symmetry

MDPI

Article

Exact Solutions for Coupled Variable Coefficient KdV Equation via Quadratic Jacobi's Elliptic Function Expansion

Xiaohua Zeng [1], Xiling Wu [1], Changzhou Liang [1], Chiping Yuan [2,*] and Jieping Cai [1,*]

1 School of Economics and Trade, Guangzhou Xinhua University, Dongguan 523133, China
2 Institute of Guangdong, Hong Kong and Macao Development Studies, Sun Yat-sen University, Guangzhou 510275, China
* Correspondence: yuanchip@sysu.edu.cn (C.Y.); cjp@xhsysu.edu.cn (J.C.)

Abstract: The exact traveling wave solutions to coupled KdV equations with variable coefficients are obtained via the use of quadratic Jacobi's elliptic function expansion. The presented coupled KdV equations have a more general form than those studied in the literature. Nine couples of quadratic Jacobi's elliptic function solutions are found. Each couple of traveling wave solutions is symmetric in mathematical form. In the limit cases $m \to 1$, these periodic solutions degenerate as the corresponding soliton solutions. After the simple parameter substitution, the trigonometric function solutions are also obtained.

Keywords: coupled KdV equations; variable coefficients; quadratic Jacobi's elliptic function; soliton; traveling wave solution

check for updates

Citation: Zeng, X.; Wu, X.; Liang, C.; Yuan, C.; Cai, J. Exact Solutions for Coupled Variable Coefficient KdV Equation via Quadratic Jacobi's Elliptic Function Expansion. *Symmetry* **2023**, *15*, 1021. https://doi.org/10.3390/sym15051021

Academic Editor: Abraham A. Ungar

Received: 8 April 2023
Revised: 27 April 2023
Accepted: 30 April 2023
Published: 4 May 2023

1. Introduction

Constructing exact solutions to nonlinear evolution equations has attracted great research interest because they can be used to explain the evolution of complex nonlinear processes in different areas, such as fluid dynamics, physics, economics, and finance. Among the variety of nonlinear evolution equations, the most famous, the Korteweg–de Vries (KdV) equation, represents many nonlinear systems. To date, extensive research has been carried out on the complex nonlinear waves of KdV equations. Many methods have been developed to solve these equations, such as inverse scattering transformation [1], Darboux–Bäcklund transformation [2,3], Hirota's bilinear method [4,5], the first integral method [6], the homogeneous balance principle [7,8], the F-expansion method [9–11], the Wronskian method [12], the variational method [13], and Painlevé analysis [14]. On the other hand, KdV equations have different forms in different models, and the relevant methods were developed to study different types of KdV equations. In 2016, a simplified version of Hirota's method was used to investigate three extended higher-order KdV-type equations [15]. Later, various solutions were constructed for a new integrable nonlocal modified KdV equation with distinct physical structures [16,17]. In Ref. [18], multiple soliton solutions ranging from king type, single soliton and double soliton to multiple solitons were offered for space–time fractional modified KdV equations. These methods can also be used to obtain soliton, periodical, and rogue wave solutions for the Ivancevic equation [19–22], which defines the option-pricing wave function in terms of the stock price and time.

In recent years, the variable coefficient KdV equations were found to be more realistic models than their constant coefficient counterparts for describing the physical phenomena and physical properties behind them. Therefore, research on the variable coefficient KdV equation has become a much discussed topic in recent years. Usually, the variable coefficient KdV equations are much more complicated and difficult to solve. Fortunately, these methods for studying the constant coefficient KdV equation are also effective for solving the variable coefficient KdV equation. In the past, the Hirota bilinear method was used to represent multi-soliton solutions of the variable coefficient coupled KdV equation and analyze the dynamic characteristics [23,24]. In Ref. [25], a modified sine–cosine method was used to construct the exact periodic solutions and soliton solutions for two families of fifth-order KdV equations with variable coefficients and linear damping terms. In Ref. [26], the bell polynomial method was proved to be a powerful mathematical tool for solving the KdV equation with variable coefficients to reach the N-soliton solutions. Additionally, the periodic wave solutions were obtained by using the Riemann function method. In Refs. [27,28], new multi-soliton solutions were obtained by using bilinearization on a new modified KdV with time-dependent coefficients. Recently, the multivariate transformation technique was used to construct periodic and decay mode solutions for the generalized variable-coefficient KdV equation [29]. In 2019, the inverse scattering transform was extended to a super KdV equation with an arbitrary variable coefficient by using Kulish and Zeitlin's approach [30]. A symbolic computational method, the simplified Hirota's method, and a long-wave method can also be used to reach various exact solutions for the KdV equation with the extension of time-dependent coefficients [31]. Very recently, Liu and his coworkers used an auxiliary equation method to solve the coupled KdV equations with variable coefficients [32]. They obtained a series of new exact solutions under the condition that only variable coefficients are integrals. A question then naturally arises: can exact solutions be obtained for the variable coefficient coupled KdV equations with a more general form? In this paper, variable coefficient KdV equations with more general forms are solved by using the quadratic Jacobi's elliptic function expansion method. Several new families of exact solutions, i.e., Jacobi's elliptic function solutions and trigonometric function solutions, are obtained for the variable coefficient coupled KdV equations in a more general form. The existence condition and characteristics of these solutions are also presented.

The organization of the paper is as follows. In Section 2, the theoretical model, i.e., the coupled variable coefficient KdV equations in general forms, is presented. By using the method of quadratic Jacobi's elliptic function expansion, traveling wave elliptic function solutions are obtained. In Section 3, a specific example is used to obtain nine types of exact Jacobi's elliptic function solutions. We also show that these exact solutions can be transformed into trigonometric function solutions by simple parameter substitution. Section 4 gives a discussion of the results, and Section 5 presents a conclusion to the paper.

