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Positive Periodic Solutions for a First-Order Nonlinear Neutral Differential Equation with Impulses on Time Scales

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Abstract: In this article, we discuss the existence of a positive periodic solution for a first-order nonlinear neutral differential equation with impulses on time scales. Based on the Leggett–Williams fixed-point theorem and Krasnoselskii's fixed-point theorem, some sufficient conditions are established for the existence of positive periodic solution. An example is given to show the feasibility and application of the obtained results. Since periodic solutions are solutions with symmetry characteristics, the existence conditions for periodic solutions also imply symmetry.

Keywords: fixed-point theorem; time scales; existence; periodic solution

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1. Introduction

Nonlinear neutral functional differential equations on time scales have been studied by many authors due to their wide applications. In 2010, Wang, Li, and Fei [1] studied the following nonlinear neutral functional differential equation with impulses on time scales:

$$\begin{cases} (x(t)+c(t)x(t-r_1))^{\Delta} = a(t)g(x(t))x(t) - \sum_{i=1}^{n} \lambda_i f_i(t, x(t-\tau_i(t))),\ t \neq t_j,\ t \in \mathbb{T}, \\ x(t_j^-) - x(t_j^+) = I_j(x(t_j)),\ t = t_j, \end{cases} \tag{1}$$

where $j = 1, 2, \cdots, q$, $\mathbb{T}$ is a periodic time scale, $a \in C(\mathbb{T}, \mathbb{R}^+)$, $c \in C(\mathbb{T}, [0,1))$, $\tau_i \in C(\mathbb{T}, \mathbb{R})$ and all of them are ω—periodic functions, $f_i \in C(\mathbb{T} \times \mathbb{R}^+, \mathbb{R}^+)$ is nondecreasing with respect to x, and ω—periodic with respect to its first argument, $g \in C(\mathbb{R}^+, \mathbb{R}^+)$ is a bounded function, $I_j \in C(\mathbb{R}, \mathbb{R}^+)$ is a bounded function. Under $c \in [0,1)$ and other conditions, using the Leggett–Williams fixed-point theorem, the authors obtained some sufficient conditions for guaranteeing the existence of three positive periodic solutions to (1). In this paper, we will extend the range of $c(t)$ to $(-\infty, 1) \cup (1, +\infty)$ and obtain the existence of positive periodic solutions for a first-order neutral differential equation on time scales. Many biological models, physical models, and economic models are described by first-order neutral differential equations. Researchers have conducted significant research on the above equations for a long time. From the continuation theorem of the coincidence degree principle, Sella [2] studied a first-order neutral functional differential equation. Using the Leggett–Williams fixed-point theorem, Wang and Dai [3] considered the existence of three periodic solutions of nonlinear neutral functional differential equations. Luo et al. [4] investigated the existence of positive periodic solutions for the first-order neutral differential equation with time-varying delays. In a very recent paper, Candan [5] dealt with the existence of positive periodic solutions for the first-order neutral differential equation by using Krasnoselskii's fixed-point theorem.

In 1988, Hilger [6] first introduced the theory of time scales for unifying continuous and discrete analysis. After that, the study for differential equations on time scales has attracted many researchers' attention, resulting in significant research results. In [7], Ardjouni and Djoudi studied the existence of periodic solutions for nonlinear neutral dynamic equations with variable delay on a time scale by using a large contraction theorem. Based on pinning impulsive control, finite-time synchronization of nonlinear complex dynamical networks on time scales was obtained in [8]. Using matrix-valued Lyapunov functions, Babenko [9] considered the consensus tracking investigation for multi-agent systems on the time scale. Liu and Zhang [10] dealt with uniqueness and stability results for functional differential equations on time scales which greatly extend the research range of dynamic equations. For more results about dynamic equations, see [11–18].

Pulse exists widely in nature, and its research is of great significance to reveal the essence of the system and control the behavior of the system. Therefore, the study of impulsive differential equations has important practical value. For important research on impulsive differential equations or systems, Akgl and Zafer [19] obtained prescribed asymptotic behavior of second-order impulsive differential equations via principal and nonprincipal solutions; principal and nonprincipal solutions of impulsive differential equations were studied in [20]; Akgöl and Zafer [21] obtained asymptotic integration of second-order impulsive differential equations; the authors of [22] investigated multiplicity results for second order impulsive differential equations by variational methods. For more details about impulsive differential equations, see [23–27].

