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# On Relational Weak $\left(F_{\Re}^{m},\eta\right)$-Contractive Mappings and Their Applications

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**Abstract:** In this article, we introduce the concept of weak $\left(F_{\Re}^{m},\eta\right)$-contractions on relation-theoretic m-metric spaces and establish related fixed point theorems, where $\eta$ is a control function and $\Re$ is a relation. Then, we detail some fixed point results for cyclic-type weak $\left(F_{\Re}^{m},\eta\right)$-contraction mappings. Finally, we demonstrate some illustrative examples and discuss upper and lower solutions of Volterra-type integral equations of the form $\xi(\alpha) = \int_0^{\alpha} A(\alpha,\sigma,\xi(\sigma))m\sigma + \Psi(\alpha),\ \ \alpha \in [0,1]$.

**Keywords:** relation theoretic $M$-metric space; weak $\left(F_{\Re}^{m};\eta\right)$-contractions; integral equation; fixed point

**MSC:** 47H10; 54H25

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## 1. Introduction and Preliminaries

The classical Banach contraction theorem [1] is an important and fruitful tool in nonlinear analysis. In the past few decades, many authors have extended and generalized the Banach contraction mapping principle in several ways (see [2–12]). On the other hand, several authors, such as Boyd and Wong [13], Browder [14], Wardowski [15], Jleli and Samet [16], and many other researchers have extended the Banach contraction principle by employing different types of control functions (see [17–21] and the references therein). Alam et al. [22] introduced the concept of the relation-theoretic contraction principle and proved some well known fixed-point results in this area. Afterward, many researchers focused on fixed-point theorems in relation-theoretic metric spaces. Here, we will present some basic knowledge of relation-theoretic metric spaces (see more detail in [23–26]). Furthermore, Sawangsup et al. [27] introduced the concept of the $(F,\gamma)_{\Re}$-contractive of mappings to extend $F$-contractions in metric spaces endowed with binary relations. One of the latest extensions of metric spaces and partial metric spaces [10] was given in paper [28], which completed the concept of $m$-metric spaces. Using this concept, several researchers have proven some fixed point results in this area (see [20,29–33]). Subsequently, since every $F$-contraction mapping is contractive and also continuous, Secelean et al. [34] proved that the continuity of an $F$-contraction can be obtained from condition $F_2$. After that, Imdad et al. [35] introduced the idea of a new type of $F$-contraction by dropping the condition of $F_1$ and replacing condition $(F_3)$ with the continuity of $F$. They also proved some new fixed point results in relation to theoretic metric spaces.

In this paper, we introduce weak $\left(F_{\Re}^{m},\eta\right)$-contractive mappings and cyclic-type weak $\left(F_{\Re}^{m},\eta\right)$-contractions and provide some new fixed point theorems for such mappings in relation to theoretic m-metric spaces. Finally, as an application, we discuss the lower and upper solutions of Volterra-type integral equations.

Throughout this article, $\mathbb{N}$ indicates a set of all natural numbers, $\mathbb{R}$ indicates a set of real numbers and $\mathbb{R}^+$indicates a set of positive real numbers. We also denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Henceforth, $U$ will denote a non-empty set and the self mapping $\gamma : U \to U$ with a Picard sequence based on an arbitrary $\xi_0$ in $U$ is given by $\xi_n = \gamma(\xi_{n-1}) = \gamma^n(\xi_0)$, where all $n$ are members of $\mathbb{N}$ and $\gamma^n$ denotes the $n^{th}$-iteration of $\gamma$.

The notion of $m$-metric spaces was introduced by Asadi et al. [28] as a real generalization of a partial metric space and they supported their claim by providing some constructive examples. For more detail, see, e.g., [29,31].

**Definition 1** ([28])**.** *An m-metric space on a non-empty set $U$ is a mapping $m : U \times U \to \mathbb{R}^+$ such that for all $\xi, \Im, \aleph \in U$,*

(*i*) $\xi = \Im \Longleftrightarrow m(\xi, \xi) = m(\Im, \Im) = m(\xi, \Im)$ (*$T_0$-separation axiom*);
(*ii*) $m_{\xi\Im} \le m(\xi, \Im)$ (*minimum self distance axiom*);
(*iii*) $m(\xi, \Im) = m(\Im, \xi)$ (*symmetry*);
(*iv*) $m(\xi, \Im) - m_{\xi\Im} \le (m(\xi, \aleph) - m_{\xi\aleph}) + (m(\aleph, \Im) - m_{\aleph\Im})$ (*modified triangle inequality*)

*where*

$$\begin{aligned} m_{\xi\Im} &= \min\{m(\xi, \xi), m(\Im, \Im)\}; \\ M_{\xi\Im} &= \max\{m(\xi, \xi), m(\Im, \Im)\}. \end{aligned}$$

*The pair $(U, m)$ is called an m-metric space on nonempty $U$.*

**Lemma 1** ([28])**.** *Each partial metric forms an m-metric space but the converse is not true.*

Among the classical examples of an m-metric space is a pair $(U, m)$, where $U = \{\xi, \Im, \aleph\}$ and $m$ is a self mapping on $U$ given by $m(\xi, \xi) = 1$, $m(\Im, \Im) = 9$ and $m(\aleph, \aleph) = 5$. It is clear that $m$ is an m-metric space. Note that $m$ does not form a partial metric space.

Every m-metric space $m$ on $U$ generates a $T_0$ topology, e.g., $\tau_m$, on $U$ which is based on a collection of $m$-open balls:

$$\{B_m(\xi, \epsilon) : \xi \in U, \epsilon > 0\},$$

where

$$B_m(\xi, \epsilon) = \{\Im \in U : m(\xi, \Im) < m_{\xi\Im} + \epsilon\} \text{for all } \xi \in U,\ \varepsilon > 0.$$

If $m$ is an m-metric space on $U$, then the functions $m^w$ and $m^s : U \times U \to \mathbb{R}^+$ given by

$$m^w(\xi, \Im) = m(\xi, \Im) - 2m_{\xi\Im} + M_{\xi\Im},$$

$$m^s = \begin{cases} m(\xi, \Im) - m_{\xi\Im}, \text{if } \xi \neq \Im \\ 0, \text{if } \xi = \Im. \end{cases},$$

define ordinary metrics on $U$. It is easy to see that $m^w$ and $m^s$ are equivalent metrics on $U$.

**Definition 2** ([28])**.** *Let $\{\xi_n\}$ be a sequence in an m-metric space $(U, m)$, then*

(*i*) *$\{\xi_n\}$ is said to be convergent with respect to $\tau_m$ to $\xi$ if and only if*

$$\lim_{\mu \to \infty} (m(\xi_n, \xi) - m_{\xi_n\xi}) = 0. \textit{ for all } n \in \mathbb{N}.$$

(*ii*) *If $\lim_{n,m\to\infty}(m(\xi_n, \xi_m) - m_{\xi_n\xi_m})$ and $\lim_{n,m\to\infty}(M_{\xi_n,\xi_m} - m_{\xi_n\xi_m})$ for all $n, m \in \mathbb{N}$ exists and is finite, then the sequence $\{\xi_n\}$ in a m-metric space $(U, m)$ is m-Cauchy.*
(*iii*) *If every m-Cauchy $\{\xi_n\}$ in $U$ is m-convergent with respect to $\tau_m$ to $\xi$ in $U$ such that*

$$\lim_{n\to\infty} m(\xi_n, \xi) - m_{\xi_n\xi} = 0, \textit{ and } \lim_{n\to\infty} (M_{\xi_n,\xi} - m_{\xi_n\xi}) = 0. \textit{ for all } n \in \mathbb{N},$$

*then $(U, m)$ is said to be complete.*

$(iv)$ *$\{\xi_n\}$ is an m-Cauchy sequence if and only if it is a Cauchy sequence in the metric space $(U, m^w)$,*

$(v)$ *$(U, m)$ is M-complete if and only if $(U, m^w)$ is complete.*

Denote $\nabla(F)$ by the collection of all mappings $F : (0, \infty) \to R$ satisfying [15]:

$(F_1)$ $F(\xi) < F(\Im)$ for all $\xi < \Im$;

$(F_2)$ For each sequence $\{\xi_n\}$ of positive numbers

$$\lim_{n\to\infty} \xi_n = 0 \text{ if } \lim_{n\to\infty} F(\xi_n) = -\infty;$$

$(F_3)$ There exists $p \in (0, 1)$ such that $\lim_{n\to 0^+} \xi^p F(\xi) = 0$.

As in [27], we denote $\nabla(\rho)$and $\nabla(\pi)$ (where $\rho$ and $\pi$ are two new control functions) by the collection of all mappings $F : (0, \infty) \to R$, $\eta : (0, \infty) \to R$, respectively, satisfying:

$(F_2)$ For each sequence $\{\xi_n\}$ of positive numbers, $\lim_{n\to\infty} \xi_n = 0$ if $\lim_{n\to\infty} F(\xi_n) = -\infty$;

$(F_3)$ $F$ is lower semicontinuous;

$(\eta_1)$ For each sequence $\{\xi_n\}$ of positive numbers, $\lim_{n\to\infty} \xi_n = 0$ if $\lim_{n\to\infty} \eta(\xi_n) = -\infty$;

$(\eta_2)$ $\eta$ is right upper semicontinuous.