2. Theoretical Model and Methods

We consider the coupled variable coefficient KdV equations in the following form:

$$U_t + \alpha(t)UU_x + \beta(t)VV_x + \gamma(t)U_{xxx} = 0, \tag{1}$$

$$V_t + \delta(t)UV_x + \varepsilon(t)VU_x + \gamma(t)V_{xxx} = 0, \tag{2}$$

where U and V are the amplitudes of two waves counter-propagating on shallow water surfaces, α and β are nonlinear coefficients, δ and ε are the coupled nonlinear coefficients and γ is the dispersion coefficient. All the coefficients are time-dependent, and they satisfy $\delta + \varepsilon - \alpha = \beta\sigma^2$, where σ is a constant. It should be noted that the coupled variable coefficient KdV Equations (1) and (2) have more general forms than those studied in Ref. [32].

We search for traveling wave elliptic function solutions to Equations (1) and (2) in the form:

$$(x,t) = f(\xi) = a_0 + a_1F(\xi) + a_2F^2(\xi), \tag{3}$$

$$V(x,t) = g(\xi) = b_0 + b_1F(\xi) + b_2F^2(\xi), \tag{4}$$

where $a_i(i = 0,1,2)$ and $b_i(i = 0,1,2)$ are constants which are to be determined later, $F(\xi)$ is a Jacobi's elliptic solution of ξ, which satisfies $F'^2(\xi) = q_0 + q_2F(\xi)^2 + q_4F(\xi)^4$, and

$\xi = \omega \int_0^t \delta(\tau) d\tau + \lambda x$, where $\lambda(\lambda \neq 0)$ is an arbitrary constant. Substituting (3), (4) with (1), (2), we have

$$\omega\delta f' + \lambda\alpha f f' + \lambda\beta g g' + \gamma\lambda^3 f''' = 0, \tag{5}$$

$$\omega\delta g' + \lambda\delta f g' + \gamma\lambda^3 g''' + \varepsilon\lambda g f' = 0. \tag{6}$$

Substituting the properties of Jacobi's elliptic function $F''(\zeta) = q_2 F(\zeta) + 2q_4 F(\zeta)^3$ with Equations (5) and (6), we have

$$\begin{aligned} 2\lambda\left(12\gamma\lambda^2 a_2 q_4 + \beta b_2^2 + \alpha a_2^2\right)F^3 + 3\lambda\left(\alpha a_1 a_2 + \beta b_1 b_2 + 2\gamma\lambda^2 a_1 q_4\right)F^2 + \left(2\omega\delta a_2 + \lambda\alpha a_1^2 + 2\lambda\alpha a_0 a_2 + \lambda\beta b_1^2\right.\\ \left.+2\lambda\beta b_0 b_2 + 8\gamma\lambda^3 a_2 q_2\right)F + \omega\delta a_1 + \lambda\alpha a_0 a_1 + \lambda\beta b_0 b_1 + \gamma\lambda^3 a_1 q_2 = 0 \end{aligned}$$

$$\begin{aligned} 2\lambda b_2\left(\delta a_2 + \varepsilon a_2 + 12 q_4\gamma\lambda^2\right)F^3 + \left[(\lambda\delta + \varepsilon\lambda)(a_2 b_1 + 2a_1 b_2) + 6\gamma\lambda^3 b_1 q_4\right]F^2 + \left[2\omega\delta b_2 + \lambda\delta(a_1 b_1 + 2a_0 b_2)\right.\\ \left.+\varepsilon\lambda(a_1 b_1 + 2a_2 b_0) + 8\gamma\lambda^3 b_2 q_2\right]F + \omega\delta b_1 + \lambda\delta a_0 b_1 + \gamma\lambda^3 b_1 q_2 + \varepsilon\lambda a_1 b_0 = 0 \end{aligned}$$

When all the coefficients of $F^k (k = 0, 1, 2, 3)$ are set to zero, the following equations should be satisfied:

$$12\gamma\lambda^2 a_2 q_4 + \beta b_2^2 + \alpha a_2^2 = 0, \tag{7}$$

$$\alpha a_1 a_2 + \beta b_1 b_2 + 2\gamma\lambda^2 a_1 q_4 = 0, \tag{8}$$

$$2\omega\delta a_2 + \lambda\alpha a_1^2 + 2\lambda\alpha a_0 a_2 + \lambda\beta b_1^2 + 2\lambda\beta b_0 b_2 + 8\gamma\lambda^3 a_2 q_2 = 0, \tag{9}$$

$$\omega\delta a_1 + \lambda\alpha a_0 a_1 + \lambda\beta b_0 b_1 + \gamma\lambda^3 a_1 q_2 = 0, \tag{10}$$

$$b_2\left(\delta a_2 + \varepsilon a_2 + 12 q_4\gamma\lambda^2\right) = 0, \tag{11}$$

$$(\lambda\delta + \varepsilon\lambda)(a_2 b_1 + 2a_1 b_2) + 6\gamma\lambda^3 b_1 q_4 = 0, \tag{12}$$

$$2\omega\delta b_2 + \lambda\delta(a_1 b_1 + 2a_0 b_2) + \varepsilon\lambda(a_1 b_1 + 2a_2 b_0) + 8\gamma\lambda^3 b_2 q_2 = 0, \tag{13}$$

$$\omega\delta b_1 + \lambda\delta a_0 b_1 + \gamma\lambda^3 b_1 q_2 + \varepsilon\lambda a_1 b_0 = 0 \tag{14}$$

To obtain the exact solutions of the coupled KdV equations, we assume that $a_2 \neq 0$, $b_2 \neq 0$, and $\omega \neq 0$. With these assumptions, Equation (11) can be reduced to

$$a_2 = \frac{-12 q_4 \lambda^4 \gamma}{(\delta + \varepsilon)} = -12 k q_4 \lambda^2, \tag{15}$$

where $k = \gamma/(\delta + \varepsilon)$ is a constant.

