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Controlled S-Metric-Type Spaces and Applications to Fractional Integrals

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Abstract: In this paper, we introduce controlled S-metric-type spaces and give some of their properties and examples. Moreover, we prove the Banach fixed point theorem and a more general fixed point theorem in this new space. Finally, using the new results, we give two applications on Riemann–Liouville fractional integrals and Atangana–Baleanu fractional integrals.

Keywords: fixed point; S-metric; controlled metric type; fractional integral

MSC: 2010 Primary 47H10; Secondary 54H25

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Citation: Ekiz Yazici, N.; Ege, O.; Mlaiki, N.; Mukheimer, A. Controlled S-Metric-Type Spaces and Applications to Fractional Integrals. *Symmetry* **2023**, *15*, 1100. https://doi.org/10.3390/sym15051100

Academic Editor: Juan Luis García Guirao

Received: 28 April 2023
Revised: 11 May 2023
Accepted: 15 May 2023
Published: 17 May 2023

1. Introduction

Fixed point theory is an exciting branch of mathematics. It can be seen as a mixture of analysis, topology, and geometry. Since this theory helps to solve some mathematical problems, it is a rapidly developing field. Fixed point theory deals with the existence and uniqueness of fixed points of functions, and the main problem is how to find them. The first step in this field was taken in 1922 by Stefan Banach's theorem known as the "Banach Contraction Principle" [1] in the literature. Since then, new fixed point theorems have been proved by many scientists.

Similarly, metric spaces are important for the area of fixed point. For a function to be a metric function, it must satisfy three conditions. One of these is the symmetry condition. Different types of generalizations of a metric space have been produced over time. Some of them, such as G-metric space, were defined by Mustafa and Sims [2]; D-metric spaces wre defined by Dhage [3]; and D^*-metric spaces were introduced by Sedghi et al. [4]. Subsequently, some fixed point theorems have been proved in these spaces [5–7]. Additionally, Sedghi et al. [8] introduced the notion of a S-metric space and represented some of its properties.

After introducing the b-metric space [9,10] as a new generalization of the metric space, an extended b-metric was defined in [11]. Mlaiki et al. [12] provided one of the generalizations of the b-metric space. They introduced the concept of a controlled metric-type space by applying the control function $\delta : X \times X \to [1, \infty)$ to the triangle inequality. Many authors have since given different fixed point results and examples in this space. For example, some of them are given in [13,14]. Various generalizations of controlled metric-type spaces have also been defined [15–18]. Our motivation for this work is the S-metric spaces and controlled metric-type spaces we just mentioned.

This paper is organized as follows: In the Section 2, we give the required background. In Section 3, we introduce a controlled S-metric-type space via the S-metric and control function. This section also includes some properties of this space and some fixed point results with supporting examples. Some examples in these sections also use symmetric functions. In the last section of the paper, we give two applications. These applications are the Riemann–Liouville fractional integral and the Atangana–Baleanu fractional integral. These two fractional integrals were chosen because the existence of their solutions is of great importance in applied mathematics and engineering.

2. Preliminaries

In the following, we recall some definitions and results from the literature.

Definition 1 ([8])**.** *Let Θ be a nonempty set. A function $S : \Theta^3 \to [0, \infty)$ is called S-metric if for each $\theta, \eta, \zeta, a \in \Theta$, it satisfies*
(i) $S(\theta, \eta, \zeta) \geq 0$;
(ii) $S(\theta, \eta, \zeta) = 0$ if and only if $\theta = \eta = \zeta$;
(iii)$S(\theta, \eta, \zeta) \leq S(\theta, \theta, a) + S(\eta, \eta, a) + S(\zeta, \zeta, a)$.

Example 1 ([8])**.** *Let $\Theta = \mathbb{R}$ and $\|.\|$ be a norm on Θ. Then,*

$$S(\theta, \eta, \zeta) = \|\eta + \zeta - 2\theta\| + \|\eta - \zeta\|$$

is an S-metric on Θ.

Definition 2 ([12])**.** *Let Θ be a nonempty set and $\alpha : \Theta \times \Theta \to [1, \infty)$ be a function. The controlled metric type on Θ is $d : \Theta \times \Theta \to [0, \infty)$ that satisfies the following conditions, for all $\theta, \eta, \zeta \in \Theta$:*
(1) $d(\theta, \eta) = 0$ if and only if $\theta = \eta$;
(2) $d(\theta, \eta) = d(\eta, \theta)$;
(3) $d(\theta, \eta) \leq \alpha(\theta, \zeta) d(\theta, \zeta) + \alpha(\zeta, \eta) d(\eta, \zeta)$.
Then, the pair (Θ, d) is called a controlled metric-type space.

