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A Mathematical Theoretical Study of a Coupled Fully Hybrid $(\mathbf{k}, \mathbf{\Phi})$-Fractional Order System of BVPs in Generalized Banach Spaces

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Abstract: In this paper, we study a coupled fully hybrid system of $(\mathbf{k}, \mathbf{\Phi})$–Hilfer fractional differential equations equipped with non-symmetric $(\mathbf{k}, \mathbf{\Phi})$–Riemann-Liouville ($\mathcal{RL}$) integral conditions. To prove the existence and uniqueness results, we use the Krasnoselskii and Perov fixed-point theorems with Lipschitzian matrix in the context of a generalized Banach space ($\mathcal{GBS}$). Moreover, the Ulam–Hyers ($\mathcal{UH}$) stability of the solutions is discussed by using the Urs's method. Finally, an illustrated example is given to confirm the validity of our results.

Keywords: $(\mathbf{k}, \mathbf{\Phi})$–Hilfer fractional derivative; existence; nonlinear analysis; Ulam stability; generalized Banach spaces; Lipschitzian matrix

MSC: 34A08; 26A33; 34A34

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1. Introduction

Fractional differential equations (FDEs) are equations that include fractional-order derivatives instead of classical integer-order derivatives. There are several types of fractional derivative definitions that have appeared in area of fractional calculus, for example, the Riemann-Liouville ($\mathcal{RL}$), Hadamard, Grunwald-Letnikov, Caputo, Caputo–Fabrizo, and Atangana–Baleanu–Caputo derivatives [1–5]. Indeed, FDEs have a large flood of applications in different scopes such as chemistry, physics, finance, engineering, and infectious disease. The combination of FDEs and other analytical and numerical methods can be found in many works such as impulsive FDEs [6,7], implicit hybrid FDEs [8–10], mathematical modelings with the help of FDEs [11–15], neutral FDEs [16,17], p-Laplacian FDEs [18], variable order time-fractional FDEs [19], random and fuzzy FDEs [20,21], integro-differential inclusions [22,23], and references therein.

In 2012, $\mathcal{RL}$-fractional integral was extended by Mubben et al. [24] to k-$\mathcal{RL}$-fractional integral. Later, in 2018, Kwun et al. [25] introduced the $(\mathbf{k}, \mathbf{\Phi})$–$\mathcal{RL}$ definition for these operators; then, Kucche and Mali presented the most generalized operator named the $(\mathbf{k}, \mathbf{\Phi})$–Hilfer fractional operator [26], which attracted the attention of many authors such as Samadi et al. [27] who studied the existence of solutions for the coupled $(\mathbf{k}, \mathbf{\Phi})$–Hilfer nonlinear FDEs with $(\mathbf{k}, \mathbf{\Phi})$–$\mathcal{RL}$ integral conditions. Additionally, Tariboon et al. [28] employed the Krasnoselskii, Banach, and Leray-Schauder theorems to study the qualitative properties of $(\mathbf{k}, \mathbf{\Phi})$–Hilfer FDEs and inclusions with multi-point boundary conditions. Recently, in [29], Kamsrisuk et al. investigated the existence and uniqueness results of multi-point non-local $(\mathbf{k}, \mathbf{\Phi})$–Hilfer FDEs via the fixed-point method.

Over the course of many years, $\mathcal{UH}$ stability was utilized to examine the behavior of solutions for FDEs, and it can be discussed by employing fixed-point methods or by comparing the distance between the solutions of the primary equation and the so-called linearization equation, which relates to the primary equation. We also study this notion for solutions to our proposed system with a special technique. We mention some papers devoted to the study of $\mathcal{UH}$ stability [30–33].

Inspired by the aforementioned works, in this paper, we study the following coupled fully hybrid system of $(\mathbf{k}, \mathbf{\Phi})$–Hilfer FDEs:

$$\begin{cases} {}^{\mathrm{k,H}}\mathrm{D}^{\alpha_1,\beta_1,\Phi}\omega_1(\kappa) = \mathcal{F}_1(\kappa, \mu(\kappa), \nu(\kappa)), \\ {}^{\mathrm{k,H}}\mathrm{D}^{\alpha_2,\beta_2,\Phi}\omega_2(\kappa) = \mathcal{F}_2(\kappa, \mu(\kappa), \nu(\kappa)), \end{cases} \quad \kappa \in \mathrm{J} := [a, b], \tag{1}$$

with $(\mathbf{k}, \mathbf{\Phi})$–fractional integrals conditions

$$\begin{cases} \mu(a) - \phi_1 = \displaystyle\int_a^{\tau_1} \frac{\Phi'(\sigma)(\Phi(\tau_1) - \Phi(\sigma))^{\frac{\delta_1}{k}-1}}{k\Gamma_k(\delta_1)} \mathcal{H}_1(\sigma, \mu(\sigma), \nu(\sigma))\mathrm{d}\sigma, \\ \nu(a) - \phi_2 = \displaystyle\int_a^{\tau_2} \frac{\Phi'(\sigma)(\Phi(\tau_2) - \Phi(\sigma))^{\frac{\delta_2}{k}-1}}{k\Gamma_k(\delta_2)} \mathcal{H}_1(\sigma, \mu(\sigma), \nu(\sigma))\mathrm{d}\sigma, \end{cases} \quad \beta_i \leq \alpha_i, \tau_i \in \mathrm{J}, \tag{2}$$

where

$$\omega_1 = \frac{\mu(\kappa) - \mu(a)}{\mathcal{G}_1(\kappa, \mu(\kappa), \nu(\kappa))}, \qquad \omega_2 = \frac{\nu(\kappa) - \nu(a)}{\mathcal{G}_2(\kappa, \mu(\kappa), \nu(\kappa))},$$

${}^{\mathrm{k,H}}\mathrm{D}^{\alpha_i,\beta_i,\Phi}$ is the $(\mathbf{k}, \mathbf{\Phi})$–Hilfer fractional derivative of orders $\alpha_i \in (0, 1]$ and types $\beta_i \in [0, 1]$, ${}^{\mathrm{k}}\mathrm{I}_{a^+}^{\delta_i,\Phi}$ is the $(\mathbf{k}, \mathbf{\Phi})$–$\mathcal{RL}$-fractional integrals of order $\delta_i > 0$, $\phi_i \in \mathbb{R}$ and $\mathcal{G}_i : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \backslash \{0\}$, and $\mathcal{F}_i, \mathcal{H}_i : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, 2$ are continuous functions.

This research is the first paper in which we analyze the uniqueness and existence properties in connection to solutions of a coupled fully hybrid system of $(\mathbf{k}, \mathbf{\Phi})$–Hilfer-fractional BVPs of FDEs with newly defined $(\mathbf{k}, \mathbf{\Phi})$–Hilfer-fractional operators. In view of the nature of these operators, our results will cover all of the previous studies in special cases. It is sufficient that we take $k = 1$ and $\Phi(\kappa) = \kappa$; then, we obtain the classical standard Hilfer fractional derivative. The main technique of this paper for the existence property is to use of Lipschitzian matrices and the Perov theorem. Additionally, another contribution of this paper is that the criterion of Urs is used for studying the $\mathcal{UH}$ stability in combination with $(\mathbf{k}, \mathbf{\Phi})$–Hilfer-fractional operators. These items constitute the novelty of this paper in comparison to other studies.

This paper is organized as follows: several definitions and preliminaries in connection to these new operators are given in Section 2. The existence and uniqueness of the solutions of the $(\mathbf{k}, \mathbf{\Phi})$–Hilfer-fractional fully hybrid BVPs of FDEs (1)–(2) are proved in $\mathcal{GBS}$ by employing fixed-point theorem techniques in Section 3. In addition, the $\mathcal{UH}$ stability of the solution is established. In Section 4, an application of the main results is illustrated and examined by an example.

2. Background Notions

In this section, we present some notions and definitions that will be used to investigate the desired results.

Assume that $C([a, b], \mathbb{R}^n)$ is used for the description of the Banach space of each continuous function $\mu : [a, b] \to \mathbb{R}^n$ with the norm $\|\mu\| = \sup_{\kappa \in [a,b]} \|\mu(\kappa)\|$. Let $\mu, \nu \in \mathbb{R}^n$ with $\mu = (\mu_1, \mu_2, \ldots, \mu_n), \nu = (\nu_1, \nu_2, \ldots, \nu_n)$. Then, $\mu \leq \nu$ means $\mu_i \leq \nu_i, i = 1, \ldots, n$, and if

$c \in \mathbb{R}$, then $\mu \leq c$ means $\mu_i \leq c, i = 1, \ldots, n$. Set $\mathbb{R}^n_+ = \{\mu \in \mathbb{R}^n : \mu_i \in \mathbb{R}_+, i = 1, \ldots, n\}$. Moreover, we take

$$|\mu| = (|\mu_1|, |\mu_2|, \ldots, |\mu_n|),$$
$$\max(\mu, \nu) = (\max(\mu_1, \nu_1), \max(\mu_2, \nu_2), \ldots, \max(\mu_n, \nu_n)).$$

Definition 1 ([25])**.** *Consider a (increasing) function Φ from $[a, b]$ into $\mathbb{R}$ s.t. $\Phi'(\kappa) \neq 0, \forall \kappa \in [a, b]$. Then, the $(\mathbf{k}, \mathbf{\Phi})$-$\mathcal{RL}$-fractional integral of order $\alpha > 0$ for the function $\mathfrak{h} \in L^1([a, b], \mathbb{R})$ is*

$$^{\mathrm{k}}\mathrm{I}^{\alpha;\Phi}_{a+} h(\kappa) = \frac{1}{k\Gamma_k(\alpha)} \int_a^\kappa \Phi'(u)(\Phi(\kappa) - \Phi(u))^{\frac{\alpha}{k}-1} \mathfrak{h}(u) du, k > 0,$$

where the k-Gamma function Γ_k is formulated by

$$\Gamma_k(z) = \int_0^\infty s^{z-1} e^{-\frac{s^k}{k}} ds, z \in \mathbb{C}, \Re(z) > 0,$$

with some properties such as

$$\lim_{k \to 1} \Gamma_k(\alpha) = \Gamma(\alpha), \Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1} \Gamma\left(\frac{\alpha}{k}\right) \text{ and } \Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha).$$

Definition 2 ([26])**.** *Suppose that $k \in \mathbb{R}^+ = (0, \infty), \Phi \in C^n([a, b], \mathbb{R}), \Phi'(\kappa) \neq 0, \forall \kappa \in [a, b]$. Then, the $(\mathbf{k}, \mathbf{\Phi})$-Hilfer derivative of order $\alpha \in (n-1, n]$ with the type $\beta \in [0, 1]$ for the function $\mathfrak{h} \in \mathrm{C}^n([a, b], \mathbb{R})$ is given as*

$$^{\mathrm{k,H}}\mathrm{D}^{\alpha,\beta;\Phi} \mathfrak{h}(\kappa) = {}^{\mathrm{k}}\mathrm{I}^{\beta(nk-a);\Phi}_{a+} \left(\frac{k}{\Phi'(\kappa)} \frac{\mathrm{d}}{\mathrm{d}\kappa} \right)^n \mathrm{I}^{(1-\beta)(nk-\alpha);\Phi}_{a+} \mathfrak{h}(\kappa), \quad n = \left\lceil \frac{\alpha}{k} \right\rceil.$$

Remark 1. *For $n - 1 < \frac{\theta_k}{k} \leq n$ s.t. $\frac{\alpha}{k} \in (n-1, n]$ and $\beta \in [0, 1]$ s.t. $\theta_k = \alpha + \beta(nk - \alpha)$ and $\beta(nk - \alpha) = \theta_k - \alpha$, the $(\mathbf{k}, \mathbf{\Phi})$-Hilfer fractional derivative can be reformultaed in the sense of $(\mathbf{k}, \mathbf{\Phi})$-$\mathcal{RL}$-fractional derivative as the following form:*

$$^{\mathrm{k,H}}\mathrm{D}^{\alpha,\beta;\Phi} \mathfrak{h}(\kappa) = {}^{\mathrm{k}}\mathrm{I}^{\theta_k-\alpha;\Phi}_{a+} \left(\frac{k}{\Phi'(\kappa)} \frac{\mathrm{d}}{\mathrm{d}\kappa} \right)^n {}^{\mathrm{k}}\mathrm{I}^{nk-\theta_k;\Phi}_{a+} \mathfrak{h}(\kappa) = {}^{\mathrm{k}}\mathrm{I}^{\theta_k-\alpha;\Phi}_{a+} \left({}^{\mathrm{k,RL}}\mathrm{D}^{\theta_k;\Phi} \mathfrak{h} \right)(\kappa).$$

In the next lemmas, we provide some properties of $(\mathbf{k}, \mathbf{\Phi})$-fractional operators.