Substituting the condition that $\delta + \varepsilon - \alpha = \beta\sigma^2$ with Equation (7), we have

$$b_2 = 12 k\sigma\lambda^2 q_4. \tag{16}$$

Equations (8), (10) and (12) can be satisfied simultaneously only when a_1 and b_1 are both zero. Therefore, we can obtain from Equation (9) that

$$\omega = \frac{\lambda\left(\beta b_0\sigma - \alpha a_0 - 4\gamma\lambda^2 q_2\right)}{\delta}. \tag{17}$$

Inserting Equation (17) into Equation (13), we obtain

$$(\delta-\alpha)b_0=(\alpha\sigma-\delta)a_0. \tag{18}$$

Finally, we obtain explicit solutions to coupled KdV equations:

$$U(x,t)=f(\xi)=a_0-12kq_4\lambda^2F^2(\xi), \tag{19}$$

$$V(x,t)=g(\xi)=\frac{(\alpha\sigma-\delta)}{(\delta-\alpha)}a_0+12k\lambda^2q_4\left(\frac{\delta+\varepsilon-\alpha}{\beta}\right)^{\frac{1}{2}}F^2(\xi), \tag{20}$$

where

$$\xi=\frac{\lambda\left(\beta b_0\sigma-\alpha a_0-4\gamma\lambda^2q_2\right)}{\delta}\int_0^t\delta(\tau)d\tau+\lambda x. \tag{21}$$

We then illustrate the Jacobi's elliptic function solutions of the coupled KdV equations with variable coefficients. For proper values of the parameters q_0, q_2 and q_4, the ordinary differential equation $F'^2(\zeta)=q_0+q_2F(\xi)^2+q_4F(\xi)^4$ can be easily solved, and the corresponding Jacobi's elliptic function solutions are summarized in Table 1. Equations (19) and (20) admit 12 different kinds of Jacobi's elliptic functions. In the real financial markets or the physical world, the amplitude of the wave function should be finite; thus, those elliptic functions ending with "s" (ns, cs, ds) are not the candidates to be the amplitude functions. All those functions ending with "n" (sn, cn, dn) can definitely be chosen as the amplitude function in any order, and all those ending with "d" or "c" can be the amplitude function in some orders. By substituting the values of q_0, q_2 and q_4 and the corresponding Jacobi's elliptic function solutions with the equation, a series of wave solutions will be obtained.

Table 1. Values of (q_0, q_2, q_4) and corresponding Jacobi's elliptic function $F(\xi)$.

q_0	q_2	q_4	F'^2=q_0+q_2F^2+q_4F^4	$F(\xi)$
1	$-(1+m^2)$	m^2	$F'^2=(1-F^2)(1-m^2F^2)$	$sn\xi, cd\xi=\frac{cn\xi}{dn\xi}$
$1-m^2$	$2m^2-1$	$-m^2$	$F'^2=(1-F^2)(m^2F^2+1-m^2)$	$cn\xi$
m^2-1	$2-m^2$	-1	$F'^2=(1-F^2)(F^2+m^2-1)$	$dn\xi$
m^2	$-(1+m^2)$	1	$F'^2=(1-F^2)(m^2-F^2)$	$ns\xi, dc\xi=\frac{dn\xi}{cn\xi}$
$-m^2$	$2m^2-1$	$1-m^2$	$F'^2=(1-F^2)[(m^2-1)F^2-m^2]$	$nc\xi=\frac{1}{cn\xi}$
-1	$2-m^2$	m^2-1	$F'^2=(1-F^2)[(1-m^2)F^2-1]$	$nd\xi=\frac{1}{dn\xi}$
1	$2-m^2$	$1-m^2$	$F'^2=(1+F^2)[(1-m^2)F^2+1]$	$sc\xi=\frac{sn\xi}{cn\xi}$
1	$2m^2-1$	$-m^2(1-m^2)$	$F'^2=(1+m^2F^2)[1+(m^2-1)F^2]$	$sd\xi=\frac{sn\xi}{dn\xi}$
$1-m^2$	$2-m^2$	1	$F'^2=(1+F^2)[F^2+1-m^2]$	$cs\xi=\frac{cn\xi}{sn\xi}$
$-m^2+m^4$	$2m^2-1$	1	$F'^2=(F^2+m^2)[F^2+m^2-1]$	$ds\xi=\frac{dn\xi}{sn\xi}$

In the following, we provide a specific example to illustrate the properties of the Jacobi's function solutions of the coupled KdV equations.

3. Results

To provide a specific example, we discuss the case $\alpha(t)=4-2\cos t$, $\beta(t)=5-\cos t$, $\gamma=3-\cos t$, and $\delta(t)=6-2\cos t$, $\varepsilon(t)=3-\cos t$. In this case, $k=1/3$ and $\sigma=1$.