Example 2 ([12])**.** *Take $\Theta = \{0, 1, 2\}$ and choose the function d as*

$$d(x, y) = \begin{cases} 0, & x = y; \\ 1, & x = 1, y = 0 \text{ or } y = 1, x = 0; \\ \frac{1}{2}, & x = 0, y = 2 \text{ or } x = 2, y = 0; \\ \frac{2}{5}, & x = 1, y = 2 \text{ or } x = 2, y = 1. \end{cases}$$

Take $\alpha : \Theta \times \Theta \to [1, \infty)$ defined by

$$\alpha(0, 0) = \alpha(1, 1) = \alpha(2, 2) = \alpha(0, 2) = 1$$

$$\alpha(1, 2) = \frac{5}{4}, \ \alpha(0, 1) = \frac{11}{10}$$

and let α be symmetric. It is clear that d is a controlled metric type.

3. Main Results

This section includes the new results of this paper.

Definition 3. *Let Θ be a nonempty set and $\alpha : \Theta \times \Theta \to [1, \infty)$ be a function. If a function $S : \Theta^3 \to [0, \infty)$ satisfies the following conditions for all $\theta, \eta, \zeta, a \in \Theta$, then it is called a controlled S-metric type and the pair (Θ, S) is said to be a controlled S-metric type space.*
(i) $S(\theta, \eta, \zeta) \geq 0$;
(ii) $S(\theta, \eta, \zeta) = 0 \Leftrightarrow \theta = \eta = \zeta$;
(iii) $S(\theta, \eta, \zeta) \leq \alpha(\theta, a) S(\theta, \theta, a) + \alpha(\eta, a) S(\eta, \eta, a) + \alpha(\zeta, a) S(\zeta, \zeta, a)$.

Example 3. *Let Θ be a nonempty set and (Θ, d) be a controlled metric-type space with a symmetric function $\alpha : \Theta \times \Theta \to [1, \infty)$. Take $S : \Theta^3 \to [0, \infty)$ as*

$$S(\theta, \eta, \zeta) = d(\theta, \zeta) + d(\eta, \zeta).$$

The conditions (i) and (ii) clearly hold. We only prove condition (iii).

$$\begin{aligned} S(\theta,\eta,\zeta) &= d(\theta,\zeta)+d(\eta,\zeta) \\ &\leq \alpha(\theta,a)d(\theta,a)+\alpha(\eta,a)d(\eta,a)+2\alpha(\zeta,a)d(\zeta,a) \\ &\leq 2\alpha(\theta,a)d(\theta,a)+2\alpha(\eta,a)d(\eta,a)+2\alpha(\zeta,a)d(\zeta,a) \\ &= \alpha(\theta,a)S(\theta,\theta,a)+\alpha(\eta,a)S(\eta,\eta,a)+\alpha(\zeta,a)S(\zeta,\zeta,a). \end{aligned}$$

Consequently, S is a controlled S-metric type.

Example 4. *Let* Θ *be a nonempty set and* (Θ, d) *be a controlled metric-type space with a symmetric function* $\alpha : \Theta \times \Theta \to [1, \infty)$*. Take* $S : \Theta^3 \to [0, \infty)$ *as*

$$S(\theta,\eta,\zeta) = d(\theta,\eta)+d(\theta,\zeta)+d(\eta,\zeta).$$

It is easy to see that S is a controlled S-metric type.

Example 5. *Choose* $\Theta = \{1, 2, ...\}$*. Take a function* $S : \Theta \times \Theta \times \Theta \to [0, \infty)$ *defined by*

$$S(\theta,\eta,\zeta) = \begin{cases} 0, & \Leftrightarrow \theta=\eta=\zeta, \\ \frac{1}{\theta}, & \text{if } \theta=\eta \text{ is even and } \zeta \text{ is odd}, \\ \frac{1}{\zeta}, & \text{if } \zeta \text{ is even and } \theta=\eta \text{ is odd}, \\ 1, & \text{otherwise}. \end{cases}$$

Consider $\alpha : \Theta \times \Theta \to [1, \infty)$ *as*

$$\alpha(\theta,\eta) = \begin{cases} \theta, & \text{if } \theta \text{ is even and } \eta \text{ is odd} \\ \eta, & \text{if } \eta \text{ is even and } \theta \text{ is odd}. \\ 1, & \text{otherwise} \end{cases}$$

It is easy to show that conditions (i) and (ii) are satisfied. We should prove condition (iii).
Case 1: If $\theta = \eta = \zeta$*, then (iii) is satisfied.*
Case 2: All cases except Case 1, where θ, η, ζ *are in the domain set of the function S, are listed in the table below .*