Lemma 1 ([26])**.** *Let $\mathfrak{h} \in C^n([a, b], \mathbb{R})$. With the same assumptions given in the above remark, we have*

$$^{\mathrm{k}}\mathrm{I}^{\theta_k;\Phi}_{a+} \left({}^{\mathrm{k,RL}}\mathrm{D}^{\theta_k;\Phi} \mathfrak{h} \right)(\kappa) = {}^{\mathrm{k}}\mathrm{I}^{\alpha;\Phi}_{a+} \left({}^{\mathrm{k,H}}\mathrm{D}^{\alpha,\beta;\Phi} \mathfrak{h} \right)(\kappa).$$

Lemma 2 ([26])**.** *For $k > 0$ and with the above assumptions, let $\mathfrak{h} \in C^n([a, b], \mathbb{R})$ and ${}^{\mathrm{k}}\mathrm{I}^{nk-\alpha;\Phi}_{a+} \mathfrak{h} \in C^n([a, b], \mathbb{R})$. Then,*

$$^{\mathrm{k}}\mathrm{I}^{\alpha;\Phi}_{a+} \left({}^{\mathrm{k,RL}}\mathrm{D}^{\alpha;\Phi} \mathfrak{h}(\kappa) \right) = \mathfrak{h}(\kappa) - \sum_{j=1}^{n} \frac{(\Phi(\kappa) - \Phi(a))^{\frac{\alpha}{k}-j}}{\Gamma_k(\alpha - jk + k)} \left[\left(\frac{k}{\Phi'(\kappa)} \frac{\mathrm{d}}{\mathrm{d}\kappa} \right)^{n-j} {}^{\mathrm{k}}\mathrm{I}^{nk-\alpha;\Phi}_{a+} \mathfrak{h}(\kappa) \right]_{z=a}.$$

Lemma 3 ([26])**.** *Let $\zeta, k \in \mathbb{R}^+$ and $\eta \in \mathbb{R}$ s.t. $\frac{\eta}{k} > -1$. We have*

(i) $\quad {}^{\mathrm{k}}\mathrm{I}^{\zeta,\Phi}_{a+} (\Phi(\kappa) - \Phi(a))^{\frac{\eta}{k}} = \frac{\Gamma_k(\eta + k)}{\Gamma_k(\eta + k + \zeta)} (\Phi(\kappa) - \Phi(a))^{\frac{\eta+\zeta}{k}}.$

(ii) $\quad {}^{\mathrm{k,H}}\mathrm{D}^{\zeta,\Phi} (\Phi(\kappa) - \Phi(a))^{\frac{\eta}{k}} = \frac{\Gamma_k(\eta + k)}{\Gamma_k(\eta + k - \zeta)} (\Phi(\kappa) - \Phi(a))^{\frac{\eta-\zeta}{k}}.$

Definition 3 ([34]). *A real square matrix $\mathbb{A}$ convergent to zero iff its spectral radius $\rho(\mathbb{A})$ is precise less than 1; this means that $|\Lambda| < 1$ with $\det(\mathbb{A} - \Lambda\mathbb{I}) = 0$ for each $\Lambda \in \mathbb{C}$ and $\mathbb{I}$ represents the unit matrix of $\mathbb{A}_{n\times n}(\mathbb{R})$.*

Theorem 1 ([34]). *Let $\mathbb{A}$ be a non-negative square matrix. Then, the following items are equivalent:*
(i) *As $n \to \infty$, $\mathbb{A}^n \to 0$;*
(ii) *The spectral radius $\rho(\mathbb{A}) < 1$;*
(iii) *$(\mathbb{I} - \mathbb{A})$ is non-singular and $(\mathbb{I} - \mathbb{A})^{-1} = \mathbb{I} + \mathbb{A} + \cdots + \mathbb{A}^n + \cdots$;*
(iv) *The matrices $\mathbb{I} - \mathbb{A}$ and $(\mathbb{I} - \mathbb{A})^{-1}$ are non-singular and non-negative, respectively.*

Definition 4 ([35,36]). *Let the generalized metric space be denoted by (Ξ, d). If there is a matrix $\mathbb{A}$ converging to zero, then the mapping $\Pi \colon \Xi \to \Xi$ is contractive, where*

$$\forall \mu, \nu \in \Xi, \quad \mathrm{d}(\Pi(\mu), \Pi(\nu)) \leq \mathbb{A}\mathrm{d}(\mu, \nu).$$

Now, we recall two fixed-point theorems that will be used in the next sections.

Theorem 2 ([35,37]). *Let complete generalized metric space be (Ξ, d). If the mapping $\Pi \colon \Xi \to \Xi$ is a contractive with Lipschitz's matrix $\mathbb{A}$, then Π possesses one and only one fixed point μ_0, and $\forall \mu \in \Xi$, we obtain*

$$\forall k \in \mathbb{N}, \quad \mathrm{d}(\Pi^k(\mu), \mu_0) \leq \mathbb{A}^k(\mathbb{I} - \mathbb{A})^{-1}\mathrm{d}(\mu, \Pi(\mu)).$$

Theorem 3 ([38]). *Assume that Ψ be a convex, closed, non-empty subset of a $\mathcal{GBS}$ Ξ. Let Π and Υ map Ψ into Ξ such that*
(i) $\Pi\mu + \Upsilon\nu \in \Psi$, $\forall \mu, \nu \in \Psi$;
(ii) *The mapping Π is continuous and compact;*
(iii) *The mapping Υ is an $\mathbb{A}$-contraction.*

Then, $\Pi x + \Upsilon x = x$ possesses at least one solution on Ψ.

3. Qualitative Results

Throughout this section, we prove the existence, uniqueness, and $\mathcal{UH}$ stability of solutions for the coupled fully hybrid system of $(\mathbf{k}, \mathbf{\Phi})$–Hilfer BVPs (1)–(2).

Now, in order to establish qualitative results of the mentioned system (1)–(2), we need to provide the following lemma. In this lemma, we derive the main structure of solution in form of an integral equation.

Lemma 4. *If the solution of the fully hybrid $(\mathbf{k}, \mathbf{\Phi})$–Hilfer BVP given by*

$$\begin{cases} {}^{\mathrm{k,H}}\mathrm{D}^{\alpha,\beta,\Phi}\omega(\kappa) = \mathcal{F}(\kappa, \mu(\kappa)), & \kappa \in \mathrm{J} := [a, b], \\ \mu(a) - \phi_1 = {}^{\mathrm{k}}\mathrm{I}_{a+}^{\delta_1;\Phi}\mathcal{H}(\tau, \mu(\tau)), & \beta \leq \alpha \in (0, 1], \tau \in \mathrm{J}, \end{cases} \quad \omega(\kappa) = \frac{\mu(\kappa) - \mu(a)}{\mathcal{G}(\kappa, \mu(\kappa))}, \tag{3}$$

exists, then it is equivalent to the integral equation

$$\mu(\kappa) = \phi_1 + \int_a^\tau \frac{\Phi'(\sigma)(\Phi(\tau) - \Phi(\sigma))^{\frac{\delta}{k}-1}}{k\Gamma_k(\delta)}\mathcal{H}(\sigma, \mu(\sigma))\mathrm{d}\sigma$$
$$+ \mathcal{G}(\kappa, \mu(\kappa)) \int_a^\kappa \frac{\Phi'(\sigma)(\Phi(\kappa) - \Phi(\sigma))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}\mathcal{F}(\sigma)\mathrm{d}\sigma. \tag{4}$$

Proof. Let $\mu(\kappa)$ be a solution of the problem (3). Integrating on (3) and then, using the properties of the fractional operators, we obtain

$$\mu(\kappa) = \mu(a) + \mathcal{G}(\kappa, \mu(\kappa)) \int_a^\kappa \frac{\Phi'(\sigma)(\Phi(\kappa) - \Phi(\sigma))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)} \mathcal{F}(\sigma, \mu(\sigma))d\sigma,$$

and (4) can be obtained. Secondly, let μ be a solution of the integral Equation (4). Then,

$$\mu(\kappa) - \mu(a) = \mathcal{G}(\kappa, \mu(\kappa))^{\mathrm{k}}\mathrm{I}_{a+}^{\alpha;\Phi}\mathcal{F}(\sigma, \mu(\sigma))d\sigma, \quad \frac{\mu(\kappa) - \mu(a)}{\mathcal{G}(\kappa, \mu(\kappa))} = {}^{\mathrm{k}}\mathrm{I}_{a+}^{\alpha;\Phi}\mathcal{F}(\sigma, \mu(\sigma))d\sigma,$$

and

$$\begin{aligned} {}^{\mathrm{k,H}}\mathrm{D}^{\alpha,\beta,\Phi}\frac{\mu(\kappa) - \mu(a)}{\mathcal{G}(\kappa, \mu(\kappa))} &= {}^{\mathrm{k}}\mathrm{I}_{a+}^{\beta(\mathrm{k}-\alpha);\Phi}\left(\frac{k}{\Phi'(\kappa)}\frac{d}{d\kappa}\right){}^{\mathrm{k}}\mathrm{I}_{a+}^{(1-\beta)(\mathrm{k}-\alpha);\Phi}\frac{\mu(\kappa) - \mu(a)}{\mathcal{G}(\kappa, \mu(\kappa))} \\ &= \left(\frac{k}{\Phi'(\tau)}\frac{\mathrm{d}}{\mathrm{dt}}\right){}^{\mathrm{k}}\mathrm{I}_{a+}^{k-\alpha;\Phi}\ {}^{\mathrm{k}}\mathrm{I}_{a+}^{\alpha;\Phi}\mathcal{F}(\sigma, \mu(\sigma)) = \mathcal{F}(\sigma, \mu(\sigma)). \end{aligned}$$

This completes the proof. □

In view of Lemma 4, we need to present the following lemma, which plays a key role in the main theorems. In fact, this lemma shows the solution of the given system via two integral equations.

