symmetry

MDPI

Article

Double-Controlled Quasi M-Metric Spaces

Irshad Ayoob [1,2], Ng Zhen Chuan [2] and Nabil Mlaiki [1,*]

[1] Department of Mathematics and Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia
[2] School of Mathematics, Universiti Sains Malaysia, Gelugor 11800, Malaysia
* Correspondence: nmlaiki2012@gmail.com or nmlaiki@psu.edu.sa

Abstract: One of the well-studied generalizations of a metric space is known as a partial metric space. The partial metric space was further generalized to the so-called M-metric space. In this paper, we introduce the Double-Controlled Quasi M-metric space as a new generalization of the M-metric space. In our new generalization of the M-metric space, the symmetry condition is not necessarily satisfied and the triangle inequality is controlled by two binary functions. We establish some fixed point results, along with the examples and applications to illustrate our results.

Keywords: quasi-metric space; partial metric space; M-metric space; Double-Controlled Quasi M-metric spaces; fixed point theorem

MSC: 47H10; 54H25

check for updates

Citation: Ayoob, I.; Chuan, N.Z.; Mlaiki, N. Double-Controlled Quasi M-Metric Spaces. *Symmetry* **2023**, *15*, 893. https://doi.org/10.3390/sym15040893

Academic Editors: Alexander Zaslavski and Youssef N. Raffoul

Received: 15 February 2023
Revised: 5 April 2023
Accepted: 6 April 2023
Published: 10 April 2023

1. Introduction

Over the past few decades, numerous researchers have focused on fixed point theory. This is due to its application in the existence and uniqueness of solutions to differential and integral equations, engineering, mathematical economics, dynamical systems, neural networks, and many other fields. The classic result of fixed points that has been extensively studied by researchers is the result of Banach [1]. A few examples of existing concepts where the Banach fixed point theorem has been studied include cone metric space [2–4], partial symmetric space [5], partial JS-metric space [6], M-metric space [7], M_b-metric space [8], extended M_b-metric space [9], rectangular M-metric space [10], and others. Various types of contraction mappings in which fixed points in extended metric spaces have been studied include Banach contraction mapping, Kannan contraction mapping, Ciric contraction mapping, and several others [11–13].

To further generalize the underlying metric spaces, Czerwik [14] and Bakhtin [15] introduced the concept of b-metric spaces by adding a constant to the right-hand side of the triangle inequality, resulting in a fascinating generalization of metric spaces with a different topology. Kamran et al. [16] extended this definition to so-called extended b-metric spaces in 2017, and established related fixed point theorems. In 2018, Mlaiki et al. [17] further generalized this concept to so-called controlled metric spaces by using a binary control function on the right side of the triangle inequality, and established a corresponding Banach fixed point result.

Abdeljawad et al. [18] introduced a further generalization of controlled metric spaces, called Double-Controlled metric-type spaces, in which two binary control functions are used on the right side of the triangle inequality. Furthermore, the authors established the corresponding Banach- and Kannan-type fixed point results in Double-Controlled metric-type spaces. The Double-Controlled metric-type space is defined [18] in the following way.

Definition 1 ([18])**.** *Let $\mathcal{X}$ be a non-empty set and $\zeta_1, \zeta_2 : \mathcal{X} \times \mathcal{X} \to [1, \infty)$. A function $\zeta : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is called a Double-Controlled metric type if it satisfies:*

1. $\zeta(\eta,\theta) = 0$ *if, and only if* $\eta = \theta$ *for all* $\eta, \theta \in \mathcal{X}$
2. $\zeta(\eta,\theta) = \zeta(\theta,\eta)$ *for all* $\eta, \theta \in \mathcal{X}$
3. $\zeta(\eta,\mu) \leq \zeta_1(\eta,\theta)\zeta(\eta,\theta) + \zeta_2(\theta,\mu)\zeta(\theta,\mu)$ *for all* $\eta, \theta, \mu \in \mathcal{X}$.

The pair $(\mathcal{X}, \zeta)$ *is a called a Double-Controlled metric-type space.*

In Ref. [19], Wilson proposed a *Quasi*-metric space (also known as an asymmetric metric space) as an extension of metric space. This is a metric space $(\mathcal{X}, \eta)$, but η does not have to be symmetric. Quasi-metric spaces have been used in a variety of recent advances in applied mathematics, including models for material failure [20], shape-memory alloys [21], problems regarding the existence and uniqueness of Hamilton–Jacobi equations [22], and automated taxonomy construction [23].

We recall the definition of *Quasi*-metric space.

Definition 2 ([19])**.** *Let* $\mathcal{X}$ *be a nonempty set. A Quasi-metric on* $\mathcal{X}$ *is a function* $\eta : \mathcal{X}^2 \to [0, +\infty)$ *such that for all* $\mu, \omega, w \in \mathcal{X}$

1. $\eta(\mu,\omega) = 0$ *if, and only if* $\mu = \omega$,
2. $\eta(\mu,\omega) \leq \eta(\mu,w) + \eta(w,\omega)$.

A pair $(\mathcal{X}, \eta)$ *is called a Quasi-metric space.*

In general, any *Quasi*-metric space is a metric space, although the converse is not always true. Topological terms like convergence, Cauchyness, completeness, and continuity are different in quasi-metric spaces from those used in metric spaces. The reader may consult [24] for these ideas in *Quasi*-metric spaces. Several researchers [25–27] have studied fixed point theory in the context of *Quasi*-metric spaces.

To further generalize Double-Controlled metric-type spaces, Shoaib et al. [28] introduced so-called Double-Controlled Quasi metric-type spaces, defined in the following manner:

Definition 3 ([28])**.** *Let* $\mathcal{X}$ *be a non-empty set and* $\zeta_1, \zeta_2 : \mathcal{X} \times \mathcal{X} \to [1, \infty)$. *A function* $\zeta : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ *is called a Double-Controlled quasi-metric type if it satisfies:*

1. $\zeta(\eta,\theta) = 0$ *if, and only if* $\eta = \theta$ *for all* $\eta, \theta \in \mathcal{X}$,
2. $\zeta(\eta,\mu) \leq \zeta_1(\eta,\theta)\zeta(\eta,\theta) + \zeta_2(\theta,\mu)\zeta(\theta,\mu)$ *for all* $\eta, \theta, \mu \in \mathcal{X}$.

The pair $(\mathcal{X}, \zeta)$ *is called a Double-Controlled quasi metric-type space.*

The difference between a Double-Controlled metric-type space and Double-Controlled quasi metric-type space is that the symmetry condition is not necessarily satisfied in the latter.

As a further generalization of metric spaces, Matthews [29] introduced the notion of partial metric spaces as an extension of metric spaces and established the Banach-type fixed point theorem in the same space. Several researchers such as O'Neill [30], Bukatin and Scott [31,32], Escardo [33], Romaguera and Schellekens [34,35], and Waszkiewicz [36,37] have studied the connection between domain theory and partial metrics.

We state the definition of partial metric space.

Definition 4 ([29])**.** *Let* $\mathcal{X}$ *be a nonempty set. A partial metric on* $\mathcal{X}$ *is a function* $\mathcal{J} : \mathcal{X}^2 \to [0, +\infty)$ *such that for all* $\mu, \omega, w \in \mathcal{X}$

1. $\mathcal{J}(\mu,\mu) = \mathcal{J}(\omega,\omega) = \mathcal{J}(\mu,\omega)$ *if, and only if* $\mu = \omega$,
2. $\mathcal{J}(\mu,\mu) \leq \mathcal{J}(\mu,\omega)$,
3. $\mathcal{J}(\mu,\omega) = \mathcal{J}(\omega,\mu)$,
4. $\mathcal{J}(\mu,w) \leq \mathcal{J}(\mu,\omega) + \mathcal{J}(\omega,w) - \mathcal{J}(w,w)$.

A pair $(\mathcal{X}, \mathcal{J})$ *is called a partial metric space.*

In Ref. [7], Asadi et al. extended the definition of a partial metric space to a M-metric space. The authors in Ref. [7] also established that every partial metric space is a M-metric space; however, every M-metric space need not be a partial metric space. We need the following notations to state the definition of a M-metric space.

Notation 1 ([7])**.**

1. $m_{\mu,\omega} := \min\{\mathcal{N}(\mu,\mu), \mathcal{N}(\omega,\omega)\}$.
2. $\mathcal{N}_{\mu,\omega} := \max\{\mathcal{N}(\mu,\mu), \mathcal{N}(\omega,\omega)\}$.

Definition 5 ([7])**.** *Let $\mathcal{X}$ be a nonempty set. A M-metric on $\mathcal{X}$ is a function $\mathcal{N} : \mathcal{X}^2 \to [0,+\infty)$ such that for all $\mu, \omega, w \in \mathcal{X}$*

1. $\mathcal{N}(\mu,\mu) = \mathcal{N}(\omega,\omega) = \mathcal{N}(\mu,\omega)$ *if, and only if* $\mu = \omega$,
2. $m_{\mu,\omega} \leq \mathcal{N}(\mu,\omega)$,
3. $\mathcal{N}(\mu,\omega) = \mathcal{N}(\omega,\mu)$,
4. $(\mathcal{N}(\mu,\omega) - m_{\mu,\omega}) \leq (\mathcal{N}(\omega,w) - m_{\omega,w}) + (\mathcal{N}(w,\omega) - m_{w,\omega})$.

A pair $(\mathcal{X},\mathcal{N})$ is called a M-metric space.