Cases	θ	η	ζ
1.	even	even	even
2.	even	even	odd
2.1.	even=	even	odd
3.	even	odd	even
4.	even	odd	odd
5.	odd	even	even
6.	odd	even	odd
7.	odd	odd	even
7.1.	odd=	odd	even
8.	odd	odd	odd

In that case, (Θ, S) *is a controlled S-metric-type space.*

Definition 4. *Let* (Θ, S) *be a controlled S-metric-type space and* $\{\theta_n\}_{n\geq 0}$ *be a sequence in* Θ*.*

(1) *A sequence* $\{\theta_n\}$ *is convergent to some* $\theta \in \Theta$ *if, for each* $\epsilon > 0$*, there exists* $N = N(\epsilon) \in \mathbb{N}$ *such that* $S(\theta_n, \theta_n, \theta) < \epsilon$ *for all* $n \geq N$*.*
(2) *A sequence* $\{\theta_n\}$ *in* Θ *is called a Cauchy sequence if, for each* $\epsilon > 0$*, there exists* $N = N(\epsilon) \in \mathbb{N}$ *such that* $S(\theta_n, \theta_n, \theta_m) < \epsilon$ *for all* $m, n \geq N$*.*

(3) *A controlled S-metric-type space* (Θ, S) *is said to be complete if every Cauchy sequence is convergent.*

Lemma 1. *In a controlled S-metric-type space, we have* $S(\theta,\theta,\eta) = S(\eta,\eta,\theta)$.

Proof. By the definition of a controlled S-metric type, we have

$$S(\theta,\theta,\eta) \le \alpha(\theta,\theta)S(\theta,\theta,\theta) + \alpha(\theta,\theta)S(\theta,\theta,\theta) + \alpha(\eta,\theta)S(\eta,\eta,\theta). \tag{1}$$

Similarly,

$$S(\eta,\eta,\theta) \le \alpha(\eta,\eta)S(\eta,\eta,\eta) + \alpha(\eta,\eta)S(\eta,\eta,\eta) + \alpha(\theta,\eta)S(\theta,\theta,\eta). \tag{2}$$

From (1) and (2), we obtain

$$S(\theta,\theta,\eta) \le \alpha(\eta,\theta)S(\eta,\eta,\theta) \le \alpha(\theta,\eta)S(\theta,\theta,\eta). \tag{3}$$

Since $\alpha(\theta,\eta) \ge 1$, we conclude that $S(\theta,\theta,\eta) = S(\eta,\eta,\theta)$. □

Theorem 1. *Let* (Θ, S) *be a controlled S-metric-type space and* $T : \Theta \to \Theta$ *be a mapping such that*

$$S(T\theta, T\theta, T\eta) \le LS(\theta,\theta,\eta) \tag{4}$$

for all $\theta, \eta, \zeta \in \Theta$, *where* $L \in (0,1)$. *For* $\theta_0 \in \Theta$, *take* $\theta_n = T^n\theta_0$. *Suppose that*

$$\sup_{m\ge 1} \lim_{i\to\infty} \frac{\alpha(\theta_{i+1},\theta_{i+2})}{\alpha(\theta_i,\theta_{i+1})}\alpha(\theta_{i+1},\theta_m) < \frac{1}{L}. \tag{5}$$

In addition, assume that, for every $\theta \in \Theta$

$$\lim_{n\to\infty} \alpha(\theta_n,\theta) \text{ and } \lim_{n\to\infty} \alpha(\theta,\theta_n) \text{ exist and are finite.} \tag{6}$$

Then, T has a unique fixed point.

Proof. Consider the sequence $\{\theta_n = T^n\theta_0\}$. By using (4), we have

$$S(\theta_n,\theta_n,\theta_{n+1}) = S(T^n\theta_0, T^n\theta_0, T^n\theta_1) \le L^n S(\theta_0,\theta_0,\theta_1) \text{ for all } n \ge 0. \tag{7}$$