Lemma 5. *Let $\alpha_i \in (0,1]$ be fixed, $\theta_{k,i} = \alpha_i + \beta_i(k - \alpha_i)$ with $\alpha_i, k \in \mathbb{R}^+ = (0, \infty), \alpha_i < k$, and $\beta_i \in [0,1]$, and $\mathcal{G}_i \in C(\mathrm{J} \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n \backslash \{0\})$, $\mathcal{F}_i, \mathcal{H}_i \in C(\mathrm{J} \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $i = 1, 2$. Then, the solution of the coupled fully hybrid system of $(\mathbf{k}, \boldsymbol{\Phi})$–Hilfer BVPs (1)–(2) is equivalent to the following integral equations:*

$$\begin{cases} \mu(\kappa) = \phi_1 & + \displaystyle\int_a^{\tau_1} \frac{\Phi'(\sigma)(\Phi(\tau_1) - \Phi(\sigma))^{\frac{\delta_1}{k}-1}}{k\Gamma_k(\delta_1)} \mathcal{H}_1(\sigma, \mu(\sigma), \nu(\sigma))d\sigma \\ & + \mathcal{G}_1(\kappa, \mu(\kappa), \nu(\kappa)) \displaystyle\int_a^\kappa \frac{\Phi'(\sigma)(\Phi(\kappa) - \Phi(\sigma))^{\frac{\alpha_1}{k}-1}}{k\Gamma_k(\alpha_1)} \mathcal{F}_1(\sigma, \mu(\sigma), \nu(\sigma))d\sigma, \\ \nu(\kappa) = \phi_2 & + \displaystyle\int_a^{\tau_2} \frac{\Phi'(\sigma)(\Phi(\tau_2) - \Phi(\sigma))^{\frac{\delta_2}{k}-1}}{k\Gamma_k(\delta_2)} \mathcal{H}_2(\sigma, \mu(\sigma), \nu(\sigma))d\sigma \\ & + \mathcal{G}_2(\kappa, \mu(\kappa), \nu(\kappa)) \displaystyle\int_a^\kappa \frac{\Phi'(\sigma)(\Phi(\kappa) - \Phi(\sigma))^{\frac{\alpha_2}{k}-1}}{k\Gamma_k(\alpha_2)} \mathcal{F}_2(\sigma, \mu(\sigma), \nu(\sigma))d\sigma, \end{cases}, \quad \kappa \in \mathrm{J}.$$

Now, the product space $\mathbb{X} := C(\mathrm{J}, \mathbb{R}^n) \times C(\mathrm{J}, \mathbb{R}^n)$ is a $\mathcal{GBS}$ with the following norm:

$$\|(\mu, \nu)\|_{\mathbb{X}} = \begin{pmatrix} \|\mu\| \\ \|\nu\| \end{pmatrix}.$$

Additionally, let the operator $\mathbb{T} = (\mathbb{T}_1, \mathbb{T}_2) \colon \mathbb{X} \to \mathbb{X}$ define

$$\mathbb{T}(\mu, \nu) = (\mathbb{T}_1(\mu, \nu), \mathbb{T}_2(\mu, \nu)), \tag{5}$$

with

$$\begin{aligned} (\mathbb{T}_1(\mu, \nu))(\kappa) = \phi_1 &+ \int_a^{\tau_1} \frac{\Phi'(\sigma)(\Phi(\tau_1) - \Phi(\sigma))^{\frac{\delta_1}{k}-1}}{k\Gamma_k(\delta_1)} \mathcal{H}_1(\sigma, \mu(\sigma), \nu(\sigma))d\sigma \\ &+ \mathcal{G}_1(\kappa, \mu(\kappa), \nu(\kappa)) \int_a^\kappa \frac{\Phi'(\sigma)(\Phi(\kappa) - \Phi(\sigma))^{\frac{\alpha_1}{k}-1}}{k\Gamma_k(\alpha_1)} \mathcal{F}_1(\sigma, \mu(\sigma), \nu(\sigma))d\sigma, \end{aligned} \tag{6}$$

and

$$(\mathbb{T}_2(\nu,\mu))(\kappa) = \phi_2 + \int_a^{\tau_2} \frac{\Phi'(\sigma)(\Phi(\tau_2)-\Phi(\sigma))^{\frac{\delta_2}{k}-1}}{k\Gamma_k(\delta_2)} \mathcal{H}_2(\sigma,\mu(\sigma),\nu(\sigma))d\sigma \\ + \mathcal{G}_2(\kappa,\mu(\kappa),\nu(\kappa)) \int_a^{\kappa} \frac{\Phi'(\sigma)(\Phi(\kappa)-\Phi(\sigma))^{\frac{\alpha_2}{k}-1}}{k\Gamma_k(\alpha_2)} \mathcal{F}_2(\sigma,\mu(\sigma),\nu(\sigma))d\sigma. \tag{7}$$

For computational convenience, we introduce the following notations:

$$\begin{cases} \mathcal{F}_{i.\max} := \sup_{\kappa\in J} \|\mathcal{F}_i(\kappa,0,0)\|, \quad \mathcal{G}_{i.\max} := \sup_{\kappa\in J} \|\mathcal{G}_i(\kappa,0,0)\|, \\ \mathbb{A}_i := \dfrac{(\Phi(b)-\Phi(a))^{\frac{\alpha_i}{k}}}{\Gamma_k(\alpha_i+k)}, \quad \mathbb{B}_i := \dfrac{(\Phi(\tau_i)-\Phi(a))^{\frac{\delta_i}{k}}}{\Gamma_k(\delta_i+k)}, i=1,2, \\ \mathcal{A}_i = \mathrm{M}_{\mathcal{G}_i}.\mathbb{B}_i \mathrm{L}_{\mathcal{F}_i}, \quad \mathcal{B}_i = \mathrm{M}_{\mathcal{G}_1}.\mathbb{B}_1 \bar{\mathrm{L}}_{\mathcal{F}_1}, \quad \mathcal{C}_i = \mathrm{M}_{\mathcal{G}_i}.\mathbb{B}_i \mathcal{F}_{i.\max}, \quad i=1,2. \end{cases}$$

Let us list the following hypotheses:

(HP_1) For $i = 1,2$, the functions $\mathcal{F}_i$ and $\mathcal{G}_i$ are bounded on the $J \times \mathbb{R}^n \times \mathbb{R}^n$ subject to bounds $\mathrm{M}_{\mathcal{F}_i}$ and $\mathrm{M}_{\mathcal{G}_i}$, respectively.

(HP_2) $\mathrm{L}_{Z_i} > 0$ and $\bar{\mathrm{L}}_{Z_i} > 0$, $Z_i = (\mathcal{F}_i, \mathcal{G}_i, \mathcal{H}_i)$, $i = 1,2$ exist, where

$$\|\mathcal{F}_i(\kappa,\mu_1,\nu_1) - \mathcal{F}_i(\kappa,\mu_2,\nu_2)| \le \mathrm{L}_{\mathcal{F}_i}\|\mu_1-\mu_2\| + \bar{\mathrm{L}}_{\mathcal{F}_i}\|\nu_1-\nu_2\|,$$

$$\|\mathcal{G}_i(\kappa,\mu_1,\nu_1) - \mathcal{G}_i(\kappa,\mu_2,\nu_2)| \le \mathrm{L}_{\mathcal{G}_i}\|\mu_1-\mu_2\| + \bar{\mathrm{L}}_{\mathcal{G}_i}\|\nu_1-\nu_2\|,$$

$$\|\mathcal{H}_i(\kappa,\mu_1,\nu_1) - \mathcal{H}_i(\kappa,\mu_2,\nu_2)| \le \mathrm{L}_{\mathcal{H}_i}\|\mu_1-\mu_2\| + \bar{\mathrm{L}}_{\mathcal{H}_i}\|\nu_1-\nu_2\|,$$

for all $\kappa \in J$ and each $\mu_1,\nu_1,\mu_2,\nu_2 \in \mathbb{R}^n$.

(HP_3) $\bar{\mathbb{A}}_i, \bar{\mathbb{B}}_i < 1$, where

$$\bar{\mathbb{A}}_i = \left[\mathbb{A}_i \mathrm{L}_{\mathcal{H}_i} + \mathbb{B}_i\left(\mathrm{L}_{\mathcal{G}_i}\mathrm{M}_{\mathcal{F}_i} + \mathrm{L}_{\mathcal{F}_i}\mathrm{M}_{\mathcal{G}_i}\right)\right],$$

$$\bar{\mathbb{B}}_i = \left[\mathbb{A}_i \bar{\mathrm{L}}_{\mathcal{H}_i} + \mathbb{B}_i\left(\bar{\mathrm{L}}_{\mathcal{G}_i}\mathrm{M}_{\mathcal{F}_i} + \bar{\mathrm{L}}_{\mathcal{F}_i}\mathrm{M}_{\mathcal{G}_i}\right)\right].$$

Next, we are in a position to investigate and prove the uniqueness result by using Perov's fixed-point theorem.

Theorem 4. *Let the hypotheses* (HP_1)–(HP_3) *hold. Then, the coupled fully hybrid system of* $(\mathbf{k},\mathbf{\Phi})$*–Hilfer BVPs* (1)–(2) *possesses one and only one solution.*

Proof. In order to show that $\mathbb{T}$ has exactly one fixed point, we will use Perov's fixed-point theorem. Indeed, we prove that the mapping $\mathbb{T}$ is an $\mathbb{A}_{\mathbb{MAT}}$-contraction on $\mathbb{X}$. For given $(\mu_1,\nu_1),(\mu_2,\nu_2) \in \mathbb{X}$, and $\kappa \in J$, using (HP_1) and (HP_2), we can obtain

$$\begin{aligned} &\left\|\left(\mathbb{T}_1(\mu_1,\nu_1)\right)(\kappa) - \left(\mathbb{T}_1(\mu_2,\nu_2)\right)(\kappa)\right\| \\ &\le \int_a^{\tau_1} \frac{\Phi'(\sigma)(\Phi(\tau_1)-\Phi(\sigma))^{\frac{\delta_1}{k}-1}}{k\Gamma_k(\delta_1)} \left|\mathcal{H}_1(\sigma,\mu_1(\sigma),\nu_1(\sigma)) - \mathcal{H}_1(\sigma,\mu_2(\sigma),\nu_2(\sigma))\right| d\sigma \\ &+ \left|\mathcal{G}_1(\kappa,\mu_1(\kappa),\nu_1(\kappa))\right| \int_a^{\kappa} \frac{\Phi'(\sigma)(\Phi(\kappa)-\Phi(\sigma))^{\frac{\alpha_1}{k}-1}}{k\Gamma_k(\alpha_1)} \\ &\times \left|\mathcal{F}_1(\sigma,\mu_1(\sigma),\nu_1(\sigma)) - \mathcal{F}_1(\sigma,\mu_2(\sigma),\nu_2(\sigma))\right| d\sigma \end{aligned}$$

$$+\left|\mathcal{G}_1(\kappa,\mu_1(\kappa),\nu_1(\kappa))-\mathcal{G}_1(\kappa,\mu_2(\kappa),\nu_2(\kappa))\right|\int_a^{\kappa}\frac{\Phi'(\sigma)(\Phi(\kappa)-\Phi(\sigma))^{\frac{\alpha_1}{k}-1}}{k\Gamma_k(\alpha_1)}$$