Example 1. *Let $\mathcal{X} = [0,\infty)$. Then, $\mathcal{N} : \mathcal{X}^2 \to [0,+\infty)$ defined by $\mathcal{N}(\mu,\omega) = \frac{\mu+\omega}{2}$ is a M-metric on $\mathcal{X}$.*

Example 2 ([7])**.** *Let $\mathcal{X} = \{a,b,c\}$. Define*

$$\mathcal{N}(a,a) = 1, \mathcal{N}(b,b) = 9, \mathcal{N}(c,c) = 5,$$

$$\mathcal{N}(a,b) = \mathcal{N}(b,a) = 10, \mathcal{N}(a,c) = \mathcal{N}(c,a) = 7, \mathcal{N}(b,c) = \mathcal{N}(3,c) = 7$$

Then $\mathcal{N}$ is an M-metric on $\mathcal{X}$, but not a partial metric.

The M-metric spaces have been extensively studied by several researchers [8–10,38–41]. Similar to the Double-Controlled quasi metric-type space (see Definition 3), we extend the M-metric spaces to Double-Controlled Quasi M-metric spaces, and prove the related fixed point results along with the examples and applications.

We shall use the following notations:

Notation 2 ([7])**.**

1. $z_{\mu,\omega} := \min\{\zeta(\mu,\mu), \zeta(\omega,\omega)\}$.
2. $R_{\mu,\omega} := \max\{\zeta(\mu,\mu), \zeta(\omega,\omega)\}$.

Definition 6. *Let $\mathcal{X}$ be a nonempty set, and $\alpha, \tau : \mathcal{X}^2 \to [1,+\infty)$ be two maps called control functions. A Double-Controlled quasi M-metric on $\mathcal{X}$ is a function $\zeta : \mathcal{X}^2 \to [0,+\infty)$ such that for all $\mu, \omega, w \in \mathcal{X}$*

1. $\zeta(\mu,\mu) = \zeta(\omega,\omega) = \zeta(\mu,\omega) = \zeta(\omega,\mu)$ *if, and only if* $\mu = \omega$,
2. $z_{\mu,\omega} \leq \zeta(\mu,\omega)$,
3. $(\zeta(\mu,\omega) - z_{\mu,\omega}) \leq \alpha(\mu,\omega)(\zeta(\mu,w) - z_{\mu,w}) + \tau(w,\omega)(\zeta(w,\omega) - z_{w,\omega})$.

A pair $(\mathcal{X},\zeta)$ is called a Double-Controlled quasi M-metric space.

Every Double-Controlled quasi M-metric space is a M-metric space, however the converse is not true in general.

Example 3. *Let $\mathcal{X} = \{a,b,c\}, \alpha = \tau = 1$ and $\zeta : \mathcal{X} \times \mathcal{X} \longrightarrow [0,\infty)$ be defined by*

$$\zeta(a,a) = 1, \quad \zeta(b,b) = 9, \quad \zeta(c,c) = 5, \quad \zeta(a,c) = 7 = \zeta(c,a),$$

$$\zeta(b,c) = 8 = \zeta(c,b), \quad \zeta(a,b) = 10, \quad \zeta(b,a) = 11.$$

It is not difficult to verify that $(\mathcal{X}, \zeta)$ is a Double-Controlled Quasi M-metric space. Since $\zeta(a,b) \neq \zeta(b,a)$, we see that $(\mathcal{X}, \zeta)$ is not an M-metric space.

Example 4. *Let $\mathcal{X} = [0,1], \alpha = \tau = 1$ and $\zeta : \mathcal{X} \times \mathcal{X} \longrightarrow [0,\infty)$ be defined by $\zeta(\mu,\omega) = 2\mu + \omega$. Then $(\mathcal{X}, \zeta)$ is a Double-Controlled Quasi M-metric space.*

Example 5. *Let $\mathcal{X} = \{4,5,6\}$ and $\zeta, \alpha : \mathcal{X} \times \mathcal{X} \longrightarrow [0,\infty), \alpha : \mathcal{X} \times \mathcal{X} \longrightarrow [1,\infty)$ be defined by $\alpha(\mu,\omega) = \mu\omega, \tau(\mu,\omega) = 1$, and*

$$\zeta(4,4) = 1, \zeta(5,5) = \zeta(6,6) = 1, \quad \zeta(4,5) = 6 = \zeta(5,4),$$

$$\zeta(4,6) = 4 = \zeta(6,4), \quad \zeta(5,6) = 2, \quad \zeta(6,5) = 3.$$

It is not difficult to verify that $(\mathcal{X}, \zeta)$ is a Double-Controlled Quasi M-metric space, however $(\mathcal{X}, \zeta)$ is not a M-metric space. Indeed, for $\mu = 4, \omega = 5, w = 6$, we have $(\zeta(4,5) - z_{4,5}) = 5 \leq (\zeta(4,6) - z_{4,6}) + (\zeta(6,5) - z_{6,5}) = 4$, that is, condition (3) of Definition 5 is not satisfied.

Similar to the Remark 1.1 in [7], it is not difficult to see the following holds in a Double-Controlled Quasi M-metric space:

Proposition 1. *Let $(\mathcal{X}, \zeta)$ be a Double-Controlled Quasi M-metric space; then for $\mu, \omega, w \in \mathcal{X}$, we have,*

1. $0 \leq R_{\mu\omega} + z_{\mu\omega} = \zeta(\mu,\mu) + \zeta(\omega,\omega)$
2. $0 \leq R_{\mu\omega} - z_{\mu\omega} = |\zeta(\mu,\mu) - \zeta(\omega,\omega)|$
3. $R_{\mu\omega} - z_{\mu\omega} \leq \alpha(\mu,\omega)\big(R_{\mu w} - z_{\mu w}\big) + \tau(w,\omega)(R_{w\omega} - z_{w\omega})$.

2. Topology of Double-Controlled Quasi *M*-Metric Space

Definition 7. *Let $(\mathcal{X}, \zeta)$ be a Double-Controlled Quasi M-metric space. Let $g \in \mathcal{X}$ and $\epsilon > 0$. Then:*

1. *The forward open ball B^+ centered at g is defined as*

$$B^+(g,\epsilon) = \{h \in \mathcal{X} | \zeta(g,h) - z_{g,h} < \epsilon\}$$

2. *The backward open ball B^- centered at g is defined as*

$$B^-(g,\epsilon) = \{h \in \mathcal{X} | \zeta(h,g) - z_{h,g} < \epsilon\}$$

Remark 1. *It is easy to see that the Double-Controlled Quasi M-metric ζ generates T_0 forward topology τ^+ and T_0 backward topology τ^- on $\mathcal{X}$, where the base of the topology τ^+ and τ^- is given by $\{B^+(g,\epsilon) : g \in \mathcal{X}, \epsilon > 0\}$ and $\{B^-(g,\epsilon) : g \in \mathcal{X}, \epsilon > 0\}$, respectively.*

In this paper, we shall work with forward topology τ^+.

Definition 8. *Let $(\mathcal{X}, \zeta)$ be a Double-Controlled Quasi M-metric space, and $\{\theta_n\}$ be a sequence in $\mathcal{X}$.*

1. *Then the sequence $\{\theta_n\}$ converges to a point $g \in \mathcal{X}$ from the left if, and only if*

$$\lim_{n \to +\infty} (\zeta(\theta_n, g) - z_{\theta_n, g}) = 0$$

2. *Then the sequence $\{\theta_n\}$ converges to a point $g \in \mathcal{X}$ from the right if, and only if*

$$\lim_{n \to +\infty} (\zeta(g, \theta_n) - z_{g, \theta_n}) = 0$$

3. *The sequence $\{\theta_n\}$ converges to a point $g \in \mathcal{X}$ if, and only if it converges to g from the left, and from the right.*

Definition 9. *Let $(\mathcal{X}, \zeta)$ be a Double-Controlled Quasi M-metric space, and $\{\theta_n\}$ be a sequence in $\mathcal{X}$. We say that:*

1. *the sequence $\{\theta_n\}$ is left ζ-Cauchy if, and if*
$$\lim_{n,m \to +\infty} (\zeta(\theta_n, \theta_m) - z_{\theta_n,\theta_m})$$
and
$$\lim_{n,m \to +\infty} (R_{\theta_n,\theta_m} - z_{\theta_n,\theta_m})$$
exist finitely.
2. *the sequence $\{\theta_n\}$ is right ζ-Cauchy if, and only if*
$$\lim_{n,m \to +\infty} (\zeta(\theta_m, \theta_n) - z_{\theta_m,\theta_n})$$
and
$$\lim_{n,m \to +\infty} (R_{\theta_m,\theta_n} - z_{\theta_m,\theta_n})$$
exist finitely
3. *the sequence $\{\theta_n\}$ is ζ-Cauchy if, and only if it is both left ζ-Cauchy and right ζ-Cauchy.*

Definition 10. *Let $(\mathcal{X}, \zeta)$ be a Double-Controlled Quasi M-metric space, and $\{\theta_n\}$ be a ζ-Cauchy in $\mathcal{X}$. We say that:*

