−0.8	0.285219	0.285219	-1.81174×10^{-5}	$-3.5230\,8 \times 10^{-5}$
−0.7	0.357872	0.357872	-2.77101×10^{-5}	-2.83933×10^{-5}
−0.6	0.436869	0.436869	-3.76338×10^{-5}	-0.21777×10^{-4}
−0.5	0.521411	0.521411	-0.47337×10^{-4}	-1.56448×10^{-5}
−0.4	0.610642	0.610642	-5.61359×10^{-5}	-1.02418×10^{-5}
0.3	0.703661	0.703661	-6.32343×10^{-5}	-5.78561×10^{-6}
−0.2	0.799530	0.799530	-0.67759×10^{-4}	-2.45844×10^{-6}
−0.1	0.897287	0.897287	-6.88098×10^{-5}	-3.99668×10^{-7}
0	0.995950	0.995950	-6.55214×10^{-5}	2.99566×10^{-7}
0.1	1.094530	1.094530	-5.71342×10^{-5}	-3.99701×10^{-7}
0.2	1.192050	1.192050	-4.30678×10^{-5}	-2.48219×10^{-6}
0.3	1.287540	1.287540	-2.29905×10^{-5}	-5.87823×10^{-6}
0.4	1.380040	1.380040	3.11952×10^{-6}	-1.04656×10^{-5}
0.5	1.468640	1.468640	3.49319×10^{-5}	-1.60733×10^{-5}
0.6	1.552470	1.552470	7.17426×10^{-5}	-2.24876×10^{-5}
0.7	1.630690	1.630690	0.112480×10^{-3}	-2.94595×10^{-5}
0.8	1.702550	1.702550	0.155738×10^{-3}	-0.36714×10^{-4}
0.9	1.767320	1.767320	0.199841×10^{-3}	-4.39613×10^{-5}
1	1.824380	1.824380	0.242930×10^{-3}	-5.09083×10^{-5}

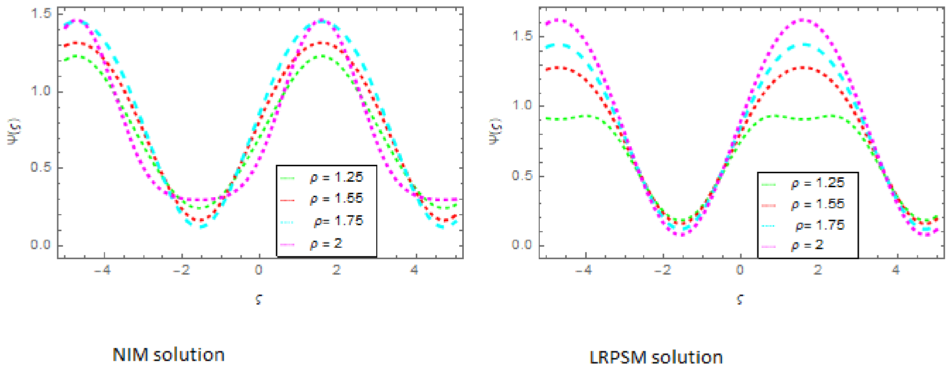

Figure 2. Two-dimensional comparison of NIM and LRPSM solution for $\psi(\varsigma, \tau)$ at different values of $\rho = 0.25$, $\rho = 0.55$, $\rho = 0.75$ and $\rho = 1$, and $\tau = 0.4$.