$$\times\left|\mathcal{F}_1(\sigma,\mu_2(\sigma),\nu_2(\sigma))\right|d\sigma$$

$$\leq\int_a^{\tau_1}\frac{\Phi'(\sigma)(\Phi(\tau_1)-\Phi(\sigma))^{\frac{\delta_1}{k}-1}}{k\Gamma_k(\delta_1)}\big(L_{\mathcal{H}_1}\|\mu_1(\sigma)-\mu_2(\sigma)\|+\bar{L}_{\mathcal{H}_1}\|\nu_1(\sigma)-\nu_2(\sigma)\|\big)d\sigma$$

$$+M_{\mathcal{G}_1}\int_a^{\kappa}\frac{\Phi'(\sigma)(\Phi(\kappa)-\Phi(\sigma))^{\frac{\alpha_1}{k}-1}}{k\Gamma_k(\alpha_1)}\big(L_{\mathcal{F}_1}\|\mu_1(\sigma)-\mu_2(\sigma)\|+\bar{L}_{\mathcal{F}_1}\|\nu_1(\sigma)-\nu_2(\sigma)\|\big)d\sigma$$

$$+\big(L_{\mathcal{G}_1}\|\mu_1(\kappa)-\mu_2(\kappa)\|+\bar{L}_{\mathcal{G}_1}\|\nu_1(\kappa)-\nu_2(\kappa)\|\big)\int_a^{\kappa}\frac{\Phi'(\sigma)(\Phi(\kappa)-\Phi(\sigma))^{\frac{\alpha_1}{k}-1}}{k\Gamma_k(\alpha_1)}M_{\mathcal{F}_1}d\sigma$$

$$\leq\frac{(\Phi(\tau_1)-\Phi(a))^{\frac{\delta_1}{k}}}{\Gamma_k(\delta_1+k)}\big(L_{\mathcal{H}_1}\|\mu_1-\mu_2\|+\bar{L}_{\mathcal{H}_1}\|\nu_1-\nu_2\|\big)$$

$$+M_{\mathcal{G}_1}\frac{(\Phi(b)-\Phi(a))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)}\big(L_{\mathcal{F}_1}\|\mu_1-\mu_2\|+\bar{L}_{\mathcal{F}_1}\|\nu_1-\nu_2\|\big)$$

$$+M_{\mathcal{F}_1}\frac{(\Phi(b)-\Phi(a))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)}\big(L_{\mathcal{G}_1}\|\mu_1-\mu_2\|+\bar{L}_{\mathcal{G}_1}\|\nu_1-\nu_2\|\big)$$

$$\leq\mathbb{A}_1\big(L_{\mathcal{H}_1}\|\mu_1-\mu_2\|+\bar{L}_{\mathcal{H}_1}\|\nu_1-\nu_2\|\big)$$

$$+M_{\mathcal{G}_1}\mathbb{B}_1\big(L_{\mathcal{F}_1}\|\mu_1-\mu_2\|+\bar{L}_{\mathcal{F}_1}\|\nu_1-\nu_2\|\big)$$

$$+M_{\mathcal{F}_1}\mathbb{B}_1\big(L_{\mathcal{G}_1}\|\mu_1-\mu_2\|+\bar{L}_{\mathcal{G}_1}\|\nu_1-\nu_2\|\big).$$

Hence,

$$\begin{aligned}\|\mathbb{T}_1(\mu_1,\nu_1)-\mathbb{T}_1(\mu_2,\nu_2)\| &\leq \Big[\mathbb{A}_1L_{\mathcal{H}_1}+\mathbb{B}_1\big(L_{\mathcal{G}_1}M_{\mathcal{F}_1}+L_{\mathcal{F}_1}M_{\mathcal{G}_1}\big)\Big]\|\mu_1-\mu_2\|\\ &+\Big[\mathbb{A}_1\bar{L}_{\mathcal{H}_1}+\mathbb{B}_1\big(\bar{L}_{\mathcal{G}_1}M_{\mathcal{F}_1}+\bar{L}_{\mathcal{F}_1}M_{\mathcal{G}_1}\big)\Big]\|\nu_1-\nu_2\|\\ &:=\bar{\mathbb{A}}_1\|\mu_1-\mu_2\|+\bar{\mathbb{B}}_1\|\nu_1-\nu_2\|.\end{aligned}$$

By the same technique, we can also obtain

$$\begin{aligned}\|\mathbb{T}_2(\mu_1,\nu_1)-\mathbb{T}_2(\mu_2,\nu_2)\| &\leq \Big[\mathbb{A}_2L_{\mathcal{H}_2}+\mathbb{B}_2\big(L_{\mathcal{G}_2}M_{\mathcal{F}_2}+L_{\mathcal{F}_2}M_{\mathcal{G}_2}\big)\Big]\|\mu_1-\mu_2\|\\ &+\Big[\mathbb{A}_2\bar{L}_{\mathcal{H}_2}+\mathbb{B}_2\big(\bar{L}_{\mathcal{G}_2}M_{\mathcal{F}_2}+\bar{L}_{\mathcal{F}_2}M_{\mathcal{G}_2}\big)\Big]\|\nu_1-\nu_2\|\\ &:=\bar{\mathbb{A}}_2\|\mu_1-\mu_2\|+\bar{\mathbb{B}}_2\|\nu_1-\nu_2\|.\end{aligned}$$

This implies that

$$\|\mathbb{T}(\mu_1,\nu_1)-\mathbb{T}(\mu_2,\nu_2)\|_{\mathbb{X}}\leq\mathbb{A}_{\mathbb{MAT}}\|(\mu_1,\nu_1)-(\mu_2,\nu_2)\|_{\mathbb{X}},$$

where

$$\mathbb{A}_{\mathbb{MAT}}=\begin{pmatrix}\bar{\mathbb{A}}_1 & \bar{\mathbb{B}}_1\\ \bar{\mathbb{A}}_2 & \bar{\mathbb{B}}_2\end{pmatrix}. \tag{8}$$

According to (HP_3), we have $\mathbb{A}^n_{\mathbb{MAT}} \to 0$ as $n \to \infty$. Thus, $\mathbb{T}$ is contractive, and due to Perov's theorem, $\mathbb{T}$ has exactly one fixed point. Thus, the coupled fully hybrid system of $(\mathbf{k}, \mathbf{\Phi})$–Hilfer BVPs (1)–(2) possesses a unique solution in $\mathbb{X}$. □

The following result is achieved based on the Krasnoselskii's Theorem 3. In fact, here we prove the existence result with the help of the Krasnoselskii's fixed-point theorem in a generalized Banach space.

Theorem 5. *Let* (HP_1) *and* (HP_2) *hold. Also, if* $\rho(\mathbb{D}_{\mathbb{MAT}})$, $\rho(\mathbb{B}_{\mathbb{MAT}} + \mathbb{D}_{\mathbb{MAT}}) < 1$, *such that*

$$\mathbb{B}_{\mathbb{MAT}} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{B}_1 \\ \mathcal{A}_2 & \mathcal{B}_2 \end{pmatrix}, \text{ and } \mathbb{D}_{\mathbb{MAT}} = \begin{pmatrix} \mathbb{A}_1 \mathrm{L}_{\mathcal{H}_i} & \mathbb{A}_1 \bar{\mathrm{L}}_{\mathcal{H}_i} \\ \mathbb{A}_2 \mathrm{L}_{\mathcal{H}_i} & \mathbb{A}_2 \bar{\mathrm{L}}_{\mathcal{H}_i} \end{pmatrix},$$

then, the coupled fully hybrid system of $(\mathbf{k}, \mathbf{\Phi})$*–Hilfer BVPs* (1)–(2) *admits at least one solution.*

Proof. In order to use Theorem (3), we need to take a set $\mathbb{Q}_\xi \subseteq \mathbb{X}$ such that $\mathbb{Q}_\xi$ is closed, convex, bounded, and defined as

$$\mathbb{Q}_\xi = \{(\mu, \nu) \in \mathbb{X} : \|(\mu, \nu)\|_{\mathbb{X}} \le \xi\},$$

with $\xi := (\xi_1, \xi_2) \in \mathbb{R}^2_+$ such that

$$\begin{cases} \xi_1 \ge \rho_1 M_1 + \rho_2 M_2, \\ \xi_2 \ge \rho_3 M_1 + \rho_4 M_2, \end{cases}$$

where M_1, M_2, and $\rho_i, i = \overline{1,4}$ are non-negative real numbers that will be specified later.

Now, consider the mappings $\mathbb{U} = (\mathbb{U}_1, \mathbb{U}_2)$ and $\mathbb{V} = (\mathbb{V}_1, \mathbb{V}_2)$ on $\mathbb{Q}_\xi$ as

$$\begin{cases} \mathbb{U}_1(\mu,\nu)(\kappa) = \mathcal{G}_1(\kappa, \mu(\kappa), \nu(\kappa)) \displaystyle\int_a^\kappa \frac{\Phi'(\sigma)(\Phi(\kappa) - \Phi(\sigma))^{\frac{\alpha_1}{k}-1}}{k\Gamma_k(\alpha_1)} \mathcal{F}_1(\sigma, \mu(\sigma), \nu(\sigma)) d\sigma, \\ \mathbb{U}_2(\mu,\nu)(\kappa) = \mathcal{G}_2(\kappa, \mu(\kappa), \nu(\kappa)) \displaystyle\int_a^\kappa \frac{\Phi'(\sigma)(\Phi(\kappa) - \Phi(\sigma))^{\frac{\alpha_2}{k}-1}}{k\Gamma_k(\alpha_2)} \mathcal{F}_2(\sigma, \mu(\sigma), \nu(\sigma)) d\sigma, \end{cases}$$

and

$$\begin{cases} \mathbb{V}_1(\mu,\nu)(\kappa) = \phi_1 + \displaystyle\int_a^{\tau_1} \frac{\Phi'(\sigma)(\Phi(\tau_1) - \Phi(\sigma))^{\frac{\delta_1}{k}-1}}{k\Gamma_k(\delta_1)} \mathcal{H}_1(\sigma, \mu(\sigma), \nu(\sigma)) d\sigma, \\ \mathbb{V}_2(\mu,\nu)(\kappa) = \phi_2 + \displaystyle\int_a^{\tau_2} \frac{\Phi'(\sigma)(\Phi(\tau_2) - \Phi(\sigma))^{\frac{\delta_2}{k}-1}}{k\Gamma_k(\delta_2)} \mathcal{H}_2(\sigma, \mu(\sigma), \nu(\sigma)) d\sigma. \end{cases}$$

It is obvious that both $\mathbb{U}$ and $\mathbb{V}$ are well-defined. Moreover, by Lemma 5 the mappings form the system (5) as

$$\mathbb{T}(\mu,\nu) := (\mathbb{U}_1(\mu,\nu), \mathbb{U}_2(\mu,\nu)) + (\mathbb{V}_1(\mu,\nu), \mathbb{V}_2(\mu,\nu)). \tag{9}$$

Our purpose is to confirm this fact that $\mathbb{U}$ and $\mathbb{V}$ fulfill all properties of Theorem 3. For better clarity, the proof is broken down into three steps.

