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Characteristics of Solitary Stochastic Structures for Heisenberg Ferromagnetic Spin Chain Equation

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Abstract: The impact of Stratonovich integrals on the solutions of the Heisenberg ferromagnetic spin chain equation using the unified solver approach is examined in this study. In particular, using arbitrary parameters, the traveling wave arrangements of rational, trigonometric, and hyperbolic functions are developed. The detailed arrangements are exceptionally critical for clarifying diverse complex wonders in plasma material science, optical fiber, quantum mechanics, super liquids and so on. Here, the Itô stochastic calculus and the Stratonovich stochastic calculus are considered. To describe the dynamic behaviour of random solutions, some graphical representations for these solutions are described with appropriate parameters.

Keywords: Heisenberg spin chain; unified solver technique; stochastic traveling wave solutions; physical applications

MSC: 34A05; 35A20; 35C05; 35Q51; 35Q62; 35Q80

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1. Introduction

Understanding the dynamical wave patterns for nonlinear partial differential equations (NPDEs) is crucial for understanding the underlying workings of complex phenomena [1–6]. In mathematical physics, the NPDEs and symmetry are closely related concepts. Particularly, symmetry plays a significant role in the study of NPDEs. Furthermore, symmetry approaches have recently been developed to get unique reduced solutions for NPDEs [7]. Vinogradov proposed four distinct strategies for studying NPDE symmetries [8]. The waves generated numerous nonlinear scientific phenomena that have applications in many domains, including chemical physics, molecular biology, fluid mechanics, engineering, solid-state physics, ecology, nuclear physics, quantum mechanics [9–14]. Numerous scholars used NPDEs to develop voyaging wave arrangements by employing a few methods [15–17]. There is not a single method to solve all of these equations due to the complexity of nonlinear waves. The random impact on the soliton solutions' spread has received an increasing amount of consideration recently. This effect is crucial in explaining many complicated issues.

Stochastic calculus is an area of mathematics that deals with stochastic processes, allowing the modeling of random systems [18,19]. The foundation of many stochastic processes is a continuous function that is not differentiable. Because differential equations requiring the use of derivative terms cannot be defined on non-smooth functions, they are excluded. Where integral equations do not require the direct definition of derivative terms, a theory of integration is required. The theory is known as Itô calculus and Stratonovich calculus in quantitative finance [20].

The Black–Scholes model, which models the random movement of an asset price, is the main application of stochastic calculus in finance. The Weiner process employs the physical process of Brownian motion (specifically, a geometric Brownian motion) as a model of asset prices. This process is represented by a stochastic differential equation, which is actually an integral equation despite its name [21,22].

A system is made up of multiple elements working in concert to accomplish a specific goal. It is referred to as a dynamical system if the output time-varying variables of this system depend on the initial conditions and some input variables. Modeling is the process of creating suitable mathematical equations that adequately describe the dynamical system. If the differential equations can be easily derived from engineering and physical circumstances, it is desirable to understand the solution behaviour [23,24]. Environmental and stochastic influences can affect nonlinear systems. Therefore, rather than studying these systems deterministically, it is best to do so stochastically. Additionally, it should be mentioned that each perturbation that occurs in these systems should be considered, because neglecting this perturbation could have negative consequences.

Heisenberg ferromagnet models are crucial in the modern magnet theory [25]. It describes the nonlinear dynamics of magnets. One of the intriguing types of nonlinear excitations that depict spin dynamics in semiclassical continuous Heisenberg systems is the magnetic soliton. The study of soliton propagation and interaction may aid in the analysis of magnetic materials' nonlinear properties. Soliton is a wave that maintains its speed and shape as it travels. That is the main reason why it piques the interest of engineers, mathematicians and physicists. Besides the deterministic type perturbations, the stochastic type perturbations do have to be taken into account from practical considerations. All these features served as strong motivation for our new approach.

A branch of mathematics called stochastic calculus works with stochastic processes. It enables the definition of an integration theory that is consistent for integrals of stochastic processes relative to stochastic processes. A new integrable nonlinear Schrodinger equation (NLSE) in the dimension of the form (2 + 1) was recently discovered to regulate the nonlinear spin dynamics of the Heisenberg ferromagnetic spin chain (HFSC) equation [25–28]

$$i\Xi_t + \mu_1\,\Xi_{xx} + \mu_2\,\Xi_{yy} + \mu_3\,\Xi_{xy} - \mu_4\mid\Xi\mid^2\Xi + i\sigma\Xi\circ B_t = 0\,,\quad i = \sqrt{-1}, \tag{1}$$

where $\mu_1 = \gamma^4(v + v_2)$, $\mu_2 = \gamma^4(v_1 + v_2)$, $\mu_3 = 2\gamma^4\,v_2$, $\mu_4 = 2\gamma^4\,A$. Here, the complex-valued function $\Xi(x, y, t)$ denotes the wave propagation, γ is a lattice parameter, v and v_1 are the coefficients of bilinear exchange interactions along the x- and y-directions. The neighboring interaction on the diagonal is represented by v_2, whereas A denoted the uniaxial crystal field anisotropy parameter [25]. The noise B_t is the time derivative of a standard Wiener process $B(t)$. The standard Wiener process is also known as Brownian motion. The parameter v_2 denotes the random neighboring interaction along the diagonal. The parameter A represents the uniaxial crystal field anisotropy.