This paper is mainly motivated by paper [1]. In particular, we will study the following first-order nonlinear neutral differential equation with impulses on time scales:

$$\begin{cases} (x(t) - c(t)x(t-\tau_0))^{\Delta} = a(t)g(x(t))x(t) - \lambda f(t, x(t-\tau(t))),\ t \neq t_j,\ t \in \mathbb{T}, \\ x(t_j^-) - x(t_j^+) = I_j(x(t_j)),\ t = t_j, \end{cases} \tag{2}$$

where $j = 1, 2, \cdots, q$, $\mathbb{T}$ is a periodic time scale, $a \in C(\mathbb{T}, \mathbb{R}^+)$, $c \in C(\mathbb{T}, (-\infty, 1) \cup (1, +\infty))$, $\tau \in C(\mathbb{T}, \mathbb{R})$ and all of them are ω—periodic functions, τ_0 is a constant, λ is a parameter, $f \in C(\mathbb{T} \times \mathbb{R}^+, \mathbb{R}^+)$ is nondecreasing with respect to x, and ω—periodic with respect to its first argument, $g \in C(\mathbb{R}^+, \mathbb{R}^+)$ is a bounded function, $I_j \in C(\mathbb{R}, \mathbb{R}^+)$ is a bounded function, $x(t_j^-)$ and $x(t_j^+)$ represent the left and right limits of $x(t_j)$, respectively, and there exists a positive constant p such that $t_{j+p} = t + \omega$ and $I_{j+p}(x(t_{j+p})) = I_j(x(t_j))$. Since $\mathbb{T}$ is a ω—periodic time scale, we let $\{[0, \omega] \cap t_j; j \in \mathbb{Z}^+\} = \{t_1, t_2, \cdots, t_q\}$.

Remark 1. *We find that the number of delays in (1) has no essential effect on the conclusions, so Equation (2) contains only one variable delay.*

Equation (2) is a neutral-type single population model with impulses on time scales, where $x(t)$ is population density at time t, g represents the relationship between the intrinsic growth rate and population density of a population, and f indicates that population growth is related to population density at all times in the past. The existing conclusions mostly use the fixed-point theorem to obtain the existence of positive periodic solutions for Equation (2). However, due to the lack of utilizing the properties of neutral operators, the application scope of the obtained conclusions is relatively small. This article combines the properties of neutral-type operators with fixed-point theorems to obtain the existence results of positive solutions, which greatly generalize the existing results of [1–3].

We list the main contributions of this paper as follows:

(1) We extend the range of positive periodic solutions for Equation (2), and when $|c| > 1$, there is also a positive periodic solution.
(2) This article provides a method for studying an impulsive differential equation on time scales using the properties of neutral-type operators, which provides a new approach for studying equations of the same type.

The following sections are organized as follows: In Section 2, some preliminaries are given. Section 3 gives the existence for three positive periodic solutions of Equation (2). Section 4 gives the existence of at least one positive periodic solution of Equation (2). In Section 5, an example is given to show the feasibility of our results. Finally, Section 6 concludes the paper.

2. Preliminaries

A time scale $\mathbb{T}$ is a closed subset of $\mathbb{R}$. For $t \in \mathbb{T}$, the forward jump σ and backward jump operator ρ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},\ \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

The point $t \in \mathbb{T}$ is said to left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a right-scattered minimum m, set $\mathbb{T}_k := \mathbb{T} < \{m\}$, otherwise $\mathbb{T}_k := \mathbb{T}$. The forward graininess $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$. A function is said to be left-dense continuous provided it is continuous at a left-dense point in $\mathbb{T}$ and its right-side limit exists at right-dense point in $\mathbb{T}$. We give the following notations: $[a,b]_{\mathbb{T}} = \{t \in \mathbb{T},\ a \leq t \leq b\}$, the intervals $[a,b)_{\mathbb{T}}$, $(a,b]_{\mathbb{T}}$ and $[a,b]_{\mathbb{T}}$ are similar to the above notation.