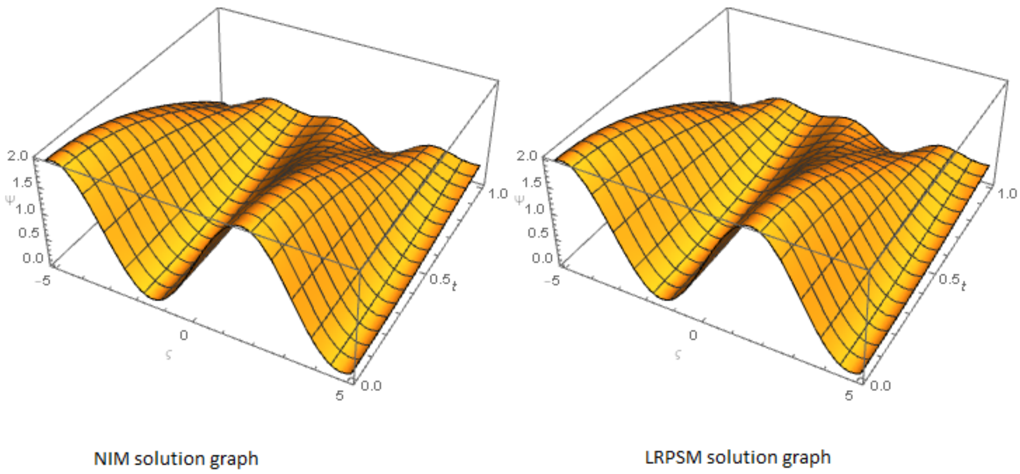

Figure 3. Comparison of NIM and LRPSM solution graphs.

Figure 4, two-dimensional comparison of NIM and LRPSM solution for $\psi(\varsigma, \tau)$ at different values of $\rho = 0.25$, $\rho = 0.55$, $\rho = 0.75$ and $\rho = 1$, and $\tau = 0.4$ of Problem 2. In Figure 5, show that the three dimensional graph of NIM and LRPSM solution of Problem 2. In table 2, compare the solutions obtained using the proposed technique and the exact method for various fractional orders ρ with $\tau = 0.04$ of Problem 2.

Problem 2. *Consider the cubic non-linear FKFG equation as:*

$$D_\tau^\rho \Psi(\varsigma, \tau) = \varsigma^2 \frac{\partial^2 \Psi(\varsigma, \tau)}{\partial \varsigma^2} - C^2 \Psi(\varsigma, \tau) + \delta \epsilon \Psi^3(\varsigma, \tau) = 0, \quad \text{where} \quad 1 < \rho \leq 2, \tag{31}$$

and consider Equation (20) with the following ICs:

$$\Psi(\varsigma, 0) = \epsilon \cos(k\varsigma), \frac{\partial \Psi(\varsigma, 0)}{\partial \varsigma} = 0. \tag{32}$$

Solution by LRPSM

Applying LT to Equation (31) and making use of Equation (32), we obtain

$$\Psi(\varsigma, s) - \frac{\epsilon \cos(k\varsigma)}{s} - \varsigma^2 \frac{1}{s^\rho} \left(\frac{\partial^2 \Psi(\varsigma, s)}{\partial \varsigma^2} \right) + C^2 \Psi(\varsigma, s) - \delta \epsilon \frac{1}{s^\rho} \left(\mathcal{L}_\tau \left[\mathcal{L}_\tau^{-1} [\Psi(\varsigma, s)] \right]^3 \right) = 0, \tag{33}$$

and so the k^{th}-truncated term series are

$$\Psi(\eta,s)=\frac{(\epsilon\cos(k\varsigma))}{s}+\sum_{n=1}^{k}\frac{f_n(\varsigma,s)}{s^{n\rho+1}},\quad n=1,2,3,4\cdots. \tag{34}$$

and the k^{th}-LRFs are:

$$\mathcal{L}_t Res_k(\varsigma,s)=\Psi_k(\varsigma,s)-\frac{\epsilon\cos(k\varsigma)}{s}-\varsigma^2\frac{1}{s^\rho}\left(\frac{\partial^2\Psi_k(\varsigma,s)}{\partial\varsigma^2}\right)+C^2\Psi_k(\varsigma,s)-\delta\epsilon\frac{1}{s^\rho}\left(\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}[\Psi_k(\varsigma,s)]\right]^3\right), \tag{35}$$