In this paper, we present some new stochastic solutions for Equation (1) via a Stratonovich sense, using the unified solver technique [29] based on He's variations technique [30–32]. This technique provides some types of wave solutions based on the physical parameters. These solutions enable critical applications in modern magnet theory [25]. The proposed technique can be used as a box-solver for a number of natural science systems. It eliminates time-consuming computations and offers critical solutions in an explicit format. This solver is straightforward, robust, functional, and convenient. To the best of our knowledge, no one has ever used the proposed method to solve Equation (1) through a Stratonovich sense.

The remainder of the article is structured as follows. Section 2 introduces the some notes about stochastic calculus. In Section 3, random solutions for the nonlinear spin dynamics of (2 + 1)-dimensional HFSC are shown. In Section 4, we discuss the obtained results. Finally, concluding remarks are reported in Section 5.

2. On the Interpretation of Stochastic Calculus

2.1. Itô Integral

Since almost all sample functions of B_t are of unbounded variation, we cannot, in general, interpret the integral:

$$\int_a^b G(s)dB(s)$$

as an ordinary Riemann–Stieltjes integral.

The task is now to define the stochastic integral:

Definition 1 ([33]). *For every $G \in L^2[a, b]$, the stochastic integral (or Itô's integral) of G with respect to the Wiener process $B(t)$ over the interval [a, b] with mean square limits is defined as:*

$$\int_a^b G(s)dB(s) = \lim_{n\to\infty} \int_a^b G_n dB,$$

where G_n is a sequence of step functions in $L^2[a, b]$ that approximate in the sense of:

$$\lim_{n\to\infty} \int_a^b |G(s) - G_n(s)|^2 ds = 0.$$

Definition 2 ([33]). *If $\lim_{n\to\infty} Y_n = Y$ exist, then the r.v. Y is called the Itô stochastic integral, or Itô integral for short, of $X(t)$ with respect to $B(t)$ over the interval T. It is denoted by:*

$$\int_a^b X(t)dB(t) = \lim_{n\to\infty} Y_n.$$

2.2. Stratonovich Integral

The Itô integral's most popular substitute is the Stratonovich. The Stratonovich integral is commonly employed in physics, despite the Itô integral typically being the preferred option in practical mathematics. The crucial characteristic of the Itô integral, which does not "see into the future", is absent from the Stratonovich integral. Since past events are the sole information available in various real-world applications, like stock price modelling, the Itô interpretation makes more sense [20]. Typically, the Itô interpretation is applied in financial mathematics. Similar to the Riemann integral, which is defined as a limit of Riemann sums, the Stratonovich integral can be defined as follows:

$$\int_a^b X(t)dB(t) = \lim_{n\to\infty} \frac{X(t_{i+1}) + X(t_i)}{2}(B(t_{i+1}) - B(t_i)).$$

The Stratonovich and Itô variants of the stochastic integral are two commonly used variants. The modelling problem primarily determines which form is appropriate; however, once that form is selected, an equivalent equation of the other type can be created utilizing the same solutions. In many sources, such as Refs. [20,21], the following relationship is utilized to switch between Stratonovich (represented by $\int_0^t \Phi \circ d\eta$) and Itô (represented by $\int_0^t \Phi d\eta$):

$$\int_0^t \Phi(\tau, Z_\tau)d\eta(\tau) = \int_0^t \Phi(\tau, Z_\tau) \circ d\eta(\tau) - \frac{1}{2}\int_0^t \Phi(\tau, Z_\tau)\frac{\partial\Phi(\tau, Z_\tau)}{\partial\tau}d\tau, \tag{2}$$

where Φ is considered to be sufficiently regular and $\{Z_t, t \geq 0\}$ is a stochastic process.

3. The Stochastic Solutions

Let the following complex wave transformation be [25]:

$$\Xi(x,y,t) = e^{i(qx+py-wt)-\sigma^2 t-\sigma B(t)}u(\zeta), \quad \zeta = \alpha_1\, x + \alpha_2\, y - \nu\, t. \tag{3}$$

Here, $u(\zeta)$ denotes the amplitude function to be determined. The parameters q and p denote, respectively, the wave numbers in the x- and y-directions, w is frequency of the pulse and ν is the group velocity of the wave packet.

From the wave transformation (3) we find that Ξ_{xx}, Ξ_{xx} and Ξ_{xx} take the form:

$$\begin{aligned} \Xi_{xx} &= [\alpha_1^2 u'' - q^2 u + 2i\alpha_1 q u']e^{i(qx+py-w)-\sigma^2 t-\sigma B(t)} \\ \Xi_{yy} &= [\alpha_2^2 u'' - p^2 u + 2i\alpha_1 p u']e^{i(qx+py-w)-\sigma^2 t-\sigma B(t)} \\ \Xi_{xy} &= [\alpha_1\alpha_2 u'' - pqu + i(\alpha_1 p + \alpha_2 q)u']e^{i(qx+py-w)-\sigma^2 t-\sigma B(t)}. \end{aligned} \tag{4}$$

For Ξ_t, we find that:

$$\Xi_t = [-\nu u' - iwu + \tfrac{1}{2}\sigma^2 u - \sigma^2 u - \sigma B_t u]e^{i(qx+py-wt)-\sigma^2 t-\sigma B(t)}, \tag{5}$$

where $\frac{1}{2}\sigma^2 u$ is the correction term. Thus,

$$\Xi_t = [-(\nu u' + iwu) - \sigma u(B_t + \frac{1}{2}\sigma)]e^{i(qx+py-w)-\sigma^2 t-\sigma B(t)}. \tag{6}$$