Now, we present some extensive examples of control functions in $\rho$ and $\eta$.

**Example 1.** *The following functions belong to $\nabla(\rho)$ and $\nabla(\pi)$*

$$(1)\ F_1(\xi) = \begin{cases} \frac{-1}{\xi}, & \text{if } \xi \in [3, \infty) \\ \frac{-1}{(\xi+1)}, & \text{if } \xi \in (3, \infty) \end{cases} \qquad (2)\ F_2(\xi) = \begin{cases} \frac{-1}{\xi} + \xi, & \text{if } \xi \in [2.8, \infty) \\ 2\xi - 3, & \text{if } \xi \in (3, \infty) \end{cases}$$
$$(3)\ \eta_1(\varpi) = \begin{cases} \frac{-1}{\xi}, & \text{if } \xi \in (0, 4.6) \\ \cos\xi, & \text{if } \xi \in [4.6, \infty) \end{cases} \qquad (4)\ \eta_2(s) = \begin{cases} \ln\left(\frac{\xi}{3} + \sin\xi\right), & \text{if } \xi \in (0, 3.2) \\ \sin\xi, & \text{if } \xi \in [3.2, \infty) \end{cases}.$$

Let $\Re = \{(\xi, \Im) \in U^2 : \xi, \Im \in U\}$ be a relation on $U$. If $(\xi, \Im) \in \Re$ then we say that $\xi \preceq \Im$ ($\xi$ precede $\Im$) under $\Re$ denoted by $\xi\Re\Im$, and the inverse of $\Re$ is denoted by $\Re^{-1} = \{(\xi, \Im) \in U^2 : (\Im, \xi) \in \Re\}$. The set $S = \Re \cup \Re^{-1} \subseteq U^2$ consequently illustrates another relation $S^*$ on $U$ given by $\xi S^* \Im \Leftrightarrow \Im S \xi$ with $\xi \neq \Im$.

As $(\gamma)_{Fix}$ denotes a set of all fixed points of $\gamma$, $\Theta([\Psi, S]) = \{\xi \in U : \xi S \gamma(\xi)\}$ and $F(\xi, \Im, \nabla)$ denotes the fashion of all paths in $\nabla$ from $\xi$ to $\Im$.

**Definition 3** ([22])**.** *Let $U \neq \phi$ and $\gamma : U \to U$, and $\Re$ is a binary relation on $U$. Then, $\Re$ is $\gamma$-closed if for any $\Omega, \Im \in U$,*

$$\xi\Re\Im \Rightarrow \gamma(\xi)\Re\gamma(\Im).$$

**Definition 4** ([22])**.** *Let $U \neq \phi$ and $\Re$ be a binary relation on $U$. Then, $\Re$ is transitive if $\xi\Re\aleph \in$ and $\aleph\Re\Im \Rightarrow \aleph\Re\Im$ for all $\xi, \Im, \aleph \in U$.*

**Definition 5** ([22])**.** *Let $\xi, \Im \in U$. A path of length $n \in \mathbb{N}$ in $\Re$: $\xi \to \Im$ is a finite sequence $\{t_0, t_1, t_2, \ldots, t_n\} \subseteq U$ such that*

$(i)$ *$t_0 = \xi$ and $t_n = \Im$;*

$(ii)$ *$(t_j, t_{j+1}) \in \Re$ for all $j$ in this set $\{0, 1, 2, \ldots, n-1\}$.*
*Consider that a class of all paths from $\xi$ to $\Im$ in $\Re$ is written as $\nabla(\xi, \Im, \Re)$. Note that a path of length $n$ involves $n + 1$ elements of $U$, although they are not necessarily distinct.*

**Definition 6** ([36])**.** *Let $(U, m)$ be a relation theoratic m-metric space endowed with binary relation $\Re$ on $U$, which is regular if for each sequences $\{\xi_n\}$ in $U$, we have*

$$\left.\begin{array}{c} \xi_n\Re\xi_{n+1} \text{ for all } n \in \mathbb{N} \\ \lim_{n\to\infty}(m(\xi_n, \xi) - m_{\xi_n\xi}) = 0 \text{ i.e., } \xi_n \stackrel{t_m}{\to} \xi \in \Re \end{array}\right\} \Rightarrow \xi_n\Re\xi \text{ for all } n \in \mathbb{N}.$$

**Definition 7** ([36]). *Let $(U, m)$ be a relation theoratic m-metric space endowed with binary relation $\Re$ on $U$. A sequence $\xi_n \in U$ is called $\Re$-preserving if $\xi_n \Re \xi_{n+1}$.*

**Definition 8** ([36]). *Let $(U, m)$ be a relation theoratic m-metric space endowed with binary relation $\Re$ on $U$, which is said to be $\Re$-complete if for each $\Re$-preserving m-Cauchy sequence $\{\xi_n\}$ in $U$, there exists some $\xi$ in $U$ such that*

$$\lim_{n\to\infty} m(\xi_n, \xi) - m_{\xi_n \xi} = 0, \text{ and } \lim_{n\to\infty} (M_{\xi_n, \xi} - m_{\xi_n \xi}) = 0.$$

**Definition 9** ([36]). *Let $U \neq \phi$ and $\gamma : U \to U$. Then, $\gamma$ is said to be $\Re$-continuous at $\xi$ if, for $\Re$-preserving sequence $\{\xi_n\}$ with $\xi_n \to \xi$, we have $\gamma(\xi_n) \to \gamma(\xi)$ as $\mu \to \infty$. $\gamma$ is $\Re$-continuous if it is $\Re$-continuous at each point of $U$.*

## 2. Weak $(F^m_{\Re}, \eta)$-Contractions

In this section, we introduce the concept of weak $(F^m_{\Re}, \eta)$-contraction relations and establish related fixed point theorems in relation theoretic m-metric space, where $\eta$ is a control function and $\Re$ is a relation. We begin with the following Lemma.

**Lemma 2.** *Assume that $(U, m)$ is an m-metric space and let $\{\xi_n\}$ be a sequence in $U$ such that $\lim_{n\to\infty} m(\xi_n, \xi_{n+1}) = 0$. If $\{\xi_n\}$ is not an m-Cauchy sequence in $U$, then there exists $\varepsilon > 0$ and two subsequences $\left\{\xi_{\alpha(\chi)}\right\}$ and $\left\{\xi_{\beta(\chi)}\right\}$ of positive integers such that $\{\alpha_\chi\} > \{\beta_\chi\} > \chi$ and the following sequences converges to $\varepsilon^+$ as $\chi$ converges to $+\infty$. With $M^*(\xi, \Im) = m(\xi, \Im) - m_{\xi\Im}$;*

$$\begin{aligned} &M^*\left(\xi_{\alpha(\chi)}, \xi_{\beta(\chi)}\right), M^*\left(\xi_{\alpha(\chi)}, \xi_{\beta(\chi)+1}\right), M^*\left(\xi_{\alpha(\chi)-1}, \xi_{\beta(\chi)}\right), \\ &M^*\left(\xi_{\beta(\chi)+1} \xi_{\beta(\chi)-1}\right), M^*\left(\xi_{\beta(\chi)+1}, \xi_{\beta(\chi)+1}\right). \end{aligned} \tag{1}$$

**Proof.** If $\{\xi_n\}$ is not an $m$-Cauchy sequence in $U$, there exists $\varepsilon > 0$ and two sequences $\{\alpha_\chi\}$ and $\{\beta_\chi\}$ of positive integers such that $\{\alpha_\chi\} > \{\beta_\chi\} > \chi$ and

$$M^*\left(\xi_{\alpha(\chi)}, \xi_{\beta(\chi)-1}\right) < \varepsilon, \; M^*\left(\xi_{\alpha(\chi)}, \xi_{\beta(\chi)}\right) \geq \varepsilon, \tag{2}$$

for all positive integers $\chi$. Using the triangle inequality of m-metric space, we obtain

$$\begin{aligned} \varepsilon &\leq M^*\left(\xi_{\alpha(\chi)}, \xi_{\beta(\chi)}\right) \leq M^*\left(\xi_{\alpha(\chi)}, \xi_{\beta(\chi)}\right) + M^*\left(\xi_{\alpha(\chi)-1}, \xi_{\beta(\chi)}\right) \\ &< M^*\left(\xi_{\alpha(\chi)}, \xi_{\beta(\chi)}\right) + \varepsilon. \end{aligned}$$

Thus,

$$\lim_{\chi\to\infty} M^*\left(\xi_{\alpha(\chi)}, \xi_{\beta(\chi)}\right) = \varepsilon,$$

which implies

$$\lim_{\chi\to\infty} \left(m\left(\xi_{\alpha(\chi)}, \xi_{\beta(\chi)}\right) - m_{\xi_{\alpha(\chi)}, \xi_{\beta(\chi)}}\right) = \varepsilon.$$