Definition 1 ([28]). *A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}_k$, where $\mu(t)$ is the graininess function. The set of all regressive rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ whereas the set $\mathcal{R}^+$ is given by $\{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0$ for all $t \in \mathbb{T}\}$. Let $p \in \mathcal{R}$. The set of all right-dense continuous functions on $\mathbb{T}$ is defined by $C_{rd}(\mathbb{T}, \mathbb{R})$.*

Definition 2 ([28]). *Let $g : \mathbb{T} \to \mathbb{R}$ be a function and $t \in \mathbb{T}_k$. Then, define $g^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$; there exists a neighborhood U of t such that*

$$|g(\sigma(t)) - g(s) - g^{\Delta}(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|,$$

for all $s \in U$. g^{Δ} is called to be the Delta derivative of g at t.

Definition 3 ([29]). *A time scale $\mathbb{T}$ is periodic if there exists $m > 0$ such that if $t \in \mathbb{T}$, then $t \pm m \in \mathbb{T}$. For $\mathbb{T} = \mathbb{R}$, the smallest positive m is called the period of the time scale.*

Definition 4 ([29]). *Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with the period m. The function $g : \mathbb{T} \to \mathbb{R}$ is periodic with period ω if there exists a natural number n such that $\omega = nm$, $g(t \pm \omega) = g(t)$ for all $t \in \mathbb{T}$. When $\mathbb{T} = \mathbb{R}$, g is a periodic function if ω is the smallest positive number such that $g(t \pm \omega) = g(t)$.*

Let E be a Banach space and P be a a cone in E. Denote $\rho : P \to [0, \infty)$. If ρ is continuous and

$$\rho(tx + (1-t)y) \geq t\rho(x) + (1-t)\rho(y) \text{ for all } x, y \in E \text{ and } t \in [0,1],$$

we call ρ a nonnegative continuous concave functional on P. Given numbers α_1 and α_4 with $0 < \alpha_1 < \alpha_4$, ρ is a nonnegative continuous concave functional on P. Define the following sets:

$$P_{\alpha_1} = \{x \in P : ||x|| < \alpha_1\},\ \overline{P_{\alpha_1}} = \{x \in P : ||x|| \leq \alpha_1\},\ P(\rho, \alpha_1, \alpha_4) = \{x \in P : \alpha_1 \leq \rho(x), ||x|| \leq \alpha_4\}.$$

Lemma 1 ([30]). *(Leggett–Williams fixed-point theorem) Let $T : \overline{P_{\alpha_4}} \to \overline{P_{\alpha_4}}$ be completely continuous and ρ is a nonnegative continuous concave functional on P with $\rho(x) \leq ||x||$ for $x \in \overline{P_{\alpha_4}}$. Assume that there exist constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ with $0 < \alpha_1 < \alpha_2 < \alpha_3 \leq \alpha_4$ such that*
(1) $\{x \in P(\rho, \alpha_2, \alpha_3) : \rho(x) > \alpha_2\} \neq \emptyset$ and $\rho(Tx) > \alpha_2$ for $x \in P(\rho, \alpha_2, \alpha_3)$;
(2) $||Tx|| < \alpha_1$ for $x \in \overline{P_{\alpha_1}}$;
(3) $\rho(Tx) > \alpha_2$ for $x \in P(\rho, \alpha_2, \alpha_4)$ with $||Tx|| > \alpha_3$. Then, T has at least three fixed points x_1, x_2, and x_3, satisfying

$$x_1 \in \overline{P_{\alpha_1}},\ x_2 \in \{x \in P(\rho, \alpha_2, \alpha_3) : \rho(x) > \alpha_2\},\ x_3 \in \overline{P_{\alpha_4}} \setminus \left(P(\rho, \alpha_2, \alpha_3) \cup \overline{P_{\alpha_1}}\right).$$

Lemma 2 ([31]). *(Krasnoselskii's fixed-point theorem) Let B be a Banach space and K be a cone in B. Assume that Ω_1 and Ω_2 are open bounded subsets of B with $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$, and let*

$$T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

$$||Tu|| \leq ||u||,\ \ u \in K \cap \partial\Omega_1 \text{ and } ||Tu|| \geq ||u||,\ \ u \in K \cap \partial\Omega_2,$$

or

$$||Tu|| \geq ||u||,\ \ u \in K \cap \partial\Omega_1 \text{ and } ||Tu|| \leq ||u||,\ \ u \in K \cap \partial\Omega_2.$$