For all natural numbers $n < m$, we have

$$\begin{aligned}
S(\theta_n,\theta_n,\theta_m) &\le \alpha(\theta_n,\theta_{n+1})S(\theta_n,\theta_n,\theta_{n+1}) + \alpha(\theta_n,\theta_{n+1})S(\theta_n,\theta_n,\theta_{n+1}) + \alpha(\theta_m,\theta_{n+1})S(\theta_m,\theta_m,\theta_{n+1}) \\
&= 2\alpha(\theta_n,\theta_{n+1})S(\theta_n,\theta_n,\theta_{n+1}) + \alpha(\theta_{n+1},\theta_m)S(\theta_{n+1},\theta_{n+1},\theta_m) \\
&\le 2\alpha(\theta_n,\theta_{n+1})S(\theta_n,\theta_n,\theta_{n+1}) + \alpha(\theta_{n+1},\theta_m)[\alpha(\theta_{n+1},\theta_{n+2})S(\theta_{n+1},\theta_{n+1},\theta_{n+2}) \\
&+ \alpha(\theta_{n+1},\theta_{n+2})S(\theta_{n+1},\theta_{n+1},\theta_{n+2}) + \alpha(\theta_{n+2},\theta_m)S(\theta_{n+2},\theta_{n+2},\theta_m)] \\
&\le 2\alpha(\theta_n,\theta_{n+1})S(\theta_n,\theta_n,\theta_{n+1}) + 2\alpha(\theta_{n+1},\theta_m)\alpha(\theta_{n+1},\theta_{n+2})S(\theta_{n+1},\theta_{n+1},\theta_{n+2}) \\
&+ \alpha(\theta_{n+1},\theta_m)\alpha(\theta_{n+2},\theta_m)S(\theta_{n+2},\theta_{n+2},\theta_m) \\
&\le 2\alpha(\theta_n,\theta_{n+1})S(\theta_n,\theta_n,\theta_{n+1}) + 2\alpha(\theta_{n+1},\theta_m)\alpha(\theta_{n+1},\theta_{n+2})S(\theta_{n+1},\theta_{n+1},\theta_{n+2}) \\
&+ \alpha(\theta_{n+1},\theta_m)\alpha(\theta_{n+2},\theta_m)[2\alpha(\theta_{n+2},\theta_{n+3})S(\theta_{n+2},\theta_{n+2},\theta_{n+3}) \\
&+ \alpha(\theta_{n+3},\theta_m)S(\theta_{n+3},\theta_{n+3},\theta_m)] \\
&= 2\alpha(\theta_n,\theta_{n+1})S(\theta_n,\theta_n,\theta_{n+1}) + 2\alpha(\theta_{n+1},\theta_m)\alpha(\theta_{n+1},\theta_{n+2})S(\theta_{n+1},\theta_{n+1},\theta_{n+2})
\end{aligned}$$

$$
\begin{aligned}
&+ \quad 2\alpha(\theta_{n+1},\theta_m)\alpha(\theta_{n+2},\theta_m)\alpha(\theta_{n+2},\theta_{n+3})S(\theta_{n+2},\theta_{n+2},\theta_{n+3}) \\
&+ \quad \alpha(\theta_{n+1},\theta_m)\alpha(\theta_{n+2},\theta_m)\alpha(\theta_{n+3},\theta_m)S(\theta_{n+3},\theta_{n+3},\theta_m) \\
&\leq \quad \ldots \\
&\leq \quad 2\alpha(\theta_n,\theta_{n+1})S(\theta_n,\theta_n,\theta_{n+1}) + 2\sum_{i=n+1}^{m-2}\left[\prod_{j=n+1}^{i}\alpha(\theta_j,\theta_m)\right]\alpha(\theta_i,\theta_{i+1})S(\theta_i,\theta_i,\theta_{i+1}) \\
&+ \quad 2\prod_{i=n+1}^{m-1}\alpha(\theta_i,\theta_m)S(\theta_{m-1},\theta_{m-1},\theta_m) \\
&\leq \quad 2\alpha(\theta_n,\theta_{n+1})L^nS(\theta_0,\theta_0,\theta_1) + 2\sum_{i=n+1}^{m-2}\left[\prod_{j=n+1}^{i}\alpha(\theta_j,\theta_m)\right]\alpha(\theta_i,\theta_{i+1})L^iS(\theta_0,\theta_0,\theta_1) \\
&+ \quad 2\prod_{i=n+1}^{m-1}\alpha(\theta_i,\theta_m)L^{m-1}S(\theta_0,\theta_0,\theta_1) \\
&\leq \quad 2\alpha(\theta_n,\theta_{n+1})L^nS(\theta_0,\theta_0,\theta_1) \\
&+ \quad 2\sum_{i=n+1}^{m-2}\left[\prod_{j=n+1}^{i}\alpha(\theta_j,\theta_m)\right]\alpha(\theta_i,\theta_{i+1})L^iS(\theta_0,\theta_0,\theta_1) \\
&+ \quad 2\prod_{i=n+1}^{m-1}\alpha(\theta_i,\theta_m)L^{m-1}\alpha(\theta_{m-1},\theta_m)S(\theta_0,\theta_0,\theta_1) \\
&\leq \quad 2\alpha(\theta_n,\theta_{n+1})L^nS(\theta_0,\theta_0,\theta_1) \\
&+ \quad 2\sum_{i=n+1}^{m-1}\left[\prod_{j=n+1}^{i}\alpha(\theta_j,\theta_m)\right]\alpha(\theta_i,\theta_{i+1})L^iS(\theta_0,\theta_0,\theta_1) \\
&\leq \quad 2\alpha(\theta_n,\theta_{n+1})L^nS(\theta_0,\theta_0,\theta_1) \\
&+ \quad 2\sum_{i=n+1}^{m-1}\left[\prod_{j=0}^{i}\alpha(\theta_j,\theta_m)\right]\alpha(\theta_i,\theta_{i+1})L^iS(\theta_0,\theta_0,\theta_1).
\end{aligned}
$$