Furthermore,

$$\lim_{\chi\to\infty} m_{\xi_{\alpha(\chi)}, \xi_{\beta(\chi)}} = 0.$$

Hence,

$$\lim_{\chi\to\infty} m\left(\xi_{\alpha(\chi)}, \xi_{\beta(\chi)}\right) = \varepsilon. \tag{3}$$

Again, using the triangle inequality,

$$\begin{aligned} M^*\left(\xi_{\alpha(\chi)},\xi_{\beta(\chi)}\right) &\leq M^*\left(\xi_{\alpha(\chi)},\xi_{\beta(\chi)+1}\right)+M^*\left(\xi_{\alpha(\chi)+1},\xi_{\beta(\chi)+1}\right) \\ &\quad +M^*\left(\xi_{\alpha(\chi)+1},\xi_{\beta(\chi)}\right), \end{aligned}$$

and

$$\begin{aligned} M^*\left(\xi_{\alpha(\chi)+1},\xi_{\beta(\chi)+1}\right) &\leq M^*\left(\xi_{\alpha(\chi)},\xi_{\beta(\chi)+1}\right)+M^*\left(\xi_{\alpha(\chi)},\xi_{\beta(\chi)}\right) \\ &\quad +M^*\left(\xi_{\alpha(\chi)+1},\xi_{\beta(\chi)}\right). \end{aligned}$$

Taking $\chi \to +\infty$ in the above inequality and from (3), we have

$$\lim_{\chi\to\infty} M^*\left(\xi_{\alpha(\chi)+1},\xi_{\beta(\chi)+1}\right)=\varepsilon.$$

□

Now, we introduce the concept of weak $\left(F^m_{\Re},\eta\right)$-contractions.

**Definition 10.** *Given a relation theoretic m-metric space $(U,m)$ endowed with binary relation $\Re$ on $U$. Suppose*

$$\Xi=\{\xi S^*\Im : m(\xi,\Im)>0\}.$$

*We can say that a self mapping $\gamma : U \to U$ is a weak $\left(F^m_{\Re},\eta\right)$-contraction if there exists $F^m_{\Re}\in\nabla(\rho)$, $\eta\in\nabla(\pi)$ and*

$$\tau+F^m_{\Re}(m(\gamma(\xi),\gamma(\Im)))\leq \eta(m(\xi,\Im)), \tag{4}$$

*for all $(\xi,\Im)\in\Xi$.*

Our main result is demonstrated in the following.

**Theorem 1.** *Let $(U,m)$ be a complete relation theoretic m-metric space endowed with transitive binary relation $\Re$ on $U$, $\gamma : U \to U$, satisfying the following conditions:*

- $(i)$ $\Theta([\gamma,\Re])$ *is non-empty;*
- $(ii)$ $\Re$ *is $\gamma$-closed;*
- $(iii)$ $\gamma$ *is $\Re$-continuous;*
- $(iv)$ $\gamma$ *is a weak $\left(F^m_{\Re},\eta\right)$-contraction mapping with $F^m_{\Re}(\xi)>\eta(\xi)$ for all $\xi>0$ .*

Then, $\gamma$ possesses a fixed point in $U$.

**Proof.** Let $\xi_0\in\Theta([\gamma,\Re])$. Define a sequence $\{\xi_{n+1}\}$ in $U$ by $\xi_{n+1}=\gamma(\xi_n)=\gamma^{n+1}(\xi_0)$ for each $n\in\mathbb{N}$. If there exists a member $n_0$ of $\mathbb{N}$ such that $\gamma(\xi_{n_0})=\xi_{n_0}$, then $\gamma$ has a fixed point $\xi_{n_0}$ and the proof is complete. Let

$$\xi_{n+1}\neq\xi_n, \tag{5}$$

for all member $n$ of $\mathbb{N}$ such that $m(\xi_{n+1},\xi_n)>0$. Since $\gamma(\Omega_0)S^*\Omega_0$, and by the $\gamma$-closedness of $\Re$, $\Omega_{n+1}S^*\Omega_n$ for all $n\in\mathbb{N}$. Thus, $(\xi_n,\xi_{n+1})\in\Xi$ and from $(iv)$ we obtain

$$\begin{aligned} F^m_{\Re}(m(\xi_{n+1},\xi_n)) &= F^m_{\Re}(m(\gamma(\xi_n),\gamma(\xi_{n-1}))) \\ &\leq F^m_{\Re}(m(\xi_n,\xi_{n-1}))-\tau \end{aligned}$$

Let $\delta_n = m(\xi_n, \xi_{n+1})$ for all $n \in \mathbb{N}$. Then, $\delta_\mu > 0$ for all $n \in \mathbb{N}$, and using (5), one obtains

$$F^m_{\Re}(\delta_n) \leq (\delta_{n-1}) - \tau < F^m_{\Re}(\delta_{n-1}) - \tau \leq \eta(\delta_{n-2}) - 2\tau \leq \ldots \leq \eta(\delta_{n-2}) - n\tau.$$

From the above inequality, we obtain $\lim_{n\to\infty} F^m_{\Re}(\delta_n) = -\infty$. Then, by $(F_2)$, we have

$$\lim_{n\to\infty} \delta_n = 0. \tag{6}$$

From (3) and (6), we have $\xi_{n+1} \neq \xi_n$ for all $n, m \in \mathbb{N}$ with $n \neq m$. Now, we shall prove that $\{\xi_n\}$ is am $m$-Cauchy sequence in $(U, m)$. Assume, in contrast, that $\{\xi_n\}$ is not an $m$-Cauchy sequence. By Lemmas 2.1 and 2.6, there exists $\varepsilon > 0$ and two subsequences $\left\{\xi_{\alpha(\chi)}\right\}$ and $\left\{\xi_{\beta(\chi)}\right\}$ of $\{\xi_n\}$ such that $\left\{\xi_{\alpha(\chi)}\right\} > \left\{\xi_{\beta(\chi)}\right\} > \chi$ and

$$\begin{aligned}\lim_{\chi\to\infty} m\left(\xi_{\alpha(\chi)}, \xi_{\beta(\chi)}\right) &= \varepsilon \\ \lim_{\chi\to\infty} m\left(\xi_{\alpha(\chi)-1}, \xi_{\beta(\chi)-1}\right) &= \varepsilon.\end{aligned}$$

Since $\Re$ is a transitive relation, $\left(\xi_{\alpha(\chi)-1}, \xi_{\beta(\chi)-1}\right) \in \Re$. From condition $(iv)$, we have

$$\tau + F^m_{\Re}\left(m\left(\xi_{\alpha(\chi)}, \xi_{\beta(\chi)}\right)\right) \leq \eta\left(m\left(\xi_{\alpha(\chi)-1}, \xi_{\beta(\chi)-1}\right)\right)$$

and so

$$\begin{aligned}\tau + \lim_{\chi\to\infty}\inf F^m_{\Re}\left(m\left(\xi_{\alpha(\chi)}, \xi_{\beta(\chi)}\right)\right) &\leq \lim_{\chi\to\infty}\inf \eta\left(m\left(\xi_{\alpha(\chi)-1}, \xi_{\beta(\chi)-1}\right)\right) \\ &\leq \lim_{\chi\to\infty}\sup \eta\left(m\left(\xi_{\alpha(\chi)-1}, \xi_{\beta(\chi)-1}\right)\right).\end{aligned}$$

Thus,

$$\begin{aligned}\tau + F^m_{\Re}(\varepsilon^*) &\leq \eta(\varepsilon^*) \\ &< F^m_{\Re}(\varepsilon^*)\end{aligned}$$

is a contradiction; hence, $\{\xi_n\}$ is an $m$-Cauchy sequence in $(U, m)$. Since $(U, m)$ is $\Re$-complete, there exists $\xi^* \in U$ such that $\{\xi_\mu\}$ converges to $\xi^*$ with respect to $t_m$; that is, $m(\xi_n, \xi^*) - m_{\xi_n,\xi^*} \to 0$ as $n \to \infty$. Now, the $\Re$-continuity of $\gamma$ implies that

$$\xi = \lim_{n\to\infty} \xi_{n+1} = \lim_{n\to\infty} \gamma(\xi_n) = \gamma(\xi).$$