Step 1: $\mathbb{U}(\mu,\nu) + \mathbb{V}(\bar{\mu},\bar{\nu}) \in \mathbb{Q}_\xi, \forall (\mu,\nu), (\bar{\mu},\bar{\nu}) \in \mathbb{Q}_\xi$.
In fact, from (HP_2), for $(\mu,\nu), (\bar{\mu},\bar{\nu}) \in \mathbb{X}$, $\forall \kappa \in \mathrm{J}$, we can obtain

$$\|\mathbb{U}_1(\mu,\nu)(\kappa)\|$$

$$\le \|\mathcal{G}_1(\kappa, \mu(\kappa), \nu(\kappa))\|$$

$$\times \int_a^\kappa \frac{\Phi'(\sigma)(\Phi(\kappa)-\Phi(\sigma))^{\frac{\alpha_1}{k}-1}}{k\Gamma_k(\alpha_1)} \left(\left\|\mathcal{F}_1(\sigma,\mu(\sigma),\nu(\sigma))-\mathcal{F}_1(\sigma,0,0)\right\|+\left\|\mathcal{F}_1(\sigma,0,0)\right\|\right)d\sigma$$

$$\leq M_{\mathcal{G}_1}.\int_a^\kappa \frac{\Phi'(\sigma)(\Phi(\kappa)-\Phi(\sigma))^{\frac{\alpha_1}{k}-1}}{k\Gamma_k(\alpha_1)} \Big[\left(L_{\mathcal{F}_1}\|\mu(\sigma)\|+\bar{L}_{\mathcal{F}_1}\|\nu(\sigma)\|\right)+\mathcal{F}_{1.\max}\Big]d\sigma$$

$$\leq M_{\mathcal{G}_1}.\mathbb{B}_1\Big[\left(L_{\mathcal{F}_1}\|\mu\|+\bar{L}_{\mathcal{F}_1}\|\nu\|\right)+\mathcal{F}_{1.\max}\Big]$$

$$\leq M_{\mathcal{G}_1}.\mathbb{B}_1 L_{\mathcal{F}_1}\|\mu\|+M_{\mathcal{G}_1}.\mathbb{B}_1\bar{L}_{\mathcal{F}_1}\|\nu\|+M_{\mathcal{G}_1}.\mathbb{B}_1\mathcal{F}_{1.\max}.$$

Hence,

$$\|\mathbb{U}_1(\mu,\nu)\| \leq \mathcal{A}_1\|\mu\|+\mathcal{B}_1\|\nu\|+\mathcal{C}_1. \tag{10}$$

By similar procedure, we obtain

$$\|\mathbb{U}_2(\mu,\nu)\| \leq \mathcal{A}_2\|\mu\|+\mathcal{B}_2\|\nu\|+\mathcal{C}_2. \tag{11}$$

Thus, inequalities (10) and (11) imply that

$$\left\|\mathbb{U}(\mu,\nu)\right\|_{\mathbb{X}} := \begin{pmatrix} \left\|\mathbb{U}_1(\mu,\nu)\right\| \\ \left\|\mathbb{U}_2(\mu,\nu)\right\| \end{pmatrix} \leq \mathbb{B}_{\mathbb{MAT}}\begin{pmatrix} \|\mu\| \\ \|\nu\| \end{pmatrix} + \begin{pmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \end{pmatrix}, \tag{12}$$

where

$$\mathbb{B}_{\mathbb{MAT}} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{B}_1 \\ \mathcal{A}_2 & \mathcal{B}_2 \end{pmatrix}.$$

In a similar way, we obtain

$$\left\|\mathbb{V}(\bar{\mu},\bar{\nu})\right\|_{\mathbb{X}} := \begin{pmatrix} \left\|\mathbb{V}_1(\bar{\mu},\bar{\nu})\right\| \\ \left\|\mathbb{V}_2(\bar{\mu},\bar{\nu})\right\| \end{pmatrix} \leq \mathbb{D}_{\mathbb{MAT}}\begin{pmatrix} \|\bar{\mu}\| \\ \|\bar{\nu}\| \end{pmatrix} + \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{13}$$

where

$$\mathbb{D}_{\mathbb{MAT}} = \begin{pmatrix} \mathbb{A}_1 L_{\mathcal{H}_i} & \mathbb{A}_1 \bar{L}_{\mathcal{H}_i} \\ \mathbb{A}_2 L_{\mathcal{H}_i} & \mathbb{A}_2 \bar{L}_{\mathcal{H}_i} \end{pmatrix}.$$

Recombine (12) and (13), which implies that

$$\left\|\mathbb{U}(\mu,\nu)\right\|_{\mathbb{X}} + \left\|\mathbb{V}(\bar{\mu},\bar{\nu})\right\|_{\mathbb{X}} \leq \mathbb{B}_{\mathbb{MAT}}\begin{pmatrix} \|\mu\| \\ \|\nu\| \end{pmatrix} + \mathbb{D}_{\mathbb{MAT}}\begin{pmatrix} \|\bar{\mu}\| \\ \|\bar{\nu}\| \end{pmatrix} + \begin{pmatrix} \mathcal{C}_1+\phi_1 \\ \mathcal{C}_2+\phi_2 \end{pmatrix}. \tag{14}$$

Therefore, we check for $\xi = (\xi_1,\xi_2) \in \mathbb{R}_+^2$ such that $\mathbb{U}(\mu,\nu)+\mathbb{V}(\bar{\mu},\bar{\nu}) \in \mathbb{Q}_\xi$. Regarding this, in view of (14), it is sufficient to verify that

$$\mathbb{C}_{\mathbb{MAT}}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \leq \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix},$$

where $\mathbb{C}_{\mathbb{MAT}} = \mathbb{B}_{\mathbb{MAT}} + \mathbb{D}_{\mathbb{MAT}}$, and

$$\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \begin{pmatrix} \mathcal{C}_1+\phi_1 \\ \mathcal{C}_2+\phi_2 \end{pmatrix}.$$

Equivalently,

$$\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \leq (\mathbb{I}-\mathbb{C}_{\mathbb{MAT}})\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \tag{15}$$

Since the spectral radius of $\mathbb{C}_{\mathbb{MAT}}$ is <1, according to Theorem 1, we have the matrix $(\mathbb{I}-\mathbb{C}_{\mathbb{MAT}})$ is non-singular and $(\mathbb{I}-\mathbb{C}_{\mathbb{MAT}})^{-1}$ has positive elements. So, (15) is equal to

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \geq (\mathbb{I}-\mathbb{A}_{\mathbb{MAT}})^{-1}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}.$$

In addition, if we take

$$(\mathbb{I}-\mathbb{A}_{\mathbb{MAT}})^{-1} = \begin{pmatrix} \rho_1 & \rho_2 \\ \rho_3 & \rho_4 \end{pmatrix},$$

thus, we find

$$\begin{cases} \xi_1 \geq \rho_1 M_1 + \rho_2 M_2, \\ \xi_2 \geq \rho_3 M_1 + \rho_4 M_2. \end{cases}$$

Therefore, $\mathbb{G}(\mu,\nu)+\mathbb{H}(\bar{\mu},\bar{\nu}) \in \mathbb{Q}_{\xi}$.

Step 2: The mapping $\mathbb{V}$ is $\mathbb{D}_{\mathbb{MAT}}$-contraction on $\mathbb{Q}_{\xi}$.
Indeed, $\forall\kappa \in J$ and for any $(\mu_1,\nu_1),(\mu_2,\nu_2) \in \mathbb{Q}_{\xi}$, by a similar procedure in the proof of Theorem 4, it is not difficult to verify that

$$\left\|\mathbb{V}(\mu_1,\nu_1)-\mathbb{V}(\mu_2,\nu_2)\right\|_{\mathbb{X},\mathfrak{B}} \leq \mathbb{D}_{\mathbb{MAT}}\|(\mu_1,\nu_1)-(\mu_2,\nu_2)\|_{\mathbb{X}}.$$

Since the spectral radius of $\mathbb{D}_{\mathbb{MAT}}$ is <1, the mapping $\mathbb{V}$ is an $\mathbb{D}_{\mathbb{MAT}}$-contraction on $\mathbb{Q}_{\xi}$.

Step 3: The mapping $\mathbb{U}$ is continuous and compact.
By the continuity of $\mathbb{G}_1$ and $\mathbb{G}_2$, we deduce that $\mathbb{U}$ is continuous. Moreover, we show that $\mathbb{U}$ is uniformly bounded on $\mathbb{Q}_{\xi}$. From (12), and $\forall(\mu,\nu) \in \mathbb{Q}_{\xi}$, we find that

$$\|\mathbb{U}(\mu,\nu)\|_{\mathbb{X}} := \begin{pmatrix} \|\mathbb{U}_1(\mu,\nu)\| \\ \|\mathbb{U}_2(\mu,\nu)\| \end{pmatrix} \leq \mathbb{B}_{\mathbb{MAT}}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \end{pmatrix} < \infty.$$

This means that the mapping $\mathbb{U}$ is uniformly bounded on $\mathbb{Q}_{\xi}$.

At the last step, we are going to prove that $\mathbb{U}(\mathbb{Q}_{\xi})$ is equicontinuous. From the hypotheses (HP_1) and (HP_2), for $(\mu,\nu) \in \mathbb{Q}_{\xi}$, and $\kappa_1 \leq \kappa_2$ for any $\kappa_1,\kappa_2 \in J$, we obtain

$$\begin{aligned}
&\|\mathbb{U}_1(\mu,\nu)(\kappa_2)-\mathbb{U}_1(\mu,\nu)(\kappa_1)\| \\
=&\Bigg|\mathcal{G}_1(\kappa_2,\mu(\kappa_2),\nu(\kappa_2))\int_a^{\kappa_2}\frac{\Phi'(\sigma)(\Phi(\kappa_2)-\Phi(\sigma))^{\frac{\alpha_1}{k}-1}}{k\Gamma_k(\alpha_1)}\mathcal{F}_1(\sigma,\mu(\sigma),\nu(\sigma))d\sigma \\
&-\mathcal{G}_1(\kappa_1,\mu(\kappa_1),\nu(\kappa_1))\int_a^{\kappa_1}\frac{\Phi'(\sigma)(\Phi(\kappa_1)-\Phi(\sigma))^{\frac{\alpha_1}{k}-1}}{k\Gamma_k(\alpha_1)}\mathcal{F}_1(\sigma,\mu(\sigma),\nu(\sigma))d\sigma\Bigg| \\
=&\Bigg|\mathcal{G}_1(\kappa_2,\mu(\kappa_2),\nu(\kappa_2))\int_a^{\kappa_2}\frac{\Phi'(\sigma)(\Phi(\kappa_2)-\Phi(\sigma))^{\frac{\alpha_1}{k}-1}}{k\Gamma_k(\alpha_1)}\mathcal{F}_1(\sigma,\mu(\sigma),\nu(\sigma))d\sigma \\
&-\mathcal{G}_1(\kappa_1,\mu(\kappa_1),\nu(\kappa_1))\int_a^{\kappa_1}\frac{\Phi'(\sigma)(\Phi(\kappa_1)-\Phi(\sigma))^{\frac{\alpha_1}{k}-1}}{k\Gamma_k(\alpha_1)}\mathcal{F}_1(\sigma,\mu(\sigma),\nu(\sigma))d\sigma \\
&+\mathcal{G}_1(\kappa_1,\mu(\kappa_1),\nu(\kappa_1))\int_a^{\kappa_2}\frac{\Phi'(\sigma)(\Phi(\kappa_2)-\Phi(\sigma))^{\frac{\alpha_1}{k}-1}}{k\Gamma_k(\alpha_1)}\mathcal{F}_1(\sigma,\mu(\sigma),\nu(\sigma))d\sigma
\end{aligned}$$

