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Some New Fractional Hadamard and Pachpatte-Type Inequalities with Applications via Generalized Preinvexity

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Abstract: The term convexity associated with the theory of inequality in the sense of fractional analysis has a broad range of different and remarkable applications in the domain of applied sciences. The prime objective of this article is to investigate some new variants of Hermite–Hadamard and Pachpatte-type integral inequalities involving the idea of the preinvex function in the frame of a fractional integral operator, namely the Caputo–Fabrizio fractional operator. By employing our approach, a new fractional integral identity that correlates with preinvex functions for first-order differentiable mappings is presented. Moreover, we derive some refinements of the Hermite–Hadamard-type inequality for mappings, whose first-order derivatives are generalized preinvex functions in the Caputo–Fabrizio fractional sense. From an application viewpoint, to represent the usability of the concerning results, we presented several inequalities by using special means of real numbers. Integral inequalities in association with convexity in the frame of fractional calculus have a strong relationship with symmetry. Our investigation provides a better image of convex analysis in the frame of fractional calculus.

Keywords: preinvex function; n-polynomial preinvex function; Hermite–Hadamard inequality; Caputo–Fabrizio operator

MSC: 26A51; 26A33; 26D07; 26D10; 26D15

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1. Introduction

The topic of convexity has grown over the past few decades into one of the most fascinating and pragmatic areas to explore the substantial class of problems emerging in both pure and applied sciences. Inventive methods and calculations have taken several diverse paths for the study of a convex function. The term convexity provides a meaningful structure for developing and initiating mathematical techniques for the investigation of complicated mathematical problems. The combined study of convexity and inequalities has captivated the experts in the field of inequality and subsequently numerous unique versions of the classical Hadamard inequality have been presented and investigated in the literature. The subject of convexity has a magnificent correlation with the theory of inequalities. Many known and valuable inequalities are consequences of convex functions. In the literature, there are some famous inequalities, namely the Jensen and Hadamard inequalities, that explain and explore the meaning of convex functions in a geometrical sense. For the literature, see the references [1–12].

The term fractional calculus handles the investigation regarding fractional integration over a complex domain and its applications. Fractional calculus is the asserted assigning of the integration of an arbitrary non-integer order. Due to its practical uses, it has recently retained and attracted the interest and attention of many mathematicians. In the literature, probably, J. Liouville was the first mathematician to provide the logical definition of a fractional derivative and he published approximately nine papers on fractional calculus between 1832 and 1837, and the last one was published in 1855. During the years 1802–1829, probably, N.H. Abel was the first to offer the first application of fractional calculus. On the other hand, the idea of symmetry is a magnificent framework that is employed to portray the surroundings and issues of the real world, as well as to establish the interaction between applied and mathematical sciences, such as engineering and physics. Therefore, the concept of symmetry exists in fractional calculus as in many other fields. Sarikaya et al. [13], for the first time, presented a fractional analog of the Hadamard inequality employing the R-L fractional operator. Fractional operators assume an important part in the advancement of fractional calculus. The main motivation for studying fractional integral inequality is because of the ceaseless investigations of some famous inequalities, such as the Ostrowski, Simpson, Hadamard inequalities, etc., via different types of fractional operators as mentioned earlier by various mathematicians. In recent years, many researchers put effort and mind into fractional analysis to carry a novel version and new notions to applied mathematics with different trademarks and attributes in the literature. Fractional analysis is important in modeling; engineering; mathematical biology; financial modeling; fluid flow; transform theory; predicting some natural phenomena such as landslides, tsunamis, and floods; weather forecasting; the health sector industry; and image processing. Due to its widespread importance, the study of fractional calculus has grown into an absorbing field for scientists and mathematicians. Fractional integral inequality theory has several uses and is important in applied mathematics. The Caputo–Fabrizio and Atangana–Baleanu are non-singular kernel derivatives. The non-locality of problems in the physical realm can be treated more successfully with these fractional derivatives. These operators are an attempt to maintain the fractional calculus momentum and to add the most appropriate operators to the conversation. For the literature, see the references [14–17].

The prime aim of this paper is to investigate and examine the Caputo–Fabrizio fractional variant of the Hadamard- and Pachpatte-type inequalities pertaining to preinvex functions in a polynomial sense. In light of the aforementioned theme and as encouraged by extensive research endeavors, the construction of this work is arranged in the following manner. First of all, in Section 2, we add and elaborate some known definitions, remarks, and theorems because all these are needed in our investigation in further sections. In Section 3, we derive the Hadamard inequality via the n-polynomial preinvex function pertaining to the Caputo–Fabrizio fractional integral operator. In Sections 4 and 5, we investigate a new integral equality, and employing this equality, we present some novel refinements of the Hermite–Hadamard-type inequality involving the prerinvex function in a polynomial sense pertaining to a fractional integral operator. In Section 6, we construct a new variant of the Pachpatte-type inequality involving the Caputo–Fabrizio operator via the n-polynomial preinvex function. Finally, in Section 7, we provide some means-type applications in support of the established results.

2. Preliminaries

For the sake of readers' interest, clarity, significance, and quality, it will be advisable to explore and concentrate on a couple of definitions, remarks, and theorems in this section. The prime goal of this portion is to explore and investigate certain related concepts and definitions that are necessary for our analysis in upcoming sections of this manuscript. The term convex function and the general form of the Hermite–Hadamard inequality in the frame of classical calculus are examined and explored first. Furthermore, we recall the n-polynomial convex function and related theorem, invex set, and preinvex function with examples. In addition, the Sobolev space and Caputo–Fabrizio fractional integral operator are added. We sum up this section by remembering the n-polynomial preinvex function, which will be necessary in our investigations.

In 1905, Jensen explored the convex function in the following way:

Definition 1 ([18,19])**.** *A function $\mathcal{H} : [\mathsf{w}_1, \mathsf{w}_2] \to \mathbb{R}$ is called convex if*

$$\mathcal{H}(\mu x + (1-\mu)y) \leq \mu\mathcal{H}(x) + (1-\mu)\mathcal{H}(y),$$

holds true for every $x, y \in [w_1, w_2]$ *and* $\mu \in [0, 1]$.

Theorem 1 (see [20]). *Consider* $\mathcal{H} : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ *to be a convex function with* $w_1 < w_2$ *and* $w_1, w_2 \in \mathbb{I}$. *Then, the following inequality holds true:*

$$\mathcal{H}\left(\frac{w_1 + w_2}{2}\right) \leq \frac{1}{w_2 - w_1} \int_{w_1}^{w_2} \mathcal{H}(x)dx \leq \frac{\mathcal{H}(w_1) + \mathcal{H}(w_2)}{2}. \tag{1}$$

For some recent generalizations, see [21–23] and the references cited therein.

In 2020, İşcan [24] introduced a new family of generalized convex functions in Definition 2.

Definition 2. *Assume that* $\mathcal{H} : \mathbb{I} \to \mathbb{R}$. *Then,* $\mathcal{H}$ *is an n-polynomial convex function if*

$$\mathcal{H}(\mu x + (1-\mu)y) \leq \frac{1}{n} \sum_{s=1}^{n} \left(1 - (1-\mu)^s\right) \mathcal{H}(x) + \frac{1}{n} \sum_{s=1}^{n} \left(1 - \mu^s\right) \mathcal{H}(y),$$

holds true for all $x, y \in \mathbb{I}$, $n \in \mathbb{N}$, *and* $\mu \in [0, 1]$.

Theorem 2 (see [24]). *Let* $\mathcal{H} : \mathbb{I} \subseteq \mathbb{R} \setminus \{0\}$ *be an n-polynomial convex function. If* $w_1, w_2 \in \mathbb{I}$ *with* $w_1 < w_2$ *and* $\mathcal{H} \in L[w_1, w_2]$, *then*

$$\frac{2^{-1}n}{n + 2^{-n} - 1} \mathcal{H}\left(\frac{w_1 + w_2}{2}\right) \leq \frac{1}{w_2 - w_1} \int_{w_1}^{w_2} \mathcal{H}(x)dx \leq \frac{\mathcal{H}(w_1) + \mathcal{H}(w_2)}{n} \sum_{s=1}^{n} \frac{s}{s+1}. \tag{2}$$

Remark 1. *It is worthy to note that the inequality* (2) *is a more general inequality and that inequality* (1) *is a specific instance of inequality* (2) *when "*$n = 1$*".*

Definition 3 ([25]). *Let us assume that* $\mathbb{I} \subset \mathbb{R}^n$, *and then* $\mathbb{I}$ *is invex with respect to* $\xi(.,.)$, *if* $\forall\, w_1, w_2 \in \mathbb{I}$, *and* $\mu \in [0, 1]$

$$w_1 + \mu\xi(w_2, w_1) \in \mathbb{I}.$$

Remark 2. *It is obvious that every invex set is not a convex set, but the converse holds true [26].*

Definition 4. *A function* $\mathcal{H} : \mathbb{I} \longrightarrow \mathbb{R}$ *is said to be a preinvex function with respect to the bifunction* ξ, *if the following inequality holds true.*

$$\mathcal{H}(w_1 + \mu\xi(w_2, w_1)) \leq (1-\mu)\mathcal{H}(w_1) + \mu\mathcal{H}(w_2),\ \forall w_1, w_2 \in \mathbb{I}, \mu \in [0, 1].$$

Remark 3. *It is clear that the concept of preinvexity is more generalized as compared to the convex function. So, we say that every convex function is a preinvex function, but every preinvex function is not a convex function [27].*
For example, $\mathcal{H}(\mu) = -|\mu|$, $\forall \mu \in \mathbb{R}$, *is a preinvex function with respect to*

$$\xi(w_2, w_1) = \begin{cases} w_2 - w_1 & if\ \ w_1 w_2 \geq 0, \\ w_1 - w_2 & if\ \ w_1 w_2 < 0, \end{cases}$$

but not a convex function.

The following condition-C was introduced for the first time and used by Mohan and Neogy [28] in 1995. This condition has played a decisive role in the development of the theory of optimization and inequalities, see [29,30] and references therein.

Condition-C [28]: Assume that $\mathbb{I}$ is an open invex subset of $\mathbb{R}^n$ with respect to $\xi : \mathbb{I} \times \mathbb{I} \to \mathbb{R}$. $\forall\ \mathrm{w}_1, \mathrm{w}_2 \in \mathbb{I}$ and $\mu \in [0,1]$

$$\begin{aligned} \xi(\mathrm{w}_2, \mathrm{w}_2 + \mu\xi(\mathrm{w}_1, \mathrm{w}_2)) &= -\mu\xi(\mathrm{w}_1, \mathrm{w}_2) \\ \xi(\mathrm{w}_1, \mathrm{w}_2 + \mu\xi(\mathrm{w}_1, \mathrm{w}_2)) &= (1-\mu)\ \xi(\mathrm{w}_1, \mathrm{w}_2), \end{aligned} \tag{3}$$

$\forall\ \mathrm{w}_1, \mathrm{w}_2 \in \mathbb{I}$ and $\mu_1, \mu_2 \in [0,1]$.

From the above equations, we have

$$\xi(\mathrm{w}_2 + \mu_2\ \xi(\mathrm{w}_1, \mathrm{w}_2)\ ,\ \mathrm{w}_2 + \mu_1\xi(\mathrm{w}_1, \mathrm{w}_2)) = (\mu_2 - \mu_1)\xi(\mathrm{w}_1, \mathrm{w}_2).$$

In the year 2014, Noor [31] obtained the Hadamard inequality via a preinvex function, which is given as follows:

Theorem 3 (see [31])**.** *Assume that $\mathcal{H} : \mathbb{I} = [\mathrm{w}_1, \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)] \to (0, \infty)$ is a preinvex function on $\mathbb{I}^\circ$ and $\mathrm{w}_1, \mathrm{w}_2 \in \mathbb{I}^\circ$ with $\mathrm{w}_1 < \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)$. Then,*

$$\mathcal{H}\left(\frac{2\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)}{2}\right) \leq \frac{1}{\xi(\mathrm{w}_2, \mathrm{w}_1)} \int_{\mathrm{w}_1}^{\mathrm{w}_1 + \xi(\nu_{12}, \mathrm{w}_1)} \mathcal{H}(x)dx \leq \frac{\mathcal{H}(\mathrm{w}_1) + \mathcal{H}(\mathrm{w}_2)}{2}. \tag{4}$$

Interested readers can likewise refer to [32–37] and the references included therein for some recent trends identified with preinvexity. Here, we offer a few crucial definitions from the theory of fractional calculus which are implemented in the results that follow.

Definition 5 (see [38])**.** *Let $p \in [1, \infty)$ and $(\mathrm{w}_1, \mathrm{w}_2)$ be an open subset of $\mathbb{R}$, and then the Sobolev space $H^p(\mathrm{w}_1, \mathrm{w}_2)$ is defined by*

$$H^p(\mathrm{w}_1, \mathrm{w}_2) = \{\mathcal{H} \in L^2(\mathrm{w}_1, \mathrm{w}_2) : \Delta^u \mathcal{H} \in L^2(\mathrm{w}_1, \mathrm{w}_2), \forall\ \ |u| \leq p\}.$$

Definition 6 (see [39])**.** *Let $\mathcal{H} \in H'(\mathrm{w}_1, \mathrm{w}_2)$, $\lambda \in [0,1]$ and $\mathrm{w}_1 < \mathrm{w}_2$, and then the Caputo–Fabrizio fractional integral in the left sense becomes*

$$\left({}^{\mathcal{CF}}_{\mathrm{w}_1}\mathrm{I}^{\lambda}\mathcal{H}\right)(\mu) = \frac{(1-\lambda)}{\xi(\lambda)}\mathcal{H}(\mu) + \frac{\lambda}{\xi(\lambda)} \int_{\mathrm{w}_1}^{\mu} \mathcal{H}(x)dx.$$

Similarly, the right fractional integral is defined as

$$\left({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\mathrm{w}_2}\mathcal{H}\right)(\mu) = \frac{(1-\lambda)}{\xi(\lambda)}\mathcal{H}(\mu) + \frac{\lambda}{\xi(\lambda)} \int_{\mu}^{\mathrm{w}_2} \mathcal{H}(x)dx.$$

Note: where $\xi(\lambda) > 0$ is a normalization function verifying this condition $\xi(0) = \xi(1) = 1$. Recent generalizations of inequalities in an integral sense via fractional operators can exist in the literature (see, for example, [40–43]) and the references included therein.