Therefore, $\xi$ is a fixed point of $\gamma$. □

**Example 2.** *Let $U = [0, \infty)$ and $m$ be a relation theoretic $m$-metric space defined by $m(\xi, \Im) = \frac{\xi+\Im}{2}$ for all $\xi, \Im \in U$. Then, $(U, m)$ is a complete $m$-metric space. Consider a sequence $\{\varpi_n\} \subseteq U$ given by $\varpi_n = \frac{n(n+1)(n+2)}{3}$ for all $\mu \in \mathbb{N}$. Set a binary relation $\Re$ on $U$ by $\Re=\{(1,1)\} \cup \{(1, \varpi_\Gamma) : \Gamma \in \mathbb{N}\} \cup \{(\varpi_\Gamma, \varpi_\Lambda) : \Gamma < \Lambda$ for each $\Gamma, \Lambda \in \mathbb{N}\}$. Define a mapping $\gamma : U \to U$ by*

$$\gamma(\xi) = \begin{cases} \xi, & \text{if } \xi \in [0,1] \\ ceil(\ln \xi), & \text{if } \xi \in [1, \varpi_1] \\ \left(\frac{\xi - \varpi_1}{\varpi_2 - \varpi_1}\right) + 1, & \text{if } \xi \in [\varpi_1, \varpi_2] \\ \frac{\varpi_{n-1}(\varpi_{n+1} - \xi) + \varpi_n(\xi - \varpi_n)}{\varpi_{n+1} - \varpi_n}, & \text{if } \xi \in \left[\varpi_n, \varpi_{\mu+1}\right] \text{ for all } n = 2, 3, \ldots 100. \end{cases}$$

*Obviously, $\Re$ is $\gamma$-closed and $\gamma$ is continuous. Define $F^m_{\Re}, \eta : (0,\infty) \to R$ by*

$$\begin{aligned} F^m_{\Re}(\varpi) &= \{\frac{-1}{\varpi} + \frac{4}{5}\varpi \quad \text{if } \varpi \in (0,1.1] \frac{-1}{\varpi} + \varpi \quad \text{if } \varpi \in (1.1,\infty) \text{ and} \\ \eta(\varpi) &= \{\frac{-1}{\varpi} + \frac{1}{3}\varpi \quad \text{if } \varpi \in (0,6.5) \frac{-2}{\varpi} + \varpi \quad \text{if } \varpi \in [6.5,\infty) \end{aligned}$$

*Now, we will show that $\gamma$ is a $(F^m_{\Re},\eta)$-contraction mapping. Assume that $(\xi,\Im) \in \Xi = \{\xi S^* \Im : m(\gamma(\xi),\gamma(\Im)) > 0\}$. Therefore, we will discuss four cases.*

***Case 1*** *If $\xi = 1$ and $\Im = \varpi_2$, then $m(\xi,\Im) = 4.5$ and $m(\gamma(\xi),\gamma(\Im)) = 1.5$,*

$$\begin{aligned} 2 + F^m_{\Re}(m(\gamma(\xi),\gamma(\Im))) &= 2 - \frac{1}{m(\gamma(\xi),\gamma(\Im))} + \frac{4}{5}m(\gamma(\xi),\gamma(\Im)) \\ &\leq -\frac{2}{m(\xi,\Im)} + m(\xi,\Im) = \eta(m(\xi,\Im)) \end{aligned}$$

***Case 2*** *If $\xi = 1$ and $\Im = \varpi_\Gamma$ for all $\Gamma > 2$, then $m(\xi,\Im) = \left|\frac{1+\varpi_\Gamma}{2}\right| \geq 10.5$ and $m(\gamma(\xi),\gamma(\Im)) = \left|\frac{1+\varpi_{\Gamma-1}}{2}\right| \geq 4.5$,*

$$\begin{aligned} 2\left|\frac{1+\varpi_{\Gamma-1}}{2}\right| - \left|\frac{1+\varpi_\Gamma}{2}\right| &< 2\left|\frac{1+\varpi_{\Gamma-1}}{2}\right| < \left|\frac{1+\varpi_\Gamma}{2}\right|\left|\frac{1+\varpi_{\Gamma-1}}{2}\right| \\ &< \left|\frac{1+\varpi_\Gamma}{2}\right|\left|\frac{1+\varpi_{\Gamma-1}}{2}\right|\left(\left|\frac{1+\varpi_\Gamma}{2}\right|\left|\frac{1+\varpi_{\Gamma-1}}{2}\right| - 2\right) \end{aligned}$$

*which implies*

$$2 + \frac{2}{\left|\frac{1+\varpi_\Gamma}{2}\right|} - \frac{1}{\left|\frac{1+\varpi_{\Gamma-1}}{2}\right|} \leq \left|\frac{1+\varpi_\Gamma}{2}\right| - \left|\frac{1+\varpi_{\Gamma-1}}{2}\right|,$$

*and thus,*

$$2 - \frac{1}{\left|\frac{1+\varpi_{\Gamma-1}}{2}\right|} - \left|\frac{1+\varpi_{\Gamma-1}}{2}\right| \leq -\frac{2}{\left|\frac{1+\varpi_p}{2}\right|} - \left|\frac{1+\varpi_\Gamma}{2}\right|.$$

*Then,*

$$\begin{aligned} 2 + F^m_{\Re}(m(\gamma(\xi),\gamma(\Im))) &= 2 - \frac{1}{m(\gamma(\xi),\gamma(\Im))} + m(\gamma(\xi),\gamma(\Im)) \\ &\leq -\frac{2}{m(\xi,\Im)} + m(\xi,\Im) = \eta(m(\xi,\Im)). \end{aligned}$$

***Case 3*** *If $\xi = \varpi_1$ and $\Im = \varpi_2$, then $m(\xi,\Im) = 5$ and $m(\gamma(\xi),\gamma(\Im)) = 1$,*

$$\begin{aligned} 2 + F^m_{\Re}(m(\gamma(\xi),\gamma(\Im))) &= 2 - \frac{1}{m(\gamma(\xi),\gamma(\Im))} + \frac{4}{5}m(\gamma(\Omega),\gamma(\Im)) \\ &\leq -\frac{2}{m(\xi,\Im)} + m(\xi,\Im) = \eta(m(\xi,\Im)). \end{aligned}$$

***Case 4*** *If $\xi = \varpi_\Gamma$ and $\Im = \varpi_\Lambda$ for all $\Gamma$ and $\Lambda$ in $\mathbb{N}$ and $(\Gamma,\Lambda)$ is not equal to $(1,2)$ with $\Gamma < \Lambda$, then $m(\xi,\Im) = \left|\frac{\varpi_\Gamma+\varpi_\Lambda}{2}\right| \geq 14$ and $m(\gamma(\xi),\gamma(\Im)) = \left|\frac{\varpi_{\Gamma-1}+\varpi_{\Lambda-1}}{2}\right| \geq 7$,*

$$\begin{aligned} 2\left|\frac{\varpi_{\Gamma-1}+\varpi_{\Gamma-1}}{2}\right| - \left|\frac{\varpi_\Gamma+\varpi_\Lambda}{2}\right| &< 2\left|\frac{\varpi_{\Gamma-1}+\varpi_{\Lambda-1}}{2}\right| < \left|\frac{\varpi_\Gamma+\varpi_\Lambda}{2}\right|\left|\frac{\varpi_{\Gamma-1}+\varpi_{\Lambda-1}}{2}\right| \\ &< \left|\frac{\varpi_\Gamma+\varpi_\Lambda}{2}\right|\left|\frac{\varpi_{\Gamma-1}+\varpi_{\Lambda-1}}{2}\right|\left(\left|\frac{\varpi_\Gamma+\varpi_\Lambda}{2}\right|\left|\frac{\varpi_{\Gamma-1}+\varpi_{\Lambda-1}}{2}\right| - 2\right), \end{aligned}$$

*which implies*

$$2+\frac{2}{\left|\frac{\varpi_{\Gamma}+\varpi_{\Lambda}}{2}\right|}-\frac{1}{\left|\frac{\varpi_{\Gamma-1}+\varpi_{\Lambda-1}}{2}\right|}\leq\left|\frac{\varpi_{\Gamma}+\varpi_{\Lambda}}{2}\right|-\left|\frac{\varpi_{\Gamma-1}+\varpi_{\Lambda-1}}{2}\right|.$$

*Then,*

$$2-\frac{1}{\left|\frac{\varpi_{\Gamma-1}+\varpi_{\Lambda-1}}{2}\right|}+\left|\frac{\varpi_{\Gamma-1}+\varpi_{\Lambda-1}}{2}\right|\leq-\frac{2}{\left|\frac{\varpi_{\Gamma}+\varpi_{\Lambda}}{2}\right|}+\frac{2}{\left|\frac{\varpi_{\Gamma}+\varpi_{\Lambda}}{2}\right|}.$$

*Hence,*

$$\begin{aligned}2+F^{m}_{\Re}(m(\gamma(\xi),\gamma(\Im))) &= 2-\frac{1}{m(\gamma(\xi),\gamma(\Im))}+m(\gamma(\xi),\gamma(\Im))\\ &\leq -\frac{2}{m(\xi,\Im)}+m(\xi,\Im)=\eta(m(\xi,\Im)).\end{aligned}$$

*Therefore, from all cases, we deduce that*

$$\tau+F^{m}_{\Re}(m(\gamma(\xi),\gamma(\Im)))\leq\eta(m(\xi,\Im)),$$

*for all $\xi,\Im\in\Xi$. Then, $\gamma$ is a weak $\left(F^{m}_{\Re},\eta\right)$-contraction mapping with $\tau=2$. Furthermore, there exists $\xi_0=1$ in $U$ such that $\Omega_0 S^*\gamma(\Omega_0)$ and the class $\Theta([\gamma,\Re])$ is non-empty. Thus, all conditions of Theorem 2.3 hold and $\gamma$ has a fixed point.*