3.1. Jacobi-sn Function and Jacobi-cd Function Solutions

First, we discuss the case $q_0=1, q_2=-1-m^2$ and $q_4=m^2$. In this case, the Jacobi's elliptical function can be chosen as the m-order *sn*-function or cd-function, i.e.,

$F(\xi) = \mathrm{sn}(\xi, m)$ or $F(\xi) = \mathrm{cd}(\xi, m) = \mathrm{cn}(\xi, m)/\mathrm{dn}(\xi, m)$. Substituting these expressions with Equations (19)–(21), we have

$$U_1(x,t) = f(\xi) = a_0 - 12km^2\lambda^2\mathrm{sn}^2(\xi), \tag{22}$$

$$V_1(x,t) = g(\xi) = \frac{(\alpha\sigma - \delta)}{(\delta - \alpha)}a_0 + 12k\lambda^2 m^2\left(\frac{\delta + \varepsilon - \alpha}{\beta}\right)^{\frac{1}{2}}\mathrm{sn}^2(\xi), \tag{23}$$

and

$$U_2(x,t) = a_0 - 12km^2\lambda^2\mathrm{cd}^2(\xi), \tag{24}$$

$$V_2(x,t) = \frac{(\alpha\sigma - \delta)}{(\delta - \alpha)}a_0 + 12k\lambda^2 m^2\left(\frac{\delta + \varepsilon - \alpha}{\beta}\right)^{\frac{1}{2}}\mathrm{cd}^2(\xi), \tag{25}$$

where

$$\xi = \frac{\lambda\left(\beta b_0\sigma - \alpha a_0 + 4\gamma\lambda^2\left(1 + m^2\right)\right)}{\delta}\int_0^t \delta(\tau)d\tau + \lambda x. \tag{26}$$

When $m = 0$, a coupled constant solution is obtained. In the limit case when $m = 1$, $sn(\xi) \to \tanh(\xi)$. Therefore, Equations (22) and (23) can be read as

$$U_1(x,t) = f(\xi) = a_0 - 12k\lambda^2\tanh^2(\xi), \tag{27}$$

$$V_1(x,t) = g(\xi) = \frac{(\alpha\sigma - \delta)}{(\delta - \alpha)}a_0 + 12k\lambda^2\left(\frac{\delta + \varepsilon - \alpha}{\beta}\right)^{\frac{1}{2}}\tanh^2(\xi). \tag{28}$$

3.2. Jacobi-cn Function Solutions

In the following, we consider the case wherein $q_0 = 1 - m^2, q_2 = 2m^2 - 1$, and $q_4 = -m^2$. In this case, the Jacobi's elliptical function is the m-order cn function, i.e., $F(\xi) = \mathrm{cn}(\xi, m)$. Substituting these forms into Equations (19)–(21), we have

$$U_3(x,t) = a_0 + 12km^2\lambda^2\mathrm{cn}^2(\xi), \tag{29}$$

$$V_3(x,t) = \frac{(\alpha\sigma - \delta)}{(\delta - \alpha)}a_0 - 12k\lambda^2 m^2\left(\frac{\delta + \varepsilon - \alpha}{\beta}\right)^{\frac{1}{2}}\mathrm{cn}^2(\xi), \tag{30}$$

where

$$\xi = \frac{\lambda\left[\beta b_0\sigma - \alpha a_0 - 4\gamma\lambda^2\left(2m^2 - 1\right)\right]}{\delta}\int_0^t \delta(\tau)d\tau + \lambda x. \tag{31}$$

Similarly with the counterparts of Jacobi sn-function solutions, the Jacobi cn-function solutions of U and V also reduce to constant solutions when $m \to 0$. Furthermore, a family of periodical wave solutions can be obtained when $0 < m < 1$. When $m = 1$, a family of secant solutions can be obtained, and they read as

$$U_3(x,t) = a_0 + 12k\lambda^2\mathrm{sech}^2(\xi), \tag{32}$$

$$V_3(x,t) = \frac{(\alpha\sigma - \delta)}{(\delta - \alpha)}a_0 - 12k\lambda^2\left(\frac{\delta + \varepsilon - \alpha}{\beta}\right)^{\frac{1}{2}}\mathrm{sech}^2(\xi). \tag{33}$$

3.3. Jacobi-dn Function Solutions

If $q_0 = m^2 - 1, q_2 = 2 - m^2$, and $q_4 = -1$, the Jacobi's elliptical function is the m-order dn function, i.e., $F(\xi) = \mathrm{dn}(\xi, m)$. Substituting these forms with Equations (19)–(21), we have

$$U_4(x,t) = f(\xi) = a_0 + 12k\lambda^2 \mathrm{dn}^2(\xi), \tag{34}$$

$$V_4(x,t) = g(\xi) = \frac{(\alpha\sigma - \delta)}{(\delta - \alpha)} a_0 - 12k\lambda \left(\frac{\delta + \varepsilon - \alpha}{\beta}\right)^{\frac{1}{2}} \mathrm{dn}^2(\xi), \tag{35}$$

where

$$\xi = \frac{\lambda\left(\beta b_0 \sigma - \alpha a_0 - 8\gamma\lambda^2 + 4\gamma\lambda^2 m^2\right)}{\delta} \int_0^t \delta(\tau) d\tau + \lambda x. \tag{36}$$

3.4. Jacobi-dc Function Solutions

If $q_0 = m^2, q_2 = -\left(1 + m^2\right)$, and $q_4 = 1$, the Jacobi's elliptical function is the m-order *dc* function, i.e., $F(\xi) = \mathrm{dc}(\xi, m) = \mathrm{dn}(\xi, m)/\mathrm{cn}(\xi, m)$. Therefore, the family of exact function reads as

$$U_5(x,t) = f(\xi) = a_0 - 12k\lambda^2 \mathrm{dc}^2(\xi), \tag{37}$$

$$V_5(x,t) = g(\xi) = \frac{(\alpha\sigma - \delta)}{(\delta - \alpha)} a_0 + 12k\lambda \left(\frac{\delta + \varepsilon - \alpha}{\beta}\right)^{\frac{1}{2}} \mathrm{dc}^2(\xi), \tag{38}$$

where

$$\xi = \frac{\lambda\left(\beta b_0 \sigma - \alpha a_0 - 4\gamma\lambda^2 - 4\gamma\lambda^2 m^2\right)}{\delta} \int_0^t \delta(\tau) d\tau + \lambda x. \tag{39}$$