Then, T has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

3. Three Positive Periodic Solutions for Equation (2)

Throughout this paper, we need the following assumptions:
(H_1) $a \in C(\mathbb{T}, \mathbb{R}^+)$, $c \in C(\mathbb{T}, (-\infty, 1) \cup (1, +\infty))$, $\tau \in C(\mathbb{T}, \mathbb{R})$ all of which are ω—periodic functions;
(H_2) $f \in C(\mathbb{T} \times \mathbb{R}^+, \mathbb{R}^+)$ is nondecreasing with respect to x, ω—periodic with respect to its first argument;
(H_3) $g \in C(\mathbb{R}^+, \mathbb{R}^+)$ is a bounded function with $0 < l \leq g(x) \leq L$, $I_j \in C(\mathbb{R}, \mathbb{R}^+)$ is a bounded function with $||I_j|| \leq d_j$, $j = 1, 2, \cdots, q$, where l, L, and d_j are given constants.

Let

$$PC(\mathbb{T}) = \{x : x|_{(t_j, t_{j+1})_{\mathbb{T}}},\ x(t_j^+) = x(t_j),\ j = 1, 2, \cdots, q\}.$$

Let

$$E = \{x : x \in PC(\mathbb{T}),\ x(t) = x(t + \omega)\}$$

with the norm

$$||x|| = \sup_{t \in [0,\omega]_{\mathbb{T}}, x \in E} |x(t)|,$$

then E is a Banach space. Let $A : E \to E$ be defined by

$$(Ax)(t) = x(t) - c(t)x(t - \tau_0).$$

Lemma 3. *If $|c(t)| \neq 1$, then operator A has continuous inverse A^{-1} on E, satisfying*

(1)

$$[A^{-1}f](t) = \begin{cases} f(t) + \sum\limits_{j=1}^{\infty} \prod\limits_{i=1}^{j} c(t - (i-1)\tau_0) f(t - j\tau_0),\ c^M < 1,\ \forall f \in E, \\ -\frac{f(t+\sigma)}{c(t+\tau_0)} - \sum\limits_{j=1}^{\infty} \prod\limits_{i=1}^{j+1} \frac{1}{c(t+i\tau_0)} f(t + j\tau_0 + \tau_0),\ c^m > 1,\ \forall f \in E, \end{cases}$$

(2)

$$||A^{-1}f|| \leq \begin{cases} \frac{1}{1-c^M}||f||,\ c^M < 1,\ \forall f \in E, \\ \frac{1}{c^m - 1}||f||,\ c^m > 1,\ \forall f \in E, \end{cases}$$

where $c^M = \max_{t \in [0,\omega]_{\mathbb{T}}} |c(t)|$, $c^m = \min_{t \in [0,\omega]_{\mathbb{T}}} |c(t)|$.

Proof. The proof is similar to the proof of Lemma 2.1 in [32]; we omit it. □

Remark 2. *Let $\Phi : E \to E$ be defined by*

$$(\Phi x)(t) = x(t) + c(t)x(t - \tau_0).$$

In [1], the authors obtained the following results:
If $0 \le c(t) < 1$, then Φ has a bounded inverse ϕ^{-1} on E, and for all $x \in E$

$$(\Phi^{-1}x)(t) = \sum_{j\ge 0} \prod_{0\le i\le j-1} (-1)^k c(t - i\tau_0)x(t - j\tau_0)$$

and $||\Phi x|| \le \frac{||x||}{1-c^M}$. Obviously, Lemma 3 greatly extends the results of [1].

From Lemma 3, and letting $(Ax)(t) = u(t)$, Equation (2) can be rewritten by

$$\begin{cases} u^\Delta(t) = a(t)g((A^{-1}u)(t))(A^{-1}u)(t) - \lambda f(t,(A^{-1}u)(t-\tau(t))),\ t \ne t_j,\ t \in \mathbb{T}, \\ (A^{-1}u)(t_j^-) - (A^{-1}u)(t_j^+) = I_j((A^{-1}u)(t_j)),\ t = t_j. \end{cases} \tag{3}$$