1. *$(\mathcal{X}, \zeta)$ is left ζ-complete, with respect to forward topology τ^+, if every left ζ-Cauchy sequence converges to a point $g \in \mathcal{X}$ such that*
$$\lim_{n \to +\infty} \big(\zeta(\theta_n, g) - z_{\theta_n,g}\big) = 0$$
and
$$\lim_{n \to +\infty} \big(R_{\theta_n,g} - z_{\theta_n,g}\big) = 0.$$
2. *$(\mathcal{X}, \zeta)$ is right ζ-complete, with respect to forward topology τ^+, if every left ζ-Cauchy sequence converges to a point $g \in \mathcal{X}$ such that*
$$\lim_{n \to +\infty} \big(\zeta(g, \theta_n) - z_{g,\theta_n}\big) = 0$$
and
$$\lim_{n,m \to +\infty} \big(R_{g,\theta_n} - z_{g,\theta_n}\big) = 0.$$
3. *$(\mathcal{X}, \zeta)$ is ζ-complete, with respect to forward topology τ^+, if, and only if $(\mathcal{X}, \zeta)$ is both left ζ-complete and right ζ-complete.*

Definition 11. *Let $(\mathcal{X}, \zeta)$ be a Double-Controlled Quasi M-metric space, and a map $F : \mathcal{X} \longrightarrow \mathcal{X}$. We say that:*

1. *F is left ζ-continuous if, and only if for each sequence $\{\theta_n\}$ in $\mathcal{X}$ converging to $g \in \mathcal{X}$ from the left implies that $\{F\theta_n\}$ converges to Fg from the left, that is, we have,*
$$\lim_{n \to +\infty} \big(\zeta(\theta_n, g) - z_{\theta_n,g}\big) = 0 \quad \Longrightarrow \lim_{n \to +\infty} \big(\zeta(F\theta_n, Fg) - z_{F\theta_n,Fg}\big) = 0$$
2. *F is right ζ-continuous if, and only for each sequence $\{\theta_n\}$ in $\mathcal{X}$ converging to $g \in \mathcal{X}$ from the right implies that $\{F\theta_n\}$ converges to Fg from the right, that is, we have,*
$$\lim_{n \to +\infty} \big(\zeta(g, \theta_n) - z_{g,\theta_n}\big) = 0 \quad \Longrightarrow \lim_{n \to +\infty} \big(\zeta(Fg, F\theta_n) - z_{Fg,F\theta_n}\big) = 0$$
3. *F is ζ-continuous if it is both left and right ζ-continuous.*

The proof of the following result is similar to Lemma (3.5) in [9].

Lemma 1. *Let $(\mathcal{X},\zeta)$ be a Quasi M-metric space where ζ is continuous in the usual Euclidean metric. Suppose the self-mapping $F:\mathcal{X}\longrightarrow\mathcal{X}$ satisfies*

$$\zeta(Fg,Fh)\le k\zeta(g,h)$$

for some $k\in[0,1)$. Define a sequence $\{\theta_n\}\in\mathcal{X}$ by $\theta_n=F\theta_{n-1}$. If $\{\theta_n\}$ converges to a point $s\in\mathcal{X}$ from the left (or right), then $\{F\theta_n\}$ converges to $Fs\in\mathcal{X}$ from the left (or right), in the sense of Definition 8. That is,

$$\lim_{n\to+\infty}\zeta(\theta_n,s)-z_{\theta_n,s}=0,$$

implies

$$\lim_{n\to+\infty}(\zeta(F\theta_n,Fs)-z_{F\theta_n,Fs})=0.$$

3. Main Result

The following result is analogous to the classical Banach contraction principle.

Theorem 1. *Let $(\mathcal{X},\zeta)$ be a complete Double-Controlled Quasi M-metric space. Suppose that $F\colon\mathcal{X}\to\mathcal{X}$ is a self-map satisfying*

$$\zeta(Fg,Fh)\le k\zeta(g,h), \tag{1}$$

for all $g,h\in\mathcal{X}$, where $k\in(0,1)$. For $\theta\in X$, define the sequence $\theta_n=F^n\theta$. Suppose that

$$\sup_{m\ge1}\lim_{i\to\infty}\frac{\alpha(\theta_{i+1},\theta_{i+2})}{\alpha(\theta_i,\theta_{i+1})}\tau(\theta_{i+1},\theta_m)<\frac{1}{k}. \tag{2}$$

In addition, assume that, for every $\theta\in\mathcal{X}$,

$$\lim_{n\to\infty}\alpha(\theta,\theta_n),\quad \lim_{n\to\infty}\tau(\theta_n,\theta) \tag{3}$$

exist, and are finite. Then, F has a unique fixed point.

Proof. Fix $\theta_0\in\mathcal{X}$ and define a sequence $\{\theta_n\}$ in $\mathcal{X}$ inductively by taking $\theta_n=F\theta_{n-1}, n\ge0$.

$$\begin{aligned}\zeta(\theta_n,\theta_{n+1})&=\zeta(F\theta_{n-1},F\theta_n)\\&\le k\zeta(\theta_{n-1},\theta_n)\\&=k\zeta(F\theta_{n-2},F\theta_{n-1})\\&\le k^2\zeta(\theta_{n-2},\theta_{n-1})\\&\ \vdots\\&\le k^n\zeta(\theta_0,\theta_1)\end{aligned} \tag{4}$$

That is,

$$\zeta(\theta_n,\theta_{n+1})\le k^n\zeta(\theta_0,\theta_1) \tag{5}$$

Similarly, we have

$$\zeta(\theta_{n+1},\theta_n)\le k^n\zeta(\theta_1,\theta_0) \tag{6}$$

Now, consider $n,m\in\mathbb{N}$ where $n<m$. Then using the triangular inequality repeatedly, we have

$$
\begin{aligned}
\zeta(\theta_n,\theta_m)-z_{\theta_n,\theta_m} &\le \alpha(\theta_n,\theta_{n+1})\big(\zeta(\theta_n,\theta_{n+1})-z_{\theta_n,\theta_{n+1}}\big)+\tau(\theta_{n+1},\theta_m)\big(\zeta(\theta_{n+1},\theta_m)-z_{\theta_{n+1},\theta_m}\big)\\
&\le \alpha(\theta_n,\theta_{n+1})\big(\zeta(\theta_n,\theta_{n+1})-z_{\theta_n,\theta_{n+1}}\big)\\
&+\tau(\theta_{n+1},\theta_m)\left[\begin{array}{c}\alpha(\theta_{n+1},\theta_{n+2})\big(\zeta(\theta_{n+1},\theta_{n+2})-z_{\theta_{n+1},\theta_{n+2}}\big)\\ +\tau(\theta_{n+2},\theta_m)\big(\zeta(\theta_{n+2},\theta_m)-z_{\theta_{n+2},\theta_m}\big)\end{array}\right]\\
&\le \alpha(\theta_n,\theta_{n+1})\big(\zeta(\theta_n,\theta_{n+1})-z_{\theta_n,\theta_{n+1}}\big)\\
&+\tau(\theta_{n+1},\theta_m)\alpha(\theta_{n+1},\theta_{n+2})\big[\big(\zeta(\theta_{n+1},\theta_{n+2})-z_{\theta_{n+1},\theta_{n+2}}\big)\big]\\
&+\tau(\theta_{n+1},\theta_m)\tau(\theta_{n+2},\theta_m)\big[\zeta(\theta_{n+2},\theta_m)-z_{\theta_{n+2},\theta_m}\big]\\
&\vdots\\
&\le \alpha(\theta_n,\theta_{n+1})\big(\zeta(\theta_n,\theta_{n+1})-z_{\theta_n,\theta_{n+1}}\big)\\
&+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\tau(\theta_j,\theta_m)\right)\alpha(\theta_i,\theta_{i+1})\big[\big(\zeta(\theta_i,\theta_{i+1})-z_{\theta_i,\theta_{i+1}}\big)\big]\\
&+\prod_{k=n+1}^{m-1}\tau(\theta_k,\theta_m)\big[\big(\zeta(\theta_{m-1},\theta_m)-z_{\theta_{m-1},\theta_m}\big)\big]\\
&\le \alpha(\theta_n,\theta_{n+1})(\zeta(\theta_n,\theta_{n+1}))\\
&+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\tau(\theta_j,\theta_m)\right)\alpha(\theta_i,\theta_{i+1})\zeta(\theta_i,\theta_{i+1})\\
&+\prod_{k=n+1}^{m-1}\tau(\theta_k,\theta_m)\zeta(\theta_{m-1},\theta_m) \qquad (7)\\
&\le \alpha(\theta_n,\theta_{n+1})k^n\zeta(\theta_0,\theta_1)\\
&+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\tau(\theta_j,\theta_m)\right)\alpha(\theta_i,\theta_{i+1})k^i\zeta(\theta_0,\theta_1)\\
&+\left(\prod_{i=n+1}^{m-1}\tau(\theta_i,\theta_m)\right)k^{m-1}(\zeta(\theta_0,\theta_1))\\
&\le \alpha(\theta_n,\theta_{n+1})k^n\zeta(\theta_0,\theta_1)\\
&+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\tau(\theta_j,\theta_m)\right)\alpha(\theta_i,\theta_{i+1})k^i\zeta(\theta_0,\theta_1)\\
&+\left(\prod_{j=n+1}^{m-1}\tau(\theta_i,\theta_m)\right)k^{m-1}\alpha(\theta_{m-1},\theta_m)\zeta(\theta_0,\theta_1)\\
&= \alpha(\theta_n,\theta_{n+1})k^n\zeta(\theta_0,\theta_1)\\
&+\sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\tau(\theta_j,\theta_m)\right)\alpha(\theta_i,\theta_{i+1})k^i\zeta(\theta_0,\theta_1)\\
&\le \alpha(\theta_n,\theta_{n+1})k^n\zeta(\theta_0,\theta_1)\\
&+\sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i}\tau(\theta_j,\theta_m)\right)\alpha(\theta_i,\theta_{i+1})k^i\zeta(\theta_0,\theta_1)
\end{aligned}
$$