**Theorem 2.** *Theorem 1 remains true if the condition $(ii)$ is replaced by the following:*
$(ii)'$ $(X,\kappa,\nabla)$ *is regular.*

**Proof.** Similar to the argument of Theorem 1 we will show the sequence $\{\xi_n\}$ is $m$-cauchy and converges to some $\xi$ in $U$ such that $m(\xi_n,\xi)-m_{\xi_n,\xi}$ as $n\to\infty$. Now,

$$\begin{aligned}\lim_{n\to\infty}m(\xi_n,\xi) &= \lim_{n\to\infty}m_{\xi_n,\xi}=\lim_{n\to\infty}\min\{m(\xi_n,\xi_n),m(\xi,\xi)\}=m(\xi,\xi)\\ &= \lim_{n,m\to\infty}m(\xi_n,\xi_m)=0\text{ and }\lim_{n,m\to\infty}m_{\xi_n,\xi_m}=0.\end{aligned}$$

As $\xi_n S^*\xi_{n+1}$, then $\xi_n S^*\xi$ for all $n\in\mathbb{N}$. Set $L=\{n\in\mathbb{N}:\gamma(\xi_n)=\gamma(\xi)\}$. We have two cases dependent on $L$.

**Case 1:** If $\{L$ is finite$\}$, then there exists $n_0\in\mathbb{N}$ such that $\gamma(\xi_n)\neq\gamma(\xi)$ for every $n\geq n_0$. Moreover, $\xi_n S^*\xi$ and $\gamma(\xi_n)S^*\gamma(\xi)$ for all $n\geq n_0$. Since $\gamma$ is a weak $\left(F^{m}_{\Re},\eta\right)$-contraction mapping, we have

$$\tau+F^{m}_{\Re}\left(m\left(\gamma\left(\xi_\mu\right),\gamma(\xi)\right)\right)\leq\eta\left(m\left(\xi_\mu,\xi\right)\right).$$

Since, $\lim_{n\to\infty}m(\xi_n,\xi)=0$,

$$\lim_{n\to\infty}F^{m}_{\Re}(m(\xi_n,\xi))=-\infty.$$

Hence,

$$\lim_{n\to\infty}F^{m}_{\Re}(m(\gamma(\xi_n),\gamma(\xi)))=-\infty.$$

Therefore, $\lim_{n\to\infty}m(\gamma(\xi_n),\gamma(\xi))=0$ and $\gamma(\xi)=\xi$, where $\xi$ is a fixed point of $\gamma$.

**Case 2:** If $\{\,L$ is infinite$\}$, then there exists a subsequence $\left\{\xi_{n_k}\right\}\subset\{\xi_n\}$ such that $\xi_{n_k+1}=\gamma\left(\xi_{n_k}\right)=\gamma(\xi)$ for all $k\in\mathbb{N}$. Thus, $\gamma\left(\xi_{n_k}\right)\to\gamma(\xi)$ with respect to $t_m$ as $\xi_n\to\xi$, then $\gamma(\xi)=\xi$, i.e., $\gamma$ has a fixed point. Hence, the proof is complete. □

Now, we discuss various results to ensure the uniqueness of the fixed points:

**Theorem 3.** *If $\digamma(\xi, \Im, \nabla) \neq \phi$ for all $\xi, \Im \in (\gamma)_{Fix}$ in Theorem 1 and Theorem 2, then $\gamma$ possesses a unique fixed point.*

**Proof.** Let $\xi, \Im \in$Fix$(\gamma)$ such that $\xi \neq \Im$. Since $\digamma(\xi, \Im, \nabla) \neq \phi$, then there exists a path $(\{a_0, a_1, \ldots a_n\})$ of some finite length $\mu$ in $\nabla$ from $\xi$ to $\Im$ (with $a_s \neq a_{s+1}$ for all $s \in [0, p-1]$). Then, $a_0 = \xi$, $a_k = \Im$, $a_s S^* a_{s+1}$ for every $s \in [0, p-1]$. As $a_s \in \gamma(U)$, $\gamma(a_s) = a_s$ for all $s \in [0, p-1]$ and since $F^m_\eta(\xi) > \eta(\xi)$, we obtain

$$F^m_R(m(a_s, a_{s+1})) = F^m_{\Re}(m(\gamma(a_s), \gamma(a_{s+1}))) \leq \eta(m(a_s, a_{s+1}))$$

Since $F^m_{\Re}(a) > \eta(a)$ for all $a > 0$,

$$F^m_{\Re}(m(a_s, a_{s+1})) < F^m_{\Re}(m(a_s, a_{s+1})).$$

Hence, $\gamma$ possesses a unique fixed point. □

**Theorem 4.** *Let $(U, m)$ be a complete relation theoretic m-metric space endowed with a transitive binary relation $\Re$ on $U$. Let $\gamma : U \to U$ satisfy the following:*

$(i)$ *The class $\Theta([\gamma, \Re])$ is nonempty;*
$(ii)$ *The binary relation $\Re$ is $\gamma$-closed;*
$(iii)$ *The mapping $\gamma$ is $\Re$-continuous;*
$(iv)$ *There exists $F^m_{\Re} \in \nabla(\rho)$, $\eta \in \nabla(\pi)$ and $\xi > 0$ such that*

$$\tau + F^m_{\Re}\Big(\kappa\Big(m(\xi), \gamma^2(\xi)\Big)\Big) \leq \eta(m(\xi, \gamma(\xi)))$$

*for all $\xi \in U$, with $\gamma(\xi) S^* \gamma^2(\xi)$ and $F^m_\eta(\xi) > \eta(\xi)$ for all $\xi > 0$.*

Then, $\gamma$ has a fixed point.
Furthermore, if the following conditions are satisfied:

$(v)$ $(iv)^{'}$
$(vi)$ $\xi \in (\gamma^n)_{\text{Fix}}$ (for some $n \in \mathbb{N}$) which implies that $\xi S^* \gamma(\xi)$.

Then, $(\gamma^n)_{\text{Fix}} = (\gamma)_{\text{Fix}}$ for each $n$ is a member of $\mathbb{N}$.

**Proof.** Let $\xi_0 \in \Theta([\gamma, \Re])$, i.e., $\xi_0 S^* \gamma(\xi_0)$, then, from $(ii)$, we obtain $\xi_n S^* \xi_{n+1}$ for each $n \in \mathbb{N}$. Denote $\xi_{n+1} = \gamma(\xi_n) = \gamma^{n+1}(\xi_0)$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $\gamma(\xi_{n_0}) = \xi_{n_0}$, then $\gamma$ has a fixed point $\xi_{n_0}$. Now, assume that

$$\xi_{n+1} \neq \xi_n, \tag{7}$$

for every $n \in \mathbb{N}$. Then, $\xi_n S^* \xi_{n+1}$(for all $n \in \mathbb{N}$). Continuing this process and from $(iv)$ we have,

$$F^m_{\Re}\Big(m\Big(\gamma(\xi_{n-1}), \gamma^2(\xi_{n-1})\Big)\Big) \leq F^m_{\Re}(m(\xi_{n-1}, \gamma(\xi_{n-1}))) \leq m(\xi_{n-1}, \xi_n) - \tau,$$

for all $n \in \mathbb{N}$, which implies,

$$\begin{aligned} F^m_{\Re}(m(\xi_n, \xi_{n+1})) &\leq \eta(m(\xi_{n-1}, \xi_n)) - \tau \\ &< F^m_{\Re}(m(\xi_{n-2}, \xi_{n-1})) - \tau \\ &\leq \eta(m(\xi_{n-1}, \xi_n)) - 2\tau \\ &\ldots \\ &\leq \eta(m(\xi_0, \xi_1)) - n\tau. \end{aligned}$$

Setting $n \to \infty$ in the above inequality, we deduce that $\lim_{n\to\infty} F_{\Re}^{m}(m(\xi_n, \xi_{n+1})) = -\infty$. Since $F_{\Re}^{m} \in \nabla(\rho)$, then

$$\lim_{n\to\infty} m(\xi_n, \xi_{n+1}) = 0. \tag{8}$$

From conditions (7) and (8), we have $\xi_{n+1} \neq \xi_n$ for all $n, m \in \mathbb{N}$ with $n \neq m$. Now, we will prove that $\{\xi_n\}$ is an $m$-Cauchy sequence in $(U, m)$. Assume, in contrast, that $\{\xi_n\}$ is not an $m$-Cauchy sequence; then, by Lemma 2 and (6) , there exists $\varepsilon > 0$ and two subsequences $\left\{\xi_{\alpha(\chi)}\right\}$ and $\left\{\xi_{\beta(\chi)}\right\}$ of $\{\xi_n\}$ such that $\{\alpha(\chi)\} > \{\beta(\chi)\} > \chi$ and

$$\begin{aligned} \lim_{\chi\to\infty} m\left(\xi_{\alpha(\chi)}, \xi_{\beta(\chi)}\right) &= \varepsilon \text{ and} \\ \lim_{\chi\to\infty} m\left(\xi_{\alpha(\chi)-1}, \xi_{\beta(\chi)-1}\right) &= \varepsilon. \end{aligned}$$