Using the relation to swab from Itô to Stratonovich (2) the above equation will be

$$\begin{aligned} \Xi_t &= -[(\nu u' + iwu) + \sigma u \circ B_t]e^{i(qx+py-wt)-\sigma^2 t-\sigma B(t)} \\ &= -(\nu u' + iwu)e^{i(qx+py-wt)-\sigma^2 t-\sigma B(t)} - \sigma\Xi \circ B_t. \end{aligned} \tag{7}$$

Inserting Equations (3) and (4) into Equation (1) and then decomposing the result into real and imaginary parts, gives:

$$Lu'' + Mu^3 + Nu + i[\nu - (2\alpha_1\mu_1 + \alpha_2\mu_3)q - (2\alpha_2\mu_2 + \alpha_1\mu_3)p] = 0, \tag{8}$$

where

$$L = \mu_1\alpha_1^2 + \mu_2\alpha_2^2 + \alpha_1\alpha_2\mu_3, M = -\mu_4, N = w - (\mu_1 q^2 + \mu_2 p^2 + \mu_1\mu_2 pq). \tag{9}$$

The real part is displayed as:

$$Lu'' + Mu^3 + Nu = 0, \tag{10}$$

and the imaginary part is written as

$$\nu - (2\alpha_1\mu_1 + \alpha_2\mu_3)q - (2\alpha_2\mu_2 + \alpha_1\mu_3)p = 0. \tag{11}$$

In view of the unified solver method [29], the random solutions of Equation (1) are:
Family I:

$$\begin{aligned} u_{1,2}(x,y,t) &= \pm\sqrt{\frac{-2N}{M}}\,\mathrm{sech}\left(\pm\sqrt{-\frac{N}{L}}(\alpha_1\, x + \alpha_2\, y - \nu\, t)\right) \\ &= \pm\sqrt{\frac{2w - 2(\mu_1 q^2 + \mu_2 p^2 + \mu_1\mu_2 pq)}{\mu_4}} \\ &\quad \mathrm{sech}\left(\pm\sqrt{\frac{(\mu_1 q^2 + \mu_2 p^2 + \mu_1\mu_2 pq) - w}{\mu_1\alpha_1^2 + \mu_2\alpha_2^2 + \alpha_1\alpha_2\mu_3}}(\alpha_1\, x + \alpha_2\, y - \nu\, t)\right). \end{aligned} \tag{12}$$

Therefore, the solutions for Equation (1) are

$$\Xi_{1,2}(x,y,t) = \pm\sqrt{\frac{2w-2(\mu_1 q^2+\mu_2 p^2+\mu_1\mu_2 pq)}{\mu_4}}\, e^{i(qx+py-wt)-\sigma^2 t-\sigma B(t)} \operatorname{sech}\left(\pm\sqrt{\frac{(\mu_1 q^2+\mu_2 p^2+\mu_1\mu_2 pq)-w}{\mu_1\alpha_1^2+\mu_2\alpha_2^2+\alpha_1\alpha_2\mu_3}}(\alpha_1\, x+\alpha_2\, y-\nu\, t)\right). \quad (13)$$

Family II:

$$\begin{aligned} u_{3,4}(x,y,t) &= \pm\sqrt{\frac{-35\,N}{18\,M}}\operatorname{sech}^2\left(\pm\sqrt{-\frac{5\,N}{12\,L}}\,(\alpha_1\, x+\alpha_2\, y-\nu\, t)\right) \\ &= \pm\sqrt{\frac{35w-35(\mu_1 q^2+\mu_2 p^2+\mu_1\mu_2 pq)}{18\mu_4}} \operatorname{sech}^2\left(\pm\sqrt{\frac{5(\mu_1 q^2+\mu_2 p^2+\mu_1\mu_2 pq)-5w}{12(\mu_1\alpha_1^2+\mu_2\alpha_2^2+\alpha_1\alpha_2\mu_3)}}\,(\alpha_1 x+\alpha_2 y-\nu\, t)\right). \end{aligned} \quad (14)$$

Therefore, the solutions for Equation (1) are

$$\Xi_{3,4}(x,y,t) = \pm\sqrt{\frac{35w-35(\mu_1 q^2+\mu_2 p^2+\mu_1\mu_2 pq)}{18\,\mu_4}}\, e^{i(qx+py-wt)-\sigma^2 t-\sigma B(t)} \operatorname{sech}^2\left(\pm\sqrt{\frac{5(\mu_1 q^2+\mu_2 p^2+\mu_1\mu_2 pq)-5w}{12\,(\mu_1\alpha_1^2+\mu_2\alpha_2^2+\alpha_1\alpha_2\mu_3)}}\,(\alpha_1\, x+\alpha_2\, y-\nu\, t)\right). \quad (15)$$

Family III:

$$\begin{aligned} u_{5,6}(x,y,t) &= \pm\sqrt{\frac{-N}{M}}\tanh\left(\pm\sqrt{\frac{N}{2L}}(\alpha_1\, x+\alpha_2\, y-\nu\, t)\right) \\ &= \pm\sqrt{\frac{w-(\mu_1 q^2+\mu_2 p^2+\mu_1\mu_2 pq)}{\mu_4}} \tanh\left(\pm\sqrt{\frac{w-(\mu_1 q^2+\mu_2 p^2+\mu_1\mu_2 pq)}{2(\mu_1\alpha_1^2+\mu_2\alpha_2^2+\alpha_1\alpha_2\mu_3)}}(\alpha_1\, x+\alpha_2\, y-\nu\, t)\right). \end{aligned} \quad (16)$$