We have used $\alpha(g,h)\ge 1, \tau(g,h)\ge 1$ and $\zeta(g,h)-z_{g,h}\le\zeta(g,h)$ for all $g,h\in\mathcal{X}$. Let

$$S_p=\sum_{i=0}^{p}\left(\prod_{j=0}^{i}\tau(\theta_j,\theta_m)\right)\alpha(\theta_i,\theta_{i+1})k^i$$

The inequality (7) may be written as

$$\zeta(\theta_n, \theta_m) - k_{\theta_n,\theta_m} \leq \zeta(\theta_0, \theta_1)[\alpha(\theta_n, \theta_{n+1})k^n + (S_{m-1} - S_n)] \tag{8}$$

Letting

$$G_i = \left(\prod_{j=0}^{i} \tau(\theta_j, \theta_m)\right)\alpha(\theta_i, \theta_{i+1})k^i,$$

then

$$G_{i+1} = \left(\prod_{j=0}^{i+1} \tau(\theta_j, \theta_m)\right)\alpha(\theta_{i+1}, \theta_{i+2})k^{i+1},$$

so that we have

$$\frac{G_{i+1}}{G_i} = \frac{\alpha(\theta_{i+1}, \theta_{i+2})}{\alpha(\theta_i, \theta_{i+1})}\tau(\theta_{i+1}, \theta_m)k$$

Therefore, by Condition (2) in Theorem 1, we obtain

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{G_{i+1}}{G_i} = \sup_{m\geq 1}\lim_{i\to\infty}\frac{\alpha(\theta_{i+1}, \theta_{i+2})}{\alpha(\theta_i, \theta_{i+1})}\tau(\theta_{i+1}, \theta_m)k < 1$$

Therefore, by the Ratio test, we conclude that the sequence $\{S_n\}$ is Cauchy in the usual sense. Since $k \in [0,1)$, letting $n, m \to \infty$ in the inequality (8), we conclude that

$$\lim_{n,m\to+\infty}(\zeta(\theta_n, \theta_m) - z_{\theta_n,\theta_m}) = 0. \tag{9}$$

Similarly, using (6), we can establish that

$$\lim_{n,m\to+\infty}(\zeta(\theta_m, \theta_n) - z_{\theta_m,\theta_n}) = 0. \tag{10}$$

For $n > m$, we have

$$\begin{aligned} \zeta(\theta_n, \theta_n) &= \zeta(F\theta_{n-1}, F\theta_{n-1}) \\ &\leq k\zeta(\theta_{n-1}, \theta_{n-1}) \\ &\vdots \\ &\leq k^{n-m}\zeta(\theta_m, \theta_m) \end{aligned} \tag{11}$$

The inequality (11) implies that

$$R_{\theta_n,\theta_m} = \max\{\zeta(\theta_n, \theta_n), \zeta(\theta_m, \theta_m)\} = \zeta(\theta_n, \theta_n).$$

Hence, we get

$$\begin{aligned} R_{\theta_n,\theta_m} - z_{\theta_n,\theta_m} &\leq R_{\theta_n,\theta_m} \\ &= \zeta(\theta_n, \theta_n) \\ &= \zeta(F\theta_{n-1}, F\theta_{n-1}) \\ &\leq k\zeta(\theta_{n-1}, \theta_{n-1}) \\ &\vdots \\ &\leq k^n\zeta(\theta_0, \theta_0) \end{aligned} . \tag{12}$$

Letting $n \to \infty$, we deduce that

$$\lim_{n,m\to+\infty}(R_{\theta_n,\theta_m} - z_{\theta_n,\theta_m}) = 0 \tag{13}$$

Similarly, we can establish that

$$\lim_{n,m\to+\infty}(R_{\theta_m,\theta_n} - z_{\theta_m,\theta_n}) = 0 \tag{14}$$

By (9), (10), (13) and (14), we conclude that $\{\theta_n\}$ is ζ-Cauchy in $\mathcal{X}$. Since $\mathcal{X}$ is ζ-complete, $\{\theta_n\}$ converges to a point $\theta \in \mathcal{X}$ so that we have

$$\lim_{n\to+\infty}\zeta(\theta_n,\theta) - z_{\theta_n,\theta} = 0 \tag{15}$$

and

$$\lim_{n\to+\infty}\zeta(\theta,\theta_n) - z_{\theta,\theta_n} = 0. \tag{16}$$

Next, we prove that $F\theta = \theta$.

By the Lemma 1, F is ζ-continuous. The Definition 10 and Equation (15) implies that

$$\lim_{n\to+\infty}\left(\zeta(\theta_{n-1},F\theta) - z_{\theta_{n-1},F\theta}\right) = \lim_{n\to+\infty}(\zeta(F\theta_n,F\theta) - z_{F\theta_n,F\theta}) = 0 \tag{17}$$

By the triangular inequality, we have

$$\zeta(\theta,F\theta) - z_{\theta,F\theta} \le \alpha(\theta,\theta_n)(\zeta(\theta,\theta_n) - z_{\theta,\theta_n}) + \tau(\theta_n,F\theta)(\zeta(\theta_n,F\theta) - z_{\theta_n,F\theta}). \tag{18}$$

Taking the limit in the above inequality, and using (3), (16) and (17), we obtain

$$\zeta(\theta,F\theta) - z_{\theta,F\theta} \le 0 \tag{19}$$

By the definition of a Double-Controlled Quasi M-metric space, we have

$$z_{\theta,F\theta} - \zeta(\theta,F\theta) \le 0 \tag{20}$$

The inequalities (19) and (20) imply

$$\zeta(\theta,F\theta) = z_{\theta,F\theta} \tag{21}$$

Now, by Condition (1) of Theorem 1, we have $\zeta(F\theta,F\theta) \le k\zeta(\theta,\theta) < \zeta(\theta,\theta)$. This implies

$$R_{\theta,F\theta} = \max\{\zeta(\theta,\theta),\zeta(F\theta,F\theta)\} = \zeta(\theta,\theta) \tag{22}$$

and

$$z_{\theta,F\theta} = \min\{\zeta(\theta,\theta),\zeta(F\theta,F\theta)\} = \zeta(F\theta,F\theta) \tag{23}$$

By (21) and (23), we obtain

$$\zeta(\theta,F\theta) = \zeta(F\theta,F\theta) \tag{24}$$

Now,

$$\begin{aligned}\zeta(\theta_n,\theta_n) &= \zeta(F\theta_{n-1},F\theta_{n-1})\\ &\le k\zeta(\theta_{n-1},\theta_{n-1})\\ &\ \vdots\\ &\le k^n\zeta(\theta_0,\theta_0)\end{aligned} \tag{25}$$

This implies,

$$\lim_{n\to+\infty}\zeta(\theta_n,\theta_n) = 0 \tag{26}$$

By Equation (26), we get

$$\begin{aligned}\lim_{n\to+\infty} z_{\theta_n,\theta} &= \lim_{n\to+\infty} \min\{\zeta(\theta_n,\theta_n),\zeta(\theta,\theta)\} \\ &= \min\{0,\zeta(\theta,\theta)\} \\ &= 0\end{aligned}. \tag{27}$$

By Proposition 1, we have

$$\zeta(\theta_n,\theta_n)+\zeta(\theta,\theta) = R_{\theta_n,\theta}+z_{\theta_n,\theta}$$

or

$$\begin{aligned}\zeta(\theta,\theta) &= R_{\theta_n,\theta}+z_{\theta_n,\theta}-\zeta(\theta_n,\theta_n) \\ &= (R_{\theta_n,\theta}-z_{\theta_n,\theta})+2z_{\theta_n,\theta}-\zeta(\theta_n,\theta_n)\end{aligned}. \tag{28}$$

Since $(\mathcal{X},\zeta)$ is ζ-complete, by Definition 10,

$$\lim_{n\to+\infty}(R_{\theta_n,\theta}-z_{\theta_n,\theta}) = 0 \tag{29}$$

Using (26), (27), and (29) in (28), we obtain

$$\zeta(\theta,\theta) = 0 \tag{30}$$

By the Equations (21), (22), and (30), we have

$$\zeta(\theta,F\theta) = z_{\theta,F\theta} \le R_{\theta,F\theta} = 0 \tag{31}$$

Since $\zeta(\theta,F\theta)\ge 0$, this implies,

$$\zeta(\theta,F\theta) = 0 \tag{32}$$

Similarly, we may prove

$$\zeta(F\theta,\theta) = 0.$$

The Equations (24), (30), and (32) imply

$$\zeta(\theta,\theta) = \zeta(F\theta,F\theta) = \zeta(\theta,F\theta) = \zeta(\theta,F\theta) = 0 \tag{33}$$

which further implies $\theta = F\theta$ so that θ is a fixed point of F.