In this paper, we generalize an equality-employing Caputo–Fabrizio operator which is presented by Dragomir et al. [44].

Theorem 4 (see [39])**.** *Assume that $\mathcal{H} : \mathbb{I} \to \mathbb{R}$ is a convex function on $\mathbb{I}$. If $\mathrm{w}_1, \mathrm{w}_2 \in \mathbb{I}$ with $\mathrm{w}_1 < \mathrm{w}_2$ and $\mathcal{H} \in \mathcal{L}[\mathrm{w}_1, \mathrm{w}_2]$, then*

$$\begin{aligned} &\mathcal{H}\left(\frac{\mathrm{w}_1 + \mathrm{w}_2}{2}\right) \leq \frac{\xi(\lambda)}{\lambda(\mathrm{w}_2 - \mathrm{w}_1)}\left[\left({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\mathrm{w}_1}\mathcal{H}\right)(k) + \left({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\mathrm{w}_2}\mathcal{H}\right)(k) - \frac{2(1-\lambda)}{\xi(\lambda)}\mathcal{H}(k)\right] \\ &\leq \frac{\mathcal{H}(\mathrm{w}_1) + \mathcal{H}(\mathrm{w}_2)}{2}, \end{aligned} \tag{5}$$

where $\lambda \in [0,1]$ and $k \in [\mathrm{w}_1, \mathrm{w}_2]$.

Definition 7 ([45])**.** *Assume that $n \in \mathbb{N}$. A non-negative function $\mathcal{H} : \mathbb{I} \to \mathbb{R}$ is an n-polynomial preinvex on $\mathbb{I}$, if*

$$\mathcal{H}(\mathrm{w}_1 + \mu\xi(\mathrm{w}_2, \mathrm{w}_1)) \leq \frac{1}{n}\sum_{\mathrm{s}=1}^{n}[1 - \mu^{\mathrm{s}}]\mathcal{H}(\mathrm{w}_1) + \frac{1}{n}\sum_{\mathrm{s}=1}^{n}[1 - (1-\mu)^{\mathrm{s}}]\mathcal{H}(\mathrm{w}_2), \tag{6}$$

for all $\mathrm{w}_1, \mathrm{w}_2 \in \mathbb{I}$, and $\mu \in [0,1]$.

3. Hermite–Hadamard Inequality Involving Generalized Preinvex Function Pertaining to Caputo–Fabrizio Operator

Since the idea of the subject of convex analysis was initially raised over a century ago, numerous remarkable inequalities have been demonstrated in the field of convex analysis. The Hermite–Hadamard inequality is the most famous and widely utilized inequality in the domain of convex theory. This inequality was first proposed by Hermite and Hadamard. It has a fascinating geometric interpretation and a broad range of applicable importance. The concept of this inequality excited a lot of mathematicians to explore and examine the classical inequalities using the different senses of convexities. For example, Kirmaci [46], Mehreen [47], and Xi [48] presented some new variants of this inequality pertaining to convex functions. Ozcan [49], Dragomir [50], and Hudzik [51] utilized the concept of generalized convexity, namely the s-convex function, and proved a new type of this inequality. Rashid [52] and Butt [53] constructed this inequality via a new class of convexity in the polynomial sense.

The prime objective of this portion is to derive and prove the Hermite–Hadamard inequality via a preinvex function in a polynomial sense pertaining to the Caputo–Fabrizio operator. For this, first, we derive the Hermite–Hadamard inequality via an n-polynomial preinvex function.

Theorem 5. *Let $\mathbb{X}^\circ \subseteq \mathbb{R}$ be an open invex subset with respect to $\xi : \mathbb{X}^\circ \times \mathbb{X}^\circ \times (0,1] \to \mathbb{R}$. Let $\mathrm{w}_1, \mathrm{w}_2 \in \mathbb{X}^\circ$, $\mathrm{w}_1 < \mathrm{w}_2$ with $\mathrm{w}_1 \leq \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)$. Suppose $\mathcal{H} : [\mathrm{w}_1, \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)] \to \mathbb{R}$ is an $\mathfrak{n}$-polynomial preinvex function and satisfies condition-C, then the following Hermite–Hadamard-type inequalities hold:*

$$\begin{aligned}
&\frac{1}{2}\left(\frac{n}{n + 2^{-n} - 1}\right)\mathcal{H}\left(\mathrm{w}_1 + \frac{1}{2}\xi(\mathrm{w}_2, \mathrm{w}_1)\right) \\
&\leq \frac{1}{\xi(\mathrm{w}_2, \mathrm{w}_1)}\int_{\mathrm{w}_1}^{\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)}\mathcal{H}(x)dx \leq \frac{\mathcal{H}(\mathrm{w}_1) + \mathcal{H}(\mathrm{w}_2)}{n}\sum_{\mathrm{s}=1}^{n}\frac{\mathrm{s}}{\mathrm{s}+1}.
\end{aligned} \tag{7}$$

Proof. From the definition of the n-polynomial preinvex function of $\mathcal{H}$, we have that

$$\mathcal{H}(\mathrm{w}_1 + \mu\xi(\mathrm{w}_2, \mathrm{w}_1)) \leq \frac{1}{n}\sum_{\mathrm{s}=1}^{n}[1 - \mu^{\mathrm{s}}]\mathcal{H}(\mathrm{w}_1) + \frac{1}{n}\sum_{\mathrm{s}=1}^{n}[1 - (1-\mu)^{\mathrm{s}}]\mathcal{H}(\mathrm{w}_2),$$

$$\int_0^1 \mathcal{H}(\mathrm{w}_1 + \mu\xi(\mathrm{w}_2, \mathrm{w}_1))d\mu \leq \frac{\mathcal{H}(\mathrm{w}_1)}{\mathfrak{n}}\sum_{\mathrm{s}=1}^{n}\int_0^1[1 - \mu^{\mathrm{s}}]d\mu + \frac{\mathcal{H}(\mathrm{w}_2)}{n}\sum_{\mathrm{s}=1}^{n}\int_0^1[1 - (1-\mu)^{\mathrm{s}}]d\mu.$$

However,

$$\int_0^1 \mathcal{H}(\mathrm{w}_1 + \mu\xi(\mathrm{w}_2, \mathrm{w}_1))d\mu = \frac{1}{\xi(\mathrm{w}_2, \mathrm{w}_1)}\int_{\mathrm{w}_1}^{\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)}\mathcal{H}(x)dx$$

so

$$\frac{1}{\xi(\mathrm{w}_2, \mathrm{w}_1)}\int_{\mathrm{w}_1}^{\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)}\mathcal{H}(x)dx \leq \frac{\mathcal{H}(\mathrm{w}_1) + \mathcal{H}(\mathrm{w}_2)}{n}\sum_{\mathrm{s}=1}^{n}\frac{\mathrm{s}}{\mathrm{s}+1}.$$

This completes the right side of the above inequality. For the left side, use the property of the $\mathfrak{n}$-polynomial preinvex function and condition-C for ξ and integrate over $[0,1]$.

$$\begin{aligned}
&\mathcal{H}\Big(\mathsf{w}_1+\frac{1}{2}\xi(\mathsf{w}_2,\mathsf{w}_1)\Big)\\
&=\mathcal{H}(\mathsf{w}_1+\mu\xi(\mathsf{w}_2,\mathsf{w}_1))+\frac{1}{2}\xi(\mathsf{w}_1+(1-\mu)\xi(\mathsf{w}_2,\mathsf{w}_1),\mathsf{w}_1+\mu\xi(\mathsf{w}_2,\mathsf{w}_1))\\
&\le\frac{1}{n}\sum_{\mathsf{s}=1}^{n}\left[1-\left(\frac{1}{2}\right)^{\mathsf{s}}\right]\left[\int_0^1\mathcal{H}(\mathsf{w}_1+\mu\xi(\mathsf{w}_2,\mathsf{w}_1))d\mu+\int_0^1\mathcal{H}(\mathsf{w}_1+(1-\mu)\xi(\mathsf{w}_2,\mathsf{w}_1))d\mu\right]\\
&\le\frac{1}{n}\sum_{\mathsf{s}=1}^{n}\left[1-\left(\frac{1}{2}\right)^{\mathsf{s}}\right]\frac{2}{\xi(\mathsf{w}_2,\mathsf{w}_1)}\int_{\mathsf{w}_1}^{\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)}\mathcal{H}(x)dx\\
&\le\left[\frac{n+2^{-n}-1}{n}\right]\frac{2}{\xi(\mathsf{w}_2,\mathsf{w}_1)}\int_{\mathsf{w}_1}^{\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)}\mathcal{H}(x)dx.
\end{aligned}$$

This completes the proof. □

Theorem 6. *Assume that* $\mathcal{H}:\mathbb{I}=[\mathsf{w}_1,\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)]\to\mathbb{R}$ *is an n-polynomial preinvex function on* $\mathbb{I}$ *such that* $\mathsf{w}_1,\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)\in\mathbb{I}$ *with* $\xi(\mathsf{w}_2,\mathsf{w}_1)>0$*. If* $\mathcal{H}\in\mathcal{L}[\mathsf{w}_1,\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)]$*, then*

$$\begin{aligned}
&\frac{2^{-1}n}{n+2^{-n}-1}\mathcal{H}\left(\frac{2\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)}{2}\right)\\
&\le\frac{\xi(\lambda)}{\lambda\xi(\mathsf{w}_2,\mathsf{w}_1)}\left[\left({}^{\mathcal{CF}}_{\mathsf{w}_1}\mathrm{I}^{\lambda}\mathcal{H}\right)(k)+\left({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)}\mathcal{H}\right)(k)-\frac{2(1-\lambda)}{\xi(\lambda)}\mathcal{H}(k)\right]\\
&\le\frac{\mathcal{H}(\mathsf{w}_1)+\mathcal{H}(\mathsf{w}_2)}{n}\sum_{\mathsf{s}=1}^{n}\frac{\mathsf{s}}{\mathsf{s}+1},
\end{aligned}$$

where $\lambda\in[0,1]$ *and* $k\in[\mathsf{w}_1,\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)]$.

Proof. Applying the property of the n-polynomial preinvex function, it follows from the inequality (7) that

$$\begin{aligned}
\frac{2^{-1}n}{n+2^{-n}-1}\mathcal{H}\left(\frac{2\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)}{2}\right)&\le\frac{2}{\xi(\mathsf{w}_2,\mathsf{w}_1)}\int_{\mathsf{w}_1}^{\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)}\mathcal{H}(x)dx\\
&=\frac{2}{\xi(\mathsf{w}_2,\mathsf{w}_1)}\left(\int_{\mathsf{w}_1}^{k}\mathcal{H}(x)dx+\int_{k}^{\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)}\mathcal{H}(x)dx\right).
\end{aligned}\tag{8}$$

Multiplying both sides of (8) by $\frac{\lambda\xi(\mathsf{w}_2,\mathsf{w}_1)}{2\xi(\lambda)}$ gives

$$\begin{aligned}
&\frac{\lambda\xi(\mathsf{w}_2,\mathsf{w}_1)}{2\xi(\lambda)}\frac{2^{-1}n}{n+2^{-n}-1}\mathcal{H}\left(\frac{2\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)}{2}\right)\\
&\le\frac{\lambda}{\xi(\lambda)}\left(\int_{\mathsf{w}_1}^{k}\mathcal{H}(x)dx+\int_{k}^{\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)}\mathcal{H}(x)dx\right).
\end{aligned}\tag{9}$$

Adding $\frac{2(1-\lambda)}{\xi(\lambda)}\mathcal{H}(k)$ to both sides of (9)

$$\begin{aligned}
&\frac{2(1-\lambda)}{\xi(\lambda)}\mathcal{H}(k)+\frac{\lambda\xi(\mathrm{w}_2,\mathrm{w}_1)}{2\xi(\lambda)}\frac{2^{-1}n}{n+2^{-n}-1}\mathcal{H}\left(\frac{2\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)}{2}\right)\\
&\quad\leq\frac{2(1-\lambda)}{\xi(\lambda)}\mathcal{H}(k)+\frac{\lambda}{\xi(\lambda)}\left(\int_{\mathrm{w}_1}^{k}\mathcal{H}(x)dx+\int_{k}^{\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)}\mathcal{H}(x)dx\right)\\
&\quad=\left(\frac{(1-\lambda)}{\xi(\lambda)}\mathcal{H}(k)+\frac{\lambda}{\xi(\lambda)}\int_{\mathrm{w}_1}^{k}\mathcal{H}(x)dx\right)\\
&\quad+\left(\frac{(1-\lambda)}{\xi(\lambda)}\mathcal{H}(k)+\frac{\lambda}{\xi(\lambda)}\int_{k}^{\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)}\mathcal{H}(x)dx\right)\\
&\quad=\left({}^{\mathcal{CF}}_{\mathrm{w}_1}\mathrm{I}^{\lambda}\mathcal{H}\right)(k)+\left({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)}\mathcal{H}\right)(k).
\end{aligned}\tag{10}$$

On the other hand, from the inequality (7), we have

$$\frac{2}{\xi(\mathrm{w}_2,\mathrm{w}_1)}\int_{\mathrm{w}_1}^{\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)}\mathcal{H}(x)dx\leq\frac{\mathcal{H}(\mathrm{w}_1)+\mathcal{H}(\mathrm{w}_2)}{n}\sum_{\mathrm{s}=1}^{n}\frac{2\mathrm{s}}{\mathrm{s}+1}.\tag{11}$$

If we multiply (11) by $\frac{\lambda\xi(\mathrm{w}_2,\mathrm{w}_1)}{2\xi(\lambda)}$ and add $\frac{2(1-\lambda)}{\xi(\lambda)}\mathcal{H}(k)$ to the resulting inequality, we obtain

$$\begin{aligned}
&\left({}^{\mathcal{CF}}_{\mathrm{w}_1}\mathrm{I}^{\lambda}\mathcal{H}\right)(k)+\left({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)}\mathcal{H}\right)(k)\\
&\leq\frac{\lambda\xi(\mathrm{w}_2,\mathrm{w}_1)}{\xi(\lambda)}\frac{\mathcal{H}(\mathrm{w}_1)+\mathcal{H}(\mathrm{w}_2)}{n}\sum_{\mathrm{s}=1}^{n}\frac{\mathrm{s}}{\mathrm{s}+1}+\frac{2(1-\lambda)}{\xi(\lambda)}\mathcal{H}(k).
\end{aligned}\tag{12}$$