Therefore, the solutions for Equation (1) are

$$\Xi_{5,6}(x,y,t) = \pm\sqrt{\frac{w-(\mu_1 q^2+\mu_2 p^2+\mu_1\mu_2 pq)}{\mu_4}}\, e^{i(qx+py-wt)-\sigma^2 t-\sigma B(t)} \tanh\left(\pm\sqrt{\frac{w-(\mu_1 q^2+\mu_2 p^2+\mu_1\mu_2 pq)}{2(\mu_1\alpha_1^2+\mu_2\alpha_2^2+\alpha_1\alpha_2\mu_3)}}(\alpha_1\, x+\alpha_2\, y-\nu\, t)\right). \quad (17)$$

4. Results and Discussion

The (2 + 1)-dimensional HFSC equation having a noise term with Brownian function $B(t)$ is inspected. Many real random phenomena are better suited to Brownian motion than other processes. This stochastic HFSC model is converted to nonlinear ordinary differential equations via $B(t)$ function. We extracted some vital stochastic solutions for the (2 + 1)-dimensional HFSC equation via a Stratonovich sense. These solutions based on Brownian motion processes produce vital applications in the modern theory of magnets and eternal inflation in physical cosmology.

We describe the effects of the rigorous randomness factor on the structure, band width and amplitude of the provided solitary waves. When σ increases, we noticed that for the width, the wave's amplitude shrink and the wave starts to collapse, which is completed at $\sigma = 2$, as illustrated in Figure 1. In a stochastic case, the Brownian motion function $B(t)$ is given in more detail in [34]. The ability of the abrupt wave collapse which depends in the main on the influence of randomness is grow with increasing time t, as depicted in Figures 2–5. Furthermore, the dark solution (17), that represents the dissipative graph, was identified to be affected by time t and the noise term σ, as illustrated in Figures 4–6. Furthermore, as shown in Figure 6, the parameter σ causes the wave to compress and transform into a super waveform with a limited amplitude. On the other hand, we provide some 3D graphs of solution (17) in Figures 7–10 for more illustration about the collapsing of the amplitude for the waves. Namely, by increasing the value of the noise term σ, the waves' amplitude are more collapsed.

Substantially, the examined model's stochastic nonlinear solitonic structure with stochastic noise term caused the dynamical advantages of the isolated envelopes and dissipative–dispersive waves that were produced.

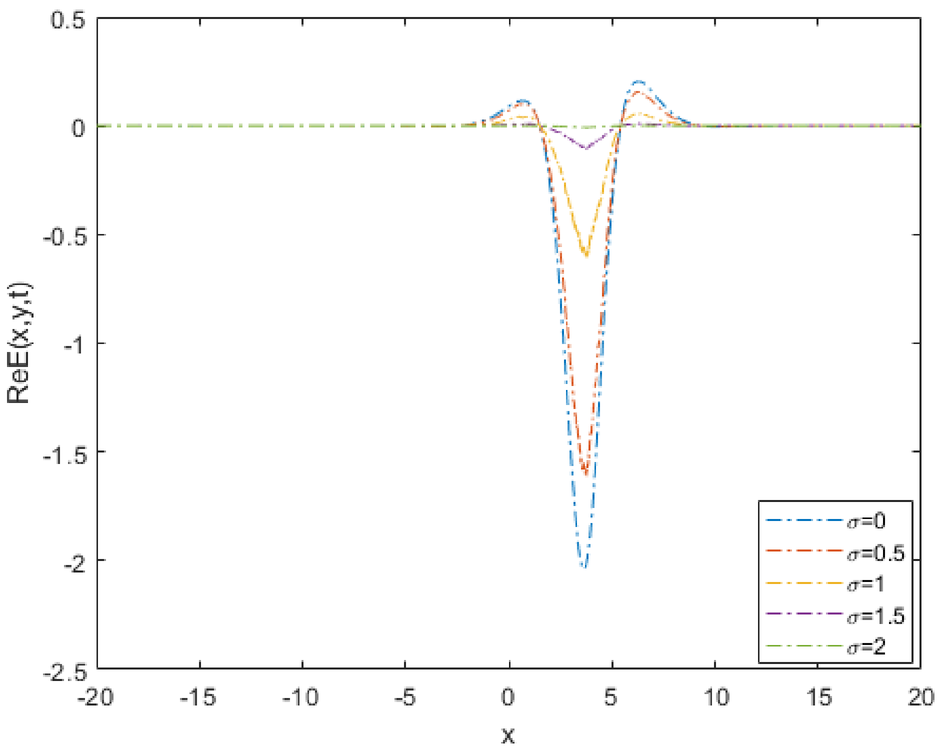

Figure 1. Plot of $\Xi_1(x, y, t)$ with $x, \sigma = 0, 0.5, 1, 1.5, 2$.