To determine $f_k(\varsigma,s)$ for $k=1,2,3,\cdots$, we substitute the n^{th}-truncated series Equation (34) into the n^{th}-Laplace residual function Equation (35), multiply the resulting equation by $s^{n\rho+1}$, and then recursively solve the relation $\lim_{s\to\infty}(s^{n\rho+1}\mathcal{L}\tau Res\psi,n(\varsigma,s))=0$ for $n=1,2,3,\cdots$. The first few terms are as follows:

$$\begin{aligned}
&f_1(\varsigma,s)=-\epsilon(C^2+k^2\varsigma^2)\cos(k\varsigma)+\delta\epsilon^4\cos^3(k\varsigma),\\
&f_2(\varsigma,s)=\epsilon\cos(k\varsigma)\Big((C^2+k^2\varsigma^2)^2+\delta\epsilon^3\cos^2(k\varsigma)(-4C^2-6k^2\varsigma^2+3\delta\epsilon^3\cos^2(k\varsigma))\Big),\\
&f_3(\varsigma,s)=\frac{3\delta\epsilon^4\cos^3(k\varsigma)(-2C^2+\delta\epsilon^3-2k^2\varsigma^2+\delta\epsilon^3\cos^3(k\varsigma))^2\Gamma(1+2\rho)}{4\Gamma^2(1+2\rho)}\\
&-\epsilon\cos(k\varsigma)\Big((C^2+k^2\varsigma^2)^3+\delta\epsilon^3(-(7C^2+24C^2k^2\varsigma^2+21k^4\varsigma^4)\cos^2(k\varsigma))\Big)\\
&-\delta\epsilon^4\cos(k\varsigma)\Big(3\delta\epsilon^3(5c^2+11k^2\varsigma^2)\cos^4(k\varsigma)-9\delta^2\epsilon^6\cos^6(k\varsigma)+24k^2\varsigma^2\sin^2(k\varsigma)\Big)\\
&-\delta\epsilon^4\cos(k\varsigma)\Big(6k^2\varsigma^2(-5\delta\epsilon^3+6k^2\varsigma^2-5\delta\epsilon^3\cos(2k\varsigma))\sin^2(k\varsigma)\Big).
\end{aligned} \tag{36}$$

Now, putting the values of $f_n(\varsigma,s)$, $n=1,2,3,\cdots$, in Equation (34), we obtain

$$\begin{aligned}
\Psi(\varsigma,s)&=\frac{\epsilon\cos(k\varsigma)}{s}+\frac{-\epsilon(C^2+k^2\varsigma^2)\cos(k\varsigma)+\delta\epsilon^4\cos^3(k\varsigma)}{s^{\rho+1}}\\
&+\frac{\epsilon\cos(k\varsigma)((C^2+k^2\varsigma^2)^2+\delta\epsilon^3\cos^2(k\varsigma)(-4C^2-6k^2\varsigma^2+3\delta\epsilon^3\cos^2(k\varsigma)))}{s^{2\rho+1}}+\cdots.
\end{aligned} \tag{37}$$

Using inverse Laplace transform, we obtain:

$$\begin{aligned}
\Psi(\varsigma,s)&=\frac{\epsilon\cos(k\varsigma)}{s}+(-\epsilon(C^2+k^2\varsigma^2)\cos(k\varsigma)+\delta\epsilon^4\cos^3(k\varsigma))\frac{\tau^\rho}{\Gamma(\rho+1)}\\
&+\frac{(\epsilon\cos(k\varsigma)((C^2+k^2\varsigma^2)^2+\delta\epsilon^3\cos^2(k\varsigma)(-4C^2-6k^2\varsigma^2+3\delta\epsilon^3\cos^2(k\varsigma))))\tau^{2\rho+1}}{\Gamma(2\rho+1)}+\cdots.
\end{aligned} \tag{38}$$

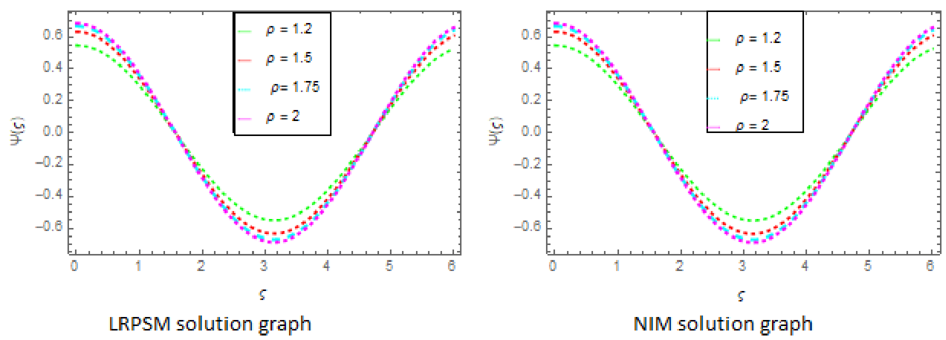

Figure 4. Three-dimensional LRPSM and NIM solution for $\psi(\varsigma,\tau)$ at different values of ρ.