Let

$$X_p = \sum_{i=0}^{p}\left(\prod_{j=0}^{i}\alpha(\theta_j,\theta_m)\right)\alpha(\theta_i,\theta_{i+1})L^i.$$

Hence, we have

$$
\begin{aligned}
S(\theta_n,\theta_n,\theta_m) &\leq (2\alpha(\theta_n,\theta_{n+1})L^n + 2(X_{m-1}-X_n))S(\theta_0,\theta_0,\theta_1) \\
&= 2S(\theta_0,\theta_0,\theta_1)(L^n\alpha(\theta_n,\theta_{n+1}) + (X_{m-1}-X_n)).
\end{aligned}
$$

From condition (5) and using the ratio test, we see that $\lim_{n\to\infty} X_n$ exists and the sequence $\{X_n\}$ is Cauchy. If we take the limit for $n, m \to \infty$, we deduce that

$$\lim_{n,m\to\infty} S(\theta_n,\theta_n,\theta_m) = 0. \tag{8}$$

Then, $\{\theta_n\}$ is a Cauchy sequence. Since (Θ, S) is a complete controlled S-metric-type space, there exists $u \in \Theta$ such that $\{\theta_n\} \to u$. We next prove that u is a fixed point of T. By the definition of a controlled S-metric type, we have

$$S(\theta_{n+1},\theta_{n+1},u) \leq 2\alpha(\theta_{n+1},\theta_n)S(\theta_{n+1},\theta_{n+1},\theta_n) + \alpha(\theta_n,u)S(\theta_n,\theta_n,u).$$

If we take the limit for $n \to \infty$, using (5), (6), and (8), we obtain

$$\lim_{n\to\infty} S(\theta_{n+1},\theta_{n+1},u) = 0. \tag{9}$$

Using the same conditions again, we have

$$\begin{aligned} S(Tu,Tu,u) &= S(u,u,Tu) \\ &\leq 2\alpha(u,\theta_{n+1})S(u,u,\theta_{n+1}) + \alpha(\theta_{n+1},Tu)S(\theta_{n+1},\theta_{n+1},Tu) \\ &\leq 2\alpha(u,\theta_{n+1})S(u,u,\theta_{n+1}) + \alpha(\theta_{n+1},Tu)LS(\theta_n,\theta_n,u). \end{aligned}$$

Taking the limit as $n \to \infty$ and considering (6) and (9), we deduce that $S(Tu,Tu,u) = 0$, that is, $Tu = u$. Finally, we show that u is unique. Assume that v is another fixed point of T and $u \neq v$. Then, we obtain

$$S(u,u,v) = S(Tu,Tu,Tv) \leq LS(u,u,v).$$

Since it is a contradiction, it must be $S(u,u,v) = 0$, i.e., $u = v$. Therefore, T has a unique fixed point. □

Now, we introduce a family of functions to investigate some new fixed point theorems.

Let $\mathfrak{M}$ be a family of all continuous functions such that $M : \mathbb{R}_+^5 \to \mathbb{R}_+$. For some $k \in [0,1)$, we consider the following conditions.

(N1) For all $\theta,\eta,\zeta \in \mathbb{R}_+$, if $\eta \leq M(\theta,\theta,0,\zeta,\eta)$ with $\zeta \leq 2\theta + \eta$, then $\eta \leq k\theta$.

(N2) For all $\eta \in \mathbb{R}_+$, if $\eta \leq M(\eta,0,\eta,\eta,0)$, then $\eta = 0$.