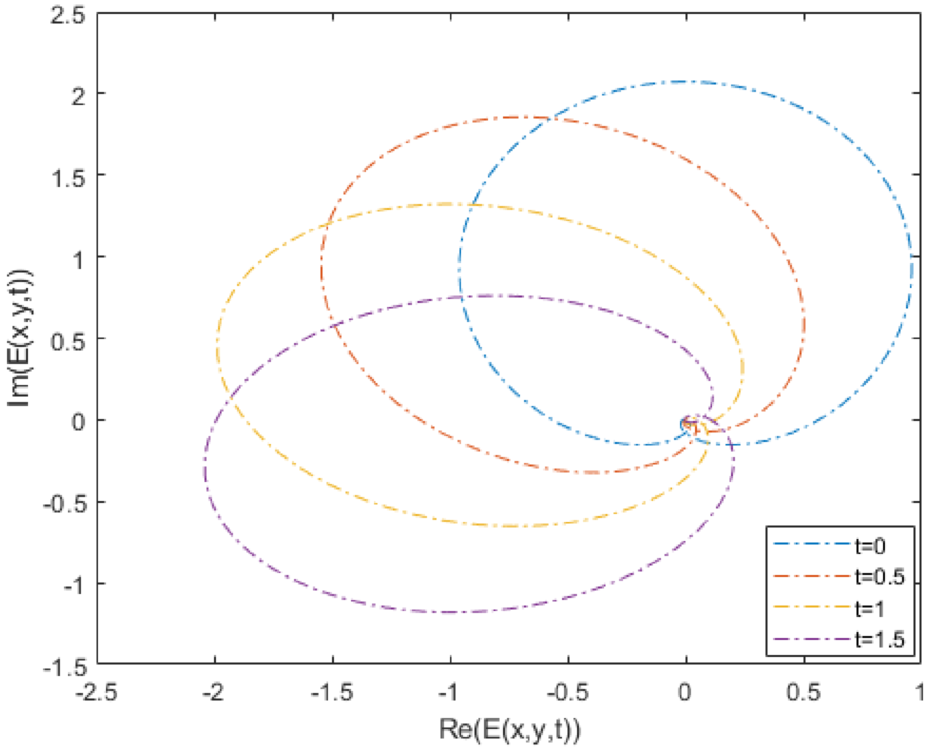

Figure 2. Trajectory of $\Xi_1(x, y, t)$, $\sigma = 0$ for $t = 0, 0.5, 1, 1.5$.

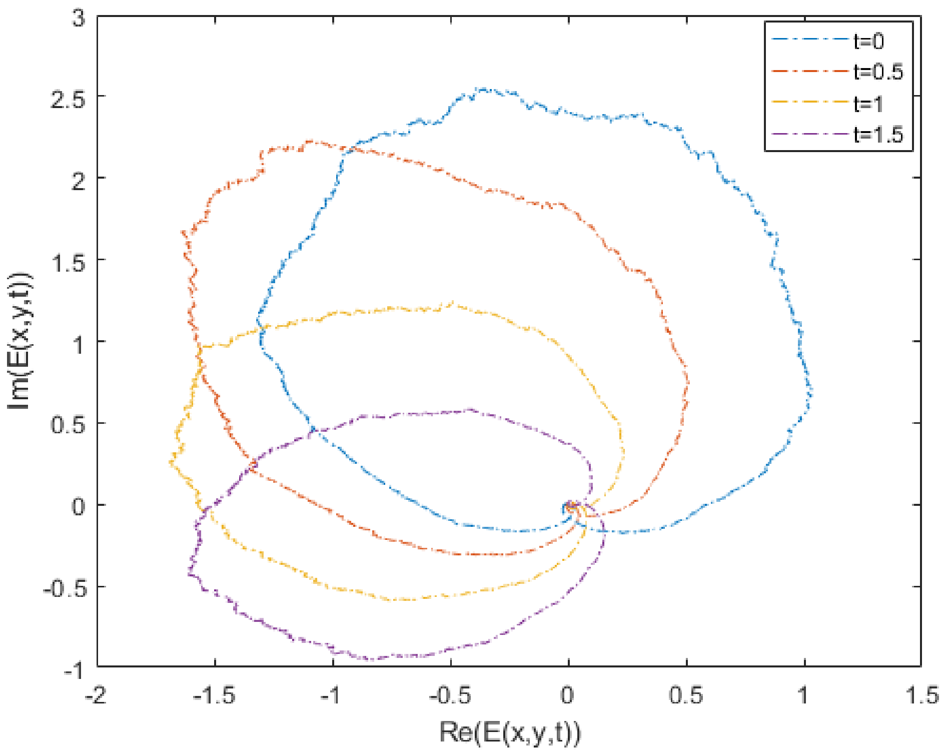

Figure 3. Trajectory of $\Xi_1(x, y, t)$, $\sigma = 0.5$ for $t = 0, 0.5, 1, 1.5$.