Solution by NIM

We obtain the equivalent integral equation of the initial-value Equations (31) and (32)

$$\Psi(\varsigma,\tau) = \epsilon\sin(k\varsigma) + I_\tau^\rho\left[\varsigma^2\frac{\partial^2\Psi(\varsigma,\tau)}{\partial\varsigma^2} - c^2\Psi(\varsigma,\tau + \delta\epsilon\Psi^3(\varsigma,\tau)\right], \tag{39}$$

where $N(\Psi) = \Psi^3$*, and using the algorithm (18) of NIM we obtain*

$$\begin{aligned}\Psi_0(\varsigma,\tau) &= \epsilon\cos(k\varsigma),\\ \Psi_1(\varsigma,\tau) &= -\frac{\tau^\rho\epsilon\cos(k\varsigma)\left(C^2 + k^2\varsigma^2 - \delta\epsilon^3\cos^2\right)}{\Gamma(\rho+1)},\end{aligned} \tag{40}$$

similarly, we can find $\Psi_2(\varsigma,\tau)$,$\Psi_3(\varsigma,\tau)$ *and so on; then, by adding we can obtain*

$$\Psi(\varsigma,\tau) = \epsilon\cos(k\varsigma) - \frac{\tau^\rho\epsilon\cos(k\varsigma)\left(C^2 + k^2\varsigma^2 - \delta\epsilon^3\cos^2\right)}{\Gamma(\rho+1)} + \cdots. \tag{41}$$