Theorem 2. *Let (Θ,S) be a complete controlled S-metric-type space and $T : \Theta \to \Theta$ satisfies*

$$S(T\theta,T\theta,T\eta) \leq M(S(\theta,\theta,\eta),S(T\theta,T\theta,\theta),S(T\theta,T\theta,\eta),S(T\eta,T\eta,\theta),S(T\eta,T\eta,\eta)) \quad (10)$$

for all $\theta,\eta,\zeta \in \Theta$ and some $M \in \mathfrak{M}$. For $\theta_0 \in \Theta$, take $\theta_n = T^n\theta_0$. Suppose that

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\alpha(\theta_{i+1},\theta_{i+2})}{\alpha(\theta_i,\theta_{i+1})}\alpha(\theta_{i+1},\theta_m) < \frac{1}{k} \quad (11)$$

where $k \in (0,1)$. Moreover, assume that, for every $\theta \in \Theta$, we have

$$\lim_{n\to\infty}\alpha(\theta_n,\theta) \text{ and } \lim_{n\to\infty}\alpha(\theta,\theta_n) \text{ exist and are finite.} \quad (12)$$

Then, we have the following:

(i) If M supplies (N1), then T has a fixed point.

(ii) If M supplies (N2) and T has a fixed point, then this fixed point is unique.

Proof. (i) Consider a sequence $\{\theta_n = T^n\theta_0\}$. Using (1) and (4), it follows that

$$\begin{aligned} S(\theta_{n+1},\theta_{n+1},\theta_{n+2}) &= S(T\theta_n,T\theta_n,T\theta_{n+1}) \\ &\leq M(S(\theta_n,\theta_n,\theta_{n+1}),S(T\theta_n,T\theta_n,\theta_n),S(T\theta_n,T\theta_n,\theta_{n+1}), \\ &\quad S(T\theta_{n+1},T\theta_{n+1},\theta_n),S(T\theta_{n+1},T\theta_{n+1},\theta_{n+1})) \\ &= M(S(\theta_n,\theta_n,\theta_{n+1}),S(\theta_{n+1},\theta_{n+1},\theta_n),S(\theta_{n+1},\theta_{n+1},\theta_{n+1}), \\ &\quad S(\theta_{n+2},\theta_{n+2},\theta_n),S(\theta_{n+2},\theta_{n+2},\theta_{n+1})) \\ &= M(S(\theta_n,\theta_n,\theta_{n+1}),S(\theta_n,\theta_n,\theta_{n+1}),0, \\ &\quad S(\theta_n,\theta_n,\theta_{n+2}),S(\theta_{n+1},\theta_{n+1},\theta_{n+2})). \end{aligned}$$

By the definition of a controlled S-metric -type space,

$$S(\theta_n,\theta_n,\theta_{n+2}) \leq 2\alpha(\theta_n,\theta_{n+1})S(\theta_n,\theta_n,\theta_{n+1}) + \alpha(\theta_{n+2},\theta_{n+1})S(\theta_{n+2},\theta_{n+2},\theta_{n+1}).$$

Since M satisfies condition (N1), we obtain

$$\begin{aligned} S(\theta_{n+1},\theta_{n+1},\theta_{n+2}) &\leq kS(\theta_n,\theta_n,\theta_{n+1}) \\ &\leq k^{n+1}S(\theta_0,\theta_0,\theta_1) \end{aligned}$$