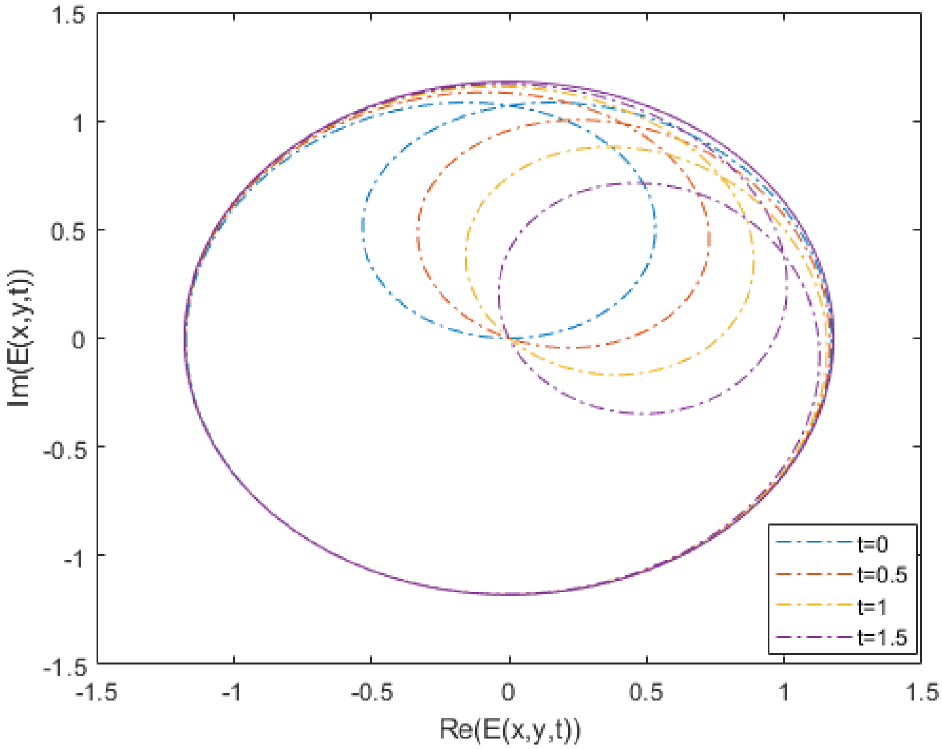

Figure 4. Trajectory of $\Xi_5(x, y, t)$, $\sigma = 0$ for $t = 0, 0.5, 1, 1.5$.

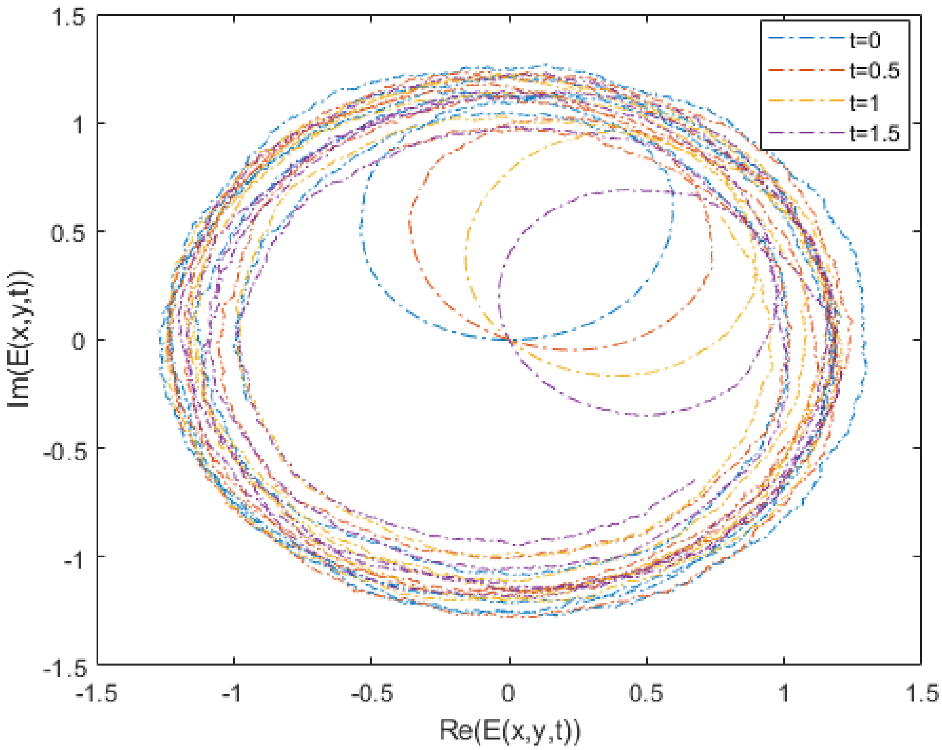

Figure 5. Trajectory of $\Xi_5(x, y, t)$, $\sigma = 0.2$ for $t = 0, 0.5, 1, 1.5$.

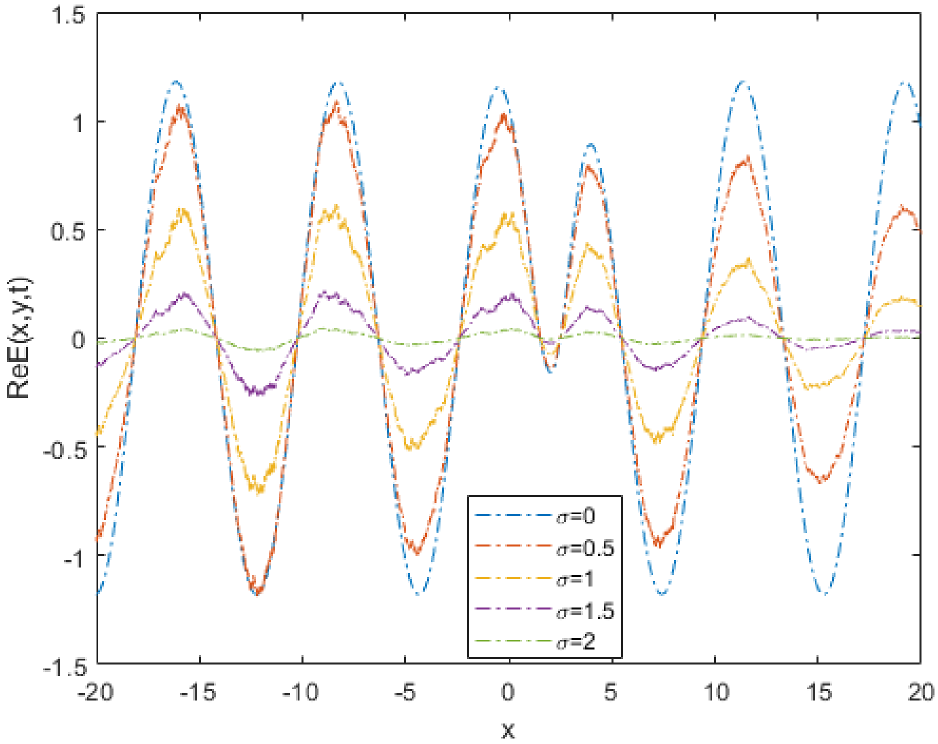

Figure 6. Plot of $\Xi_5(x,y,t)$ with $x, \sigma = 0, 0.5, 1, 1.5, 2$.