3.5. Jacobi-nc Function Solutions

If $q_0 = -m^2, q_2 = 2m^2 - 1$, and $q_4 = 1 - m^2$, the Jacobi's elliptical function should be the m-order nc function, i.e., $F(\xi) = \mathrm{dc}(\xi, m) = \mathrm{dn}(\xi, m)/\mathrm{cn}(\xi, m)$. Substituting these latter forms with Equations (19)–(21), we can read the family of the exact function as

$$U_6(x,t) = f(\xi) = a_0 - 12k\left(1 - m^2\right)\lambda^2 \mathrm{nc}^2(\xi), \tag{40}$$

$$V_6(x,t) = g(\xi) = \frac{(\alpha\sigma - \delta)}{(\delta - \alpha)} a_0 + 12k\lambda(1 - m^2) \left(\frac{\delta + \varepsilon - \alpha}{\beta}\right)^{\frac{1}{2}} \mathrm{nc}^2(\xi), \tag{41}$$

where

$$\xi = \frac{\lambda\left(\beta b_0 \sigma - \alpha a_0 - 8\gamma\lambda^2 m^2 + 4\gamma\lambda^2 m^2\right)}{\delta} \int_0^t \delta(\tau) d\tau + \lambda x. \tag{42}$$

3.6. Jacobi-nd Function Solutions

If $q_0 = -1, q_2 = 2 - m^2$, and $q_4 = m^2 - 1$, the Jacobi's elliptical function should be the m-order nd function, i.e., $F(\xi) = \mathrm{nd}(\xi, m) = 1/\mathrm{dn}(\xi, m)$. Inserting these latter forms of $F(\xi)$ into Equations (19)–(21) leads to the following periodical solutions:

$$U_7(x,t) = a_0 - 12k\left(m^2 - 1\right)\lambda^2 \mathrm{nd}^2(\xi), \tag{43}$$

$$V_7(x,t) = \frac{(\alpha\sigma - \delta)}{(\delta - \alpha)} a_0 + 12k\lambda^2(m^2 - 1) \left(\frac{\delta + \varepsilon - \alpha}{\beta}\right)^{\frac{1}{2}} \mathrm{nd}^2(\xi), \tag{44}$$

where

$$\xi = \frac{\lambda\left(\beta b_0 \sigma - \alpha a_0 - 8\gamma\lambda^2 + 4\gamma\lambda^2 m^2\right)}{\delta} \int_0^t \delta(\tau) d\tau + \lambda x. \tag{45}$$

3.7. Jacobi-sc and Jacobi-sd Function Solutions

By following the same steps as described above, we can also obtain another two types of periodical traveling solutions, i.e., Jacobi-sc and Jacobi-sd function solutions. The Jacobi-sc solutions can be read as

$$U_8(x,t) = a_0 - 12k(1-m^2)\lambda^2 \mathrm{sc}^2(\xi), \tag{46}$$

$$V_8(x,t) = \frac{(\alpha\sigma - \delta)}{(\delta - \alpha)}a_0 + 12k(1-m^2)\lambda^2\left(\frac{\delta + \varepsilon - \alpha}{\beta}\right)^{\frac{1}{2}}\mathrm{sc}^2(\xi), \tag{47}$$

where

$$\xi = \frac{\lambda\left(\beta b_0\sigma - \alpha a_0 - 8\gamma\lambda^2 + 4\gamma\lambda^2 m^2\right)}{\delta}\int_0^t \delta(\tau)d\tau + \lambda x. \tag{48}$$

The Jacobi-sd solutions are

$$U(x,t) = f(\xi) = a_0 + 12km^2(1-m^2)\lambda^2 F^2(\xi), \tag{49}$$

$$V(x,t) = g(\xi) = \frac{(\alpha\sigma - \delta)}{(\delta - \alpha)}a_0 - 12km^2(1-m^2)\lambda^2\left(\frac{\delta + \varepsilon - \alpha}{\beta}\right)^{\frac{1}{2}}F^2(\xi), \tag{50}$$

where

$$\xi = \frac{\lambda(\beta b_0\sigma - \alpha a_0 + 4\gamma(1-2m^2)\lambda^2)}{\delta}\int_0^t \delta(\tau)d\tau + \lambda x. \tag{51}$$

3.8. Trigonometric Function Solutions

It is worth noting that the trigonometric function solutions can also be obtained by using the results of this paper. If we let $\lambda = i\mu$ in Equations (27), (28), (32) and (33), and note the facts that $\tanh(ix) = i\tan(x)$ and $\mathrm{sech}(ix) = \sec(x)$, two more families of periodical traveling solutions will be obtained. For facility of demonstration, we let $\xi = \frac{\mu\left[\beta b_0\sigma - \alpha a_0 + 4\gamma\mu^2\left(2m^2-1\right)\right]}{\delta}\int_0^t \delta(\tau)d\tau + \mu x$. The solutions of the first family then read as

$$U_1(x,t) = f(\xi) = a_0 - 12k\mu^2\tan^2(\varsigma), \tag{52}$$

$$V_1(x,t) = g(\xi) = \frac{(\alpha\sigma - \delta)}{(\delta - \alpha)}a_0 + 12k\mu^2\left(\frac{\delta + \varepsilon - \alpha}{\beta}\right)^{\frac{1}{2}}\tan^2(\varsigma). \tag{53}$$

The other solution family reads as

$$U_3(x,t) = a_0 - 12k\mu^2\sec^2(\varsigma), \tag{54}$$

$$V_3(x,t) = \frac{(\alpha\sigma - \delta)}{(\delta - \alpha)}a_0 + 12k\mu^2\left(\frac{\delta + \varepsilon - \alpha}{\beta}\right)^{\frac{1}{2}}\sec^2(\varsigma). \tag{55}$$

Following the steps described above, we can also obtain other periodical trigonometric function solutions for other Jacobi's elliptic function solutions.