Upon combining (10) and (12), we attain the required result. □

Corollary 1. *Assume that $n=1$, Theorem 6 becomes*

$$\begin{aligned}
&\mathcal{H}\left(\frac{2\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)}{2}\right)\\
&\leq\frac{\xi(\lambda)}{\lambda\xi(\mathrm{w}_2,\mathrm{w}_1)}\left[\left({}^{\mathcal{CF}}_{\mathrm{w}_1}\mathrm{I}^{\lambda}\mathcal{H}\right)(k)+\left({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)}\mathcal{H}\right)(k)-\frac{2(1-\lambda)}{\xi(\lambda)}\mathcal{H}(k)\right]\leq\frac{\mathcal{H}(\mathrm{w}_1)+\mathcal{H}(\mathrm{w}_2)}{2}.
\end{aligned}$$

Remark 4. *If we put $\xi(\mathrm{w}_2,\mathrm{w}_1)=\mathrm{w}_2-\mathrm{w}_1$ in Theorem 6, it collapses to (Theorem 4, [40]).*

Remark 5. *Assume that $n=1$ and $\xi(\mathrm{w}_2,\mathrm{w}_1)=\mathrm{w}_2-\mathrm{w}_1$ in Theorem 6; it reduces to Theorem 4.*

4. Refinements of Hermite–Hadamard-Type Inequalities Using Hölder and Power Mean Inequality

In the domain of convex theory, many mathematicians have recently collaborated on innovative ideas related to this problem from multiple viewpoints. It is also completely amazing to mark that with the association and combination of convex theory, some of the standard integral inequalities in the frame of classical and fractional calculus for means can be deduced and attained from a well-known inequality, namely the Hadamard inequality. The current findings on Hermite–Hadamard inequalities for convex functions have produced a broad spectrum of developments and refinements. In 2014, Noor [31] used the concept of preinvexity and showed a novel kind of Hadamard-type inequality. After Noor's article, numerous researchers set up new expansions, estimations, and refinements of this inequality in the mode of different versions of preinvexity. For example, Barani et al. [25], for the first time, constructed some generalizations and refinements of this inequality for functions whose derivative of absolute values are preinvex. In 2016, Du et al. [54] examined a couple of refinements of this inequality in the sense of (s,m)-preinvexity.

For the comparable and identical concepts of preinvexity, we mention Wang and Liu [55], Sarikaya et al. [56], Park [57], and Wu et al. [58].

In this section, first, we investigate the lemma for a preinvex function. On the basis of the newly examined lemma, we will add the results with the addition of the Hölder inequality and its extended version, namely the Hölder–İscan inequality, the power mean inequality and its extended version, namely the improved power mean inequality. Throughout this section, the generalized preinvex function represents the n-polynomial preinvex function. To enhance the significance and quality of this section, here we add some corollaries and remarks.

Lemma 1. *Let $\mathcal{H} : \mathbb{I} = [\mathrm{w}_1, \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)] \to \mathbb{R}$ be a differentiable mapping on $\mathbb{I}^\circ$, $\mathrm{w}_1 < \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)$. If $\mathcal{H}' \in \mathcal{L}[\mathrm{w}_1, \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)]$, then*

$$\begin{aligned}
&\frac{\xi(\mathrm{w}_2, \mathrm{w}_1)}{2} \int_0^1 (1-2\mu)\mathcal{H}'(\mathrm{w}_1 + \mu\xi(\mathrm{w}_2, \mathrm{w}_1))d\mu + \frac{2(1-\lambda)}{\lambda\xi(\mathrm{w}_2, \mathrm{w}_1)}\mathcal{H}(k) \\
&= \frac{-\mathcal{H}(\mathrm{w}_1) + \mathcal{H}(\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1))}{2} + \frac{\xi(\lambda)}{\lambda\xi(\mathrm{w}_2, \mathrm{w}_1)}\Big[{}^{\mathcal{CF}}_{\mathrm{w}_1}\mathrm{I}^\lambda \mathcal{H}(k) + {}^{\mathcal{CF}}\mathrm{I}^\lambda_{\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)}\mathcal{H}(k)\Big],
\end{aligned}$$

where $\lambda \in [0, 1]$ and $k \in [\mathrm{w}_1, \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)]$.

Proof. It is easy to see that

$$\begin{aligned}
&\int_0^1 (1-2\mu)\mathcal{H}'(\mathrm{w}_1 + \mu\xi(\mathrm{w}_2, \mathrm{w}_1))d\mu \\
&= -\frac{\mathcal{H}(\mathrm{w}_1) + \mathcal{H}(\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1))}{\xi(\mathrm{w}_2, \mathrm{w}_1)} + \frac{2}{(\xi(\mathrm{w}_2, \mathrm{w}_1))^2}\left(\int_{\mathrm{w}_1}^k \mathcal{H}(x)dx + \int_k^{\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)} \mathcal{H}(x)dx\right).
\end{aligned}$$

Multiplying both sides with $\frac{\lambda(\xi(\mathrm{w}_2,\mathrm{w}_1))^2}{2\xi(\lambda)}$ and adding $\frac{2(1-\lambda)}{\xi(\lambda)}\mathcal{H}(k)$, we have

$$\begin{aligned}
&\frac{\lambda(\xi(\mathrm{w}_2, \mathrm{w}_1))^2}{2\xi(\lambda)} \int_0^1 (1-2\mu)\mathcal{H}'(\mathrm{w}_1 + \mu\xi(\mathrm{w}_2, \mathrm{w}_1))d\mu + \frac{2(1-\lambda)}{\xi(\lambda)}\mathcal{H}(k) \\
&= -\frac{\lambda\xi(\mathrm{w}_2, \mathrm{w}_1)}{\xi(\lambda)} \frac{\mathcal{H}(\mathrm{w}_1) + \mathcal{H}(\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1))}{2} + \left(\frac{(1-\lambda)}{\xi(\lambda)}\mathcal{H}(k) + \frac{\lambda}{\xi(\lambda)}\int_{\mathrm{w}_1}^k \mathcal{H}(x)dx\right) \\
&+ \left(\frac{(1-\lambda)}{\xi(\lambda)}\mathcal{H}(k) + \frac{\lambda}{\xi(\lambda)}\int_k^{\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)} \mathcal{H}(x)dx\right) \\
&= \frac{-\mathcal{H}(\mathrm{w}_1) + \mathcal{H}(\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1))}{2} \\
&+ \frac{\xi(\lambda)}{\lambda\xi(\mathrm{w}_2, \mathrm{w}_1)}\Big[{}^{\mathcal{CF}}_{\mathrm{w}_1}\mathrm{I}^\lambda \mathcal{H}(k) + {}^{\mathcal{CF}}\mathrm{I}^\lambda_{\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)}\mathcal{H}(k)\Big],
\end{aligned}$$

which completes the desired proof. □

Theorem 7. *Suppose that $\mathcal{H} : \mathbb{I} = [\mathrm{w}_1, \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)] \to \mathbb{R}$ is a differentiable function on $\mathbb{I}^\circ$, $\mathrm{w}_1 < \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)$ and $|\mathcal{H}'|$ is a generalized preinvex function on $[\mathrm{w}_1, \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)]$. If $\mathcal{H}' \in \mathcal{L}[\mathrm{w}_1, \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)]$, then*

$$\begin{aligned}
&\Bigg| -\frac{\mathcal{H}(\mathrm{w}_1) + \mathcal{H}(\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1))}{2} - \frac{2(1-\lambda)}{\lambda\xi(\mathrm{w}_2, \mathrm{w}_1)}\mathcal{H}(k) \\
&+ \frac{\xi(\lambda)}{\lambda\xi(\mathrm{w}_2, \mathrm{w}_1)}\Big[\Big({}^{\mathcal{CF}}_{\mathrm{w}_1}\mathrm{I}^\lambda \mathcal{H}\Big)(k) + \Big({}^{\mathcal{CF}}\mathrm{I}^\lambda_{\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)}\mathcal{H}\Big)(k)\Big]\Bigg| \\
&\le \frac{\xi(\mathrm{w}_2, \mathrm{w}_1)}{n} \sum_{\mathrm{s}=1}^n \left[\frac{(\mathrm{s}^2+\mathrm{s}+2)2^\mathrm{s}\mathrm{s} - 2}{(\mathrm{s}+1)(\mathrm{s}+2)2^{\mathrm{s}+1}}\right]\left(\frac{|\mathcal{H}'(\mathrm{w}_1)| + |\mathcal{H}'(\mathrm{w}_2)|}{2}\right), \qquad (13)
\end{aligned}$$

where $\lambda \in [0,1]$ *and* $k \in [\mathsf{w}_1, \mathsf{w}_1 + \xi(\mathsf{w}_2, \mathsf{w}_1)]$.

Proof. Employing Lemma 1, we have

$$\begin{aligned} & \Bigg| - \frac{\mathcal{H}(\mathsf{w}_1) + \mathcal{H}(\mathsf{w}_1 + \xi(\mathsf{w}_2, \mathsf{w}_1))}{2} - \frac{2(1-\lambda)}{\lambda \xi(\mathsf{w}_2, \mathsf{w}_1)} \mathcal{H}(k) \\ & + \frac{\xi(\lambda)}{\lambda \xi(\mathsf{w}_2, \mathsf{w}_1)} \Big[\Big({}^{\mathcal{CF}}_{\mathsf{w}_1}\mathrm{I}^{\lambda} \mathcal{H} \Big)(k) + \Big({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\mathsf{w}_1 + \xi(\mathsf{w}_2, \mathsf{w}_1)} \mathcal{H} \Big)(k) \Big] \Bigg| \\ & \leq \frac{\xi(\mathsf{w}_2, \mathsf{w}_1)}{2} \int_0^1 |1 - 2\mu| |\mathcal{H}'(\mathsf{w}_1 + \mu \xi(\mathsf{w}_2, \mathsf{w}_1))| d\mu. \end{aligned}$$

Employing the property of generalized preinvexity, we have

$$\begin{aligned} & \Bigg| - \frac{\mathcal{H}(\mathsf{w}_1) + \mathcal{H}(\mathsf{w}_1 + \xi(\mathsf{w}_2, \mathsf{w}_1))}{2} - \frac{2(1-\lambda)}{\lambda \xi(\mathsf{w}_2, \mathsf{w}_1)} \mathcal{H}(k) \\ & + \frac{\xi(\lambda)}{\lambda \xi(\mathsf{w}_2, \mathsf{w}_1)} \Big[\Big({}^{\mathcal{CF}}_{\mathsf{w}_1}\mathrm{I}^{\lambda} \mathcal{H} \Big)(k) + \Big({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\mathsf{w}_1 + \xi(\mathsf{w}_2, \mathsf{w}_1)} \mathcal{H} \Big)(k) \Big] \Bigg| \\ & \leq \frac{\xi(\mathsf{w}_2, \mathsf{w}_1)}{2} \int_0^1 |1 - 2\mu| \left(\frac{1}{n} \sum_{\mathsf{s}=1}^{n} (1 - \mu^{\mathsf{s}}) |\mathcal{H}'(\mathsf{w}_1)| + \frac{1}{n} \sum_{\mathsf{s}=1}^{n} (1 - (1-\mu)^{\mathsf{s}}) |\mathcal{H}'(\mathsf{w}_2)| \right) d\mu \\ & \leq \frac{\xi(\mathsf{w}_2, \mathsf{w}_1)}{2n} \left[|\mathcal{H}'(\mathsf{w}_1)| \sum_{\mathsf{s}=1}^{n} \int_0^1 |1 - 2\mu| (1 - \mu^{\mathsf{s}}) d\mu + |\mathcal{H}'(\mathsf{w}_2)| \sum_{\mathsf{s}=1}^{n} \int_0^1 |1 - 2\mu| (1 - (1-\mu)^{\mathsf{s}}) d\mu \right] \\ & = \frac{\xi(\mathsf{w}_2, \mathsf{w}_1)}{2n} \left[|\mathcal{H}'(\mathsf{w}_1)| \sum_{\mathsf{s}=1}^{n} \left[\frac{(\mathsf{s}^2 + \mathsf{s} + 2)2^{\mathsf{s}} - 2}{(\mathsf{s}+1)(\mathsf{s}+2)2^{\mathsf{s}+1}} \right] + |\mathcal{H}'(\mathsf{w}_2)| \sum_{\mathsf{s}=1}^{n} \left[\frac{(\mathsf{s}^2 + \mathsf{s} + 2)2^{\mathsf{s}} - 2}{(\mathsf{s}+1)(\mathsf{s}+2)2^{\mathsf{s}+1}} \right] \right] \\ & = \frac{\xi(\mathsf{w}_2, \mathsf{w}_1)}{n} \sum_{\mathsf{s}=1}^{n} \left[\frac{(\mathsf{s}^2 + \mathsf{s} + 2)2^{\mathsf{s}} - 2}{(\mathsf{s}+1)(\mathsf{s}+2)2^{\mathsf{s}+1}} \right] \left(\frac{|\mathcal{H}'(\mathsf{w}_1)| + |\mathcal{H}'(\mathsf{w}_2)|}{2} \right), \end{aligned}$$

which completes the desired proof. □

Corollary 2. *Assume that* $n = 1$ *in the inequality* (13), *and then we have*

$$\begin{aligned} & \Bigg| - \frac{\mathcal{H}(\mathsf{w}_1) + \mathcal{H}(\mathsf{w}_1 + \xi(\mathsf{w}_2, \mathsf{w}_1))}{2} - \frac{2(1-\lambda)}{\lambda \xi(\mathsf{w}_2, \mathsf{w}_1)} \mathcal{H}(k) \\ & + \frac{\xi(\lambda)}{\lambda \xi(\mathsf{w}_2, \mathsf{w}_1)} \Big[\Big({}^{CF}_{\mathsf{w}_1}\mathrm{I}^{\lambda} \mathcal{H} \Big)(k) + \Big({}^{CF}\mathrm{I}^{\lambda}_{\mathsf{w}_1 + \xi(\mathsf{w}_2, \mathsf{w}_1)} \mathcal{H} \Big)(k) \Big] \Bigg| \\ & \leq \frac{\xi(\mathsf{w}_2, \mathsf{w}_1)(|\mathcal{H}'(\mathsf{w}_1)| + |\mathcal{H}'(\mathsf{w}_2)|)}{8}. \end{aligned}$$