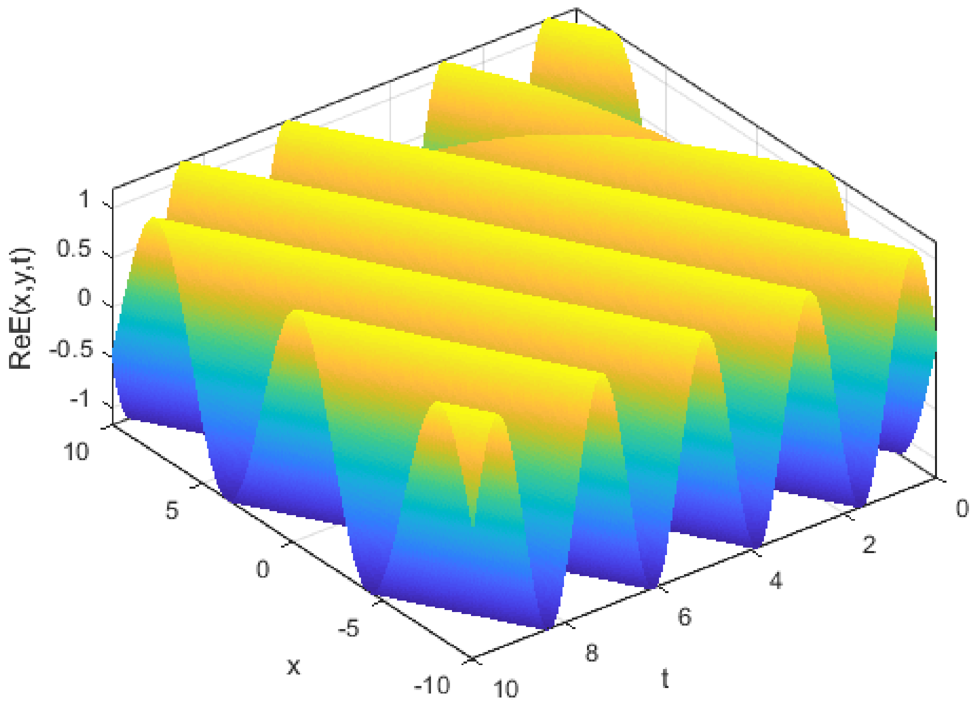

Figure 7. 3D plot of $\Xi_5(x,y,t)$ for $\sigma = 0$.

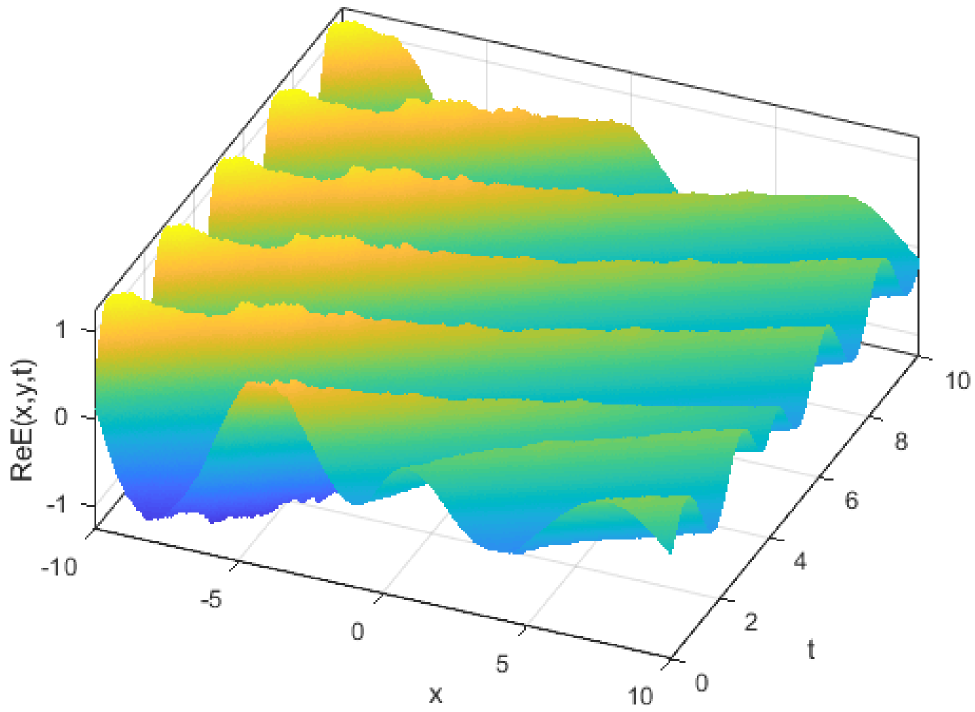

Figure 8. 3D plot of $\Xi_5(x, y, t)$ for $\sigma = 0.2$.