$$
\begin{aligned}
&- \mathcal{G}_1(\kappa_1, \mu(\kappa_1), \nu(\kappa_1)) \int_a^{\kappa_2} \frac{\Phi'(\sigma)(\Phi(\kappa_2) - \Phi(\sigma))^{\frac{\alpha_1}{k}-1}}{k\Gamma_k(\alpha_1)} \mathcal{F}_1(\sigma, \mu(\sigma), \nu(\sigma)) d\sigma \Big| \\
\leq & \Big| \mathcal{G}_1(\kappa_2, \mu(\kappa_2), \nu(\kappa_2)) - \mathcal{G}_1(\kappa_1, \mu(\kappa_1), \nu(\kappa_1)) \Big| \\
&. \int_a^{\kappa_2} \frac{\Phi'(\sigma)(\Phi(\kappa_2) - \Phi(\sigma))^{\frac{\alpha_1}{k}-1}}{k\Gamma_k(\alpha_1)} \Big| \mathcal{F}_1(\sigma, \mu(\sigma), \nu(\kappa)) \Big| d\sigma + \Big| \mathcal{G}_1(\kappa, \mu(\kappa_1), \nu(\kappa_1)) \Big| \\
&. \int_a^{\kappa_1} \frac{\Phi'(\sigma)\Big[(\Phi(\kappa_2) - \Phi(\sigma))^{\frac{\alpha_1}{k}-1} - (\Phi(\kappa_1) - \Phi(\sigma))^{\frac{\alpha_1}{k}-1}\Big]}{k\Gamma_k(\alpha_1)} \Big| \mathcal{F}_1(\sigma, \mu(\sigma), \nu(\sigma)) \Big| d\sigma \\
= & \Big| \mathcal{G}_1(\kappa_2, \mu(\kappa_2), \nu(\kappa_2)) - \mathcal{G}_1(\kappa_1, \mu(\kappa_1), \nu(\kappa_1)) - \mathcal{G}_1(\kappa_1, \mu(\kappa_2), \nu(\kappa_2)) \\
&+ \mathcal{G}_1(\kappa_1, \mu(\kappa_2), \nu(\kappa_2)) \Big| . \int_a^{\kappa_2} \frac{\Phi'(\sigma)(\Phi(\kappa_2) - \Phi(\sigma))^{\frac{\alpha_1}{k}-1}}{k\Gamma_k(\alpha_1)} \Big| \mathcal{F}_1(\sigma, \mu(\sigma), \nu(\sigma)) \Big| d\sigma \\
&+ \Big| \mathcal{G}_1(\kappa, \mu(\kappa_1), \nu(\kappa_1)) \Big| \int_a^{\kappa_1} \frac{\Phi'(\sigma)\Big[(\Phi(\kappa_2) - \Phi(\sigma))^{\frac{\alpha_1}{k}-1} - (\Phi(\kappa_1) - \Phi(\sigma))^{\frac{\alpha_1}{k}-1}\Big]}{k\Gamma_k(\alpha_1)} \Big| \mathcal{F}_1(\sigma, \mu(\sigma), \nu(\sigma)) \Big| d\sigma \\
\leq & \Big| \mathcal{G}_1(\kappa_2, \mu(\kappa_2), \nu(\kappa_2)) - \mathcal{G}_1(\kappa_1, \mu(\kappa_2), \nu(\kappa_2)) \Big| \int_a^{\kappa_2} \frac{\Phi'(\sigma)(\Phi(\kappa_2) - \Phi(\sigma))^{\frac{\alpha_1}{k}-1}}{k\Gamma_k(\alpha_1)} \Big| \mathcal{F}_1(\sigma, \mu(\sigma), \nu(\sigma)) \Big| d\sigma \\
&+ \Big| \mathcal{G}_1(\kappa_1, \mu(\kappa_2), \nu(\kappa_2)) - \mathcal{G}_1(\kappa_1, \mu(\kappa_1), \nu(\kappa_1)) \Big| \int_a^{\kappa_2} \frac{\Phi'(\sigma)(\Phi(\kappa_2) - \Phi(\sigma))^{\frac{\alpha_1}{k}-1}}{k\Gamma_k(\alpha_1)} \Big| \mathcal{F}_1(\sigma, \mu(\sigma), \nu(\sigma)) \Big| d\sigma \\
&+ \Big| \mathcal{G}_1(\kappa, \mu(\kappa_1), \nu(\kappa_1)) \Big| \int_a^{\kappa_1} \frac{\Phi'(\sigma)\Big[(\Phi(\kappa_2) - \Phi(\sigma))^{\frac{\alpha_1}{k}-1} - (\Phi(\kappa_1) - \Phi(\sigma))^{\frac{\alpha_1}{k}-1}\Big]}{k\Gamma_k(\alpha_1)} \Big| \mathcal{F}_1(\sigma, \mu(\sigma), \nu(\sigma)) \Big| d\sigma \\
\leq & \theta_1(\delta) \int_a^{\kappa_2} \frac{\Phi'(\sigma)(\Phi(\kappa_2) - \Phi(\sigma))^{\frac{\alpha_1}{k}-1}}{k\Gamma_k(\alpha_1)} \Big| \mathcal{F}_1(\sigma, \mu(\sigma), \nu(\sigma)) \Big| d\sigma \\
&+ \big(\mathrm{L}_{\mathcal{G}_1} \| \mu(\kappa_2) - \mu(\kappa_1) \| + \bar{\mathrm{L}}_{\mathcal{G}_1} \| \nu(\kappa_2) - \nu(\kappa_1) \| \big) \\
&. \int_a^{\kappa_2} \frac{\Phi'(\sigma)(\Phi(\kappa_2) - \Phi(\sigma))^{\frac{\alpha_1}{k}-1}}{k\Gamma_k(\alpha_1)} \Big| \mathcal{F}_1(\sigma, \mu(\sigma), \nu(\kappa)) \Big| d\sigma \\
&+ \Big[\big(\mathrm{L}_{\mathcal{G}_1} \| \mu(\kappa_1) \| + \bar{\mathrm{L}}_{\mathcal{G}_1} \| \nu(\kappa_1) \| \big) + \mathcal{G}_{1.\max} \Big] \\
&. \int_a^{\kappa_1} \frac{\Phi'(\sigma)\Big[(\Phi(\kappa_2) - \Phi(\sigma))^{\frac{\alpha_1}{k}-1} - (\Phi(\kappa_1) - \Phi(\sigma))^{\frac{\alpha_1}{k}-1}\Big]}{k\Gamma_k(\alpha_1)} \Big| \mathcal{F}_1(\sigma, \mu(\sigma), \nu(\sigma)) \Big| d\sigma \\
\leq & \theta_1(\delta) \frac{(\Phi(\kappa_2) - \Phi(\sigma))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1 + k)} \mathrm{M}_{\mathcal{F}_1} + \theta_1(\delta) \big(\mathrm{L}_{\mathcal{G}_1} + \bar{\mathrm{L}}_{\mathcal{G}_1} \big) \frac{(\Phi(\kappa_2) - \Phi(\sigma))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1 + k)} \mathrm{M}_{\mathcal{F}_1} \\
&+ \Big[\big(\mathrm{L}_{\mathcal{G}_1} \| \mu \| + \bar{\mathrm{L}}_{\mathcal{G}_1} \| \nu \| \big) + \mathcal{G}_{1.\max} \Big] \mathrm{M}_{\mathcal{F}_1}
\end{aligned}
$$

$$\cdot\int_a^{\kappa_1}\frac{\Phi'(\sigma)\Big[(\Phi(\kappa_2)-\Phi(\sigma))^{\frac{\alpha_1}{k}-1}-(\Phi(\kappa_1)-\Phi(\sigma))^{\frac{\alpha_1}{k}-1}\Big]}{k\Gamma_k(\alpha_1)}d\sigma$$

$$\leq\theta_1(\delta)\mathbb{B}_1\mathrm{M}_{\mathcal{F}_1}+\theta_1(\delta)\big(\mathrm{L}_{\mathcal{G}_1}+\bar{\mathrm{L}}_{\mathcal{G}_1}\big)\mathbb{B}_1\mathrm{M}_{\mathcal{F}_1}+\Big[\big(\mathrm{L}_{\mathcal{G}_1}\|\mu\|+\bar{\mathrm{L}}_{\mathcal{G}_1}\|\nu\|\big)+\mathcal{G}_{1.\max}\Big]$$

$$\cdot\frac{\mathrm{M}_{\mathcal{F}_1}}{\Gamma_k(a+k)}\Big[2(\Phi(\kappa_2)-\Phi(\kappa_1))^{\frac{\alpha_1}{k}}+\Big|(\Phi(\kappa_2)-\Phi(a))^{\frac{\alpha_1}{k}}-(\Phi(\kappa_1)-\Phi(a))^{\frac{\alpha_1}{k}}\Big|\Big]$$

$$:=\Delta_1+\bar{\Delta_1}\Big[2(\Phi(\kappa_2)-\Phi(\kappa_1))^{\frac{\alpha_1}{k}}+\Big|(\Phi(\kappa_2)-\Phi(a))^{\frac{\alpha_1}{k}}-(\Phi(\kappa_1)-\Phi(a))^{\frac{\alpha_1}{k}}\Big|\Big].$$

Similarly,

$$\|\mathbb{U}_2(\mu,\nu)(\kappa_2)-\mathbb{U}_2(\mu,\nu)(\kappa_1)\|\leq\Delta_2+\bar{\Delta_2}\Big[2(\Phi(\kappa_2)-\Phi(\kappa_1))^{\frac{\alpha_2}{k}}+\Big|(\Phi(\kappa_2)-\Phi(a))^{\frac{\alpha_2}{k}}-(\Phi(\kappa_1)-\Phi(a))^{\frac{\alpha_2}{k}}\Big|\Big].$$

Therefore,

$$\|\mathbb{U}(\mu,\nu)(\tau_2)-\mathbb{U}(\mu,\nu)(\tau_1)\|:=\begin{pmatrix}\|\mathbb{U}_1(\mu,\nu)(\tau_2)-\mathbb{U}_1(\mu,\nu)(\tau_1)\|\\ \|\mathbb{U}_2(\mu,\nu)(\tau_2)-\mathbb{U}_2(\mu,\nu)(\tau_1)\|\end{pmatrix}$$

$$\leq\begin{pmatrix}\Delta_1+\bar{\Delta_1}\Big[2(\Phi(\kappa_2)-\Phi(\kappa_1))^{\frac{\alpha_1}{k}}+\Big|(\Phi(\kappa_2)-\Phi(a))^{\frac{\alpha_1}{k}}-(\Phi(\kappa_1)-\Phi(a))^{\frac{\alpha_1}{k}}\Big|\Big]\\ \Delta_2+\bar{\Delta_2}\Big[2(\Phi(\kappa_2)-\Phi(\kappa_1))^{\frac{\alpha_2}{k}}+\Big|(\Phi(\kappa_2)-\Phi(a))^{\frac{\alpha_2}{k}}-(\Phi(\kappa_1)-\Phi(a))^{\frac{\alpha_2}{k}}\Big|\Big]\end{pmatrix}.$$

Thus, we deduce that $\mathbb{T}(\mathbb{Q}_\xi)$ is equicontinuous. Due to Arzelà–Ascoli's theorem, we conclude that the mapping $\mathbb{U}$ is compact. Hence, the requirements of Theorem 3 are fulfilled. Thus, in view of the Krasnoselskii's FPT, we derive that the mapping $\mathbb{T}=\mathbb{U}+\mathbb{V}$ defined by (9) possesses at least one fixed point $(\mu,\nu)\in\mathbb{Q}_\xi$, which is the solution of the coupled fully hybrid system of $(\mathbf{k},\mathbf{\Phi})$–Hilfer BVPs (1)–(2). □

Now, we end this section by discussing the $\mathcal{UH}$ stability of the coupled fully hybrid system of $(\mathbf{k},\mathbf{\Phi})$–Hilfer BVPs (1)–(2) by utilizing its solution in the sense of integral form given as

$$\mu(\tau)=\mathbb{T}_1(\mu,\nu)(\tau),\quad \nu(\tau)=\mathbb{T}_2(\mu,\nu)(\tau),$$

such that $\mathbb{T}_1$ and $\mathbb{T}_2$ are given in (6) and (7).