4. Discussion

Figure 1 is the Jacobi sn-function solution (as shown in Equations (22) and (23)) to the coupled KdV equations when $a_0 = 3$ and $\lambda = 1$. When $m \to 0$, Equations (22) and (23) are reduced to coupled constant solutions, i.e., $U(x,t) = a_0$ and $V(x,t) = -a_0$, which are shown in Figures 1a and 1d, respectively. When $0 < m < 1$, the Jacobi *sn*-function is a periodical function; thus, a family of periodic solutions is obtained for the coupled KdV equations.

The coupled solutions of a typical example when m = 0.5 are shown in Figure 1b and 1e, respectively. When $m = 1$, coupled solitary solutions of $U(x,t)$ and $V(x,t)$ can be obtained (Equations (27) and (28)), which are shown in Figure 1c and 1f, respectively.

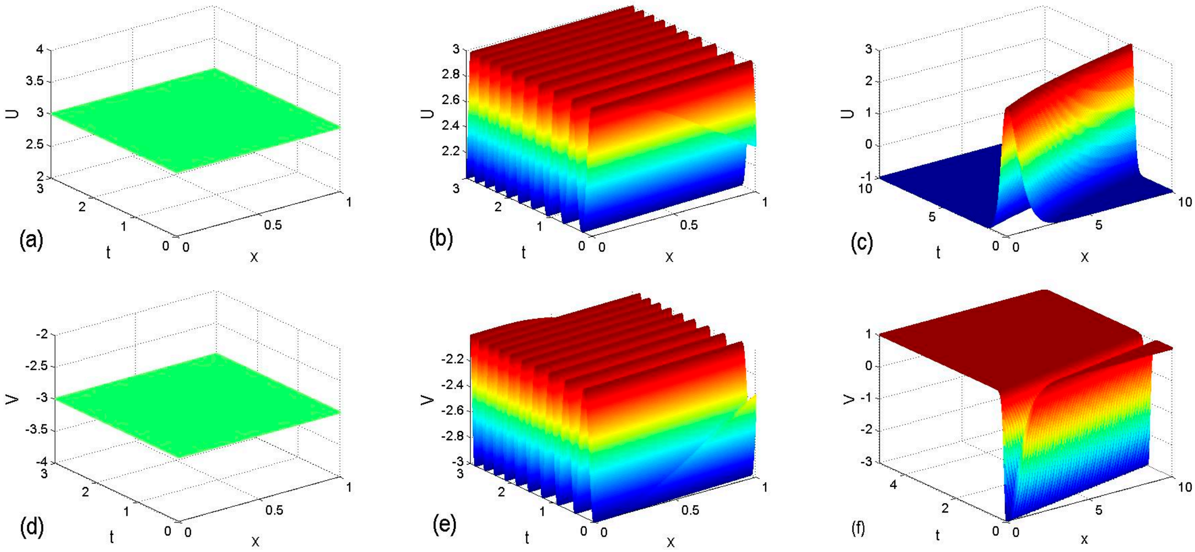

Figure 1. The Jacobi sn-function solutions of U (the upper row) and V (the lower row) to the coupled KdV equations with variable coefficients when $m \to 0$ (**a**,**d**), $m = 0.5$ (**b**,**e**), and $m \to 1$ (**c**,**f**), respectively. Other parameters are $\alpha(t) = 4 - 2\cos t$, $\beta(t) = 5 - \cos t$, $\gamma = 3 - \cos t$, and $\delta(t) = 6 - 2\cos t$, $\varepsilon(t) = 3 - \cos t$.

Figure 2 shows the Jacobi cd-function solution of Equations (24) and (25) when $a_0 = 3$, and $\lambda = 1$. Similarly with the counterparts of Jacobi sn-function solutions, the Jacobi cd-function solutions of U and V also reduce to a coupled constant solution when $m \to 0$. Furthermore, they additionally present a periodical structure when $0 < m < 1$. When $m \to 1$, a coupled soliton-like solution can be obtained, as shown in Figure 2c,e. In this case, the soliton solution travels along a curved orbit.

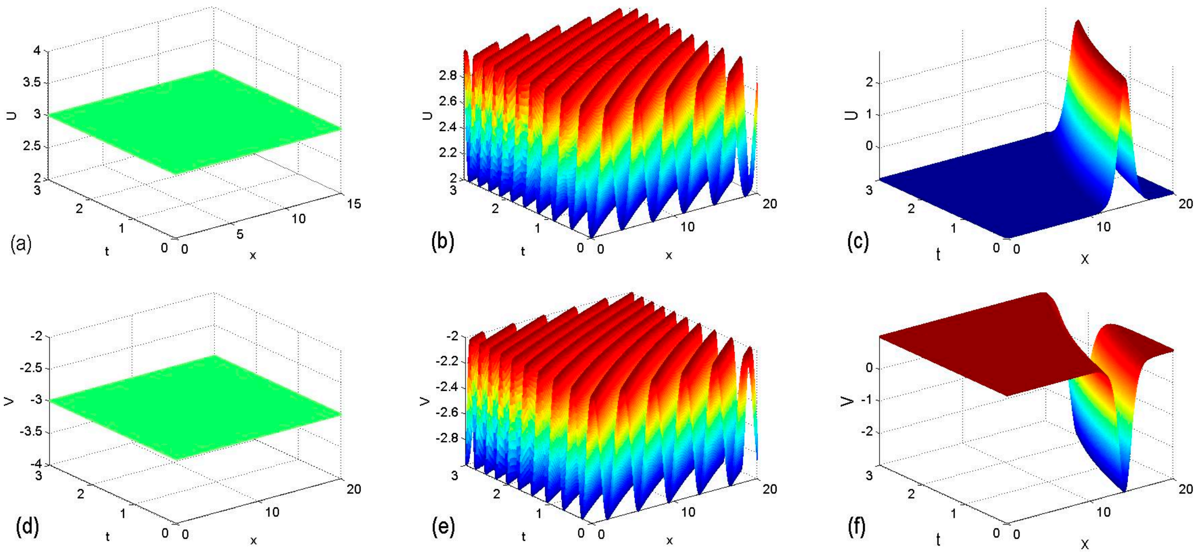

Figure 2. The Jacobi cd-function solutions of U (the upper row) and V (the lower row) to the coupled KdV equations with variable coefficients when $m \to 0$ (**a**,**d**), $m = 0.5$ (**b**,**e**), and $m \to 1$ (**c**,**f**), respectively. Other parameters are the same as those in Figure 1.