Since $\Re$ is a transitive relation, $\left(\xi_{\alpha(\chi)-1}, \xi_{\beta(\chi)-1}\right) \in \Re$. From condition $(iv)$,

$$\tau + F_{\Re}^{m}\left(m\left(\xi_{\alpha(\chi)}, \xi_{\beta(\chi)}\right)\right) \leq \eta\left(m\left(\xi_{\alpha(\chi)-1}, \xi_{\beta(\chi)-1}\right)\right)$$

and hence,

$$\begin{aligned} \tau + \lim_{\chi\to\infty} \inf F_{\Re}^{m}\left(m\left(\xi_{\alpha(\chi)}, \xi_{\beta(\chi)}\right)\right) &\leq \lim_{\chi\to\infty} \inf \eta\left(m\left(\xi_{\alpha(\chi)-1}, \xi_{\beta(\chi)-1}\right)\right) \\ &\leq \lim_{\chi\to\infty} \sup \eta\left(m\left(\xi_{\alpha(\chi)-1}, \xi_{\beta(\chi)-1}\right)\right). \end{aligned}$$

Then,

$$\begin{aligned} \tau + F_{\Re}^{m}(\varepsilon^*) &\leq \eta(\varepsilon^*) \\ &< F_{\Re}^{m}(\varepsilon^*) \end{aligned}$$

it is contradiction. Hence, $\{\xi_n\}$ is an $m$-Cauchy sequence in $(U, m)$. Since $(U, m)$ is $\Re$-complete, there exists $\xi \in U$ such that $\{\xi_n\}$ converges to $\xi^*$ with respect to $t_m$; that is, $m(\xi_n, \xi^*) - m_{\xi_n,\xi^*} \to 0$ as $n \to \infty$. By using the $\Re$-continuity of $\gamma$,

$$\xi = \lim_{n\to\infty} \xi_{n+1} = \lim_{n\to\infty} \gamma(\xi_n) = \gamma(\xi).$$

Finally, we will prove that $(\gamma^n)_{Fix} = (\gamma)_{Fix}$ where $n \in \mathbb{N}$. Assume, in contrast, that $\xi \in (\gamma^n)_{Fix}$ and $\xi \notin (\gamma)_{Fix}$ for some $n \in \mathbb{N}$. Then, from condition $(iv)'$, $m(\xi, \gamma(\xi)) > 0$ and $\xi S^* \gamma(\xi)$ . Using $(ii)$ and $(iv)$, we obtain $\gamma^n(\xi) S^* \gamma^{n+1}(\xi)$ for all $n \in \mathbb{N}$,

$$\begin{aligned} F_{\Re}^{m}(m(\xi, \gamma(\xi))) &= F_{\Re}^{m}\left(m\left(\gamma\left(\gamma^{n-1}(\xi)\right), \gamma^2\left(\gamma^{n-1}(\xi)\right)\right)\right) \leq \eta\left(m\left(\gamma\left(\gamma^{n-1}(\xi)\right), \gamma^2\left(\gamma^{n-1}(\xi)\right)\right)\right) - \tau \\ &< F_{\Re}^{m}\left(m\left(\gamma^{n-1}(\xi)\right), \gamma^n(\xi)\right) - \tau \\ &\leq \eta\left(m\left(\gamma^{n-2}(\xi)\right), \gamma^{n-1}(\xi)\right) - 2\tau \\ &< F_{\Re}^{m}\left(m\left(\gamma^{n-2}(\xi)\right), \gamma^{n-1}(\xi)\right) - 2\tau \\ &\leq \eta\left(m\left(\gamma^{n-3}(\xi)\right), \gamma^{n-2}(\xi)\right) - 3\tau \\ &\quad \cdots \\ &\leq \eta(m(\xi, \gamma(\xi))) - n\tau \end{aligned}$$

Taking $n \to \infty$ in the above inequality, we obtain

$$F_{\Re}^{m}(m(\xi, \gamma(\xi))) = -\infty$$

as a contradiction. Therefore, $(\gamma^n)_{Fix} = (\gamma)_{Fix}$ for any $n \in \mathbb{N}$. □

## 3. Cyclic-Type Weak $\left(F_{\Re}^{m}, \eta\right)$-Contraction Mappings

In 2003, Kirk et al. [37] introduced cyclic contractions in metric spaces and investigated the existence of proximity points and fixed points for cyclic contraction mappings. Inspired by [37] and our Theorems 1 and 5 we obtained the following fixed point results for cyclic-type weak $\left(F_{\Re}^{m}, \eta\right)$-contraction mappings.

**Theorem 5** ([37] **).** *Assume that $(U, m)$ is a compete m-metric space and $G$, $H$ are two non-empty closed subsets of $U$ and $\gamma : U \to U$. Suppose that the following conditions hold:*

$(i)$ $\gamma(B) \subseteq D$ *and* $\gamma(D) \subseteq B$;
$(ii)$ *There exists a constant* $k \in (0, 1)$ *such that*

$$m(\gamma(\xi), \gamma(\Im)) \leq km(\xi, \Im) \text{ for all } \xi \in B,\ \Im \in D. \tag{9}$$

*Then, $B \cap D$ is non-empty and $\xi$ in $B \cap D$ is a fixed point of $\gamma$.*

**Theorem 6.** *Let $(U, m)$ be a complete relation theoretic m-metric space endowed with a transitive binary relation $\Re$ on $U$, $G$ and $H$ are two non-empty closed subsets of $U$ and $\gamma : U \to U$. Assume that the following axioms hold:*

$(i)$ $\gamma(G) \subseteq H$ *and* $\gamma(H) \subseteq G$;
$(ii)$ *There exists* $F_{\Re}^{m} \in \nabla(\rho)$ *and* $\eta \in \nabla(\pi)$ *and* $\xi > 0$ *such that*

$$\tau + F_{\Re}^{m}(m(\gamma(\xi), \gamma(\Im))) \leq \eta(m(\xi, \Im)) \tag{10}$$

*for all $\xi$ in $G$, $\Im$ in $H$, with $F_{\eta}^{m}(\xi) > \eta(\xi)$ for all $\xi > 0$.*

Then, $\xi^* \in Z = G \cup H$ is a fixed point of $\gamma$. Moreover, $\xi \in B \cap D$.

**Proof.** From $(i)$, $Z = G \cup H$ is closed, so $Z$ is a closed subspace of $U$. Therefore, $(U, m)$ is a complete m-metric space. Set the a binary relation $\Re$ on $Z$ by

$$\Re = G \times H.$$

This implies that

$$\xi \Re \Im \in \Leftrightarrow (\xi, \Im) \in B \times D \text{ for all } \xi, \Im \in Z.$$

The set $S = \Re \cup \Re^{-1}$ is an asymmetric relation. Directly, we set $(U, m, S)$ as regular. Let $\{\xi_n\} \in Z$ be any sequence and $\xi \in Z$ be a point such that

$$\xi_n S \xi_{n+1} \text{ for all } n \in \mathbb{N}$$

and

$$\lim_{n\to\infty} m(\xi_n, \xi) = \lim_{n\to\infty} \min\{m(\xi_n, \xi_n), m(\xi, \xi)\} = m(\xi, \xi).$$

Using the definition of $S$, we have

$$(\xi_n, \xi_{n+1}) \in (B \times D) \cup (D \times B) \text{ for all } n \in \mathbb{N} \tag{11}$$

Immediately, we obtain the product of $Z \times Z$ in the m-metric space $m$ as

$$m((\xi_1, \Im_1), (\xi_2, \Im_2)) = \frac{m(\xi_1, \Im_1) + m(\xi_2, \Im_2)}{2}.$$

Since $(U, m)$ is a complete m-metric space, $(Z \times Z, m)$ is complete. Furthermore, $G \times H$ and $H \times G$ are close in $(Z \times Z, m)$ because $G$ and $H$ are closed in $(U, m)$. Applying the limit $n \to \infty$ to (11), we have $(\xi, \Im) \in (B \times D) \cup (D \times B)$. This implies that $\xi \in B \cap D$. Furthermore, from (11), we have $\xi_n \in B \cup D$. Thus, we obtain $\xi_n S^* \xi$ for all $n \in \mathbb{N}$. Therefore, our theorem is proven. Furthermore, since $\gamma$ is self mapping, from condition $(i)$, for all $\xi, \Im \in U$, we obtain

$$\begin{aligned} (\xi, \Im) \text{ in } G \times H &\Rightarrow (\gamma(\xi), \gamma(\Im)) \in H \times G \\ (\xi, \Im) \text{ in } H \times G &\Rightarrow (\gamma(\xi), \gamma(\Im)) \in G \times H. \end{aligned}$$