for $k \in (0,1)$. Thus for all $n < m$, from (1) and triangle inequality,

$$\begin{aligned} S(\theta_n,\theta_n,\theta_m) &\leq 2\alpha(\theta_n,\theta_{n+1})S(\theta_n,\theta_n,\theta_{n+1}) + \alpha(\theta_{n+1},\theta_m)S(\theta_{n+1},\theta_{n+1},\theta_m) \\ &\leq \alpha(\theta_n,\theta_{n+1})S(\theta_n,\theta_n,\theta_{n+1}) \\ &+ \alpha(\theta_{n+1},\theta_m)[2\alpha(\theta_{n+1},\theta_{n+2})S(\theta_{n+1},\theta_{n+1},\theta_{n+2}) \\ &+ \alpha(\theta_{n+2},\theta_m)S(\theta_{n+2},\theta_{n+2},\theta_m)] \\ &\leq 2\alpha(\theta_n,\theta_{n+1})S(\theta_n,\theta_n,\theta_{n+1}) \\ &+ 2\alpha(\theta_{n+1},\theta_m)\alpha(\theta_{n+1},\theta_{n+2})S(\theta_{n+1},\theta_{n+1},\theta_{n+2}) \\ &+ \alpha(\theta_{n+1},\theta_m)\alpha(\theta_{n+2},\theta_m)S(\theta_{n+2},\theta_{n+2},\theta_m) \\ &\leq \ldots \\ &\leq 2\alpha(\theta_n,\theta_{n+1})S(\theta_n,\theta_n,\theta_{n+1}) \\ &+ \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\alpha(\theta_j,\theta_m)\right)\alpha(\theta_i,\theta_{i+1})S(\theta_i,\theta_i,\theta_{i+1}) \\ &+ 2\prod_{i=n+1}^{m-1}\alpha(\theta_i,\theta_m)S(\theta_{m-1},\theta_{m-1},\theta_m) \\ &\leq 2\alpha(\theta_n,\theta_{n+1})k^nS(\theta_0,\theta_0,\theta_1) \\ &+ 2\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\alpha(\theta_j,\theta_m)\right)\alpha(\theta_i,\theta_{i+1})k^iS(\theta_0,\theta_0,\theta_1) \\ &+ 2\prod_{i=n+1}^{m-1}\alpha(\theta_i,\theta_m)k^{m-1}S(\theta_0,\theta_0,\theta_1) \\ &\leq 2\alpha(\theta_n,\theta_{n+1})k^nS(\theta_0,\theta_0,\theta_1) \\ &+ 2\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\alpha(\theta_j,\theta_m)\right)\alpha(\theta_i,\theta_{i+1})k^iS(\theta_0,\theta_0,\theta_1) \\ &+ 2\prod_{i=n+1}^{m-1}\alpha(\theta_i,\theta_m)k^{m-1}\alpha(\theta_{m-1},\theta_m)S(\theta_0,\theta_0,\theta_1) \\ &\leq 2\alpha(\theta_n,\theta_{n+1})k^nS(\theta_0,\theta_0,\theta_1) \\ &+ 2\sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\alpha(\theta_j,\theta_m)\right)\alpha(\theta_i,\theta_{i+1})k^iS(\theta_0,\theta_0,\theta_1) \\ &\leq 2\alpha(\theta_n,\theta_{n+1})k^nS(\theta_0,\theta_0,\theta_1) \\ &+ 2\sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i}\alpha(\theta_j,\theta_m)\right)\alpha(\theta_i,\theta_{i+1})k^iS(\theta_0,\theta_0,\theta_1). \end{aligned}$$

Above, we made use of $\alpha(\theta,\eta) \geq 1$. Let

$$X_p = \sum_{i=0}^{p}\left(\prod_{j=0}^{i}\alpha(\theta_j,\theta_m)\right)\alpha(\theta_i,\theta_{i+1})k^i.$$

Hence, we have

$$S(\theta_n,\theta_n,\theta_m) \leq 2S(\theta_0,\theta_0,\theta_1)[k^n\alpha(\theta_n,\theta_{n+1}) + (X_{m-1} - X_n)]. \tag{13}$$

By condition (11) and using the ratio test, we can see that $\lim_{n\to\infty} X_n$ exists and the sequence $\{X_n\}$ is Cauchy. Finally, if we take limit in the inequality (13) as $n, m \to \infty$, we conclude that

$$\lim_{n,m\to\infty} S(\theta_n, \theta_n, \theta_m) = 0,$$

that is, $\{\theta_n\}$ is a Cauchy sequence. Since (Θ, S) is a complete controlled S-metric-type space, there is an element $\theta \in \Theta$ such that $\{\theta_n\} \to \theta$.

Now, we prove that θ is a fixed point of T.

$$\begin{aligned} S(\theta_{n+1}, \theta_{n+1}, T\theta) &= S(T\theta_n, T\theta_n, T\theta) \\ &\leq M(S(\theta_n, \theta_n, \theta), S(T\theta_n, T\theta_n, \theta_n), S(T\theta_n, T\theta_n, \theta), \\ &\quad S(T\theta, T\theta, \theta_n), S(T\theta, T\theta, \theta)) \\ &= M(S(\theta_n, \theta_n, \theta), S(\theta_{n+1}, \theta_{n+1}, \theta_n), S(\theta_{n+1}, \theta_{n+1}, \theta), \\ &\quad S(T\theta, T\theta, \theta_n), S(T\theta, T\theta, \theta)). \end{aligned}$$

Since $M \in \mathfrak{M}$, using (10) and taking the limit as $n \to \infty$, we find

$$S(\theta, \theta, T\theta) \leq M(0, 0, 0, S(\theta, \theta, T\theta), S(\theta, \theta, T\theta)).$$

Since M satisfies condition (N1), $S(\theta, \theta, T\theta) \leq 0$, that is, $T\theta = \theta$.
(ii) Let θ, η be fixed points of T. It follows from (1) and (10) that

$$\begin{aligned} S(\theta, \theta, \eta) &= S(T\theta, T\theta, T\eta) \\ &\leq M(S(\theta, \theta, \eta), S(T\theta, T\theta, \theta), S(T\theta, T\theta, \eta) S(T\eta, T\eta, \theta), S(T\eta, T\eta, \eta)) \\ &= M(S(\theta, \theta, \eta), 0, S(\theta, \theta, \eta), S(\eta, \eta, \theta), 0) \\ &= M(S(\theta, \theta, \eta), 0, S(\theta, \theta, \eta), S(\theta, \theta, \eta), 0). \end{aligned}$$