Using $(A^{-1}u)(t) = u(t) + c(t)(A^{-1}u)(t-\sigma)$ and (3), we have

$$\begin{cases} u^\Delta(t) = a(t)g((A^{-1}u)(t))u(t) - a(t)H(u(t)) - \lambda f(t,(A^{-1}u)(t-\tau(t))),\ t \ne t_j,\ t \in \mathbb{T}, \\ u(t_j^-) - u(t_j^+) = (AI_jA^{-1}u)(t_j)),\ t = t_j, \end{cases} \tag{4}$$

where $H(u(t)) = -g((A^{-1}u)(t))c(t)(A^{-1}u)(t-\sigma)$. Define a cone in E by

$$P = \{u \in E : u(t) \ge k||u||\}, \tag{5}$$

where $k \in (\vartheta, \frac{r^L(1-r^l)}{1-r^L}]$, $\vartheta > \max\{\frac{c^M}{1-c^M}, \frac{\tilde{c}}{(c^m)^2-c^m}, \frac{L(c^M)^2}{lc^m(1-c^M)}, \frac{L(c^M)^2}{lc^m((c^m)^2-c^m)}\}$. The range of k is based on (6)–(9) and Theorem 3.1 of [1].

Lemma 4. *Assume that (H_1)–(H_3) hold. For $u \in P$, if $c^M < 1$ and $c(t) < 0$, or $c^m > 1$ and $-\tilde{c} \le c(t) \le -c^m$, we have*

$$\gamma_1||u|| \le (A^{-1}u)(t) \le \gamma_2||u||$$

and

$$\gamma_3||u|| \le H(u(t)) \le Lc^M\gamma_2||u||,$$

where

$$\gamma_1 = \min\left\{k - \frac{c^M}{1-c^M}, \frac{k}{\tilde{c}} - \frac{1}{(c^m)^2 - c^m}\right\},\ \gamma_2 = \max\left\{\frac{1}{1-c^M}, \frac{1}{c^m - 1}\right\},$$

$$\gamma_3 = \min\left\{lc^m k - \frac{L(c^M)^2}{1-c^M}, \frac{lc^m k}{c^M} - \frac{Lc^M}{(c^m)^2 - c^m}\right\},$$

$\tilde{c} > 0$ is a constant.

Proof. If $c^M < 1$, for $u \in P$, by Lemma 3 we have

$$\begin{aligned} (A^{-1}u)(t) &= u(t) + \sum_{j=1}^{\infty}\prod_{i=1}^{j} c(t-(i-1)\tau_0)u(t-j\tau_0) \\ &\ge k||u|| - \sum_{j=1}^{\infty}(c^M)^j||u|| \\ &= \left(k - \frac{c^M}{1-c^M}\right)||u||. \end{aligned} \tag{6}$$

If $c^m > 1$ and $-\tilde{c} \leq c(t) \leq -c^m$, for $u \in P$, by Lemma 3 we have

$$\begin{aligned} (A^{-1}u)(t) &= -\frac{u(t+\tau_0)}{c(t+\tau_0)} - \sum_{j=1}^{\infty} \prod_{i=1}^{j+1} \frac{1}{c(t+i\tau_0)} u(t+j\tau_0+\tau_0) \\ &\geq \frac{k}{\tilde{c}}||u|| - \sum_{j=1}^{\infty} \left(\frac{1}{c^m}\right)^{j+1} ||u|| \\ &= \left(\frac{k}{\tilde{c}} - \frac{1}{(c^m)^2 - c^m}\right)||u||. \end{aligned} \tag{7}$$

From (6) and (7), we have

$$(A^{-1}u)(t) \geq \gamma_1 ||u||.$$

On the other hand, again using Lemma 3, we have

$$(A^{-1}u) \leq \max\{\frac{1}{1-c^M}, \frac{1}{c^m - 1}\}||u|| = \gamma_2||u||.$$

In view of the definition of $H(u(t))$, we have

$$\begin{aligned} H(u(t)) &= -g((A^{-1}u)(t))c(t)(A^{-1}u)(t-\tau_0) \\ &\leq Lc^M \max\{\frac{1}{1-c^M}, \frac{1}{c^m-1}\}||u|| \\ &= Lc^M \gamma_2 ||u||. \end{aligned}$$

If $c^M < 1$ and $c(t) < 0$, for $u \in P$, using Lemma 3, we have

$$\begin{aligned} H(u(t)) &= -g((A^{-1}u)(t))c(t)(A^{-1}u)(t-\tau_0) \\ &= g((A^{-1}u)(t))\Big(-c(t)u(t) - c(t)\sum_{j=1}^{\infty}\prod_{i=1}^{j} c(t-(i-1)\tau_0)u(t-j\tau_0)\Big) \\ &\geq lc^m k|||u|| - Lc^M \sum_{j=1}^{\infty} (c^M)^j ||u|| \\ &= \left(lc^m k - \frac{L(c^M)^2}{1-c^M}\right)||u||. \end{aligned} \tag{8}$$