Next, we show the uniqueness of the fixed point. Suppose that F has two distinct fixed points θ and δ, such that $F\theta=\theta$ and $F\delta=\delta$. Thus, $\zeta(\theta,\delta)=\zeta(F\theta,F\delta)\le k\zeta(\theta,\delta)<\zeta(\theta,\delta)$. This implies, $\zeta(\theta,\delta)=0$. Additionally, $\zeta(\theta,\theta)=\zeta(F\theta,F\theta)\le k\zeta(\theta,\theta)<\zeta(\theta,\theta)$, which implies $\zeta(\theta,\theta)=0$. Similarly, $\zeta(\delta,\delta)=0$. Thus, we have

$$\zeta(\theta,\delta) = \zeta(\theta,\theta) = \zeta(\delta,\delta) = 0$$

which by the Definition 6 implies $\delta=\theta$. □

The following theorem is similar to the Kannan-type fixed point result.

Theorem 2. *Let $(\mathcal{X},\zeta)$ be a complete Double-Controlled Quasi M-metric space, and $F\colon \mathcal{X}\to\mathcal{X}$ be a self ζ-continuous mapping on $\mathcal{X}$ satisfying*

$$\zeta(Fg,Fh) \le k\big[\zeta(g,Fg)+\zeta(h,Fh)\big] \tag{34}$$

for all $g, h \in \mathcal{X}$, where $k \in [0, \frac{1}{2})$. For $\theta \in X$, define the sequence $\theta_n = F^n\theta$. Suppose that

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\alpha(\theta_{i+1},\theta_{i+2})}{\alpha(\theta_i,\theta_{i+1})}\tau(\theta_{i+1},\theta_m) < \frac{1}{k} \tag{35}$$

In addition, assume that, for every $\theta \in \mathcal{X}$,

$$\lim_{n\to\infty}\alpha(\theta,\theta_n), \quad \lim_{n\to\infty}\tau(\theta_n,\theta) \tag{36}$$

exist, and are finite. Then, F has a unique fixed point.

Proof. Let $\theta_0 \in \mathcal{X}$ and define a sequence $\{\theta_n\}$ in $\mathcal{X}$ inductively by taking $\theta_n = F\theta_{n-1}, n \geq 1$. Set $d_n = \zeta(\theta_n, \theta_{n+1})$ and $D_n = \zeta(\theta_{n+1}, \theta_n)$. Then we have,

$$\begin{aligned} d_n = \zeta(\theta_n,\theta_{n+1}) &= \zeta(F\theta_{n-1},F\theta_n) \\ &\leq k\big[\zeta(\theta_{n-1},F\theta_{n-1})+\zeta(\theta_n,F\theta_n)\big] \\ &= k\big[\zeta(\theta_{n-1},\theta_n)+\zeta(\theta_n,\theta_{n+1})\big] \\ &\leq k\big[d_{n-1}+d_n)\big] \end{aligned} \tag{37}$$

which implies,

$$d_n \leq \beta d_{n-1} \tag{38}$$

where $\beta = \frac{k}{1-k} < 1$ as $k \in [0, \frac{1}{2}]$.

Thus, we have

$$d_n \leq \beta d_{n-1} \leq \beta^2 d_{n-2} \leq \ldots \leq \beta^n\zeta(\theta_0,\theta_1) \tag{39}$$

Similarly, we have

$$D_n \leq \beta^n\zeta(\theta_1,\theta_0) \tag{40}$$

Now, consider $n, m \in \mathbb{N}$ where $n < m$. Then, using the triangular inequality repeatedly, we have

$$\begin{aligned} \zeta(\theta_n,\theta_m)-z_{\theta_n,\theta_m} &\leq \alpha(\theta_n,\theta_{n+1})\big(\zeta(\theta_n,\theta_{n+1})-z_{\theta_n,\theta_{n+1}}\big)+\tau(\theta_{n+1},\theta_m)\big(\zeta(\theta_{n+1},\theta_m)-z_{\theta_{n+1},\theta_m}\big) \\ &\leq \alpha(\theta_n,\theta_{n+1})\big(\zeta(\theta_n,\theta_{n+1})-z_{\theta_n,\theta_{n+1}}\big) \\ &+\tau(\theta_{n+1},\theta_m)\begin{bmatrix} \alpha(\theta_{n+1},\theta_{n+2})\big(\zeta(\theta_{n+1},\theta_{n+2})-z_{\theta_{n+1},\theta_{n+2}}\big) \\ +\tau(\theta_{n+2},\theta_m)\big(\zeta(\theta_{n+2},\theta_m)-z_{\theta_{n+2},\theta_m}\big) \end{bmatrix} \\ &\leq \alpha(\theta_n,\theta_{n+1})\big(\zeta(\theta_n,\theta_{n+1})-z_{\theta_n,\theta_{n+1}}\big) \\ &+\tau(\theta_{n+1},\theta_m)\alpha(\theta_{n+1},\theta_{n+2})\big[\big(\zeta(\theta_{n+1},\theta_{n+2})-z_{\theta_{n+1},\theta_{n+2}}\big)\big] \\ &+\tau(\theta_{n+1},\theta_m)\tau(\theta_{n+2},\theta_m)\big[\zeta(\theta_{n+2},\theta_m)-z_{\theta_{n+2},\theta_m}\big] \\ &\;\vdots \\ &\leq \alpha(\theta_n,\theta_{n+1})\big(\zeta(\theta_n,\theta_{n+1})-z_{\theta_n,\theta_{n+1}}\big) \\ &+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\tau(\theta_j,\theta_m)\right)\alpha(\theta_i,\theta_{i+1})\big[\big(\zeta(\theta_i,\theta_{i+1})-z_{\theta_i,\theta_{i+1}}\big)\big] \\ &+\prod_{k=n+1}^{m-1}\tau(\theta_k,\theta_m)\big[\big(\zeta(\theta_{m-1},\theta_m)-z_{\theta_{m-1},\theta_m}\big)\big] \end{aligned} \tag{41}$$

$$\begin{aligned}
&\leq \alpha(\theta_n,\theta_{n+1})(\zeta(\theta_n,\theta_{n+1}))\\
&+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\tau(\theta_j,\theta_m)\right)\alpha(\theta_i,\theta_{i+1})\zeta(\theta_i,\theta_{i+1})\\
&+\prod_{k=n+1}^{m-1}\tau(\theta_k,\theta_m)\zeta(\theta_{m-1},\theta_m)\\
&\leq \alpha(\theta_n,\theta_{n+1})k^n\zeta(\theta_0,\theta_1)\\
&+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\tau(\theta_j,\theta_m)\right)\alpha(\theta_i,\theta_{i+1})k^i\zeta(\theta_0,\theta_1)\\
&+\left(\prod_{i=n+1}^{m-1}\tau(\theta_i,\theta_m)\right)k^{m-1}(\zeta(\theta_0,\theta_1))\\
&\leq \alpha(\theta_n,\theta_{n+1})k^n\zeta(\theta_0,\theta_1)\\
&+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\tau(\theta_j,\theta_m)\right)\alpha(\theta_i,\theta_{i+1})k^i\zeta(\theta_0,\theta_1)\\
&+\left(\prod_{j=n+1}^{m-1}\tau(\theta_i,\theta_m)\right)k^{m-1}\alpha(\theta_{m-1},\theta_m)\zeta(\theta_0,\theta_1)\\
&= \alpha(\theta_n,\theta_{n+1})k^n\zeta(\theta_0,\theta_1)\\
&+\sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\tau(\theta_j,\theta_m)\right)\alpha(\theta_i,\theta_{i+1})k^i\zeta(\theta_0,\theta_1)\\
&\leq \alpha(\theta_n,\theta_{n+1})k^n\zeta(\theta_0,\theta_1)\\
&+\sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i}\tau(\theta_j,\theta_m)\right)\alpha(\theta_i,\theta_{i+1})k^i\zeta(\theta_0,\theta_1)
\end{aligned}$$

We have used $\alpha(g,h)\geq 1, \tau(g,h)\geq 1$ and $\zeta(g,h)-z_{g,h}\leq\zeta(g,h)$ for all $g,h\in\mathcal{X}$.

Let

$$S_p=\sum_{i=0}^{p}\left(\prod_{j=0}^{i}\tau(\theta_j,\theta_m)\right)\alpha(\theta_i,\theta_{i+1})k^i$$

The inequality (41) may be written as

$$\zeta(\theta_n,\theta_m)-k_{\theta_n,\theta_m}\leq\zeta(\theta_0,\theta_1)[\alpha(\theta_n,\theta_{n+1})k^n+(S_{m-1}-S_n)] \tag{42}$$

Letting

$$G_i=\left(\prod_{j=0}^{i}\tau(\theta_j,\theta_m)\right)\alpha(\theta_i,\theta_{i+1})k^i,$$

then

$$G_{i+1}=\left(\prod_{j=0}^{i+1}\tau(\theta_j,\theta_m)\right)\alpha(\theta_{i+1},\theta_{i+2})k^{i+1},$$

so that we have

$$\frac{G_{i+1}}{G_i}=\frac{\alpha(\theta_{i+1},\theta_{i+2})}{\alpha(\theta_i,\theta_{i+1})}\tau(\theta_{i+1},\theta_m)k$$

Therefore, by Condition (35), we obtain

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{G_{i+1}}{G_i}=\sup_{m\geq 1}\lim_{i\to\infty}\frac{\alpha(\theta_{i+1},\theta_{i+2})}{\alpha(\theta_i,\theta_{i+1})}\tau(\theta_{i+1},\theta_m)k<1$$

Therefore, by using the Ratio test, we conclude that the sequence $\{S_n\}$ is Cauchy in the usual sense. Since $k \in [0,1)$, letting $n, m \to \infty$ in the inequality (42), we conclude that

$$\lim_{n,m\to+\infty}(\zeta(\theta_n,\theta_m) - z_{\theta_n,\theta_m}) = 0. \tag{43}$$