Corollary 3. *If we put* $\xi(\mathsf{w}_2, \mathsf{w}_1) = \mathsf{w}_2 - \mathsf{w}_1$ *in the inequality* (13), *then*

$$\begin{aligned} & \Bigg| - \frac{\mathcal{H}(\mathsf{w}_1) + \mathcal{H}(\mathsf{w}_2)}{2} - \frac{2(1-\lambda)}{\lambda(\mathsf{w}_2 - \mathsf{w}_1)} \mathcal{H}(k) + \frac{\xi(\lambda)}{\lambda(\mathsf{w}_2 - \mathsf{w}_1)} \Big[\Big({}^{CF}_{\mathsf{w}_1}\mathrm{I}^{\lambda} \mathcal{H} \Big)(k) + \Big({}^{CF}\mathrm{I}^{\lambda}_{\mathsf{w}_2} \mathcal{H} \Big)(k) \Big] \Bigg| \\ & \leq \frac{\mathsf{w}_2 - \mathsf{w}_1}{n} \sum_{\mathsf{s}=1}^{n} \left[\frac{(\mathsf{s}^2 + \mathsf{s} + 2)2^{\mathsf{s}} - 2}{(\mathsf{s}+1)(\mathsf{s}+2)2^{\mathsf{s}+1}} \right] \left(\frac{|\mathcal{H}'(\mathsf{w}_1)| + |\mathcal{H}'(\mathsf{w}_2)|}{2} \right). \end{aligned}$$

Remark 6. *If we put* $\xi(\mathsf{w}_2, \mathsf{w}_1) = \mathsf{w}_2 - \mathsf{w}_1$ *and* $n = 1$ *in the inequality* (13), *then we obtain Theorem 5 in [39].*

Theorem 8. *Assume that* $\mathcal{H} : \mathbb{I} = [\mathsf{w}_1, \mathsf{w}_1 + \xi(\mathsf{w}_2, \mathsf{w}_1)] \to \mathbb{R}$ *is a differentiable function on* $\mathbb{I}^\circ$, $\mathsf{w}_1 < \mathsf{w}_1 + \xi(\mathsf{w}_2, \mathsf{w}_1)$ *and* $|\mathcal{H}'|^q$ *is a generalized preinvex function on* $[\mathsf{w}_1, \mathsf{w}_1 + \xi(\mathsf{w}_2, \mathsf{w}_1)]$*. If* $\mathcal{H}' \in \mathcal{L}[\mathsf{w}_1, \mathsf{w}_1 + \xi(\mathsf{w}_2, \mathsf{w}_1)]$*, then*

$$\begin{aligned} &\Bigg| -\frac{\mathcal{H}(\mathsf{w}_1) + \mathcal{H}(\mathsf{w}_1 + \xi(\mathsf{w}_2, \mathsf{w}_1))}{2} - \frac{2(1-\lambda)}{\lambda\xi(\mathsf{w}_2, \mathsf{w}_1)}\mathcal{H}(k) \\ &+ \frac{\xi(\lambda)}{\lambda\xi(\mathsf{w}_2, \mathsf{w}_1)}\Big[\Big({}^{\mathcal{CF}}_{\mathsf{w}_1}\mathrm{I}^{\lambda}\mathcal{H}\Big)(k) + \Big({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)}\mathcal{H}\Big)(k)\Big]\Bigg| \\ &\leq \frac{\xi(\mathsf{w}_2, \mathsf{w}_1)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{\mathsf{s}=1}^{n}\frac{\mathsf{s}}{\mathsf{s}+1}\right)^{\frac{1}{q}}\left(\frac{|\mathcal{H}'(\mathsf{w}_1)|^q + |\mathcal{H}'(\mathsf{w}_2)|^q}{2}\right)^{\frac{1}{q}}, \end{aligned} \tag{14}$$

where $\lambda \in [0,1]$ *and* $k \in [\mathsf{w}_1, \mathsf{w}_1 + \xi(\mathsf{w}_2, \mathsf{w}_1)]$.

Proof. Employing Lemma 1, we have

$$\begin{aligned} &\Bigg| -\frac{\mathcal{H}(\mathsf{w}_1) + \mathcal{H}(\mathsf{w}_1 + \xi(\mathsf{w}_2, \mathsf{w}_1))}{2} - \frac{2(1-\lambda)}{\lambda\xi(\mathsf{w}_2, \mathsf{w}_1)}\mathcal{H}(k) \\ &+ \frac{\xi(\lambda)}{\lambda\xi(\mathsf{w}_2, \mathsf{w}_1)}\Big[\Big({}^{\mathcal{CF}}_{\mathsf{w}_1}\mathrm{I}^{\lambda}\mathcal{H}\Big)(k) + \Big({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)}\mathcal{H}\Big)(k)\Big]\Bigg| \\ &\leq \frac{\xi(\mathsf{w}_2, \mathsf{w}_1)}{2}\int_0^1 |1-2\mu||\mathcal{H}'(\mathsf{w}_1 + \mu\xi(\mathsf{w}_2, \mathsf{w}_1))|d\mu. \end{aligned}$$

Employing the Hölder inequality, we have

$$\begin{aligned} &\Bigg| -\frac{\mathcal{H}(\mathsf{w}_1) + \mathcal{H}(\mathsf{w}_1 + \xi(\mathsf{w}_2, \mathsf{w}_1))}{2} - \frac{2(1-\lambda)}{\lambda\xi(\mathsf{w}_2, \mathsf{w}_1)}\mathcal{H}(k) \\ &+ \frac{\xi(\lambda)}{\lambda\xi(\mathsf{w}_2, \mathsf{w}_1)}\Big[\Big({}^{\mathcal{CF}}_{\mathsf{w}_1}\mathrm{I}^{\lambda}\mathcal{H}\Big)(k) + \Big({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)}\mathcal{H}\Big)(k)\Big]\Bigg| \\ &\leq \frac{\xi(\mathsf{w}_2, \mathsf{w}_1)}{2}\left(\int_0^1 |1-2\mu|^p d\mu\right)^{\frac{1}{p}}\left(\int_0^1 |\mathcal{H}'(\mathsf{w}_1 + \mu\xi(\mathsf{w}_2, \mathsf{w}_1))|^q d\mu\right)^{\frac{1}{q}}. \end{aligned}$$

Employing generalized preinvexity of $|\mathcal{H}'|^q$, we have

$$\begin{aligned} &\Bigg| -\frac{\mathcal{H}(\mathsf{w}_1) + \mathcal{H}(\mathsf{w}_1 + \xi(\mathsf{w}_2, \mathsf{w}_1))}{2} - \frac{2(1-\lambda)}{\lambda\xi(\mathsf{w}_2, \mathsf{w}_1)}\mathcal{H}(k) \\ &+ \frac{\xi(\lambda)}{\lambda\xi(\mathsf{w}_2, \mathsf{w}_1)}\Big[\Big({}^{\mathcal{CF}}_{\mathsf{w}_1}\mathrm{I}^{\lambda}\mathcal{H}\Big)(k) + \Big({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)}\mathcal{H}\Big)(k)\Big]\Bigg| \\ &\leq \frac{\xi(\mathsf{w}_2, \mathsf{w}_1)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{|\mathcal{H}'(\mathsf{w}_1)|^q}{n}\int_0^1 (1-\mu^{\mathsf{s}})d\mu + \frac{|\mathcal{H}'(\mathsf{w}_2)|^q}{n}\int_0^1 (1-(1-\mu)^{\mathsf{s}})d\mu\right]^{\frac{1}{q}} \\ &= \frac{\xi(\mathsf{w}_2, \mathsf{w}_1)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{|\mathcal{H}'(\mathsf{w}_1)|^q}{n}\sum_{\mathsf{s}=1}^{n}\frac{\mathsf{s}}{\mathsf{s}+1} + \frac{|\mathcal{H}'(\mathsf{w}_2)|^q}{n}\sum_{\mathsf{s}=1}^{n}\frac{\mathsf{s}}{\mathsf{s}+1}\right]^{\frac{1}{q}} \\ &= \frac{\xi(\mathsf{w}_2, \mathsf{w}_1)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{\mathsf{s}=1}^{n}\frac{\mathsf{s}}{\mathsf{s}+1}\right)^{\frac{1}{q}}\left(\frac{|\mathcal{H}'(\mathsf{w}_1)|^q + |\mathcal{H}'(\mathsf{w}_2)|^q}{2}\right)^{\frac{1}{q}}. \end{aligned}$$

This completes the proof. □

Corollary 4. *Assume that $n = 1$ in the inequality (14), and then we obtain*

$$\begin{aligned}
&\Bigg| -\frac{\mathcal{H}(\mathrm{w}_1) + \mathcal{H}(\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1))}{2} - \frac{2(1-\lambda)}{\lambda \xi(\mathrm{w}_2, \mathrm{w}_1)} \mathcal{H}(k) \\
&+ \frac{\xi(\lambda)}{\lambda \xi(\mathrm{w}_2, \mathrm{w}_1)} \left[\left({}^{CF}_{\mathrm{w}_1}\mathrm{I}^{\lambda} \mathcal{H} \right)(k) + \left({}^{CF}\mathrm{I}^{\lambda}_{\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)} \mathcal{H} \right)(k) \right] \Bigg| \\
&\leq \frac{\xi(\mathrm{w}_2, \mathrm{w}_1)}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{q}} \left(\frac{|\mathcal{H}'(\mathrm{w}_1)|^q + |\mathcal{H}'(\mathrm{w}_2)|^q}{2} \right)^{\frac{1}{q}}.
\end{aligned}$$

Corollary 5. *If we put $\xi(\mathrm{w}_2, \mathrm{w}_1) = \mathrm{w}_2 - \mathrm{w}_1$ in the inequality (14), then*

$$\begin{aligned}
&\Bigg| -\frac{\mathcal{H}(\mathrm{w}_1) + \mathcal{H}(\mathrm{w}_2)}{2} - \frac{2(1-\lambda)}{\lambda (\mathrm{w}_2 - \mathrm{w}_1)} \mathcal{H}(k) + \frac{\xi(\lambda)}{\lambda (\mathrm{w}_2 - \mathrm{w}_1)} \left[\left({}^{C\mathcal{F}}_{\mathrm{w}_1}\mathrm{I}^{\lambda} \mathcal{H} \right)(k) + \left({}^{C\mathcal{F}}\mathrm{I}^{\lambda}_{\mathrm{w}_2} \mathcal{H} \right)(k) \right] \Bigg| \\
&\leq \frac{\mathrm{w}_2 - \mathrm{w}_1}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{2}{n} \sum_{\mathrm{s}=1}^{n} \frac{\mathrm{s}}{\mathrm{s}+1} \right)^{\frac{1}{q}} \left(\frac{|\mathcal{H}'(\mathrm{w}_1)|^q + |\mathcal{H}'(\mathrm{w}_2)|^q}{2} \right)^{\frac{1}{q}}.
\end{aligned}$$

Remark 7. *If we put $\xi(\mathrm{w}_2, \mathrm{w}_1) = \mathrm{w}_2 - \mathrm{w}_1$ and $n = 1$ in the inequality (14), then we obtain Theorem 6 in [39].*

Theorem 9. *Let $\mathcal{H} : \mathbb{I} = [\mathrm{w}_1, \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)] \to R$ be a differentiable function on $\mathbb{I}^\circ$ and $\mathrm{w}_1, \mathrm{w}_2 \in \mathbb{I}$ with $\mathrm{w}_1 < \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)$ and assume that $\mathcal{H}' \in \mathcal{L}[\mathrm{w}_1, \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)]$. If $|\mathcal{H}'|^q$ is a generalized preinvex of $|\mathcal{H}'|^q$, we obtain*

$$\begin{aligned}
&\Bigg| -\frac{\mathcal{H}(\mathrm{w}_1) + \mathcal{H}(\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1))}{2} - \frac{2(1-\lambda)}{\lambda \xi(\mathrm{w}_2, \mathrm{w}_1)} \mathcal{H}(k) \\
&+ \frac{\xi(\lambda)}{\lambda \xi(\mathrm{w}_2, \mathrm{w}_1)} \left[\left({}^{C\mathcal{F}}_{\mathrm{w}_1}\mathrm{I}^{\lambda} \mathcal{H} \right)(k) + \left({}^{C\mathcal{F}}\mathrm{I}^{\lambda}_{\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)} \mathcal{H} \right)(k) \right] \Bigg| \\
&\leq \frac{\xi(\mathrm{w}_2, \mathrm{w}_1)}{2} \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left(\frac{1}{n} \sum_{\mathrm{s}=1}^{n} \frac{(\mathrm{s}^2 + \mathrm{s} + 2)2^{\mathrm{s}} - 2}{(\mathrm{s}+1)(\mathrm{s}+2)2^{\mathrm{s}+1}} \right)^{\frac{1}{q}} \left(\frac{|\mathcal{H}'(\mathrm{w}_1)|^q + |\mathcal{H}'(\mathrm{w}_2)|^q}{2} \right)^{\frac{1}{q}},
\end{aligned} \tag{15}$$

where $\lambda \in [0, 1]$ and $k \in [\mathrm{w}_1, \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)]$.