If $c^m > 1$ and $-\tilde{c} \leq c(t) \leq -c^m$, for $u \in P$, by Lemma 3 we have

$$\begin{aligned} H(u(t)) &= -g((A^{-1}u)(t))c(t)(A^{-1}u)(t-\tau_0) \\ &= g((A^{-1}u)(t))\Big(\frac{c(t)u(t+\tau_0)}{c(t+\tau_0)} + c(t)\sum_{j=1}^{\infty}\prod_{i=1}^{j+1}\frac{1}{c(t+i\tau_0)}u(t+j\tau_0+\tau_0)\Big) \\ &\geq \frac{lc^m k}{c^M}|||u|| - Lc^M \sum_{j=1}^{\infty}\left(\frac{1}{c^m}\right)^{j+1}||u|| \\ &= \left(\frac{lc^m k}{c^M} - \frac{Lc^M}{(c^m)^2 - c^m}\right)||u||. \end{aligned} \tag{9}$$

From (8) and (9), we have

$$H(u(t)) \geq \gamma_3 ||u||.$$

The proof is complete. □

Define the operator $T : P \to E$ by

$$(Tu)(t) = \int_t^{t+\omega} G(t,s)\Big(a(s)H(u(s)) + \lambda f(s,(A^{-1}u)(s-\tau(s)))\Big)ds + \sum_{j:t_j\in[t,t+\omega]_{\mathbb{T}}} G(t,t_j)e_{ag(A^{-1}u)}(\sigma(t_j),t_j)(AI_jA^{-1}u)(t_j)), \tag{10}$$

where

$$G(t,s) = \frac{e_{ag(A^{-1}u)}(t,\sigma(s))}{1-e_{ag(A^{-1}u)}(0,\omega)},\ t\in\mathbb{T},\ s\in[t,t+\omega]_{\mathbb{T}}.$$

Denote $r = e_a(0,\omega)$. Obviously

$$\frac{r^L}{1-r^L} \le G(t,s) \le \frac{1}{1-r^l}. \tag{11}$$

Remark 3. *In (4), estimates of $u(t)$ and $H(u(t))$ are crucial for obtaining the existence of positive periodic solutions. Our results greatly extend the corresponding ones of [1].*

The proof of Lemmas 5–7 is similar to the proof of Lemmas 2.8–2.10 of [1]; we omit it.

Lemma 5. *u is an ω—periodic solution of Equtaion (4) if and only if u is a fixed point of the operator T.*

Lemma 6. *Assume that (H_1)–(H_3) hold, then $TP \subset P$, and $T : P \to P$ is compact and continuous.*

Lemma 7. *Assume that (H_1)–(H_3) hold and $c^M < 1$ and $c(t) < 0$, or $c^m > 1$ and $-\tilde{c} \le c(t) \le -c^m$, then u is a positive fixed point of T in P if and only if $(A^{-1}u)(t)$ is a positive ω—periodic solution of Equation (2).*

Theorem 1. *Assume that (H_1)–(H_3) hold and $c^M < 1$ and $c(t) < 0$, or $c^m > 1$ and $-\tilde{c} \le c(t) \le -c^m$. Furthermore, suppose the following conditions hold:*
(H_4) $\alpha_0 = (1-c^M)(1-r^l) - \omega a^M Lc^M - (1-c^M)\sum_{j=1}^q d_j > 0$;
(H_5) There exist positive constants α_1, α_2, and α_4 with $0 < \alpha_1 < \alpha_2 < \alpha_4$ such that

$$\frac{\sup_{t\in[0,\omega]_{\mathbb{T}}} f(t,\frac{\alpha_1}{1-c^M})}{\frac{\alpha_1}{1-c^M}\alpha_0} < \frac{\sup_{t\in[0,\omega]_{\mathbb{T}}} f(t,\frac{\alpha_4}{1-c^M})}{\frac{\alpha_4}{1-c^M}\alpha_0} < \frac{\inf_{t\in[0,\omega]_{\mathbb{T}}} f(t,\gamma_1\alpha_2)}{\gamma_1\alpha_2\alpha_0},$$