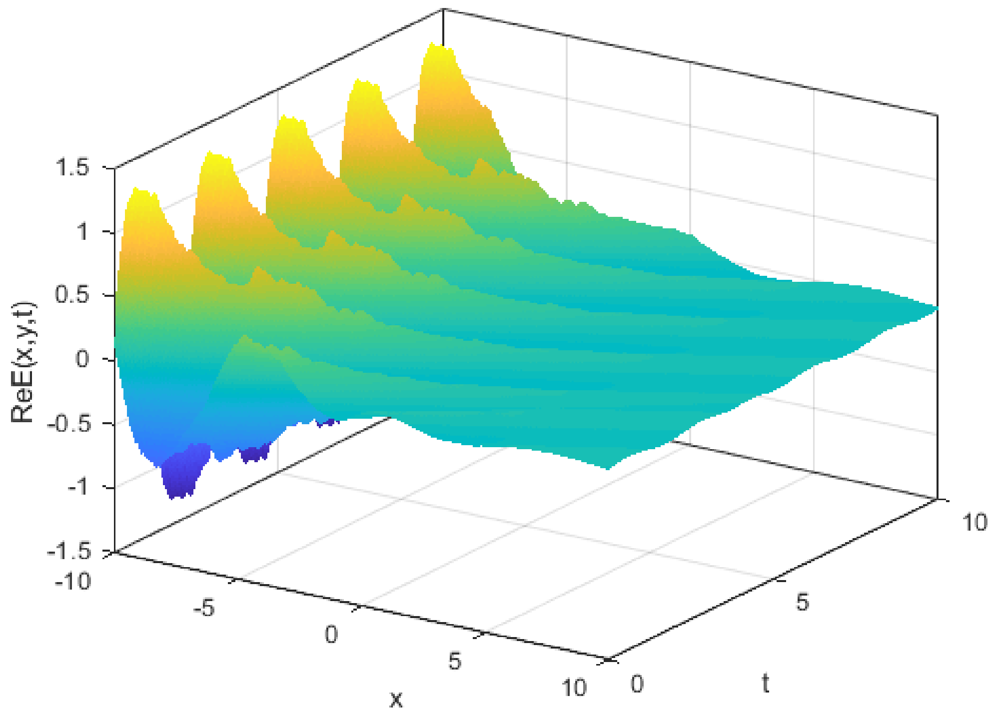

Figure 9. 3D plot of $\Xi_5(x, y, t)$ for $\sigma = 0.6$.

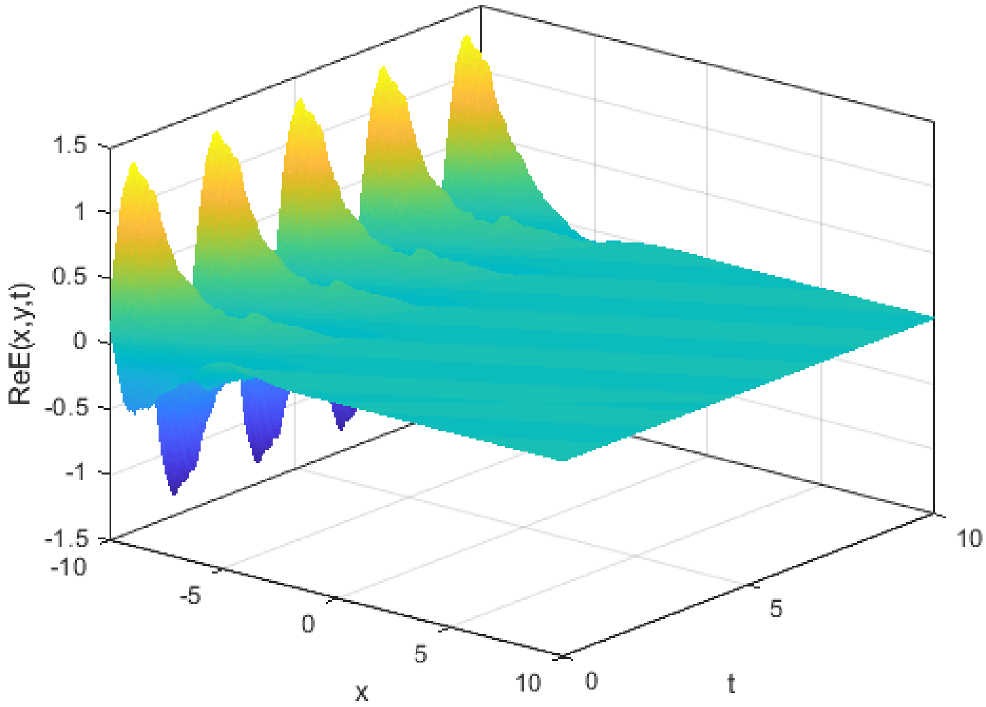

Figure 10. 3D plot of $\Xi_5(x, y, t)$ for $\sigma = 1$.

5. Conclusions

We have investigated the (2 + 1)-dimensional HFSC equation via a Stratonovich sense, using a unified technique. The proposed approach has a number of benefits, including avoiding complexity and time-consuming computations and obtaining accurate answers via physical parameters. We produced some new stochastic solutions, which play an important role in modern magnet theory. We also considered the influence of the noise term on the behaviour of solutions. It has been claimed that some modulations in collapsing dissipative and dispersive explosive formations can be shown by random stimuli. Finally, the proposed approach can be applied to other complex models, thus we will use it in our upcoming studies.

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