Proof. Employing Lemma 1, we have

$$\begin{aligned}
&\Bigg| \frac{-\mathcal{H}(\mathrm{w}_1) + \mathcal{H}(\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1))}{2} - \frac{2(1-\lambda)}{\lambda \xi(\mathrm{w}_2, \mathrm{w}_1)} \mathcal{H}(k) \\
&+ \frac{\xi(\lambda)}{\lambda \xi(\mathrm{w}_2, \mathrm{w}_1)} \left[{}^{C\mathcal{F}}_{\mathrm{w}_1}\mathrm{I}^{\lambda} \mathcal{H}(k) + {}^{C\mathcal{F}}\, \mathrm{I}^{\lambda}_{\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)} \mathcal{H}(k) \right] \Bigg| \\
&\leq \frac{\xi(\mathrm{w}_2, \mathrm{w}_1)}{2} \int_0^1 |1 - 2\mu| \left| \mathcal{H}'(\mathrm{w}_1 + \mu \xi(\mathrm{w}_2, \mathrm{w}_1)) \right| d\mu.
\end{aligned}$$

Employing the power mean inequality, we have

$$\begin{aligned}
&\Bigg| \frac{-\mathcal{H}(\mathrm{w}_1) + \mathcal{H}(\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1))}{2} - \frac{2(1-\lambda)}{\lambda \xi(\mathrm{w}_2, \mathrm{w}_1)} \mathcal{H}(k) \\
&+ \frac{\xi(\lambda)}{\lambda \xi(\mathrm{w}_2, \mathrm{w}_1)} \left[{}^{C\mathcal{F}}_{\mathrm{w}_1}\mathrm{I}^{\lambda} \mathcal{H}(k) + {}^{C\mathcal{F}}\, \mathrm{I}^{\lambda}_{\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)} \mathcal{H}(k) \right] \Bigg| \\
&\leq \frac{\xi(\mathrm{w}_2, \mathrm{w}_1)}{2} \left(\int_0^1 |1 - 2\mu| d\mu \right)^{1-\frac{1}{q}} \left(|1 - 2\mu| \left| \mathcal{H}'(\mathrm{w}_1 + \mu \xi(\mathrm{w}_2, \mathrm{w}_1)) \right|^q d\mu \right)^{\frac{1}{q}}.
\end{aligned}$$

Employing generalized preinvexity of $|\mathcal{H}'|^q$, we have

$$\begin{aligned}
&\Bigg|\frac{-\mathcal{H}(\mathsf{w}_1)+\mathcal{H}(\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1))}{2}-\frac{2(1-\lambda)}{\lambda\xi(\mathsf{w}_2,\mathsf{w}_1)}\mathcal{H}(k)\\
&+\frac{\xi(\lambda)}{\lambda\xi(\mathsf{w}_2,\mathsf{w}_1)}\Big[{}^{\mathcal{CF}}_{\mathsf{w}_1}\mathrm{I}^{\lambda}\mathcal{H}(k)+{}^{\mathcal{CF}}\,\mathrm{I}^{\lambda}_{\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)}\mathcal{H}(k)\Big]\Bigg|\\
&\leq\frac{\xi(\mathsf{w}_2,\mathsf{w}_1)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1|1-2\mu|\left[\frac{1}{n}\sum_{\mathsf{s}=1}^{n}[1-\mu^{\mathsf{s}}]|\mathcal{H}'(\mathsf{w}_1)|^q+\frac{1}{n}\sum_{\mathsf{s}=1}^{n}[1-(1-\mu)^{\mathsf{s}}]|\mathcal{H}'(\mathsf{w}_2)|^q\right]d\mu\right)^{\frac{1}{q}}\\
&\leq\frac{\xi(\mathsf{w}_2,\mathsf{w}_1)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\\
&\times\left(\frac{|\mathcal{H}'(\mathsf{w}_1)|^q}{n}\sum_{\mathsf{s}=1}^{n}\int_0^1|1-2\mu|[1-\mu^{\mathsf{s}}]d\mu+\frac{|\mathcal{H}'(\mathsf{w}_2)|^q}{n}\sum_{\mathsf{s}=1}^{n}\int_0^1|1-2\mu|[1-(1-\mu)^{\mathsf{s}}]d\mu\right)^{\frac{1}{q}}\\
&\leq\frac{\xi(\mathsf{w}_2,\mathsf{w}_1)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\\
&\times\left(\frac{|\mathcal{H}'(\mathsf{w}_1)|^q}{n}\sum_{\mathsf{s}=1}^{n}\frac{(\mathsf{s}^2+\mathsf{s}+2)2^{\mathsf{s}}-2}{(\mathsf{s}+1)(\mathsf{s}+2)2^{\mathsf{s}+1}}+\frac{|\mathcal{H}'(\mathsf{w}_2)|^q}{n}\sum_{\mathsf{s}=1}^{n}\frac{(\mathsf{s}^2+\mathsf{s}+2)2^{\mathsf{s}}-2}{(\mathsf{s}+1)(\mathsf{s}+2)2^{\mathsf{s}+1}}\right)^{\frac{1}{q}}.
\end{aligned}$$

By simplifying, we attain the required result. □

Corollary 6. *Assume that $n=1$, then*

$$\begin{aligned}
&\Bigg|\frac{-\mathcal{H}(\mathsf{w}_1)+\mathcal{H}(\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1))}{2}-\frac{2(1-\lambda)}{\lambda\xi(\mathsf{w}_2,\mathsf{w}_1)}\mathcal{H}(k)\\
&+\frac{\xi(\lambda)}{\lambda\xi(\mathsf{w}_2,\mathsf{w}_1)}\Big[{}^{\mathcal{CF}}_{\mathsf{w}_1}\mathrm{I}^{\lambda}\mathcal{H}(k)+{}^{\mathcal{CF}}\,\mathrm{I}^{\lambda}_{\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)}\mathcal{H}(k)\Big]\Bigg|\\
&\leq\frac{\xi(\mathsf{w}_2,\mathsf{w}_1)}{2}\left(\frac{|\mathcal{H}'(\mathsf{w}_1)|^q+|\mathcal{H}'(\mathsf{w}_2)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

Corollary 7. *If we put $\xi(\mathsf{w}_2,\mathsf{w}_1)=\mathsf{w}_2-\mathsf{w}_1$ in the above Theorem, then*

$$\begin{aligned}
&\left|\frac{\mathcal{H}(\mathsf{w}_1)+\mathcal{H}(\mathsf{w}_2)}{2}-\frac{2(1-\lambda)}{\lambda(\mathsf{w}_2-\mathsf{w}_1)}\mathcal{H}(k)+\frac{\xi(\lambda)}{\lambda(\mathsf{w}_2-\mathsf{w}_1)}\Big[{}^{\mathcal{CF}}_{\mathsf{w}_1}\mathrm{I}^{\lambda}\mathcal{H}(k)+{}^{\mathcal{CF}}\,\mathrm{I}^{\lambda}_{\mathsf{w}_2}\mathcal{H}(k)\Big]\right|\\
&\leq\frac{\mathsf{w}_2-\mathsf{w}_1}{2}(\frac{1}{2})^{1-\frac{1}{q}}\left(\frac{1}{n}\sum_{\mathsf{s}=1}^{n}\frac{(\mathsf{s}^2+\mathsf{s}+2)2^{\mathsf{s}}-2}{(\mathsf{s}+1)(\mathsf{s}+2)2^{\mathsf{s}+1}}\right)^{\frac{1}{q}}\left(\frac{|\mathcal{H}'(\mathsf{w}_1)|^q+|\mathcal{H}'(\mathsf{w}_2)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

Corollary 8. *If we put n=1 and $\xi(\mathsf{w}_2,\mathsf{w}_1)=\mathsf{w}_2-\mathsf{w}_1$ in the above Theorem, then*

$$\begin{aligned}
&\left|\frac{\mathcal{H}(\mathsf{w}_1)+\mathcal{H}(\mathsf{w}_2)}{2}-\frac{2(1-\lambda)}{\lambda(\mathsf{w}_2-\mathsf{w}_1)}\mathcal{H}(k)+\frac{\xi(\lambda)}{\lambda(\mathsf{w}_2-\mathsf{w}_1)}\Big[{}^{CF}_{\mathsf{w}_1}\mathrm{I}^{\lambda}\mathcal{H}(k)+{}^{\mathcal{CF}}\,\mathrm{I}^{\lambda}_{\mathsf{w}_2}\mathcal{H}(k)\Big]\right|\\
&\leq\frac{\mathsf{w}_2-\mathsf{w}_1}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{|\mathcal{H}'(\mathsf{w}_1)|^q+|\mathcal{H}'(\mathsf{w}_2)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

5. Refinements of Hermite–Hadamard-Type Inequalities Using Hölder Iscan and Improved Power Mean Inequality

Numerous mathematicians have recently worked together on innovative approaches to this topic from various angles and multiple perspectives in the field of convex theory and integral inequalities. İşcan [59], for the first time, explored the refinements of the Hölder inequality, namely the Hölder–Iscan integral inequality, in 2019. The improved power mean integral inequality is the refinement form of the power mean inequality. This inequality was introduced for the first time by Kadakal [60] in 2019.

In this section, on the basis of the newly examined lemma, we will add the results with the addition of the Hölder–İscan inequality and improved power mean inequality. To enhance the significance and quality of this section, here we add some corollaries and remarks.

Theorem 10. *Let* $\mathcal{H} : \mathbb{I} = [\mathrm{w}_1, \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)] \to \mathbb{R}$ *be a differentiable function on* $\mathbb{I}^\circ$*, with* $\mathrm{w}_1 < \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)$*,* $q > 1$*,* $p^{-1} + q^{-1} = 1$*, and assume that* $\mathcal{H}' \in \mathcal{L}[\mathrm{w}_1, \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)]$*. If* $|\mathcal{H}'|^q$ *is a generalized preinvex on* $[\mathrm{w}_1, \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)]$*, then*

$$\begin{aligned}
&\bigg| -\frac{\mathcal{H}(\mathrm{w}_1) + \mathcal{H}(\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1))}{2} - \frac{2(1-\lambda)}{\lambda \xi(\mathrm{w}_2, \mathrm{w}_1)} \mathcal{H}(k) \\
&+ \frac{\xi(\lambda)}{\lambda \xi(\mathrm{w}_2, \mathrm{w}_1)} \Big[\Big({}^{\mathcal{CF}}_{\mathrm{w}_1}\mathrm{I}^\lambda \mathcal{H} \Big)(k) + \Big({}^{\mathcal{CF}}\mathrm{I}^\lambda_{\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)} \mathcal{H} \Big)(k) \Big] \bigg| \\
&\leq \frac{\xi(\mathrm{w}_2, \mathrm{w}_1)}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{|\mathcal{H}'(\mathrm{w}_1)|^q}{n} \frac{(n+n^2)}{(2(2+n))} + \frac{|\mathcal{H}'(\mathrm{w}_2)|^q}{n} \sum_{\mathrm{s}=1}^{n} \frac{\mathrm{s}}{2(\mathrm{s}+2)} \right)^{\frac{1}{q}} \\
&+ \frac{\xi(\mathrm{w}_2, \mathrm{w}_1)}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{|\mathcal{H}'(\mathrm{w}_2)|^q}{n} \frac{(n+n^2)}{(2(2+n))} + \frac{|\mathcal{H}'(\mathrm{w}_1)|^q}{n} \sum_{\mathrm{s}=1}^{n} \frac{\mathrm{s}}{2(\mathrm{s}+2)} \right)^{\frac{1}{q}},
\end{aligned}$$

where $\lambda \in [0,1]$ *and* $k \in [\mathrm{w}_1, \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)]$.

Proof. Employing Lemma 1, we have

$$\begin{aligned}
&\bigg| -\frac{\mathcal{H}(\mathrm{w}_1) + \mathcal{H}(\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1))}{2} - \frac{2(1-\lambda)}{\lambda \xi(\mathrm{w}_2, \mathrm{w}_1)} \mathcal{H}(k) \\
&+ \frac{\xi(\lambda)}{\lambda \xi(\mathrm{w}_2, \mathrm{w}_1)} \Big[\Big({}^{\mathcal{CF}}_{\mathrm{w}_1}\mathrm{I}^\lambda \mathcal{H} \Big)(k) + \Big({}^{\mathcal{CF}}\mathrm{I}^\lambda_{\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)} \mathcal{H} \Big)(k) \Big] \bigg| \\
&\leq \frac{\xi(\mathrm{w}_2, \mathrm{w}_1)}{2} \int_0^1 |1 - 2\mu| |\mathcal{H}'(\mathrm{w}_1 + \mu \xi(\mathrm{w}_2, \mathrm{w}_1))| d\mu.
\end{aligned}$$

Employing the Hölder–İşcan inequality, we have

$$\begin{aligned}
&\bigg| -\frac{\mathcal{H}(\mathrm{w}_1) + \mathcal{H}(\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1))}{2} - \frac{2(1-\lambda)}{\lambda \xi(\mathrm{w}_2, \mathrm{w}_1)} \mathcal{H}(k) \\
&+ \frac{\xi(\lambda)}{\lambda \xi(\mathrm{w}_2, \mathrm{w}_1)} \Big[\Big({}^{\mathcal{CF}}_{\mathrm{w}_1}\mathrm{I}^\lambda \mathcal{H} \Big)(k) + \Big({}^{\mathcal{CF}}\mathrm{I}^\lambda_{\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)} \mathcal{H} \Big)(k) \Big] \bigg| \\
&\leq \frac{\xi(\mathrm{w}_2, \mathrm{w}_1)}{2} \left(\int_0^1 (1-\mu) |1 - 2\mu|^p d\mu \right)^{\frac{1}{p}} \left(\int_0^1 (1-\mu) |\mathcal{H}'(\mathrm{w}_1 + \mu \xi(\mathrm{w}_1, \mathrm{w}_2))|^q d\mu \right)^{\frac{1}{q}} \\
&+ \frac{\xi(\mathrm{w}_2, \mathrm{w}_1)}{2} \left(\int_0^1 \mu |1 - 2\mu|^p d\mu \right)^{\frac{1}{p}} \left(\int_0^1 \mu |\mathcal{H}'(\mathrm{w}_1 + \mu \xi(\mathrm{w}_1, \mathrm{w}_2))|^q d\mu \right)^{\frac{1}{q}}.
\end{aligned}$$