Similarly, using (40), we can prove that

$$\lim_{n,m\to+\infty}(\zeta(\theta_m,\theta_n) - z_{\theta_m,\theta_n}) = 0. \tag{44}$$

Without loss of generality, we may assume that

$$R_{\theta_n,\theta_m} = \max\{\zeta(\theta_n,\theta_n), \zeta(\theta_m,\theta_m)\} = \zeta(\theta_n,\theta_n).$$

Hence, we get

$$\begin{aligned} R_{\theta_n,\theta_m} - z_{\theta_n,\theta_m} &\leq R_{\theta_n,\theta_m} \\ &= \zeta(\theta_n,\theta_n) \\ &= \zeta(F\theta_{n-1}, F\theta_{n-1}) \\ &\leq k\big[\zeta(\theta_{n-1}, F\theta_{n-1}) + \zeta(\theta_{n-1}, F\theta_{n-1})\big] \\ &\leq k\big[\zeta(\theta_{n-1},\theta_n) + \zeta(\theta_{n-1},\theta_n)\big] \\ &= 2k\big[\zeta(\theta_{n-1},\theta_n)\big] \\ &= 2kd_{n-1} \end{aligned} \tag{45}$$

By the inequality (39), $\lim_{n\to+\infty} d_n = 0.$.
Letting $n \to \infty$ in the above inequality, we deduce that

$$\lim_{n,m\to+\infty}(R_{\theta_n,\theta_m} - z_{\theta_n,\theta_m}) = 0 \tag{46}$$

Similarly, we can establish that

$$\lim_{n,m\to+\infty}(R_{\theta_m,\theta_n} - z_{\theta_m,\theta_n}) = 0 \tag{47}$$

By (43), (44), (46) and (46), we conclude that $\{\theta_n\}$ is ζ- Cauchy in $\mathcal{X}$. Since $\mathcal{X}$ is ζ-complete, $\{\theta_n\}$ converges to a point $\theta \in \mathcal{X}$ so that we have

$$\lim_{n\to+\infty} \zeta(\theta_n,\theta) - z_{\theta_n,\theta} = 0 \tag{48}$$

and

$$\lim_{n\to+\infty} \zeta(\theta,\theta_n) - z_{\theta,\theta_n} = 0 \tag{49}$$

Now, we prove that θ is a fixed point of F.
Since F is ζ-continuous, the Definition 10 and the Equation (48) implies

$$\lim_{n\to+\infty}\big(\zeta(\theta_{n-1}, F\theta) - z_{\theta_{n-1},F\theta}\big) = \lim_{n\to+\infty}(\zeta(F\theta_n, F\theta) - z_{F\theta_n,F\theta}) = 0 \tag{50}$$

By the triangular inequality, we have,

$$\zeta(\theta, F\theta) - z_{\theta,F\theta} \leq \alpha(\theta,\theta_n)(\zeta(\theta,\theta_n) - z_{\theta,\theta_n}) + \tau(\theta_n, F\theta)(\zeta(\theta_n, F\theta) - z_{\theta_n,F\theta}). \tag{51}$$

Taking the limit in the above inequality, and using (49) and (50), we obtain

$$\zeta(\theta, F\theta) - z_{\theta,F\theta} \leq 0 \tag{52}$$

By the definition of Double-Controlled Quasi M-metric space, we have

$$z_{\theta,F\theta} - \zeta(\theta, F\theta) \leq 0 \tag{53}$$

The inequalities (52) and (53) imply

$$\zeta(\theta, F\theta) = z_{\theta,F\theta} \tag{54}$$

Now,

$$\begin{aligned} \zeta(\theta_n, \theta_n) &= \zeta(F\theta_{n-1}, F\theta_{n-1}) \\ &\leq k\big[\zeta(\theta_{n-1}, F\theta_{n-1}) + \zeta(\theta_{n-1}, F\theta_{n-1})\big] \\ &\leq k\big[\zeta(\theta_{n-1}, \theta_n) + \zeta(\theta_{n-1}, \theta_n)\big] \\ &= 2k\big[\zeta(\theta_{n-1}, \theta_n)\big] \\ &= 2kd_{n-1} \end{aligned} \tag{55}$$

This implies,

$$\lim_{n\to+\infty} \zeta(\theta_n, \theta_n) = 0 \tag{56}$$

By Equation (50), we get

$$\begin{aligned} \lim_{n\to+\infty} z_{\theta_n,\theta} &= \lim_{n\to+\infty} \min\{\zeta(\theta_n, \theta_n), \zeta(\theta, \theta)\} \\ &= \min\{0, \zeta(\theta, \theta)\} \\ &= 0 \end{aligned} \tag{57}$$

By Proposition 1, we have

$$\zeta(\theta_n, \theta_n) + \zeta(\theta, \theta) = R_{\theta_n,\theta} + z_{\theta_n,\theta}$$

or

$$\begin{aligned} \zeta(\theta, \theta) &= R_{\theta_n,\theta} + z_{\theta_n,\theta} - \zeta(\theta_n, \theta_n) \\ &= (R_{\theta_n,\theta} - z_{\theta_n,\theta}) + 2z_{\theta_n,\theta} - \zeta(\theta_n, \theta_n) \end{aligned} \tag{58}$$

Since $(\mathcal{X}, \zeta)$ is ζ-Complete, by Definition 10,

$$\lim_{n\to+\infty} (R_{\theta_n,\theta} - z_{\theta_n,\theta}) = 0 \tag{59}$$

Using (56), (57) and (59) in (58), we obtain

$$\zeta(\theta, \theta) = 0 \tag{60}$$

By Equations (54) and (60), we have

$$\begin{aligned} \zeta(\theta, F\theta) &= z_{\theta,F\theta} \\ &= \min\{\zeta(\theta, \theta), \zeta(F\theta, F\theta)\} \\ &= \min\{0, \zeta(F\theta, F\theta)\} \\ &= 0 \end{aligned} \tag{61}$$

Similarly, we may prove

$$\zeta(F\theta, \theta) = 0.$$

Using (34), we obtain

$$\begin{aligned} \zeta(F\theta, F\theta) &\leq k\big[\zeta(\theta, F\theta) + \zeta(\theta, F\theta)\big] \\ &\leq 2k\big[\zeta(\theta, F\theta)\big] \\ &= 0 \end{aligned} \tag{62}$$

This implies,

$$\zeta(F\theta, F\theta) = 0 \tag{63}$$

Therefore, by Equations (60), (61), and (63), we obtain

$$\zeta(\theta, \theta) = \zeta(\theta, F\theta) = \zeta(F\theta, \theta) = \zeta(F\theta, F\theta) = 0$$

which implies that $F\theta = \theta$.

Finally, we establish the uniqueness of the fixed point. Suppose that F has two distinct fixed points θ and δ, that $F\theta = \theta$ and $F\delta = \delta$. We have,

$$\zeta(\theta, \delta) = \zeta(F\theta, F\delta) \leq k\big[\zeta(\theta, F\theta) + \zeta(\delta, F\delta)\big] = k\big[\zeta(\theta, \theta) + \zeta(\delta, \delta)\big] = 0,$$

which implies that $\zeta(\theta, \delta) = 0$. Since θ and δ are fixed points, by Equation (56), we have $\zeta(\theta, \theta) = 0$ and $\zeta(\delta, \delta) = 0$. Therefore,

$$\zeta(\theta, \delta) = \zeta(\theta, \theta) = \zeta(\delta, \delta) = 0,$$

which implies that $\theta = \delta$. □

4. Applications

Finally, we provide a few applications of our proven theorems.

Example 6. *Let $\mathcal{X} = [0, 4]$. Define $\zeta : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ by $\zeta(\mu, \omega) = (\mu - \omega)^4$ and $\alpha(\mu, \omega) = \mu + \omega + 1, \tau(\mu, \omega) = \omega + 1$, then it is not difficult to see that $(\mathcal{X}, \zeta)$ is a complete Double-Controlled Quasi M-metric space. Let $F : \mathcal{X} \longrightarrow \mathcal{X}$ be defined as $F\theta = \frac{\theta}{5}$, then F has a unique fixed point.*

Proof. Let $\theta \in X$. Define a sequence as $\theta_n = F^n\theta = \frac{\theta}{5^n}$.