where $\alpha_0 = \frac{1-r^L}{\gamma_1 r^L} - \omega a^m lc^m$. Then, for $\lambda \in (\lambda_1,\lambda_2]$, Equation (2) has at least three positive ω—periodic solutions, where

$$\lambda_1 = \frac{\gamma_1\alpha_2\alpha_0}{\omega\inf_{t\in[0,\omega]_{\mathbb{T}}} f(t,\gamma_1\alpha_2)},\ \lambda_2 = \frac{\frac{\alpha_4}{1-c^M}\alpha_0}{\omega\sup_{t\in[0,\omega]_{\mathbb{T}}} f(t,\frac{\alpha_4}{1-c^M})}.$$

Proof. The proof of Theorem 1 is similar to the proof of Theorem 3.1 in [1]. Thus, we omit its proof. □

According to Corollary 3.1 of [1], we have the following corollary.

Corollary 1. *Assume that (H_1)–(H_4) hold and $c^M < 1$ and $c(t) < 0$, or $c^m > 1$ and $-\tilde{c} \le c(t) \le -c^m$. Furthermore, suppose the following conditions hold:*

$$\lim_{u\to\infty}\frac{\sup_{t\in[0,\omega]_{\mathbb{T}}} f(t,u)}{u} = 0$$

and

$$\lim_{u\to 0}\frac{\inf_{t\in[0,\omega]_{\mathbb{T}}} f(t,u)}{u}=0.$$

Equation (2) has at least three positive ω—periodic solutions.

4. One Positive Periodic Solution for Equation (2)

In this section, using Krasnoselskii's fixed-point theorem, we obtain that Equation (2) has at least one positive periodic solution. In the following proof, the cone P is defined by (5) and the mapping T is defined by (10). Due to results in Section 3, we easily obtain the following results:

(1) The mapping T maps P into P;

(2) The mapping $T : P \to P$ is completely continuous.

For the convenience of proof, by (11) we denote

$$\check{g}=\frac{r^L}{1-r^L}\le G(t,s)\le\frac{1}{1-r^l}=\hat{g}. \tag{12}$$

Theorem 2. *Suppose that assumptions (H_1)–(H_3) hold and $\lambda > 0$, $c^M < 1$, and $c(t) < 0$, or $c^m > 1$ and $-\tilde{c} \le c(t) \le -c^m$. Furthermore, assume that there are positive constants R_1 and R_2 with $R_1 < R_2$ such that*

$$\begin{aligned}\sup_{||\phi||=R_1,\phi\in P}\Bigg[\int_0^\omega\Big(a(s)H(\phi(s))+\lambda f(s,(A^{-1}\phi)(s-\tau(s)))\Big)ds\\ +\sum_{j:t_j\in[0,\omega]_{\mathbb{T}}} e_{ag(A^{-1}\phi)}(\sigma(t_j),t_j)(AI_jA^{-1}\phi)(t_j))\Bigg]\le\frac{R_1}{\hat{g}}\end{aligned} \tag{13}$$

and

$$\begin{aligned}\inf_{||\phi||=R_2,\phi\in P}\Bigg[\int_0^\omega\Big(a(s)H(\phi(s))+\lambda f(s,(A^{-1}\phi)(s-\tau(s)))\Big)ds\\ +\sum_{j:t_j\in[0,\omega]_{\mathbb{T}}} e_{ag(A^{-1}\phi)}(\sigma(t_j),t_j)(AI_jA^{-1}\phi)(t_j))\Bigg]\ge\frac{R_2}{\check{g}},\end{aligned} \tag{14}$$

where $\check{g}$ and $\hat{g}$ are defined by (12). Then, Equation (4) has an ω—periodic solution z with $R_1 \le ||z|| \le R_2$, i.e., Equation (2) has an ω—periodic solution $x > 0$.