Let us define the following mappings $\mathbb{S}_1,\mathbb{S}_2:\mathbb{X}\to C(\mathrm{J},\mathbb{R})$ as:

$$\begin{cases}{}^{k,H}\mathrm{D}^{\alpha_1,\beta_1,\Phi}\tilde{w}_1(\kappa)-\mathcal{F}_1(\kappa,\tilde{\mu}(\kappa),\tilde{\nu}(\kappa))=\mathbb{S}_1(\tilde{\mu},\tilde{\nu})(\kappa),\\ {}^{k,H}\mathrm{D}^{\alpha_2,\beta_2,\Phi}\tilde{w}_2(\kappa)-\mathcal{F}_2(\kappa,\tilde{\mu}(\kappa),\tilde{\nu}(\kappa))=\mathbb{S}_2(\tilde{\mu},\tilde{\nu})(\kappa),\end{cases}\quad\kappa\in\mathrm{J}.$$

In addition, we assume that the next inequalities

$$\begin{cases}\big\|\mathbb{S}_1(\tilde{\mu},\tilde{\nu})(\tau)\big\|\leq\epsilon_1,\\ \big\|\mathbb{S}_2(\tilde{\mu},\tilde{\nu})(\tau)\big\|\leq\epsilon_2,\end{cases}\quad\tau\in\mathrm{J},\tag{16}$$

for some $\epsilon_1,\epsilon_2>0$ are to be held.

Definition 5 ([39]). *The coupled fully hybrid system of* $(\mathbf{k}, \mathbf{\Phi})$*–Hilfer BVPs* (1)–(2) *is* $\mathcal{UH}$*-stable if there are constants* $\omega_i > 0, i = \overline{1,4}$ *such that* $\forall \epsilon_1, \epsilon_1 > 0$ *and for all solutions* $(\tilde{\mu}, \tilde{\nu}) \in \mathbb{X}$ *of inequality* (16), $\exists$ *a solution* $(\mu, \nu) \in \mathbb{X}$ *of* (1)–(2) *such that*

$$\begin{cases} \|\tilde{\mu}(\tau) - \mu(\tau)\| \leq \omega_1 \epsilon_1 + \omega_2 \epsilon_2, \\ \|\tilde{\nu}(\tau) - \nu(\tau)\| \leq \omega_3 \epsilon_1 + \omega_4 \epsilon_2, \end{cases} \quad \tau \in \mathrm{J}.$$

In this part of the paper, we aim to prove that the given coupled fully hybrid system (1)–(2) is $\mathcal{UH}$-stable. To do this, we use Urs's technique.

Theorem 6. *Consider the hypotheses of Theorem 4 to be held. Then, the coupled fully hybrid system of* $(\mathbf{k}, \mathbf{\Phi})$*–Hilfer BVPs* (1)–(2) *is* $\mathcal{UH}$*-stable.*

Proof. Let $(\mu, \nu) \in \mathbb{X}$ be the solution of the coupled fully hybrid system of $(\mathbf{k}, \mathbf{\Phi})$–Hilfer BVPs (1)–(2) satisfying (6) and (7). Assume that $(\tilde{\mu}, \tilde{\nu})$ is any solution verifying (16):

$$\begin{cases} {}^{\mathrm{k,H}}\mathrm{D}^{\alpha_1,\beta_1,\Phi} \tilde{\varpi}_1(\kappa) = \mathcal{F}_1(\kappa, \tilde{\mu}(\kappa), \tilde{\nu}(\kappa)) + \mathbb{S}_1(\tilde{\mu}, \tilde{\nu})(\kappa), \\ {}^{\mathrm{k,H}}\mathrm{D}^{\alpha_2,\beta_2,\Phi} \tilde{\varpi}_2(\kappa) = \mathcal{F}_2(\kappa, \tilde{\mu}(\kappa), \tilde{\nu}(\kappa)) + \mathbb{S}_2(\tilde{\mu}, \tilde{\nu})(\kappa). \end{cases} \quad \tau \in \mathrm{J}.$$

So,

$$\tilde{\mu}(\tau) = \mathbb{T}_1(\tilde{\mu}, \tilde{\nu})(\kappa) + \mathcal{G}_1(\kappa, \tilde{\mu}(\kappa), \tilde{\nu}(\kappa)) \int_a^\kappa \frac{\Phi'(\sigma)(\Phi(\kappa) - \Phi(\sigma))^{\frac{\alpha_1}{k} - 1}}{k\Gamma_k(\alpha_1)} \mathbb{S}_1(\tilde{\mu}, \tilde{\nu})(\sigma) ds, \quad (17)$$

and

$$\tilde{\nu}(\tau) = \mathbb{T}_2(\tilde{\mu}, \tilde{\nu})(\kappa) + \mathcal{G}_2(\kappa, \tilde{\mu}(\kappa), \tilde{\nu}(\kappa)) \int_a^\kappa \frac{\Phi'(\sigma)(\Phi(\kappa) - \Phi(\sigma))^{\frac{\alpha_2}{k} - 1}}{k\Gamma_k(\alpha_2)} \mathbb{S}_2(\tilde{\mu}, \tilde{\nu})(\sigma) ds. \quad (18)$$

Now, (17) and (18) give

$$\begin{aligned} \|\tilde{\mu}(\tau) - \mathbb{T}_1(\tilde{\mu}, \tilde{\nu})(\tau)\| &\leq \|\mathcal{G}_1(\kappa, \tilde{\mu}(\kappa), \tilde{\nu}(\kappa))\| \int_a^\kappa \frac{\Phi'(\sigma)(\Phi(\kappa) - \Phi(\sigma))^{\frac{\alpha_1}{k} - 1}}{k\Gamma_k(\alpha_1)} \|\mathbb{S}_1(\tilde{\mu}, \tilde{\nu})(\sigma)\| ds \\ &\leq \mathrm{M}_{\mathcal{G}_1} \mathbb{B}_1 \epsilon_1, \end{aligned} \quad (19)$$

and

$$\begin{aligned} \|\tilde{\nu}(\kappa) - \mathbb{T}_2(\tilde{\mu}, \tilde{\nu})(\kappa)\| &\leq \|\mathcal{G}_2(\kappa, \tilde{\mu}(\kappa), \tilde{\nu}(\kappa))\| \int_a^\kappa \frac{\Phi'(\sigma)(\Phi(\kappa) - \Phi(\sigma))^{\frac{\alpha_2}{k} - 1}}{k\Gamma_k(\alpha_2)} \|\mathbb{S}_2(\tilde{\mu}, \tilde{\nu})(\sigma)\| ds \\ &\leq \mathrm{M}_{\mathcal{G}_2} \mathbb{B}_2 \epsilon_2. \end{aligned} \quad (20)$$

Thus, by (H2) and inequalities (19) and (20), we obtain

$$\begin{aligned} \|\tilde{\mu}(\kappa) - \mu(\kappa)\| &= \|\tilde{\mu}(\kappa) - \mathbb{T}_1(\tilde{\mu}, \tilde{\nu})(\kappa) + \mathbb{T}_1(\tilde{\mu}, \tilde{\nu})(\kappa) - \mu(\kappa)\| \\ &\leq \|\tilde{\mu}(\kappa) - \mathbb{T}_1(\tilde{\mu}, \tilde{\nu})(\kappa)\| + \|\mathbb{T}_1(\tilde{\mu}, \tilde{\nu})(\kappa) - \mathbb{T}_1(\mu, \nu)(\kappa)\| \\ &\leq \mathrm{M}_{\mathcal{G}_1} \mathbb{B}_1 \epsilon_1 + \left(\bar{\mathbb{A}}_1 \|\tilde{\mu} - \mu\| + \bar{\mathbb{B}}_1 \|\tilde{\nu} - \nu\|\right). \end{aligned}$$

Hence, we obtain

$$\|\tilde{\mu} - \mu\| \leq \mathrm{M}_{\mathcal{G}_1} \mathbb{B}_1 \epsilon_1 + \left(\bar{\mathbb{A}}_1 \|\tilde{\mu} - \mu\| + \bar{\mathbb{B}}_1 \|\tilde{\nu} - \nu\|\right). \quad (21)$$

Similarly, we have

$$\|\tilde{\nu} - \nu\| \leq \mathrm{M}_{\mathcal{G}_2}\mathbb{B}_2\epsilon_2 + \left(\bar{\mathbb{A}}_2\|\tilde{\mu} - \mu\| + \bar{\mathbb{B}}_2\|\tilde{\nu} - \nu\|\right). \tag{22}$$

Inequalities (21) and(22) can be rewritten in a matrix form as

$$(\mathbb{I} - \mathbb{A}_{\mathbb{MAT}})\begin{pmatrix} \|\tilde{\mu} - \mu\| \\ \|\tilde{\nu} - \nu\| \end{pmatrix} \leq \begin{pmatrix} \mathrm{M}_{\mathcal{G}_1}\mathbb{B}_1\epsilon_1 \\ \mathrm{M}_{\mathcal{G}_2}\mathbb{B}_2\epsilon_2 \end{pmatrix}, \tag{23}$$

where $\mathbb{A}_{\mathbb{MAT}}$ is the matrix given by (8). Since the spectral radius of $\mathbb{A}_{\mathbb{MAT}}$ is < 1, by Theorem 1, we deduce that $(\mathbb{I} - \mathbb{A}_{\mathbb{MAT}})$ is non-singular and $(\mathbb{I} - \mathbb{A}_{\mathbb{MAT}})^{-1}$ possesses positive elements. Hence, (23) is equivalent to the form

$$\begin{pmatrix} \|\tilde{\mu} - \mu\| \\ \|\tilde{\nu} - \nu\| \end{pmatrix} \leq (\mathbb{I} - \mathbb{A}_{\mathbb{MAT}})^{-1}\begin{pmatrix} \mathrm{M}_{\mathcal{G}_1}\mathbb{B}_1\epsilon_1 \\ \mathrm{M}_{\mathcal{G}_2}\mathbb{B}_2\epsilon_2 \end{pmatrix},$$

which yields that

$$\begin{cases} \|\tilde{\mu} - \mu\| \leq \rho_1\mathrm{M}_{\mathcal{G}_1}\mathbb{B}_1\epsilon_1 + \rho_2\mathrm{M}_{\mathcal{G}_2}\mathbb{B}_2\epsilon_2, \\ \\ \|\tilde{\nu} - \nu\| \leq \rho_3\mathrm{M}_{\mathcal{G}_1}\mathbb{B}_1\epsilon_1 + \rho_4\mathrm{M}_{\mathcal{G}_2}\mathbb{B}_2\epsilon_2, \end{cases}$$

where $\rho_i, i = \overline{1,4}$ are the elements of $(\mathbb{I} - \mathbb{A}_{\mathbb{MAT}})^{-1}$. Consequently, the coupled fully hybrid system of $(\mathbf{k}, \mathbf{\Phi})$–Hilfer BVPs (1)–(2) is $\mathcal{UH}$-stable. □

4. Applications

We provide an example in this part to investigate and guarantee the validity of the results.