From Figures 1 and 2, one can clearly observe that each couple of traveling solutions is symmetrical in mathematical form. Along the direction of travel, these solutions are also symmetrical. Other solutions shown in Sections 3.2–3.8 have similar properties. We omit the detailed discussion for simplicity.

On the other hand, the coupled variable coefficient KdV Equations (1) and (2) can be reduced to coupled KdV equations with constant coefficients in Ref. [33] if we let $\alpha = 6$, and $\beta = -6$, $\gamma = 1$, $\delta = 3$, and $\varepsilon = 0$. These solutions to the constant-coefficient KdV equations are included in the results discussed in this paper.

5. Conclusions

The Jacobi's elliptic function expansion method is used to obtain the exact solutions to the coupled KdV equations with time-dependent variable coefficients. Several types of exact traveling wave solutions are obtained when both $(\delta + \varepsilon - \alpha)/\beta$ and $\gamma/(\delta + \varepsilon)$ are real constants. There are nine types of quadratic Jacobi's elliptic function solutions, i.e., Jacobi-sn, cn, dn, sd, cd, nd, sc, nc, and dc function solutions. Soliton-like solutions are also included in these solutions when the elliptic modulus $m \to 1$. Trigonometric function solutions can also be obtained through simple parameter substitution of the obtained Jacobi's elliptic function solutions. The result is relevant because there are no studies on the exact traveling solutions for the generalized coupled KdV equations with variable coefficients as in Equations (1) and (2). Future investigations could focus on using different research methods to solve the generalized variable coefficient coupled KdV equations and obtain different novel exact solutions. Furthermore, the method used in this paper may be applied to exploring the analytical solutions of other nonlinear partially differential equations, such as the Kadomtsev–Petviashvili equation with a variable coefficient, the variable coefficient Schrödinger equation and the variable sine-Gordon equation.

Author Contributions: Conceptualization, X.Z. and X.W.; methodology, X.Z.; software, X.Z.; validation, C.L. and X.Z.; formal analysis, X.Z.; investigation, J.C.; data curation, C.L.; writing—original draft preparation, X.Z.; writing—review and editing, X.W.; visualization, X.Z.; supervision, C.Y.; project administration, J.C.; funding acquisition, C.Y. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Guangdong Distinctive Key Disciplines construction project "public Administration" (F2017STSZD01)", the Guangdong Key construction Discipline Scientific Research Capacity Improvement Project "Research on the Crisis Response Mechanism and Policy of Guangdong SMEs in Post-epidemic Era" (2021ZDJS141), the Guangdong Key Construction Discipline Scientific Research Capacity Improvement Project "Research on the path and countermeasures of financial support for high-quality development of Guangdong's real economy" (2022ZDJS150), the Doctoral guidance program of Guangzhou Xinhua University, the Finance Course Teaching and Research Section of Guangzhou Xinhua University (2021JYS001), and the Financial Vocational Skills Training Experimental Teaching Demonstration Center (2015S001).

Data Availability Statement: Data sharing is not applicable to this paper as no datasets were generated or analyzed during the current study.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Zhang, X.; Chen, Y. Inverse scattering transformation for generalized nonlinear Schrödinger equation. *Appl. Math. Lett.* **2019**, *98*, 306–313. [CrossRef]
2. Yang, Y.Q.; Dong, H.H.; Chen, Y. Darboux-Backlund transformation and localized excitation on the periodic wave background for the nonlinear Schrödinger equation. *Wave Motion* **2021**, *106*, 102787. [CrossRef]
3. Guan, X.; Liu, W.; Zhou, Q.; Biswas, A. Darboux transformation and analytic solutions for a generalized super-NLS-mKdV equation. *Nonlinear Dyn.* **2019**, *98*, 1491–1500. [CrossRef]
4. Yang, J.R.; Mao, J.J. Soliton solutions of coupled KdV system from Hirota's bilinear direct method. *Commun. Theor. Phys.* **2008**, *49*, 22–26.
5. Liu, X.; Liu, W.; Triki, H.; Zhou, Q.; Biswas, A. Periodic attenuating oscillation between soliton interactions for higher-order variable coefficient nonlinear Schrödinger equation. *Nonlinear Dyn.* **2019**, *96*, 801–809. [CrossRef]