Employing generalized preinvexity of $|\mathcal{H}'|^q$, we have

$$\begin{aligned}
&\Bigg| -\frac{\mathcal{H}(\mathrm{w}_1)+\mathcal{H}(\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1))}{2}-\frac{2(1-\lambda)}{\lambda\xi(\mathrm{w}_2,\mathrm{w}_1)}\mathcal{H}(k)\\
&+\frac{\xi(\lambda)}{\lambda\xi(\mathrm{w}_2,\mathrm{w}_1)}\Big[\Big({}^{\mathcal{CF}}_{\mathrm{w}_1}\mathrm{I}^{\lambda}\mathcal{H}\Big)(k)+\Big({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)}\mathcal{H}\Big)(k)\Big]\Bigg|\\
&\le \frac{\xi(\mathrm{w}_2,\mathrm{w}_1)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\\
&\times\left(\frac{|\mathcal{H}'(\mathrm{w}_1)|^q}{n}\sum_{\mathrm{s}=1}^{n}\int_0^1(1-\mu)(1-\mu^{\mathrm{s}})d\mu+\frac{|\mathcal{H}'(\mathrm{w}_2)|^q}{n}\sum_{\mathrm{s}=1}^{n}\int_0^1(1-\mu)(1-(1-\mu)^{\mathrm{s}})d\mu\right)^{\frac{1}{q}}\\
&+\frac{\xi(\mathrm{w}_2,\mathrm{w}_1)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\\
&\times\left(\frac{|\mathcal{H}'(\mathrm{w}_1)|^q}{n}\sum_{\mathrm{s}=1}^{n}\int_0^1\mu(1-\mu^{\mathrm{s}})d\mu+\frac{|\mathcal{H}'(\mathrm{w}_2)|^q}{n}\sum_{\mathrm{s}=1}^{n}\int_0^1\mu(1-(1-\mu)^{\mathrm{s}})d\mu\right)^{\frac{1}{q}}\\
&=\frac{\xi(\mathrm{w}_2,\mathrm{w}_1)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{H}'(\mathrm{w}_1)|^q}{n}\sum_{\mathrm{s}=1}^{n}\frac{\mathrm{s}(\mathrm{s}+3)}{2(\mathrm{s}+1)(\mathrm{s}+2)}+\frac{|\mathcal{H}'(\mathrm{w}_2)|^q}{n}\sum_{\mathrm{s}=1}^{n}\frac{\mathrm{s}}{2(\mathrm{s}+2)}\right)^{\frac{1}{q}}\\
&+\frac{\xi(\mathrm{w}_2,\mathrm{w}_1)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{H}'(\mathrm{w}_2)|^q}{n}\sum_{\mathrm{s}=1}^{n}\frac{\mathrm{s}(\mathrm{s}+3)}{2(\mathrm{s}+1)(\mathrm{s}+2)}+\frac{|\mathcal{H}'(\mathrm{w}_1)|^q}{n}\sum_{\mathrm{s}=1}^{n}\frac{\mathrm{s}}{2(\mathrm{s}+2)}\right)^{\frac{1}{q}}\\
&=\frac{\xi(\mathrm{w}_2,\mathrm{w}_1)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{H}'(\mathrm{w}_1)|^q}{n}\frac{(n+n^2)}{(2(2+n))}+\frac{|\mathcal{H}'(\mathrm{w}_2)|^q}{n}\sum_{\mathrm{s}=1}^{n}\frac{\mathrm{s}}{2(\mathrm{s}+2)}\right)^{\frac{1}{q}}\\
&+\frac{\xi(\mathrm{w}_2,\mathrm{w}_1)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{H}'(\mathrm{w}_2)|^q}{n}\frac{(n+n^2)}{(2(2+n))}+\frac{|\mathcal{H}'(\mathrm{w}_1)|^q}{n}\sum_{\mathrm{s}=1}^{n}\frac{\mathrm{s}}{2(\mathrm{s}+2)}\right)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

Corollary 9. *Assume that $n=1$ in the inequality (14), and then we obtain*

$$\begin{aligned}
&\Bigg| -\frac{\mathcal{H}(\mathrm{w}_1)+\mathcal{H}(\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1))}{2}-\frac{2(1-\lambda)}{\lambda\xi(\mathrm{w}_2,\mathrm{w}_1)}\mathcal{H}(k)\\
&+\frac{\xi(\lambda)}{\lambda\xi(\mathrm{w}_2,\mathrm{w}_1)}\Big[\Big({}^{CF}_{\mathrm{w}_1}\mathrm{I}^{\lambda}\mathcal{H}\Big)(k)+\Big({}^{CF}\mathrm{I}^{\lambda}_{\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)}\mathcal{H}\Big)(k)\Big]\Bigg|\\
&\le\frac{\xi(\mathrm{w}_2,\mathrm{w}_1)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{2|\mathcal{H}'(\mathrm{w}_1)|^q+|\mathcal{H}'(\mathrm{w}_2)|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{H}'(\mathrm{w}_1)|^q+2|\mathcal{H}'(\mathrm{w}_2)|^q}{6}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Corollary 10. *If we put $\xi(\mathrm{w}_2,\mathrm{w}_1)=\mathrm{w}_2-\mathrm{w}_1$ in the inequality (14), then*

$$\begin{aligned}
&\Bigg| -\frac{\mathcal{H}(\mathrm{w}_1)+\mathcal{H}(\mathrm{w}_2)}{2}-\frac{2(1-\lambda)}{\xi(\lambda)}\mathcal{H}(k)+\frac{\xi(\lambda)}{\lambda(\mathrm{w}_2-\mathrm{w}_1)}\Big[\Big({}^{\mathcal{CF}}_{\mathrm{w}_1}\mathrm{I}^{\lambda}\mathcal{H}\Big)(k)+\Big({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)}\mathcal{H}\Big)(k)\Big]\Bigg|\\
&\le\frac{\mathrm{w}_2-\mathrm{w}_1}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{H}'(\mathrm{w}_1)|^q}{n}\frac{(n+n^2)}{(2(2+n))}+\frac{|\mathcal{H}'(\mathrm{w}_2)|^q}{n}\sum_{\mathrm{s}=1}^{n}\frac{\mathrm{s}}{2(\mathrm{s}+2)}\right)^{\frac{1}{q}}\\
&+\frac{\mathrm{w}_2-\mathrm{w}_1}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{H}'(\mathrm{w}_2)|^q}{n}\frac{(n+n^2)}{(2(2+n))}+\frac{|\mathcal{H}'(\mathrm{w}_1)|^q}{n}\sum_{\mathrm{s}=1}^{n}\frac{\mathrm{s}}{2(\mathrm{s}+2)}\right)^{\frac{1}{q}}.
\end{aligned}$$

Corollary 11. *If we put n=1 and $\xi(\mathsf{w}_2,\mathsf{w}_1)=\mathsf{w}_2-\mathsf{w}_1$ in the above Theorem, then*

$$\left|\frac{\mathcal{H}(\mathsf{w}_1)+\mathcal{H}(\mathsf{w}_2)}{2}-\frac{2(1-\lambda)}{\lambda(\mathsf{w}_2-\mathsf{w}_1)}\mathcal{H}(k)+\frac{\xi(\lambda)}{\lambda(\mathsf{w}_2-\mathsf{w}_1)}\left[{}^{\mathcal{CF}}_{\mathsf{w}_1}\mathrm{I}^{\lambda}\mathcal{H}(k)+{}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\mathsf{w}_2}\mathcal{H}(k)\right]\right|$$
$$\leq\frac{\mathsf{w}_2-\mathsf{w}_1}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{2|\mathcal{H}'(\mathsf{w}_1)|^q+|\mathcal{H}'(\mathsf{w}_2)|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{H}'(\mathsf{w}_1)|^q+2|\mathcal{H}'(\mathsf{w}_2)|^q}{6}\right)^{\frac{1}{q}}\right\}.$$

Theorem 11. *Let $\mathcal{H}:\mathbb{I}=[\mathsf{w}_1,\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)]\to\mathbb{R}$ be a differentiable function on $\mathbb{I}^\circ$, with $\mathsf{w}_1<\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)$, $q\geq 1$, and assume that $\mathcal{H}'\in\mathcal{L}[\mathsf{w}_1,\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)]$. If $|\mathcal{H}'|^q$ is an generalized preinvex function on $[\mathsf{w}_1,\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)]$, then*

$$\begin{aligned}&\bigg|-\frac{\mathcal{H}(\mathsf{w}_1)+\mathcal{H}(\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1))}{2}-\frac{2(1-\lambda)}{\lambda\xi(\mathsf{w}_2,\mathsf{w}_1)}\mathcal{H}(k)\\&+\frac{\xi(\lambda)}{\lambda\xi(\mathsf{w}_2,\mathsf{w}_1)}\left[\left({}^{\mathcal{CF}}_{\mathsf{w}_1}\mathrm{I}^{\lambda}\mathcal{H}\right)(k)+\left({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)}\mathcal{H}\right)(k)\right]\bigg|\\&\leq\frac{\xi(\mathsf{w}_2,\mathsf{w}_1)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\\&\times\left(\frac{|\mathcal{H}'(\mathsf{w}_1)|^q}{n}\sum_{\mathsf{s}=1}^{n}\frac{(\mathsf{s}+5)[(\mathsf{s}^2+\mathsf{s}+2)2^{\mathsf{s}}-2]}{2^{\mathsf{s}+2}(\mathsf{s}+1)(\mathsf{s}+2)(\mathsf{s}+3)}+\frac{|\mathcal{H}'(\mathsf{w}_2)|^q}{n}\sum_{\mathsf{s}=1}^{n}\frac{(\mathsf{s}^2+\mathsf{s}+2)2^{\mathsf{s}}-2}{2^{\mathsf{s}+2}(\mathsf{s}+2)(\mathsf{s}+3)}\right)^{\frac{1}{q}}\\&+\frac{\xi(\mathsf{w}_2,\mathsf{w}_1)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\\&\times\left(\frac{|\mathcal{H}'(\mathsf{w}_1)|^q}{n}\sum_{\mathsf{s}=1}^{n}\frac{(\mathsf{s}^2+\mathsf{s}+2)2^{\mathsf{s}}-2}{2^{\mathsf{s}+2}(\mathsf{s}+2)(\mathsf{s}+3)}+\frac{|\mathcal{H}'(\mathsf{w}_2)|^q}{n}\sum_{\mathsf{s}=1}^{n}\frac{(\mathsf{s}+5)[(\mathsf{s}^2+\mathsf{s}+2)2^{\mathsf{s}}-2]}{2^{\mathsf{s}+2}(\mathsf{s}+1)(\mathsf{s}+2)(\mathsf{s}+3)}\right)^{\frac{1}{q}},\end{aligned}$$

where $\lambda\in[0,1]$ and $k\in[\mathsf{w}_1,\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)]$.

Proof. First, assume that $q>1$. According to Lemma 1, we have

$$\begin{aligned}&\bigg|-\frac{\mathcal{H}(\mathsf{w}_1)+\mathcal{H}(\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1))}{2}-\frac{2(1-\lambda)}{\lambda\xi(\mathsf{w}_2,\mathsf{w}_1)}\mathcal{H}(k)\\&+\frac{\xi(\lambda)}{\lambda\xi(\mathsf{w}_2,\mathsf{w}_1)}\left[\left({}^{\mathcal{CF}}_{\mathsf{w}_1}\mathrm{I}^{\lambda}\mathcal{H}\right)(k)+\left({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)}\mathcal{H}\right)(k)\right]\bigg|\\&\leq\frac{\xi(\mathsf{w}_2,\mathsf{w}_1)}{2}\int_0^1|1-2\mu||\mathcal{H}'(\mathsf{w}_1+\mu\xi(\mathsf{w}_2,\mathsf{w}_1))|d\mu.\end{aligned}$$

Employing the property of the improved power mean inequality, we have

$$\begin{aligned}&\bigg|-\frac{\mathcal{H}(\mathsf{w}_1)+\mathcal{H}(\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1))}{2}-\frac{2(1-\lambda)}{\lambda\xi(\mathsf{w}_2,\mathsf{w}_1)}\mathcal{H}(k)\\&+\frac{\xi(\lambda)}{\lambda\xi(\mathsf{w}_2,\mathsf{w}_1)}\left[\left({}^{\mathcal{CF}}_{\mathsf{w}_1}\mathrm{I}^{\lambda}\mathcal{H}\right)(k)+\left({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\mathsf{w}_1+\xi(\mathsf{w}_2,\mathsf{w}_1)}\mathcal{H}\right)(k)\right]\bigg|\\&\leq\frac{\xi(\mathsf{w}_2,\mathsf{w}_1)}{2}\left(\int_0^1(1-\mu)|1-2\mu|d\mu\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\mu)|1-2\mu||\mathcal{H}'(\mathsf{w}_1+\mu\xi(\mathsf{w}_1,\mathsf{w}_2))|^q d\mu\right)^{\frac{1}{q}}\\&+\frac{\xi(\mathsf{w}_2,\mathsf{w}_1)}{2}\left(\int_0^1\mu|1-2\mu|d\mu\right)^{1-\frac{1}{q}}\left(\int_0^1\mu|1-2\mu||\mathcal{H}'(\mathsf{w}_1+\mu\xi(\mathsf{w}_1,\mathsf{w}_2))|^q d\mu\right)^{\frac{1}{q}}.\end{aligned}$$

Employing generalized preinvexity of $|\mathcal{H}'|^q$, we have

$$\begin{aligned}
&\Bigg| -\frac{\mathcal{H}(\mathrm{w}_1)+\mathcal{H}(\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1))}{2}-\frac{2(1-\lambda)}{\lambda\xi(\mathrm{w}_2,\mathrm{w}_1)}\mathcal{H}(k) \\
&+\frac{\xi(\lambda)}{\lambda\xi(\mathrm{w}_2,\mathrm{w}_1)}\Big[\Big({}^{\mathcal{CF}}_{\mathrm{w}_1}\mathrm{I}^{\lambda}\mathcal{H}\Big)(k)+\Big({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)}\mathcal{H}\Big)(k)\Big]\Bigg| \\
&\leq \frac{\xi(\mathrm{w}_2,\mathrm{w}_1)}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}} \\
&\times\left(\frac{|\mathcal{H}'(\mathrm{w}_1)|^q}{n}\sum_{\mathrm{s}=1}^{n}\int_0^1(1-\mu)|1-2\mu|(1-\mu^{\mathrm{s}})d\mu\right. \\
&\left.+\frac{|\mathcal{H}'(\mathrm{w}_2)|^q}{n}\sum_{\mathrm{s}=1}^{n}\int_0^1(1-\mu)|1-2\mu|(1-(1-\mu)^{\mathrm{s}})d\mu\right)^{\frac{1}{q}} \\
&+\frac{\xi(\mathrm{w}_2,\mathrm{w}_1)}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}} \\
&\times\left(\frac{|\mathcal{H}'(\mathrm{w}_1)|^q}{n}\sum_{\mathrm{s}=1}^{n}\int_0^1\mu|1-2\mu|(1-\mu^{\mathrm{s}})d\mu+\frac{|\mathcal{H}'(\mathrm{w}_2)|^q}{n}\sum_{\mathrm{s}=1}^{n}\int_0^1\mu|1-2\mu|(1-(1-\mu)^{\mathrm{s}})d\mu\right)^{\frac{1}{q}} \\
&\leq \frac{\xi(\mathrm{w}_2,\mathrm{w}_1)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}} \\
&\times\left(\frac{|\mathcal{H}'(\mathrm{w}_1)|^q}{n}\sum_{\mathrm{s}=1}^{n}\frac{(\mathrm{s}+5)[(\mathrm{s}^2+\mathrm{s}+2)2^{\mathrm{s}}-2]}{2^{\mathrm{s}+2}(\mathrm{s}+1)(\mathrm{s}+2)(\mathrm{s}+3)}+\frac{|\mathcal{H}'(\mathrm{w}_2)|^q}{n}\sum_{\mathrm{s}=1}^{n}\frac{(\mathrm{s}^2+\mathrm{s}+2)2^{\mathrm{s}}-2}{2^{\mathrm{s}+2}(\mathrm{s}+2)(\mathrm{s}+3)}\right)^{\frac{1}{q}} \\
&+\frac{\xi(\mathrm{w}_2,\mathrm{w}_1)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}} \\
&\times\left(\frac{|\mathcal{H}'(\mathrm{w}_1)|^q}{n}\sum_{\mathrm{s}=1}^{n}\frac{(\mathrm{s}^2+\mathrm{s}+2)2^{\mathrm{s}}-2}{2^{\mathrm{s}+2}(\mathrm{s}+2)(\mathrm{s}+3)}+\frac{|\mathcal{H}'(\mathrm{w}_2)|^q}{n}\sum_{\mathrm{s}=1}^{n}\frac{(\mathrm{s}+5)[(\mathrm{s}^2+\mathrm{s}+2)2^{\mathrm{s}}-2]}{2^{\mathrm{s}+2}(\mathrm{s}+1)(\mathrm{s}+2)(\mathrm{s}+3)}\right)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