Example 1. *Consider the following coupled fully hybrid system of* $(\mathbf{k}, \mathbf{\Phi})$*–Hilfer BVPs:*

$$\begin{cases} {}^{\frac{1}{2},H}\mathrm{D}^{\frac{3}{4},\frac{1}{2},\Phi}\omega_1(\kappa) = \mathcal{F}_1(\kappa, \mu(\kappa), \nu(\kappa)), \\ {}^{\frac{1}{2},H}\mathrm{D}^{\frac{1}{3},\frac{1}{4},\Phi}\omega_2(\kappa) = \mathcal{F}_2(\kappa, \mu(\kappa), \nu(\kappa)), \end{cases} \quad \kappa \in \mathrm{J} := [0,1], \tag{24}$$

with $(\mathbf{k}, \mathbf{\Phi})$*–fractional integrals conditions*

$$\begin{cases} \mu(0) - 1 = \displaystyle\int_0^{\frac{1}{3}} \frac{\Phi'(\sigma)(\Phi(\tau_1) - \Phi(\sigma))^{\frac{1}{\frac{1}{2}}-1}}{\frac{1}{2}\Gamma_{\frac{1}{2}}(1)}\mathcal{H}_1(\sigma, \mu(\sigma), \nu(\sigma))d\sigma, \\ \nu(0) - 2 = \displaystyle\int_0^{\frac{1}{4}} \frac{\Phi'(\sigma)(\Phi(\tau_2) - \Phi(\sigma))^{\frac{\frac{1}{2}}{\frac{1}{2}}-1}}{\frac{1}{2}\Gamma_{\frac{1}{2}}(\frac{1}{2})}\mathcal{H}_1(\sigma, \mu(\sigma), \nu(\sigma))d\sigma, \end{cases} \quad \beta_i \leq \alpha_i, \tau_i \in \mathrm{J}, \tag{25}$$

where

$$\omega_1 = \frac{\mu(\kappa) - \mu(0)}{\mathcal{G}_1(\kappa, \mu(\kappa), \nu(\kappa))}, \qquad \omega_2 = \frac{\nu(\kappa) - \nu(0)}{\mathcal{G}_2(\kappa, \mu(\kappa), \nu(\kappa))}.$$

Here, $\alpha_1 = \frac{3}{4}, \alpha_2 = \frac{1}{3}, \beta_1 = \frac{1}{2}, \beta_2 = \frac{1}{4}, k = \frac{1}{2}, \delta_1 = 1, \delta_2 = \frac{1}{2}, \tau_1 = \frac{1}{3}, \tau_2 = \frac{1}{4}, \phi_1 = 1, \phi_2 = 2, \Phi(\kappa) = \kappa^2, J := [0,1]$, and the functions

$$\mathcal{F}_1(\kappa, \mu(\kappa), \nu(\kappa)) = \frac{\kappa|\mu(\kappa)|}{100(1 + |\mu(\kappa)|)} + \frac{\kappa^3}{49}\sin(\nu(\kappa)) + 3\kappa;$$

$$\mathcal{F}_2(\kappa, \mu(\kappa), \nu(\kappa)) = \frac{1}{10e^\kappa}\frac{|\mu(\kappa)|}{4 + |\mu(\kappa)|} + \frac{1}{9}\cos(\nu(\kappa));$$

$$\mathcal{H}_1(\kappa, \mu(\kappa), \nu(\kappa)) = \frac{1}{20}\tan^{-1}(\mu(\kappa)) + \frac{|\nu(\kappa)|}{9(1 + |\nu(\kappa)|)} + e^\kappa;$$

$$\mathcal{H}_2(\kappa, \mu(\kappa), \nu(\kappa)) = \frac{1}{3\sqrt{\kappa+4}} \frac{|\mu(\kappa)|}{5+|\mu(\kappa)|} + \frac{1}{25+\kappa} \cos(\kappa)|\nu^2(\kappa)|;$$

$$\mathcal{G}_1(\kappa, \mu(\kappa), \nu(\kappa)) = \cos^{-1}\left(\frac{|\mu(\kappa)|}{4}\right) + \frac{\frac{1}{3}e^{-\kappa}}{1+|\nu(\kappa)|};$$

$$\mathcal{G}_2(\kappa, \mu(\kappa), \nu(\kappa)) = \frac{\sin(|\mu^3(\kappa)|)}{\sqrt{\kappa^2+9}} + \sin^{-1}\left(\frac{\kappa}{2}\right) \cos\left(|\nu^3(\kappa)|\right) + 4\kappa.$$

Obviously, the functions $\mathcal{F}_i, \mathcal{H}_i, \mathcal{G}_i, (i = 1, 2)$ are continuous. Furthermore, for all $\kappa \in \mathrm{J}$ and each $\mu_1, \nu_1, \mu_2, \nu_2 \in \mathbb{R}^n$, we have (HP_2) satisfied as follows:

$$\|\mathcal{F}_i(\kappa, \mu_1, \nu_1) - \mathcal{F}_i(\kappa, \mu_2, \nu_2)| \leq \mathrm{L}_{\mathcal{F}_i}\|\mu_1 - \mu_2\| + \bar{\mathrm{L}}_{\mathcal{F}_i}\|\nu_1 - \nu_2\|,$$

$$\|\mathcal{H}_i(\kappa, \mu_1, \nu_1) - \mathcal{H}_i(\kappa, \mu_2, \nu_2)| \leq \mathrm{L}_{\mathcal{H}_i}\|\mu_1 - \mu_2\| + \bar{\mathrm{L}}_{\mathcal{H}_i}\|\nu_1 - \nu_2\|,$$

$$\|\mathcal{G}_i(\kappa, \mu_1, \nu_1) - \mathcal{G}_i(\kappa, \mu_2, \nu_2)| \leq \mathrm{L}_{\mathcal{G}_i}\|\mu_1 - \mu_2\| + \bar{\mathrm{L}}_{\mathcal{G}_i}\|\nu_1 - \nu_2\|,$$

where $\mathrm{L}_{\mathcal{F}_1} = \frac{1}{100}, \mathrm{L}_{\mathcal{F}_2} = \frac{1}{10}, \bar{\mathrm{L}}_{\mathcal{F}_1} = \frac{1}{49}, \bar{\mathrm{L}}_{\mathcal{F}_2} = \frac{1}{9}, \mathrm{L}_{\mathcal{H}_1} = \frac{1}{20}, \mathrm{L}_{\mathcal{H}_2} = \frac{1}{6}, \bar{\mathrm{L}}_{\mathcal{H}_1} = \frac{1}{9}, \bar{\mathrm{L}}_{\mathcal{H}_2} = \frac{1}{25},$ $\mathrm{L}_{\mathcal{G}_1} = \frac{1}{4}, \mathrm{L}_{\mathcal{G}_2} = \frac{1}{3}, \bar{\mathrm{L}}_{\mathcal{G}_1} = \frac{1}{3}, \bar{\mathrm{L}}_{\mathcal{G}_2} = \frac{1}{2}.$

Additionally, (HP_1) is satisfied for

$$\mathrm{M}_{\mathcal{F}_1} = \frac{14849}{4900}, \mathrm{M}_{\mathcal{F}_2} = \frac{19}{90}, \mathrm{M}_{\mathcal{G}_1} = \frac{7}{12}, \mathrm{M}_{\mathcal{G}_2} = \frac{5}{6},$$

and we can calculate that

$$\mathbb{A}_1 = 2.12769, \mathbb{A}_2 = 1.75842, \mathbb{B}_1 = 0.0246914, \mathbb{B}_2 = 0.125.$$

Thus, we obtain

$$\bar{\mathbb{A}}_1 = \left[\mathbb{A}_1\mathrm{L}_{\mathcal{H}_1} + \mathbb{B}_1\left(\mathrm{L}_{\mathcal{G}_1}\mathrm{M}_{\mathcal{F}_1} + \mathrm{L}_{\mathcal{F}_1}\mathrm{M}_{\mathcal{G}_1}\right)\right] < 0.125235,$$

$$\bar{\mathbb{A}}_2 = \left[\mathbb{A}_2\mathrm{L}_{\mathcal{H}_2} + \mathbb{B}_2\left(\mathrm{L}_{\mathcal{G}_2}\mathrm{M}_{\mathcal{F}_2} + \mathrm{L}_{\mathcal{F}_2}\mathrm{M}_{\mathcal{G}_2}\right)\right] < 0.312283,$$

$$\bar{\mathbb{B}}_1 = \left[\mathbb{A}_1\bar{\mathrm{L}}_{\mathcal{H}_1} + \mathbb{B}_1\left(\bar{\mathrm{L}}_{\mathcal{G}_1}\mathrm{M}_{\mathcal{F}_1} + \bar{\mathrm{L}}_{\mathcal{F}_1}\mathrm{M}_{\mathcal{G}_1}\right)\right] < 0.261646,$$

$$\bar{\mathbb{B}}_2 = \left[\mathbb{A}_2\bar{\mathrm{L}}_{\mathcal{H}_2} + \mathbb{B}_2\left(\bar{\mathrm{L}}_{\mathcal{G}_2}\mathrm{M}_{\mathcal{F}_2} + \bar{\mathrm{L}}_{\mathcal{F}_2}\mathrm{M}_{\mathcal{G}_2}\right)\right] < 0.0951053.$$

Hence, all of the conditions of Theorem 4 are satisfied. Therefore, the coupled fully hybrid system of $(\mathbf{k}, \mathbf{\Phi})$–Hilfer BVPs (24)–(25) has one and only one solution. Consequently, by referring to Theorem 6, we easily conclude that the solution is $\mathcal{UH}$-stable.

5. Conclusions

This research paper was devoted to studying a coupled fully hybrid system of quadratic differential equations in the sense of the $(\mathbf{k}, \mathbf{\Phi})$–Hilfer fractional derivative with subject to the $(\mathbf{k}, \mathbf{\Phi})$–$\mathcal{RL}$ fractional integral conditions. The existence and uniqueness of solutions for such a system were established by utilizing the Perov and Krasnoselskii fixed-point theorems in $\mathcal{GBS}$. Moreover, $\mathcal{UH}$ stability was proved by Urs's technique. Finally, an example was provided for checking the validity of our results. For the next research projects, we would like to extend our methods and techniques in the context of post-quantum calculus. One can combine these methods with numerical techniques for studying the dynamics of the solutions based on (p, q)-operators.

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