We have

$$\zeta(F\mu, F\omega) = (F\mu - F\omega)^4 = \left(\frac{\mu}{5} - \frac{\omega}{5}\right)^4 \leq \frac{1}{625}(\mu - \omega)^4 = k\zeta(\mu, \omega)$$

where $k = \frac{1}{625}$. Consider

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\alpha(\theta_{i+1}, \theta_{i+2})}{\alpha(\theta_i, \theta_{i+1})}\tau(\theta_{i+1}, \theta_m) = \sup_{m\geq 1}\lim_{i\to\infty}\frac{\frac{\theta}{5^{i+1}} + \frac{\theta}{5^{i+2}} + 1}{\frac{\theta}{5^i} + \frac{\theta}{5^{i+1}} + 1}\left(\frac{\theta}{5^m} + 1\right) = \frac{\theta}{5} + 1 < 625 = \frac{1}{k}.$$

Moreover, for each, $\theta \in [0, 4]$, we have

$$\lim_{n\to\infty}\alpha(\theta, \theta_n) = \lim_{n\to\infty}\left(\theta + \frac{\theta}{5^n} + 1\right) = \theta + 1 < \infty$$

and

$$\lim_{n\to\infty}\tau(\theta_n, \theta) = \lim_{n\to\infty}\theta + 1 = \theta + 1 < \infty.$$

Therefore, all the conditions of Theorem 1 are satisfied, hence F has a unique fixed point. □

Example 7. *Consider the space of all continuous real valued functions $\mathcal{X} = C[0, 1]$, and $\zeta(r(\mu), h(\mu)) : \mathcal{X} \times \mathcal{X} \longrightarrow [0, +\infty)$ be defined as*

$$\zeta(r(\mu), h(\mu)) = \sup_{\mu\in[0,1]}|r(\mu) - h(\mu)|^2 + \sup_{\mu\in[0,1]}|r(\mu)|^2.$$

Define the control functions $\alpha, \tau : \mathcal{X} \times \mathcal{X} \to [1, +\infty)$ by

$$\alpha(r(\mu), h(\mu)) = \tau(r(\mu), h(\mu)) = 1 = 1 + \sup_{\mu \in [0,1]} |r(\mu)h(\mu)| \text{for all } r, h \in \mathcal{X}.$$

It is not difficult to see that $(\mathcal{X}, \zeta)$ is a complete Double-Controlled Quasi M-metric space.

Theorem 3. *Let $\mathcal{X} = C[0,1]$ be the complete Double-Controlled metric-like space given in Example 7. Consider the following Fredholm integral equation*

$$r(\mu) = \int_0^1 l(\mu, \omega, r(\mu)) d\omega, \tag{64}$$

where $l(\mu, \omega, r(\mu)) : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ is a given continuous function satisfying the following condition for all $r(\mu), h(\mu) \in \mathcal{X}, \mu, \omega \in [0,1]$:

1. $$|l(\mu, \omega, r(\mu)) - l(\mu, \omega, h(\mu))| \leq \sqrt{H_1(\mu)}, \quad |l(\mu, \omega, r(\mu))| \leq \sqrt{H_2(\mu)}$$

where

$$H_1(\mu) \leq kd(r(\mu), Fr(\mu)),$$

and

$$H_2(\mu) \leq kd(h(\mu), Fh(\mu)),$$

$F(r(\mu)) = \int_0^1 l(\mu, \omega, r(\mu)) d\omega$ *and* $k \in [0, \frac{1}{1 + (\sup_{\mu,\omega} |l(\mu,\omega,r(\mu))|)^2})$.

2. $l(\mu, \omega, \int_0^1 l(\mu, \omega, r(\mu))) < l(\mu, \omega, r(\mu))$ *for all* μ, ω.

Then the integral Equation (64) has a unique solution.

Proof. Let $F : C[0,1] \longrightarrow C[0,1]$ be defined by $F(r(\mu)) = \int_0^1 l(\mu, \omega, r(\mu)) d\omega$ then

$$\begin{aligned}
\zeta(Fr(\mu), Fh(\mu)) = & \sup_{\mu \in [0,1]} |Fr(\mu) - Fh(\mu)|^2 + \sup_{\mu \in [0,1]} |Fr(\mu)|^2 \\
& = \sup_{\mu \in [0,1]} |\int_0^1 l(\mu, \omega, r(\mu)) d\omega - \int_0^1 l(\mu, \omega, h(\mu)) d\omega|^2 \\
& + \sup_{\mu \in [0,1]} |\int_0^1 l(\mu, \omega, r(\mu)) d\omega|^2 \\
& \leq \sup_{\mu \in [0,1]} \int_0^1 |l(\mu, \omega, r(\mu)) d\omega - l(\mu, \omega, h(\mu))|^2 d\omega \\
& + \sup_{\mu \in [0,1]} \int_0^1 |l(\mu, \omega, r(\mu))|^2 d\omega \\
& \leq \sup_{\mu \in [0,1]} \int_0^1 \left[|l(\mu, \omega, r(\mu)) d\omega - l(\mu, \omega, h(\mu))|^2 \right. \\
& \left. + |l(\mu, \omega, r(\mu))|^2 \right] d\omega \\
& \leq \sup_{\mu \in [0,1]} \int_0^1 |\sqrt{H_1(\mu)}|^2 + |\sqrt{H_2(\mu)}|^2 d\omega \\
& \leq \sup_{\mu \in [0,1]} |H_1(\mu)| + |H_2(\mu)| \int_0^1 d\omega \\
& \leq \sup_{\mu \in [0,1]} H_1(\mu) + H_2(\mu) \\
& \leq k[d(r(\mu), Fr(\mu)) + d(h(\mu), Fh(\mu))].
\end{aligned} \tag{65}$$

Now using assumption (2) of Theorem 2 for the sequence $\theta_n = F^n\theta$, we have

$$
\begin{aligned}
(\theta_n r)(\mu) = (F^n r)(\mu) = F\Big(F^{n-1}r(\mu)\Big) &= \int_0^1 l\Big(\mu,\omega,F^{n-1}r(\mu)\Big)d\omega \\
&= \int_0^1 l\Big(\mu,\omega,F\Big(F^{n-2}r\Big)(\mu)\Big)d\omega \\
&= \int_0^1 l\Big(\mu,\omega,\int_0^1 l\Big(\mu,\omega,\Big(F^{n-2}r(\mu)\Big)\Big)\Big)d\omega \\
&< \int_0^1 l\Big(\mu,\omega,\Big(F^{n-2}r(\mu)\Big)\Big)d\omega = \Big(F^{n-1}r(\mu)\Big).
\end{aligned}
$$

Thus, we see that the sequence $(F^n r(\mu))_n$ is strictly decreasing and bounded below $\mu \in [0,1]$, and so it converges to some s. This further implies by Dini's theorem from the real analysis that$\sup_t|F^n r(\mu)|$ converges to some $s' \leqslant \sup_{\mu,\omega}|l(\mu,\omega,r(\mu))|$. Note that $\alpha(F^n r, F^m r) = \tau(F^n r, F^m r) = 1 + \sup_\mu |F^n r(\mu)||F^m r(\mu)|$ converges to $1 + l^2 \leq 1 + (\sup_{\mu,\omega}|l(\mu,\omega,r(\mu))|)^2$.

Now consider,

$$
\begin{aligned}
\sup_{m\geq 1}\lim_{i\to\infty}\frac{\alpha(\theta_{i+1},\theta_{i+2})}{\alpha(\theta_i,\theta_{i+1})}\tau(\theta_{i+1},\theta_m) &= \sup_{m\geq 1}\lim_{i\to\infty}\frac{\alpha(F^{i+1}r(\mu),F^{i+2}r(\mu))}{\alpha(F^i r(\mu),F^{i+1}r(\mu))}\tau\Big(F^{i+1}r(\mu),F^m r(\mu)\Big) \\
&= \sup_{m\geq 1}\frac{1+l^2}{1+l^2}\lim_{i\to\infty}\tau\Big(F^{i+1}r(\mu),F^m r(\mu)\Big) \\
&= \sup_{m\geq 1}\lim_{i\to\infty}\tau\Big(F^{i+1}r(\mu),F^m r(\mu)\Big) \\
&= \sup_{m\geq 1}\lim_{i\to\infty} 1 + \sup_\mu |F^{i+1}r(\mu)||F^{m+1}r(\mu)| \\
&\leq 1 + (\sup_{\mu,\omega}|l(\mu,\omega,r(\mu))|)^2 < \frac{1}{k}.
\end{aligned}
$$

Therefore, all the conditions of Theorem 2 are satisfied, which implies that the integral Equation (64) has a solution. □

5. Conclusions and Open Problems

We developed the idea of Double-Controlled Quasi M-metric space as a new generalization of M-metric space, and established fixed point results of the Banach and Kannan types along with the application. It is an open problem to establish the Banach-type fixed point results in Double-Controlled Quasi M-metric spaces for other types of contraction mappings, like Ciric contraction mapping, Riech contraction mapping, Hardy–Roger contraction mapping, and Caristi contraction mapping. Researchers have studied [42–45] mathematical control theory, fractional and differential integral equations, and functional equations by using the techniques of fixed point theory. It is of great interest to find serious applications of Double-Controlled quasi M-metric spaces to the theory of differential and integral equations. Future studies in this direction are highly suggested.

Finally, we provide a very important direction for the future work in the framework of Double-Controlled Quasi M-metric spaces. When there is no unique fixed point, one technique to generalize the fixed-point results is to investigate the geometric properties of the set of fixed points. In this direction, the fixed-circle problem (see [46]) and the fixed-figure problem (see [47]) have been introduced. More relevantly to our current studies, Maliki et al. [48] studied the fixed-disc point problem in the framework of Double-Controlled Quasi-metric-type spaces. As a future work, it is highly suggested to study the fixed-circle, fixed-ellipse, fixed-disc and other fixed-figure problems in the framework of Double-Controlled Quasi M-metric spaces.

Author Contributions: I.A.: writing—original draft, methodology; N.Z.C.: conceptualization, supervision, writing—original draft; N.M.: conceptualization, supervision, writing—original draft. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data is unavailable due to privacy restriction.

Acknowledgments: The authors, I. Ayoob and N. Mlaiki would like to thank the University of Prince Sultan for the payment of the publication fee for this paper through TAS LAB.

Conflicts of Interest: The authors have no competing interests.