The binary relation $\Re$ is $\gamma$-closed, and as $B \neq \phi$, there exists $\xi_0 \in B$ such that $\gamma(\xi_0) \in D$, i.e., $\xi_0 S^* \gamma(\xi_0)$. Therefore, all the hypotheses of Theorem (2.8) are satisfied. Hence, $(\gamma)_{Fix} \neq \phi$ and also $(\gamma)_{Fix} \subseteq B \cap D$. Finally, as $\xi S^* \Im$ for all $\xi, \Im \in G \cap H$. Hence, $G \cap H$ is $\nabla$-directed. Hence, all conditions of Theorem 3 are satisfied and $\gamma$ has a unique fixed point. □

## 4. Application

In this section, we study existence of a solution for a Volterra-type integral equation by using Theorem 2.6. Consider the following Volterra-type integral equation:

$$\xi(\alpha) = \int_0^{\alpha} A(\alpha, \sigma, \xi(\sigma)) m\sigma + \Psi(\alpha), \ \ \alpha \in [0, 1], \tag{12}$$

where $A : [0,1] \times [0,1] \times [0,1] \to [0,1]$ and $\Psi : [0,1] \to [0,1]$. Consider the Banach contraction $\delta = C([0,1],[0,1])$ of all continuous functions $\xi : [0,1] \to [0,1]$ equipped with norm $\|\xi\| = \max_{0 \leq \alpha \leq 1} |\xi(\alpha)|$. Define an m-metric space $m$ on $\delta$ by $m(\xi, \Im) = \left\| \frac{\xi + \Im}{2} \right\|$ for each $\xi, \Im$ in $\delta$. Then $(\delta, m)$ is a complete m-metric space.

**Definition 11.** *Lower and upper solutions of (9) are functions $\Lambda$ and $\Theta$ in Banach space $\delta$, respectively, such that*

$$\Lambda(\alpha) \leq \int_0^{\alpha} A(\alpha, \sigma, \xi(\sigma)) \kappa\sigma + \Psi(\alpha) \text{ and } \Theta(\alpha) \geq \int_0^{\alpha} A(\alpha, \sigma, \xi(\sigma)) m\sigma + \Psi(\alpha), \ \alpha \in [0,1]$$

In this section, we prove the existence and unique solution to the Volterra-type integral Equation (12).

**Theorem 7.** *Consider Volterra-type integral Equation (12). Assume that there is a positive real number $\tau$ such that*

$$\left| \frac{A(\alpha, \sigma, \xi) + A(\alpha, \sigma, \Im)}{2} \right| \leq \left| \frac{\xi + \Im}{2} \right| e^{-\frac{1}{\left[1 + \left| \frac{\Omega + \Im}{2} \right| \right]} - \tau}, \tag{13}$$

*for all $\alpha, \sigma$ in $[0,1]$ and $\xi, \Im$ in $\delta$. if (12) has a lower solution, then a solution exists for the integral Equation (12).*

**Proof.** We define an operator $\gamma : \delta \to \delta$, $F^m_{\Re}, \eta : R^+ \to R$ by

$$\gamma(\xi(\alpha)) = \int_0^{\alpha} A(\alpha, \sigma, \xi(\sigma)) m\sigma + \Psi(\alpha), \ \ \xi \in \delta,$$

$$\eta(\varpi) = \ln \varpi - \frac{1}{[1 + \varpi]}$$

and

$$F^m_{\Re}(\varpi) = \ln \varpi$$

for all $\varpi \in R^+$, $F^m_{\Re} \in \nabla(\rho)$ and $\eta \in \nabla(\pi)$, respectively. We can verify easily that $\gamma$ is well defined and $\preceq$ on $\Re$ is $\gamma$-closed. Note that $\xi$ is a fixed point of $\gamma$ if and only if there is a solution to (12). Now, we want to prove that $\gamma$ is a $F^m_{\Re}$-contraction mapping with $\eta$. Let

$$(\xi, \Im) \in \Xi = \{\xi S^* \Im : m(\xi, \Im) > 0, \text{ where } m \text{ is Banach space } \},$$

which implies that $\xi \preceq \Im$. Since $\Re$ is $\gamma$-closed, then $\gamma(\xi) \preceq \gamma(\Im)$,

$$\begin{aligned}
\left|\frac{\gamma(\xi(\alpha))+\gamma(\Im(\alpha))}{2}\right| &= \left|\frac{\int_0^\alpha A(\alpha,\sigma,\xi(\sigma))m\sigma+\Psi(\alpha)+\int_0^\alpha A(\alpha,\sigma,\Im(\sigma))m\sigma+\Psi(\alpha)}{2}\right| \\
&\quad \left|\frac{\int_0^\alpha A(\alpha,\sigma,\xi(\sigma))m\sigma+\Psi(\alpha)+\int_0^\alpha A(\alpha,\sigma,\Im(\sigma))m\sigma+\Psi(\alpha)}{2}\right| \\
&\le \int_0^\alpha \left|\frac{\xi+\Im}{2}\right| e^{-\frac{1}{\left[1+\left\|\frac{\xi+\Im}{2}\right\|\right]}-\tau} \\
&\le \int_0^\alpha \left|\frac{\xi+\Im}{2}\right| e^{-\frac{1}{\left[1+\left\|\frac{\xi+\Im}{2}\right\|\right]}-\tau} \\
&\le \int_0^\alpha \max_{\alpha\in[0,1]}\left|\frac{\xi+\Im}{2}\right| e^{-\frac{1}{\left[1+\left\|\frac{\xi+\Im}{2}\right\|\right]}-\tau} \\
&\le \left\|\frac{\xi+\Im}{2}\right\| e^{-\frac{1}{\left[1+\left\|\frac{\xi+\Im}{2}\right\|\right]}-\tau},
\end{aligned}$$

and so

$$\left|\frac{\gamma(\xi(\alpha))+\gamma(\Im(\alpha))}{2}\right| \le \left\|\frac{\xi+\Im}{2}\right\| e^{-\frac{1}{\left[1+\left\|\frac{\xi+\Im}{2}\right\|\right]}-\tau}.$$

Taking the supremum norm on both sides, we have

$$\left\|\frac{\gamma(\xi(\alpha))+\gamma(\Im(\alpha))}{2}\right\| \le \left\|\frac{\xi+\Im}{2}\right\| e^{-\frac{1}{\left[1+\left\|\frac{\xi+\Im}{2}\right\|\right]}-\tau}.$$

This implies that

$$\ln\left(\left\|\frac{\gamma(\xi(\alpha))+\gamma(\Im(\alpha))}{2}\right\|\right) \le \ln\left(\left\|\frac{\xi+\Im}{2}\right\| e^{-\frac{1}{\left[1+\left\|\frac{\xi+\Im}{2}\right\|\right]}-\tau}\right),$$

then

$$\ln\left(\left\|\frac{\gamma(\xi(\alpha))+\gamma(\Im(\alpha))}{2}\right\|\right) = \ln\left(\left\|\frac{\xi+\Im}{2}\right\|\right) - \frac{1}{\left[1+\left\|\frac{\xi+\Im}{2}\right\|\right]} - \tau.$$

Consequently,

$$\tau + F^m_\Re\left(\left\|\frac{\gamma(\xi)+\gamma(\Im)}{2}\right\|_{tr}\right) \le \eta\left(\left\|\frac{\xi+\Im}{2}\right\|_{tr}\right).$$

Thus,

$$\tau + F^m_\Re(m(\gamma(\xi),\gamma(\Im))) \le \eta(m(\xi,\Im)).$$

Therefore, $\gamma$ is an $(F^m_R,\eta)$-contraction and thus, Inequality (4) holds. Since $\{\xi_\mu\}$ is an $\Re$-preserving sequence $\{\xi_n\}$ in $Z([0,1])$ such that $\xi_n$ converges with respect to $t_m$ to $\xi$ for some $\xi$ in $Z([0,1])$, we obtain

$$\xi_0(\alpha) \preceq \xi_1(\alpha) \preceq \xi_2(\alpha) \preceq \ldots \preceq \xi_n(\alpha) \preceq \xi_{n+1}(\alpha) \preceq \ldots,$$

for all $\alpha \in [0,1]$. Which implies,

$$\xi_n(\alpha) \preceq \xi(\alpha) \text{ for all } \alpha \in [0,1].$$

Thus, $\xi, \Im \in (\gamma)_{Fix}$. Then, $\aleph = \max\{\xi, \Im\} \in Z([0,1])$, and thus $\xi \preceq \aleph$, $\Im \preceq \aleph$, $\xi S^* \aleph$ and $\Im S^* \aleph$. Hence, all axioms of Theorem 3 hold and the integral Equation (12) has a solution. □

**Theorem 8.** *Consider Volterra-type integral Equation (12). Assume that A is non-decreasing in the third variables; then, there is positive real number $\tau$ such that*

$$\left| \frac{A(\alpha,\sigma,\xi) + A(\alpha,\sigma,\Im)}{2} \right| \leq \left| \frac{\xi + \Im}{2} \right| e^{-\frac{1}{\left[1+\left|\frac{\xi+\Im}{2}\right|\right]} - \tau},$$