6. Lu, B. The first integral method for some time fractional differential equations. *J. Math. Anal. Appl.* **2012**, *395*, 684–693. [CrossRef]
7. Zhu, F.B.; Zang, Q.P.; Ji, J.E. Exact solutions for a Wick-type stochastic 2D KdV equation. *Appl. Math. Comput.* **2010**, *216*, 2766–2770. [CrossRef]
8. Liu, H.Z.; Xin, X.P.; Wang, Z.G.; Liu, X.Q. Backlund transformation classification, integrability and exact solutions to the generalized Burgers'-KdV equation. *Commun. Nonlinear Sci. Numer. Simul.* **2017**, *44*, 11–18. [CrossRef]
9. He, Y.H.; Zhao, Y.M.; Long, Y. New Exact Solutions for a Higher-Order Wave Equation of KdV Type Using Extended F-Expansion Method. *Math. Probl. Eng.* **2013**, *2013*, 128970. [CrossRef]
10. Wang, H.C.; Liang, J.C.; Chen, G.H.; Zhou, L.; Ling, D.X.; Liu, M.X. Exact analytical solutions of higher-order nonlinear Schrodinger equation. *Optik* **2017**, *131*, 438–445. [CrossRef]
11. Ozisik, M.; Secer, A.; Bayram, M. On solitary wave solutions for the extended nonlinear Schrodinger equation via the modified F-expansion method. *Opt. Quantum Electron.* **2023**, *55*, 215. [CrossRef]
12. Zhang, C.; Tian, B.; Xu, T.; Li, L.L.; Lu, X.; Geng, T.; Zhu, H.W. Exact Solutions to a (3+1)-Dimensional Variable-Coefficient Kadomtsev-Petviashvilli Equation via the Bilinear Method and Wronskian Technique. *Commun. Theor. Phys.* **2009**, *52*, 468–472.
13. Wazwaz, A.M. Optical bright and dark soliton solutions for coupled nonlinear Schrodinger (CNLS) equations by the variational iteration method. *Optik* **2020**, *207*, 164457. [CrossRef]
14. Dang, G. Meromorphic solutions of the seventh-order KdV equation by using an extended complex method and Painleve analysis. *ScienceAsia* **2023**, *49*, 108–115. [CrossRef]
15. Wazwaz, A.M. The simplified Hirota's method for studying three extended higher-order KdV-type equations. *J. Ocean Eng. Sci.* **2016**, *1*, 181–185. [CrossRef]
16. Wazwaz, A.M. A new integrable nonlocal modified KdV equation: Abundant solutions with distinct physical structures. *J. Ocean Eng. Sci.* **2017**, *2*, 1–4. [CrossRef]
17. Liang, Z.F.; Tang, X.Y.; Ding, W. Infinitely many nonlocal symmetries and nonlocal conservation laws of the integrable modified KdV-sine-Gordon equation. *Commun. Theor. Phys.* **2021**, *73*, 055003. [CrossRef]
18. Arefin, M.A.; Sadiya, U.; Inc, M.; Uddin, M.H. Adequate soliton solutions to the space–time fractional telegraph equation and modified third-order KdV equation through a reliable technique. *Opt. Quant. Electron.* **2022**, *54*, 309. [CrossRef]
19. Ivancevic, V.G. Adaptive-Wave Alternative for the Black-Scholes Option Pricing Model. *Cogn. Comput.* **2010**, *2*, 17–30. [CrossRef]
20. Jena, R.M.; Chakraverty, S.; Baleanu, D. A novel analytical technique for the solution of time-fractional Ivancevic option pricing model. *Phys. A* **2020**, *550*, 124380. [CrossRef]
21. Chen, Y.Q.; Tang, Y.H.; Manafian, J.; Rezazadeh, H.; Osman, M.S. Dark wave, rogue wave and perturbation solutions of Ivancevic option pricing model. *Nonlinear Dyn.* **2021**, *105*, 2539–2548. [CrossRef]
22. Zeng, X.; Liang, C.; Yuan, C. Solitary Wave and Singular Wave Solutions for Ivancevic Option Pricing Model. *Math. Probl. Eng.* **2022**, *2022*, 4599194. [CrossRef]
23. Zhang, Y.; Cheng, Z. Exact Soliton Solutions and Quasi-Periodic Wave Solutions to The Forced Variable-Coefficient KdV Equation. *Int. J. Mod. Phys. B* **2012**, *26*, 1250072. [CrossRef]
24. Zhang, S.; Cai, B. Multi-soliton solutions of a variable-coefficient KdV hierarchy. *Nonlinear Dyn.* **2014**, *78*, 1593–1600. [CrossRef]
25. Triki, H.; Wazwaz, A.M. Traveling wave solutions for fifth-order KdV type equations with time-dependent coefficients. *Commun. Nonlinear Sci. Numer. Simul.* **2014**, *19*, 404–408. [CrossRef]
26. Ahmed, E.A.; Wael, S.; Seadawy, A.; Maowad, S.M.; EL-Kalaawy, O.H. Bifurcation; bilinear forms, conservation laws and soliton solutions of the temporal-second-order KdV equation. *Int. J. Mod. Phys. B* **2022**, *36*, 2250181. [CrossRef]
27. Gurses, M.; Pekcan, A. Nonlocal KdV equations. *Phys. Lett. A* **2020**, *384*, 126894. [CrossRef]
28. Zhang, S.; Zhang, L. Bilinearization and new multi-soliton solutions of mKdV hierarchy with time-dependent coefficients. *Open Phys.* **2016**, *14*, 69–75.
29. Zhang, Y.; Liu, J. Periodic and decay mode solutions of the generalized variable-coefficient Korteweg-de Vries equation. *Mod. Phys. Lett. B* **2019**, *33*, 1950234. [CrossRef]
30. Zhang, S.; You, C. Inverse scattering transform for a supersymmetric Korteweg-de Vries equation. *Therm. Sci.* **2019**, *23*, S677–S684.
31. Ismael, H.F.; Murad, M.A.S.; Bulut, H. Various exact wave solutions for KdV equation with time-variable coefficients. *J. Ocean Eng. Sci.* **2022**, *7*, 409–418. [CrossRef]
32. Liu, W.; Fu, Z.; Tian, C. The exact solutions to coupled KdV equations with variable coefficients by auxiliary equation method, Journal of Northwest University. *Nat. Sci. Ed.* **2020**, *50*, 955–959.
33. Yao, S.W.; Zafar, A.; Urooj, A.; Tariq, B.; Shakeel, M.; Inc, M. Novel solutions to the coupled KdV equations and the coupled system of variant Boussinesq equations. *Results Phys.* **2023**, *45*, 106249. [CrossRef]