References

1. Banach, S. Sur les operations dans les ensembles abstraits et leur applications aux equations intgrales. *Fundam. Math.* **1922**, *3*, 133181. [CrossRef]
2. Long-Guang, H.; Xian, Z. Cone metric spaces and fixed point theorems of contractive mappings. *J. Math. Anal. Appl.* **2007**, *332*, 1468–1476.
3. Shatanawi, W. On w-compatible mappings and common coupled coincidence point in cone metric spaces. *Appl. Math. Lett.* **2012**, *25*, 925–931. [CrossRef]
4. Shatanawi, W.; Rajic, R.V.; Radenovic, S.; Al-Rawashhdeh, A. Mizoguchi-Takahashi-type theorems in tvs-cone metric spaces. *FIxed Point Theory Appl.* **2012**, *11*, 106. [CrossRef]
5. Asim, M.; Khan, A.R.; Imdad, M. Fixed point results in partial symmetric spaces with an application. *Axioms* **2019**, *8*, 13. [CrossRef]
6. Asim, M.; Imdad, M. Partial JS-metric spaces and fixed fixed point results. *Indian J. Math.* **2019**, *61*, 175–186.
7. Asadi, M.; Karapınar, E.; Salimi, P. New extension of p-metric spaces with some fixed-point results on M-metric spaces. *J. Inequal. Appl.* **2014**, *2014*, 18. [CrossRef]
8. Mlaiki, N.; Zarrad, A.; Souayah, N.; Mukheimer, A. Thabet Abdeljawed Fixed point theorems in M_b-metric spaces. *J. Math. Anal.* **2016**, *7*, 1–9.
9. Mlaiki, N.; Yilmaz Özgür, N.; Mukheimer, A.; Taş, N. A new extension of the M_b-metric spaces. *J. Math. Anal.* **2018**, *9*, 118–133.
10. Özgür, N.Y.; Mlaiki, N.; Tas, N.; Souayah, N. A new generalization of metric spaces: Rectangular M-metric spaces. *Math. Sci.* **2018**, *12*, 223–233. [CrossRef]
11. Al-Rawashdeh, A.; Hassen, A.; Abdelbasset, F.; Sehmim, S.; Shatanawi, W. On common fixed points for $\alpha - F$-contractions and applications. *J. Nonlinear Sci. Appl.* **2016**, *9*, 3445–3458. [CrossRef]
12. Shatanawi, W.; Mustafa, Z.; Tahat, N. Some coincidence point theorems for nonlinear contraction in ordered metric spaces. *Fixed Point Theory Appl.* **2011**, *2011*, 68. [CrossRef]
13. Shatanawi, W. Some fixed point results for a generalized Ψ-weak contraction mappings in orbitally metric spaces. *Chaos Solitons Fractals* **2012**, *45*, 520–526. [CrossRef]
14. Czerwik, S. Contraction mappings in b-metric spaces. *Acta Math. Inform. Univ. Ostra.* **1993**, *1*, 5–11.
15. Bakhtin, I.A. The contraction mapping principle in almost metric spaces. *Funct. Anal.* **1989**, *30*, 26–37.
16. Kamran, T.; Samreen, M.; Ain, Q.U.L. A Generalization of b-metric space and some fixed point theorems. *Mathematics* **2017**, *5*, 19. [CrossRef]
17. Mlaiki, N.; Aydi, H.; Souayah, N.; Abdeljawad, T. Controlled metric-type spaces and the related contraction principle. *Mathematics* **2018**, *6*, 194. [CrossRef]
18. Abdeljawad, T.; Mlaiki, N.; Aydi, H.; Souayah, N. Double-Controlled metric-type spaces and some fixed point results. *Adv. Theory Nonlinear Anal. Appl.* **2021**, *5*, 320. [CrossRef]
19. Wilson, W.A. On quasi metric spaces. *Am. J. Math.* **1931**, *53*, 675–684. [CrossRef]
20. Rieger, R.Z.; Zimmer, J. Young measure flow as a model for damage. *Z. Angew. Math. Phys.* **2009**, *60*, 1–32. [CrossRef]
21. Mielke, A.; Roubcek, T. A rate-independent model for inelastic behavior of shape-memory alloys. *Multiscale Model. Simul.* **2003**, *1*, 571–597. [CrossRef]
22. Mennucci, A. On asymmetric distances. *Anal. Geom. Metr. Spaces* **2014**, *2*, 115–153. [CrossRef]
23. Woon, W.L.; Madnick, S. Asymmetric information distances for automated taxonomy construction. *Knowl. Inf. Syst.* **2009**, *21*, 91–111. [CrossRef]
24. Jleli, M.; Samet, B. Remarks on G-metric spaces and fixed point theorems. *Fixed Point Theory Appl.* **2012**, *2012*, 210. [CrossRef]
25. Chandok, S.; Manrao, S. Exitence of Fixed Points in Quasi Metric Spaces. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* **2020**, *69*, 266–275. [CrossRef]
26. Sharma, S.; Chandok, S. Existence of best proximity point in quasi partial metric space with an application. *J. Anal.* **2022**, *30*, 1–6. [CrossRef]

27. Sharma, R.K.; Chandok, S. The Generalized Hyperstability of General Linear Equation in Quasi-2-Banach Space. *Acta Math. Sci.* **2022**, *42*, 1357–1372. [CrossRef]
28. Shoaib, A.; Kazi, S.; Tassaddiq, A.; Alshoraify, S.S.; Rasham, T. Double-Controlled Quasi-metric-type Spaces and Some Results. *Complexity* **2020**, *2020*, 3734126. [CrossRef]
29. Matthews, S. Partial metric topology. *Ann. N. Y. Acad. Sci.* **1994**, *728*, 183–197 [CrossRef]
30. O'Neill, S.J. Partial metrics, valuations and domain theory. *Ann. N. Y. Acad. Sci.* **1996**, *806*, 304–315. [CrossRef]
31. Bukatin, M.A.; Scott, J.S. Towards computing distances between programs via Scott domains. In *Logical Foundations of Computer Sicence, Lecture Notes in Computer Science*; Adian S., Nerode, A., Eds.; Springer: Berlin/Heidelberg, Germany, 1997; Volume 1234, pp. 33–43.
32. Bukatin, M.A.; Shorina, S.Y. Partial metrics and co-continuous valuations. In *Foundations of Software Science and Computation Structures, Lecture Notes in Computer Science*; Nivat, M., Ed.; Springer: Berlin/Heidelberg, Germany, 1998; Volume 1378, pp. 33–43.
33. Escardo, M.H. PCF extended with real numbers. *Theor. Comput. Sci.* **1996**, *162*, 79–115. [CrossRef]
34. Schellekens, M. A characterization of partial metrizability: Domains are quantifiable. *Theoret. Comput. Sci.* **2003**, *305*, 409–432. [CrossRef]
35. Schellekens, M. The correpondence between partial metrics and semivaluations. *Theoret. Comput. Sci.* **2004**, *315*, 135–149. [CrossRef]
36. Waszkierwicz, P. Quantitative continuous domains. *Appl. Categor. Struct.* **2003**, *11*, 41–67. [CrossRef]
37. Waszkierwicz, P. The local triangle axiom in topology and domain theory. *Appl. Gen. Topol.* **2003**, *4*, 47–70. [CrossRef]
38. Asadi, M. Fixed point theorems for Meir-Keler mapping type in M-metric space with applications. *Fixed Point Theory Appl.* **2015**, *2015*, 210. [CrossRef]
39. Monfared, H.; Asadi, M.; Azhini, M. Coupled fixed point theorems for generalized contractions in ordered M-metric spaces. *Results Fixed Point Theory Appl.* **2018**, *2018*, 2018004. [CrossRef]
40. Monfared, H.; Azhini, M.; Asadi, M. Fixed point results on M-metric spaces. *J. Math. Anal.* **2016**, *7*, 85–101.
41. Shukla, S. Partial b-Metric Spaces and Fixed Point Theorems. *Mediterr. J. Math.* **2014**, *11*, 703–711. [CrossRef]
42. Vijayakumar, V.; Sooppy Nisar, K.; Chalishajar, D.; Shukla, A.; Malik, M.; Alsaadi, A.; Aldosary, S.F. A Note on Approximate Controllability of Fractional Semilinear Integrodifferential Control Systems via Resolvent Operators. *Fractal Fract.* **2022**, *6*, 73. [CrossRef]
43. Chandok, S.; Sharma, R.K.; Radenoviç, S. Multivalued problems via orthogonal contraction mappings with application to fractional differential equation. *J. Fixed Point Theory Appl.* **2021**, *23*, 14. [CrossRef]
44. Chandok, S.; Paunoviç, L.; Radenoviç, S. Incomplete Metric With The Application For Integral Equations of Volterra Type. *Math. Probl. Eng.* **2022**, *2022*, 5959903. [CrossRef]
45. Sharma, R.K.; Chandok, S. Well-posedness and Ulam's Stability of Functional Equations in $\mathcal{F}$-metric space with an application. *Filomat* **2022**, *36*, 16. [CrossRef]
46. Özgür, N.Y.; Taş, N. Some fixed-circle theorems on metric spaces. *Bull. Malays. Math. Sci. Soc.* **2019**, *42*, 1433–1449. [CrossRef]
47. Özgür, N.; Taş, N. Geometric properties of fixed points and simulation functions. *arXiv* **2021**, arXiv:2102.05417.
48. Mlaiki, N.; Taş, N.; Haque, S.; Rizk, D. Some Fixed-Disc Results in Double-Controlled Quasi-metric-type Spaces. *Fractal Fract.* **2022**, *6*, 107. [CrossRef]