*for all $\alpha, \sigma$ in $[0,1]$ and $\xi, \Im$ in $\delta$. If (12) has an upper solution, then a solution exists for the integral Equation (12).*

**Proof.** Define a binary relation on Banach space as follows

$$(\xi, \Im) \in \Xi = \{\xi S^* \Im \text{ with } \alpha(\xi) \succeq \alpha(\Im) \ : m(\xi, \Im) > 0, \text{ where } m \text{ is a Banach space}\}.$$

Now, due to the proof of the above Theorem, then all conditions of Theorem 8 and integral Equation (12) have unique solutions. □

**Example 3.** *Assume that a function*

$$\xi(\alpha) = \frac{\alpha}{2}, \textit{ for all } \alpha \textit{ in } [0,1]$$

*is a solution of Equation (12)*

$$\xi(\alpha) = \frac{3}{2}(\alpha) - (1+\alpha)\ln(1+\alpha) + \int_0^\alpha \ln(1+\xi(\sigma)) m\sigma, \textit{ for all } \alpha \textit{ in } [0,1]. \tag{14}$$

**Proof.** Let $\gamma$ be a self operator from $\delta$ to $\delta$, which is given by

$$\gamma(\xi(\alpha)) = \frac{3}{2}(\alpha) - (1+\alpha)\ln(1+\alpha) + \int_0^\alpha \ln(1+\xi(\sigma)) m\sigma, \text{ for all } \alpha \text{ in } [0,1].$$

Now, we take $\tau \in [0.0091, \infty)$,

$$A(\alpha,\sigma,\xi) = \ln(1+\xi(\sigma))$$

and

$$\Psi(\alpha) = \frac{3}{2}(\alpha) - (1+\alpha)\ln(1+\alpha).$$

Observe that given function $A(\alpha,\sigma,\xi) = \ln(1+\xi(\sigma))$ in the third variable is non-decreasing and that $\frac{\alpha}{2} \leq \frac{3}{2}(\alpha) - (1+\alpha)\ln(1+\alpha) + \int_0^\sigma \ln(1+\xi(\sigma)) m\sigma$ for all $\alpha$ in $[0,1]$ such that $\xi(\alpha) = \frac{\alpha}{2}$ is a lower solution of (16), then the following below inequality holds,

$$\left| \frac{A(\alpha,\sigma,\xi) + A(\alpha,\sigma,\Im)}{2} \right| \leq \left| \frac{\xi + \Im}{2} \right| e^{-\frac{1}{\left[1+\left|\frac{\xi+\Im}{2}\right|\right]} - \tau}. \tag{15}$$

Now, from the non-decreasing function $\alpha \mapsto e^{-\frac{1}{\left[1+\left|\frac{\alpha}{2}\right|\right]} - 0.091}$, we have

$$\left| \frac{\ln(1+\xi) + \ln(1+\Im)}{2} \right| \leq \left| \frac{\xi + \Im}{2} \right| e^{-\frac{1}{\left[1+\left|\frac{\xi+\Im}{2}\right|\right]} - 0.091}.$$

Hence, all conditions of Theorem 7 hold and the integral Equation (12) has a unique solution $\xi(\alpha) = \frac{\alpha}{2}$ for all $\alpha$ in $[0,1]$. □

**Example 4.** *Assume that a function*

$$\xi(\alpha) = \alpha, \textit{ for all } \alpha \in [0,1]$$

*is a solution of Equation (12):*

$$\xi(\sigma) = \alpha - (1-\alpha)\ln(2-\alpha) - \ln(2) + \int_0^\alpha \ln(2-\xi(\sigma))m\sigma, \textit{ for all } \alpha \textit{ in } [0,1]. \tag{16}$$

**Proof.** In view of the above example, the following below inequality holds for all $\xi, \Im$ in $[0,1]$ and $\tau = 0.091$

$$\left|\frac{\ln(2-\xi)+\ln(2-\Im)}{2}\right| \leq \left|\frac{\xi+\Im}{2}\right| e^{-\frac{1}{\left[1+\left|\frac{\xi+\Im}{2}\right|\right]}-\tau}.$$

Using the arguments of the above example, we can say that the all conditions of Theorem 8 hold. Hence, the integral Equation (12) has a unique solution $\xi(\alpha) = \alpha$ for all $\alpha$ in $[0,1]$. □

Finally, we give an example different to the above example and others given in the literature [38] which satisfies all conditions of Theorem 15.

**Example 5.** *Assume that a function*

$$\xi(\alpha) = \frac{1}{3}\alpha, \textit{ for all } \alpha \textit{ in } [0,1]$$

*is a solution of Equation (12):*

$$\xi(\alpha) = \frac{5}{3}\alpha - \frac{\alpha}{1+\alpha} + \int_0^\alpha \left(\frac{\xi(\sigma)}{1+\xi(\sigma)}\right)m\sigma, \textit{ for all } \alpha \textit{ in } [0,1]. \tag{17}$$

**Proof.** Let $\gamma$ be a self operator from $\delta$ to $\delta$, which is given by

$$\gamma(\xi(\alpha)) = \frac{5}{3}\alpha - \frac{\alpha}{1+\alpha} + \int_0^\alpha \left(\frac{\xi(\sigma)}{1+\xi(\sigma)}\right)m\sigma, \text{ for all } \alpha \text{ in } [0,1].$$

Now, we take $\tau \in [0.091, \infty)$,

$$A(\alpha,\sigma,\xi) = \frac{\xi(\sigma)}{1+\xi(\sigma)}$$

and

$$\Psi(\alpha) = \frac{5}{3}\alpha - \frac{\alpha}{1+\alpha}.$$

Observe that given the function $A(\alpha,\sigma,\xi) = \frac{\xi(\sigma)}{1+\xi(\sigma)}$ in the third variable is non-decreasing and that $\frac{1}{3}\alpha \leq \frac{5}{3}\alpha - \frac{\alpha}{1+\alpha} + \int_0^\alpha \left(\frac{\xi(\sigma)}{1+\xi(\sigma)}\right)m\sigma$ for all $\alpha$ in $[0,1]$ such that $\xi(\alpha) = \frac{1}{3}\alpha$ is a lower solution of (16), then the following below inequality holds:

$$\left|\frac{A(\alpha,\sigma,\xi)+A(\alpha,\sigma,\Im)}{2}\right| \leq \left|\frac{\xi+\Im}{2}\right| e^{-\frac{1}{\left[1+\left|\frac{\xi+\Im}{2}\right|\right]}-\tau}. \tag{18}$$

Now, from the non-decreasing function $\alpha \mapsto e^{-\frac{1}{\left[1+\left|\frac{\alpha}{2}\right|\right]}-0.9}$, we have

$$\left|\frac{\frac{\xi}{1+\xi}+\frac{\Im}{1+\Im}}{2}\right| \leq \left|\frac{\xi+\Im}{2}\right| e^{-\frac{1}{\left[1+\left|\frac{\xi+\Im}{2}\right|\right]}-0.9}.$$

Hence, all axioms of Theorem 7 hold and the integral Equation (12) has a unique solution $\xi(\alpha) = \frac{\alpha}{3}$ for all $\alpha$ in $[0,1]$. □

**Example 6.** *Assume that a function*

$$\xi(\alpha) = \frac{3}{5}\alpha + \frac{1}{3}, \textit{ for all } \alpha \in [0,1]$$

*is a solution of Equation (12):*

$$\xi(\sigma) = \frac{3}{5}\alpha + \frac{1}{3} - (1-\alpha)(2-\alpha) + 2 + \int_0^\alpha (1+\xi(\sigma))m\sigma, \textit{ for all } \alpha \textit{ in } [0,1]. \quad (19)$$

**Proof.** In view of the above example, the following below inequality holds for all $\xi, \Im$ in $[0,1]$ and $\tau = 0.9$

$$\left|\frac{1+\xi+1+\Im}{2}\right| \leq \left|\frac{\xi+\Im}{2}\right| e^{-\frac{1}{\left[1+\left|\frac{\xi+\Im}{2}\right|\right]}-\tau}.$$

Using the arguments of the above example, we can say that the all conditions of Theorem 7 hold. Hence, the integral Equation (12) has a unique solution $\xi(\alpha) = \frac{3}{5}\alpha + \frac{1}{3}$ for all $\alpha$ in $[0,1]$. □

## 5. Conclusions

In this article, we have introduced the notion of weak $\left(F^m_{\Re}, \eta\right)$-contractions and proved related fixed point theorems in relation theoretic m-metric space endowed with a relation $\Re$ using a control function $\eta$. Examples and applications to Volterra-type integral equations are given to validate our main results. Analogously, such results can be extended to generalized distance spaces (such as symmetric spaces, $m_b m$-spaces, $rmm$-spaces, $rm_b m$-spaces, $pm$-spaces and $p_b m$-spaces) endowed with relations.

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