Proof. Let $z \in P$ and $||z|| = R_1$. By (13) we have

$$\begin{aligned}||Tz||\le&\hat{g}\Bigg[\int_0^\omega\Big(a(s)H(z(s))+\lambda f(s,(A^{-1}z)(s-\tau(s)))\Big)ds\\ &+\sum_{j:t_j\in[0,\omega]_{\mathbb{T}}} e_{ag(A^{-1}z)}(\sigma(t_j),t_j)(AI_jA^{-1}z)(t_j))\Bigg]\\ &\le R_1=||z||,\end{aligned}$$

where $z \in P \cap \partial\Omega_1$, $\Omega_1 = \{z \in E : ||z|| < R_1\}$. Similar to the above proof, in view of (14), for $z \in P$ and $||z|| = R_2$, we have

$$\begin{aligned}||Tz||\ge&\check{g}\Bigg[\int_0^\omega\Big(a(s)H(z(s))+\lambda f(s,(A^{-1}z)(s-\tau(s)))\Big)ds\\ &+\sum_{j:t_j\in[0,\omega]_{\mathbb{T}}} e_{ag(A^{-1}z)}(\sigma(t_j),t_j)(AI_jA^{-1}z)(t_j))\Bigg]\\ &\ge R_2=||z||,\end{aligned}$$

where $z \in P \cap \partial\Omega_2$, $\Omega_2 = \{z \in E : ||z|| < R_2\}$. By Lemma 2, T has a fixed point z in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$ with $R_1 \leq ||z|| \leq R_2$. Hence, Equation (4) has an ω—periodic solution z with $R_1 \leq ||z|| \leq R_2$, i.e., Equation (2) has an ω—periodic solution $x = A^{-1}z > 0$. □

Remark 4. *Under simple conditions, we establish existence criteria of at least one positive periodic solution for Equation (2) by using Krasnoselskii's fixed-point theorem. The existence conditions of positive periodic solution of Theorem 2 are easier to verify.*

5. Example

Consider the following equation of model (2):

$$\begin{cases} (x(t) - (|\sin t| + 1.5)x(t-\pi))^{\Delta} = \frac{2}{\pi}\left(\frac{1}{4} + \frac{1}{4}e_{-x}(\sigma(t), t)\right)x(t) - \lambda x^{\frac{1}{2}}(t)\ln(x(t - e_{|\sin t|}(\sigma(t), t) + 1), \\ t \neq t_j,\ j = 1, 2, \cdots, q,\ t \in \mathbb{T}, \\ x(t_j^-) - x(t_j^+) = I_j(x(t_j)) = 27(\frac{1}{10})^j x(t_j),\ t = t_j, \end{cases} \tag{15}$$

where $\mathbb{T} = \mathbb{R}$, $\lambda > 0$ is a parameter,

$$c(t) = (|\sin t| + 1.5),\ a(t) = \frac{2}{\pi},\ g(x(t)) = \frac{1}{4} + \frac{1}{4}e_{-x}(\sigma(t), t),$$

$$f(t, x(t - \tau(t))) = x^{\frac{1}{2}}(t)\ln(x(t - \tau(t)) + 1),\ \tau(t) = e_{|\sin t|}(\sigma(t), t).$$

It is easy to see that $\omega = \pi,\ c^m = 1.5,\ c^M = 2.5,\ l = 0.25,\ L = 0.5,\ r = 0.136$. We also have

$$||I_j|| \leq 27(\frac{1}{10})^j = d_j \text{ and } \sum_{j=1}^{q} d_j \leq \sum_{j=1}^{\infty} d_j = 3.$$

Hence,

$$(1 - c^M)(1 - r^l) - \omega a^M L c^M - (1 - c^M)\sum_{j=1}^{q} d_j \approx 1.41 > 0.$$

Then, all conditions of Corollary 1 hold. Hence, Equation (15) has at least three positive π—periodic solutions.

Remark 5. *When $c^M > 1$, Equation (15) has at least three positive π—periodic solutions. However, in [1], when $c^M > 1$, Equation (15) does not necessarily have three positive periodic solutions. Hence, our results extend the corresponding results belonging to [1].*

6. Conclusions and Discussions

In past decades, the dynamic equation on time scales has aroused extensive interest in researchers because it unifies discrete analysis and continuous analysis. In this paper, the existence of a positive periodic solution for a first-order nonlinear neutral differential equation with impulses on time scales is discussed by using a conclusion about neutral-type operators obtained by the author in the early stage. The conclusions of this paper greatly extend the existing conclusions.

The methods of this paper can be extended to investigate other types of dynamic equations on time scales such as stochastic differential equations, high-order differential equations, fractional differential equations, and so on. We hope some authors can use the methods provided in this article to conduct more in-depth research on various types of dynamic equations on time scales.

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