Corollary 12. *Assume that $n=1$ in the inequality (14), and then we obtain*

$$\begin{aligned}
&\Bigg| -\frac{\mathcal{H}(\mathrm{w}_1)+\mathcal{H}(\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1))}{2}-\frac{2(1-\lambda)}{\lambda\xi(\mathrm{w}_2,\mathrm{w}_1)}\mathcal{H}(k) \\
&+\frac{\xi(\lambda)}{\lambda\xi(\mathrm{w}_2,\mathrm{w}_1)}\Big[\Big({}^{CF}_{\mathrm{w}_1}\mathrm{I}^{\lambda}\mathcal{H}\Big)(k)+\Big({}^{CF}\mathrm{I}^{\lambda}_{\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)}\mathcal{H}\Big)(k)\Big]\Bigg| \\
&\leq \frac{\xi(\mathrm{w}_2,\mathrm{w}_1)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left\{\left(\frac{3|\mathcal{H}'(\mathrm{w}_1)|^q+|\mathcal{H}'(\mathrm{w}_2)|^q}{16}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{H}'(\mathrm{w}_1)|^q+3|\mathcal{H}'(\mathrm{w}_2)|^q}{16}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Corollary 13. *If we put $\xi(\mathrm{w}_2,\mathrm{w}_1)=\mathrm{w}_2-\mathrm{w}_1$ in the inequality (14), then*

$$
\begin{aligned}
&\bigg| -\frac{\mathcal{H}(\mathrm{w}_1)+\mathcal{H}(\mathrm{w}_2)}{2} - \frac{2(1-\lambda)}{\lambda(\mathrm{w}_2-\mathrm{w}_1)}\mathcal{H}(k) + \frac{\xi(\lambda)}{\lambda(\mathrm{w}_2-\mathrm{w}_1)}\left[\left({}^{\mathcal{CF}}_{\mathrm{w}_1}\mathrm{I}^{\lambda}\mathcal{H}\right)(k) + \left({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)}\mathcal{H}\right)(k)\right]\bigg| \\
&\leq \frac{\mathrm{w}_2-\mathrm{w}_1}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}} \\
&\times \left(\frac{|\mathcal{H}'(\mathrm{w}_1)|^q}{n}\sum_{\mathrm{s}=1}^{n}\frac{(\mathrm{s}+5)[(\mathrm{s}^2+\mathrm{s}+2)2^{\mathrm{s}}-2]}{2^{\mathrm{s}+2}(\mathrm{s}+1)(\mathrm{s}+2)(\mathrm{s}+3)} + \frac{|\mathcal{H}'(\mathrm{w}_2)|^q}{n}\sum_{\mathrm{s}=1}^{n}\frac{(\mathrm{s}^2+\mathrm{s}+2)2^{\mathrm{s}}-2}{2^{\mathrm{s}+2}(\mathrm{s}+2)(\mathrm{s}+3)}\right)^{\frac{1}{q}} \\
&+ \frac{\mathrm{w}_2-\mathrm{w}_1}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}} \\
&\left(\frac{|\mathcal{H}'(\mathrm{w}_1)|^q}{n}\sum_{\mathrm{s}=1}^{n}\frac{(\mathrm{s}^2+\mathrm{s}+2)2^{\mathrm{s}}-2}{2^{\mathrm{s}+2}(\mathrm{s}+2)(\mathrm{s}+3)} + \frac{|\mathcal{H}'(\mathrm{w}_2)|^q}{n}\sum_{\mathrm{s}=1}^{n}\frac{(\mathrm{s}+5)[(\mathrm{s}^2+\mathrm{s}+2)2^{\mathrm{s}}-2]}{2^{\mathrm{s}+2}(\mathrm{s}+1)(\mathrm{s}+2)(\mathrm{s}+3)}\right)^{\frac{1}{q}}.
\end{aligned}
$$

Corollary 14. *Assume that n=1 and $\xi(\mathrm{w}_2,\mathrm{w}_1)=\mathrm{w}_2-\mathrm{w}_1$ in the above Theorem, and then*

$$
\begin{aligned}
&\left|\frac{\mathcal{H}(\mathrm{w}_1)+\mathcal{H}(\mathrm{w}_2)}{2} - \frac{2(1-\lambda)}{\lambda(\mathrm{w}_2-\mathrm{w}_1)}\mathcal{H}(k) + \frac{\xi(\lambda)}{\lambda(\mathrm{w}_2-\mathrm{w}_1)}\left[{}^{\mathcal{CF}}_{\mathrm{w}_1}\mathrm{I}^{\lambda}\mathcal{H}(k) + {}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\mathrm{w}_2}\mathcal{H}(k)\right]\right| \\
&\leq \frac{\mathrm{w}_2-\mathrm{w}_1}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left\{\left(\frac{3|\mathcal{H}'(\mathrm{w}_1)|^q+|\mathcal{H}'(\mathrm{w}_2)|^q}{16}\right)^{\frac{1}{q}} + \left(\frac{|\mathcal{H}'(\mathrm{w}_1)|^q+3|\mathcal{H}'(\mathrm{w}_2)|^q}{16}\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

6. Pachpatte-Type Inequality via *n*-Polynomial Preinvex Function Pertaining to Caputo–Fabrizio Fractional Integral Operator

The term convexity has increasingly attracted significant interest in recent years due to its relation and association with the subject of inequality. In the domain of convex analysis, many inequalities are presented due to implementations of the convexity concept in applied mathematics. Numerous mathematicians have also addressed the term preinvexity, and countless publications have been written that offer fresh estimates, extensions, and generalizations. The remarkable Pachpatte-type inequality for preinvex functions is dramatically improved by these investigations and the research. A key idea in the development of extended convex programming is preinvexity. In 2009, Nian Li [61] addressed the Pachpatte-type integral inequalities of the concept related to time scales. In 2021, the work of Butt [62] was the first to explore Pachpatte–Mercer-type inequalities that involve harmonic convexity in the frame of fractional calculus. In 2022, Sahoo [63] introduced this inequality for a center–radius order involving a preinvex function in the field of interval analysis. This inequality involving a fractional operator with the exponential kernel was studied by Sahoo [64] in 2022. In 2022, Tariq et al. [65] was the first to examine an inequality that involves a non-conformable operator using the concept of generalized preinvexity. In light of the aforementioned literature, we are going to explore and examine the Pachpatte-type inequality for the Caputo–Fabrizio operator. To improve the significance and quality of this section, a corollary and several remarks are added.

Theorem 12. *Assume that $\lambda\in[0,1]$ and $k\in[\mathrm{w}_1,\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)]$. Let $\mathcal{H},\mathcal{G}:\mathbb{I}=[\mathrm{w}_1,\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)]\to\mathbb{R}$ be two functions such that $\mathrm{w}_1,\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)\in\mathbb{I}$ with $\xi(\mathrm{w}_2,\mathrm{w}_1)>0$ and $\mathcal{H}\in\mathcal{L}[\mathrm{w}_1,\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)]$. If $\mathcal{H}$ is an n_1-polynomial preinvex function and $\mathcal{G}$ is an n_2-polynomial preinvex function, then*

$$
\begin{aligned}
&\frac{\xi(\lambda)}{\lambda\xi(\mathrm{w}_2,\mathrm{w}_1)}\left[\left({}^{\mathcal{CF}}_{\mathrm{w}_1}\mathrm{I}^{\lambda}\mathcal{H}\mathcal{G}\right)(k) + \left({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)}\mathcal{H}\mathcal{G}\right)(k) - \frac{2(1-\lambda)}{\xi(\lambda)}\mathcal{H}(k)\mathcal{G}(k)\right] \\
&\leq \mathbb{M}(\mathrm{w}_1,\mathrm{w}_2)+\mathbb{N}(\mathrm{w}_1,\mathrm{w}_2),
\end{aligned}
$$

where

$$\mathbb{M}(\mathrm{w}_1,\mathrm{w}_2) = \Delta_1(\mu)\mathcal{H}(\mathrm{w}_1)\mathcal{G}(\mathrm{w}_1) + \Delta_4(\mu)\mathcal{H}(\mathrm{w}_2)\mathcal{G}(\mathrm{w}_2),$$
$$\mathbb{N}(\mathrm{w}_1,\mathrm{w}_2) = \Delta_2(\mu)\mathcal{H}(\mathrm{w}_2)\mathcal{G}(\mathrm{w}_1) + \Delta_3(\mu)\mathcal{H}(\mathrm{w}_1)\mathcal{G}(\mathrm{w}_2)$$

and

$$\Delta_1(\mu) = \frac{1}{n_1n_2}\sum_{\mathrm{s}=1}^{n_1}[1-\mu^{\mathrm{s}}]\sum_{\mathrm{s}=1}^{n_2}[1-\mu^{\mathrm{s}}], \qquad \Delta_2(\mu) = \frac{1}{n_1n_2}\sum_{\mathrm{s}=1}^{n_1}[1-\mu^{\mathrm{s}}]\sum_{\mathrm{s}=1}^{n_2}[1-(1-\mu)^{\mathrm{s}}]$$
$$\Delta_3(\mu) = \frac{1}{n_1n_2}\sum_{\mathrm{s}=1}^{n_1}[1-(1-\mu)^{\mathrm{s}}]\sum_{\mathrm{s}=1}^{n_2}[1-\mu^{\mathrm{s}}], \quad \Delta_4(\mu) = \frac{1}{n_1n_2}\sum_{\mathrm{s}=1}^{n_1}[1-(1-\mu)^{\mathrm{s}}]\sum_{\mathrm{s}=1}^{n_2}[1-(1-\mu)^{\mathrm{s}}].$$

Proof. Let $\mathcal{H},\mathcal{G}$ be the n_1,n_2-polynomial preinvex functions, respectively, and then for $\mu\in[0,1]$

$$\mathcal{H}(\mathrm{w}_1+\mu\xi(\mathrm{w}_2,\mathrm{w}_1)) \le \frac{1}{n_1}\sum_{\mathrm{s}=1}^{n_1}[1-\mu^{\mathrm{s}}]\mathcal{H}(\mathrm{w}_1) + \frac{1}{n_1}\sum_{\mathrm{s}=1}^{n_1}[1-(1-\mu)^{\mathrm{s}}]\mathcal{H}(\mathrm{w}_2). \tag{16}$$
$$\mathcal{G}(\mathrm{w}_1+\mu\xi(\mathrm{w}_2,\mathrm{w}_1)) \le \frac{1}{n_2}\sum_{\mathrm{s}=1}^{n_2}[1-\mu^{\mathrm{s}}]\mathcal{G}(\mathrm{w}_1) + \frac{1}{n_2}\sum_{\mathrm{s}=1}^{n_2}[1-(1-\mu)^{\mathrm{s}}]\mathcal{G}(\mathrm{w}_2). \tag{17}$$

Multiplying the above inequalities (16) and (17) gives

$$\begin{aligned}
&\mathcal{H}(\mathrm{w}_1+\mu\xi(\mathrm{w}_2,\mathrm{w}_1))\mathcal{G}(\mathrm{w}_1+\mu\xi(\mathrm{w}_2,\mathrm{w}_1))\\
&\le \frac{1}{n_1n_2}\sum_{\mathrm{s}=1}^{n_1}[1-\mu^{\mathrm{s}}]\sum_{\mathrm{s}=1}^{n_2}[1-\mu^{\mathrm{s}}]\mathcal{H}(\mathrm{w}_1)\mathcal{G}(\mathrm{w}_1)\\
&+\frac{1}{n_1n_2}\sum_{\mathrm{s}=1}^{n_1}[1-\mu^{\mathrm{s}}]\sum_{\mathrm{s}=1}^{n_2}[1-(1-\mu)^{\mathrm{s}}]\mathcal{H}(\mathrm{w}_1)\mathcal{G}(\mathrm{w}_2)\\
&+\frac{1}{n_1n_2}\sum_{\mathrm{s}=1}^{n_1}[1-(1-\mu)^{\mathrm{s}}]\sum_{\mathrm{s}=1}^{n_2}[1-\mu^{\mathrm{s}}]\mathcal{H}(\mathrm{w}_2)\mathcal{G}(\mathrm{w}_1)\\
&+\frac{1}{n_1n_2}\sum_{\mathrm{s}=1}^{n_1}[1-(1-\mu)^{\mathrm{s}}]\sum_{\mathrm{s}=1}^{n_2}[1-(1-\mu)^{\mathrm{s}}]\mathcal{H}(\mathrm{w}_2)\mathcal{G}(\mathrm{w}_2)\\
&= \Delta_1(\mu)\mathcal{H}(\mathrm{w}_1)\mathcal{G}(\mathrm{w}_1) + \Delta_2(\mu)\mathcal{H}(\mathrm{w}_2)\mathcal{G}(\mathrm{w}_1)\\
&\quad + \Delta_3(\mu)\mathcal{H}(\mathrm{w}_1)\mathcal{G}(\mathrm{w}_2) + \Delta_4(\mu)\mathcal{H}(\mathrm{w}_2)\mathcal{G}(\mathrm{w}_2).
\end{aligned} \tag{18}$$