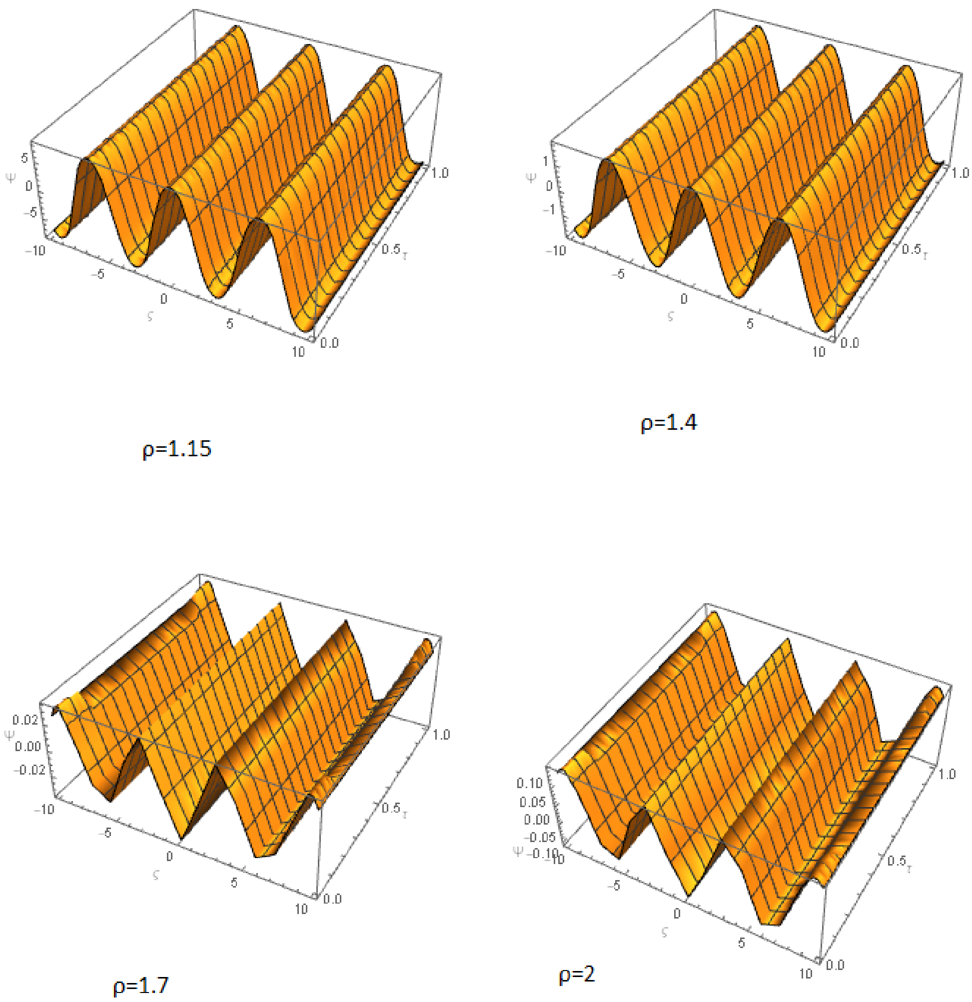

Figure 5. Three-dimensional NIM solution for $\psi(\varsigma,\tau)$ at different values of ρ.

Table 2. For example, we compare the solutions obtained using the proposed technique and the exact method for various fractional orders of ρ with $\tau = 0.04$.

ς	NIM	LRPSM	NIM Absolute Error	LRPSM Absolute Error
-1	0.161830	0.161830	-2.450×10^{-7}	-5.848×10^{-6}
-0.9	0.428538	0.428538	-8.269×10^{-7}	-5.201×10^{-6}
-0.8	0.384187	0.384187	-2.811×10^{-6}	-4.523×10^{-6}
-0.7	0.246887	0.246887	-3.771×10^{-6}	-3.839×10^{-6}
-0.6	0.535480	0.535480	-4.763×10^{-6}	-1.217×10^{-5}
-0.5	0.432147	0.432147	-1.473×10^{-5}	-2.564×10^{-6}
-0.4	0.511532	0.511532	-6.613×10^{-6}	-2.024×10^{-6}
0.3	0.612678	0.612678	-7.323×10^{-6}	-4.785×10^{-7}
-0.2	0.688420	0.688420	-1.677×10^{-5}	-3.458×10^{-7}
-0.1	0.786217	0.786217	-7.880×10^{-6}	-4.996×10^{-8}
0	0.885960	0.885960	-7.552×10^{-6}	3.995×10^{-8}
0.1	1.189150	1.189150	-4.713×10^{-6}	-4.997×10^{-8}
0.2	1.289131	1.289131	-3.306×10^{-6}	-3.482×10^{-7}
0.3	1.376450	1.376450	-3.299×10^{-6}	-4.878×10^{-7}
0.4	1.470030	1.470030	4.119×10^{-7}	-2.046×10^{-6}
0.5	1.358941	1.358941	4.493×10^{-6}	-2.607×10^{-5}
0.6	1.664810	1.664810	8.174×10^{-6}	-3.248×10^{-5}
0.7	1.781581	1.781581	1.1124×10^{-4}	-3.945×10^{-5}
0.8	1.812441	1.812441	1.1557×10^{-4}	-1.367×10^{-5}
0.9	1.805231	1.805231	1.1998×10^{-4}	-5.396×10^{-6}
1	1.786200	1.786200	1.2328×10^{-4}	-6.090×10^{-6}

6. Conclusions

This paper presents a robust analysis of fractional-order non-linear Klein–Fock–Gordon equations using two powerful analytic methods. The obtained analytical results have been rigorously calculated to confirm the reliability and validity of the suggested techniques. The figures demonstrate a remarkable correlation between the obtained and actual solutions, providing strong evidence to validate and test the accuracy of the proposed methods. Notably, our approaches offer a highly efficient and practical means to address a wide range of non-linear systems involving fractional-order partial differential equations. Furthermore, the substantial reduction in computational requirements further enhances the broad applicability of our methods. These findings highlight the remarkable accuracy of our proposed techniques, which are shown to closely match the actual answers and outperform existing methodologies. Hence, our suggested approaches represent an effective and powerful strategy to solve complex fractional-order partial differential equation non-linear systems.

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