Integrate the above last inequality (18) on both sides regarding μ with $[0,1]$ which results in

$$\begin{aligned}
&\frac{1}{\xi(\mathrm{w}_2,\mathrm{w}_1)}\int_{\mathrm{w}_1}^{\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)}\mathcal{H}(x)\mathcal{G}(x)dx\\
&\le \int_0^1[\Delta_1(\mu)\mathcal{H}(\mathrm{w}_1)\mathcal{G}(\mathrm{w}_1) + \Delta_2(\mu)\mathcal{H}(\mathrm{w}_2)\mathcal{G}(\mathrm{w}_1) + \Delta_3(\mu)\mathcal{H}(\mathrm{w}_1)\mathcal{G}(\mathrm{w}_2) + \Delta_4(\mu)\mathcal{H}(\mathrm{w}_2)\mathcal{G}(\mathrm{w}_2)]d\mu\\
&= \mathbb{M}(\mathrm{w}_1,\mathrm{w}_2) + \mathbb{N}(\mathrm{w}_1,\mathrm{w}_2).
\end{aligned}$$

That is,

$$\frac{1}{\xi(\mathrm{w}_2,\mathrm{w}_1)}\left[\int_{\mathrm{w}_1}^{k}\mathcal{H}(x)\mathcal{G}(x)dx + \int_{k}^{\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)}\mathcal{H}(x)\mathcal{G}(x)dx\right] \le \mathbb{M}(\mathrm{w}_1,\mathrm{w}_2) + \mathbb{N}(\mathrm{w}_1,\mathrm{w}_2). \tag{19}$$

Now, multiplying (19) by $\frac{\lambda\xi(\mathrm{w}_2,\mathrm{w}_1)}{\xi(\lambda)}$ and adding $\frac{2(1-\lambda)}{\xi(\lambda)}\mathcal{H}(k)\mathcal{G}(k)$ to the resulting inequality, we obtain

$$\frac{\lambda}{\xi(\lambda)}\left[\int_{\mathrm{w}_1}^{k}\mathcal{H}(x)\mathcal{G}(x)\mathrm{d}x+\int_{k}^{\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)}\mathcal{H}(x)\mathcal{G}(x)\mathrm{d}x\right]+\frac{2(1-\lambda)}{\xi(\lambda)}\mathcal{H}(k)\mathcal{G}(k)$$
$$\leq\frac{\lambda\xi(\mathrm{w}_2,\mathrm{w}_1)}{\xi(\lambda)}[\mathbb{M}(\mathrm{w}_1,\mathrm{w}_2)+N(\mathrm{w}_1,\mathrm{w}_2)]+\frac{2(1-\lambda)}{\xi(\lambda)}\mathcal{H}(k)\mathcal{G}(k).$$

Hence,

$${}^{\mathcal{CF}}_{\mathrm{w}_1}\mathrm{I}^{\lambda}\mathcal{H}(k)\mathcal{G}(k)+{}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)}\mathcal{H}(k)\mathcal{G}(k)$$
$$\leq\frac{\lambda\xi(\mathrm{w}_2,\mathrm{w}_1)}{\xi(\lambda)}[\mathbb{M}(\mathrm{w}_1,\mathrm{w}_2)+N(\mathrm{w}_1,\mathrm{w}_2)]+\frac{2(1-\lambda)}{\xi(\lambda)}\mathcal{H}(k)\mathcal{G}(k).$$

This is the required result. □

Corollary 15. *Assume that $n_1=n_2=1$ in Theorem 12, then*

$$\frac{2\xi(\lambda)}{\lambda\xi(\mathrm{w}_2,\mathrm{w}_1)}\left[\left({}^{CF}_{\mathrm{w}_1}\mathrm{I}^{\lambda}\mathcal{H}\mathcal{G}\right)(k)+\left({}^{CF}\mathrm{I}^{\lambda}_{\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)\mathcal{H}\mathcal{G}}\right)(k)-\frac{2(1-\lambda)}{\xi(\lambda)}\mathcal{H}(k)\mathcal{G}(k)\right]$$
$$\leq\frac{2}{3}M(\mathrm{w}_1,\mathrm{w}_2)+\frac{1}{3}N(\mathrm{w}_1,\mathrm{w}_2).$$

Remark 8. *If we put $\xi(\mathrm{w}_2,\mathrm{w}_1)=\mathrm{w}_2-\mathrm{w}_1$ in Theorem 12, it reduces to Theorem 5 in [40].*

Remark 9. *If we put $\xi(\mathrm{w}_2,\mathrm{w}_1)=\mathrm{w}_2-\mathrm{w}_1$ and $n_1=n_2=1$ in Theorem 12, it reduces to Theorem 3 in [39].*

7. Applications to Means

Our work is the incorporation of fractional calculus and convexity theory. It is evident from the literature that both concepts have many applications in different domains of science, starting from optimization to fluid dynamics. To be more specific, in our investigation, we will make use of some mean types, i.e., arithmetic, geometric, and harmonic means inequalities, as applications with regards to the Hadamard inequality pertaining to the Caputo–Fabrizio fractional operator for a generalized preinvex function. All these applications have useful importance in the field of statistics, probability, stochastic processes, machine learning, numerical approximations, circuit theory, and engineering. In this portion, we explore the means as applications for two positive numbers $\mathrm{w}_1,\mathrm{w}_2$ with $\mathrm{w}_1<\mathrm{w}_2$, which are given as follows:

1. The arithmetic mean

$$\mathcal{A}=\mathcal{A}(\mathrm{w}_1,\mathrm{w}_2)=\frac{\mathrm{w}_1+\mathrm{w}_2}{2},\quad \mathrm{w}_1,\mathrm{w}_2\in\mathbb{R}.$$

2. The generalized logarithmic mean

$$\mathcal{L}=\mathcal{L}_r^r(\mathrm{w}_1,\mathrm{w}_2)=\frac{\mathrm{w}_2^{r+1}-\mathrm{w}_1^{r+1}}{(r+1)(\mathrm{w}_2-\mathrm{w}_1)}.$$

Now, utilizing the results in Section 4, we discuss and explore the results to investigate some integral inequalities regarding special means. So, here, we choose $\xi(\lambda)=\xi(1)=1$.

Proposition 1. *Let $\mathrm{w}_1,\mathrm{w}_2\in\mathbb{R}^+,\mathrm{w}_1<\mathrm{w}_1+\xi(\mathrm{w}_2,\mathrm{w}_1)$, then*

$$\begin{aligned}&\left| -\mathcal{A}(\mathrm{w}_1^2, (\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1))^2) + \mathcal{L}_2^2(\mathrm{w}_1, \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)) \right| \\ &\leq \frac{\xi(\mathrm{w}_2, \mathrm{w}_1)}{n} \sum_{\mathrm{s}=1}^{n} \left[\frac{(\mathrm{s}^2 + \mathrm{s} + 2)2^{\mathrm{s}} - 2}{(\mathrm{s}+1)(\mathrm{s}+2)2^{\mathrm{s}+1}} \right] [|\mathrm{w}_1| + |\mathrm{w}_2|]. \end{aligned} \tag{20}$$

Proof. In Theorem 7, if we choose $\mathcal{H}(z) = z^2$ with $\lambda = 1$ and $\xi(\lambda) = \xi(1) = 1$, then we obtain the required inequality (20) . □

Corollary 16. *Assume that $n = 1$ in Proposition 1, and then we obtain*

$$\left| -\mathcal{A}(\mathrm{w}_1^2, (\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1))^2) + \mathcal{L}_2^2(\mathrm{w}_1, \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)) \right| \leq \frac{\xi(\mathrm{w}_2, \mathrm{w}_1)}{4} [|\mathrm{w}_1| + |\mathrm{w}_2|].$$

Corollary 17. *Assume that $\xi(\mathrm{w}_2, \mathrm{w}_1) = \mathrm{w}_2 - \mathrm{w}_1$ in Proposition 1, and then*

$$\left| -\mathcal{A}(\mathrm{w}_1^2, \mathrm{w}_2^2) + \mathcal{L}_2^2(\mathrm{w}_1, \mathrm{w}_2) \right| \leq \frac{(\mathrm{w}_2 - \mathrm{w}_1)}{n} \sum_{\mathrm{s}=1}^{n} \left[\frac{(\mathrm{s}^2 + \mathrm{s} + 2)2^{\mathrm{s}} - 2}{(\mathrm{s}+1)(\mathrm{s}+2)2^{\mathrm{s}+1}} \right] [|\mathrm{w}_1| + |\mathrm{w}_2|].$$

Proposition 2. *Assume that $\mathrm{w}_1, \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1) \in \mathbb{R}^+$and $\mathrm{w}_1 < \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)$, and then*

$$\begin{aligned}&\left| -\mathcal{A}(e_1^{\mathrm{w}}, e^{(\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1))}) + \mathcal{L}(e_1^{\mathrm{w}}, e^{(\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1))}) \right| \\ &\leq \frac{\xi(\mathrm{w}_2, \mathrm{w}_1)}{n} \sum_{\mathrm{s}=1}^{n} \left[\frac{(\mathrm{s}^2 + \mathrm{s} + 2)2^{\mathrm{s}} - 2}{(\mathrm{s}+1)(\mathrm{s}+2)2^{\mathrm{s}+1}} \right] \left[\frac{e_1^{\mathrm{w}} + e_2^{\mathrm{w}}}{2} \right]. \end{aligned} \tag{21}$$

Proof. In Theorem 7, if we choose $\mathcal{H}(z) = e^z$ with $\lambda = 1$ and $\xi(\lambda) = \xi(1) = 1$, then we obtain the required inequality (21). □

Corollary 18. *Assume that $n = 1$ in Proposition 2, and then we obtain*

$$\left| -\mathcal{A}(e_1^{\mathrm{w}}, e^{(\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1))}) + \mathcal{L}(e_1^{\mathrm{w}}, e^{(\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1))}) \right| \leq \frac{\xi(\mathrm{w}_2, \mathrm{w}_1)}{4} \left[\frac{e_1^{\mathrm{w}} + e_2^{\mathrm{w}}}{2} \right].$$

Corollary 19. *Assume that $\xi(\mathrm{w}_2, \mathrm{w}_1) = \mathrm{w}_2 - \mathrm{w}_1$ in Proposition 2, and then*

$$\left| -\mathcal{A}(e_1^{\mathrm{w}}, e^{\mathrm{w}_2}) + \mathcal{L}(e_1^{\mathrm{w}}, e^{\mathrm{w}_2}) \right| \leq \frac{(\mathrm{w}_2 - \mathrm{w}_1)}{n} \sum_{\mathrm{s}=1}^{n} \left[\frac{(\mathrm{s}^2 + \mathrm{s} + 2)2^{\mathrm{s}} - 2}{(\mathrm{s}+1)(\mathrm{s}+2)2^{\mathrm{s}+1}} \right] \left[\frac{e_1^{\mathrm{w}} + e_2^{\mathrm{w}}}{2} \right].$$

Proposition 3. *Assume that $\mathrm{w}_1, \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1) \in \mathbb{R}^+, \mathrm{w}_1 < \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)$, then*

$$\begin{aligned}&\left| -\mathcal{A}(\mathrm{w}_1^t, (\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1))^t) - \mathcal{L}_t^t(\mathrm{w}_1, \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)) \right| \\ &\leq \frac{\xi(\mathrm{w}_2, \mathrm{w}_1)}{n} \sum_{\mathrm{s}=1}^{n} \left[\frac{(\mathrm{s}^2 + \mathrm{s} + 2)2^{\mathrm{s}} - 2}{(\mathrm{s}+1)(\mathrm{s}+2)2^{\mathrm{s}+1}} \right] \left[\frac{|\mathrm{w}_1^{t-1}| + |\mathrm{w}_2^{t-1}|}{2} \right]. \end{aligned} \tag{22}$$

Proof. In Theorem 7, if we choose $\mathcal{H}(z) = z^t$ with $\lambda = 1$ and $\xi(\lambda) = \xi(1) = 1$, then we obtain the required inequality (22). □

Corollary 20. *Assume that $n = 1$ in Proposition 3, and then*

$$\left| -\mathcal{A}(\mathrm{w}_1^t, (\mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1))^t) - \mathcal{L}_t^t(\mathrm{w}_1, \mathrm{w}_1 + \xi(\mathrm{w}_2, \mathrm{w}_1)) \right| \leq \frac{\xi(\mathrm{w}_2, \mathrm{w}_1)}{8} \left[|\mathrm{w}_1^{t-1}| + |\mathrm{w}_2^{t-1}| \right].$$

Corollary 21. *If we put* $\xi(\mathrm{w}_2, \mathrm{w}_1) = \mathrm{w}_2 - \mathrm{w}_1$ *in Proposition 3, then*

$$\left| -\mathcal{A}(\mathrm{w}_1^t, \mathrm{w}_2^t) - \mathcal{L}_t^t(\mathrm{w}_1, \mathrm{w}_2) \right| \leq \frac{(\mathrm{w}_2 - \mathrm{w}_1)}{8} \left[|\mathrm{w}_1^{t-1}| + |\mathrm{w}_2^{t-1}| \right].$$

8. Conclusions

Fractional calculus has a more significant impact and offers more accurate solutions when analyzing computer models. That is to say, fractional calculus is more flexible and dynamic than classical calculus because of inherited traits and the premise of memory. Fractional calculus is widely used in engineering, mathematical biology, inequality theory, modeling and simulation, and applied mathematics. Numerous authors and scholars from a broad spectrum of scientific disciplines have expressed excitement about the investigation of fractional calculus. In this paper, the following have been performed:

(1) The authors presented some generalizations of the Hermite–Hadamard- and Pachpatte-type integral inequalities involving a generalized preinvex function in the sense of the Caputo–Fabrizio fractional operator.
(2) Furthermore, a new Lemma is demonstrated and some results in the frame of fractional-order integrals are valid for the *n*-polynomial preinvex function.
(3) To enhance the quality and reader's interest, we explored the refinements of the Hermite–Hadamard inequality in order to lemma with the aid of Hölder and its improved version and the power mean and its improved version.
(4) Some special cases are discussed.
(5) Additionally, some applications of our discussed results are examined via special means.

The novel idea of this paper can be further presented for various inequalities involving the Hermite–Hadamard, Fejér, and Simpson types pertaining to fractional operators. Some new concepts such as fuzzy interval convexities, center–radius-order convexities and interval-valued LR convexities can be used to establish further generalizations. Interested researchers can also utilize quantum calculus, coordinated interval-valued functions, fractional calculus, etc., to see the behavior of these inequalities as well. It will be quite interesting to see how different types of new convex functions can be applied to investigate fractional versions of inequalities.

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