By the fact that M satisfies condition (N2), we have $\eta = S(\theta, \theta, \eta) = 0$, i.e., $\theta = \eta$. □

Corollary 1. *In Theorem 2, if we choose* $M(\theta, \eta, \zeta, s, t) = L\theta$*, then we obtain Theorem 1.*

4. Some Applications to Fractional Integrals

4.1. Fixed Point Approximation to the Riemann–Liouville Fractional Integrals

Some mathematicians have been interested in Riemann–Liouville integral equations and the fixed point approach method. We show that the existence and uniqueness of a solution to the Riemann–Liouville equation via this method. The Riemann–Liouville fractional integral is given in the following form:

$$ {}^{RL}_{a}I^{\nu}_{x} = \frac{1}{\Gamma(\nu)} \int_a^x f(t)(x-t)^{\nu-1} dt; \Gamma(\nu) > 0, \tag{14}$$

where $\nu \in \mathbb{R}$, $f(x) \in \Theta$ (Θ is the set of all continuous functions from $[0,1]$ onto $\mathbb{R}$) and $x, t \in [0,1]$. Using Example 3, we define distance $S : \Theta \times \Theta \times \Theta \to [0, \infty)$ by

$$S(f, g, h) = d(f, h) + d(g, h)$$

for all $f(x), g(x), h(x) \in \Theta$ and $x \in [0,1]$, where $d : \Theta \times \Theta \to [0, \infty)$ defined by

$$d(f, g) = |f(x) - g(x)|^2$$

is a controlled metric type. Then, it is clear that S is a controlled S-metric type.

Now, we can show that (14) has a unique solution under the following condition:

$$\frac{1}{\Gamma^2(x+1)} \frac{(x-t)^{\nu-1}(x-a)^{2\nu}}{|(x-t)^{\nu-1}|} < L, \text{ where } L \in (0,1) \text{ and } x \neq t.$$

Define also an operator $T : \Theta \to \Theta$ by

$$Tf(x) = \frac{1}{\Gamma(\nu)} \int_a^x f(t)(x-t)^{\nu-1} dt. \tag{15}$$

Seeing that the problem in (14) has a unique solution is the same as seeing that the integral operator in (15) has a unique fixed point. Suppose that

$$\begin{aligned}
S(Tf, Tf, Tg) &= d(Tf, Tg) + d(Tf, Tg) \\
&= 2d(Tf, Tg) \\
&= 2|Tf(x) - Tg(x)|^2 \\
&= 2\left|\frac{1}{\Gamma(\nu)} \int_a^x f(t)(x-t)^{\nu-1} dt - \frac{1}{\Gamma(\nu)} \int_a^x g(t)(x-t)^{\nu-1} dt\right|^2 \\
&\leq 2\left(\frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} dt\right)^2 |f(t) - g(t)|^2 \\
&\leq 2\frac{1}{\Gamma^2(\nu)} \left(\int_a^x |(x-t)^{\nu-1}| dt\right)^2 |f(t) - g(t)|^2 \\
&= 2\frac{1}{\Gamma^2(\nu)} \frac{(x-t)^{\nu-1}}{|(x-t)^{\nu-1}|} \left(\int_a^x |(x-t)^{\nu-1}| dt\right)^2 |f(t) - g(t)|^2 \\
&= -2\frac{1}{\Gamma^2(\nu)} \frac{(x-t)^{\nu-1}}{|(x-t)^{\nu-1}|} \left(\frac{(x-t)^{\nu}}{\nu}\Big|_a^x\right)^2 |f(t) - g(t)|^2 \\
&= 2\frac{1}{\Gamma^2(\nu)} \frac{(x-t)^{\nu-1}}{|(x-t)^{\nu-1}|} \left(\frac{(x-a)^{\nu}}{\nu}\right)^2 |f(t) - g(t)|^2 \\
&= \frac{2}{\Gamma(\nu+1)} \frac{(x-t)^{\nu-1}(x-a)^{2\nu}}{|(x-t)^{\nu-1}|} |f(t) - g(t)|^2 \\
&\leq 2Ld(f, g) \\
&= LS(f, f, g).
\end{aligned}$$

Therefore, all the hypotheses of Theorem 1 are verified; thus, T has a unique fixed point. Hence, the Riemann–Liouville fractional integral equation has a unique solution.

4.2. Fixed Point Approximation to the Atangana–Baleanu Fractional Integrals

Atangana and Baleanu [19] introduced the fractional derivative and integral operator in the form of (16) in 2016. This study has been used in many fields of science [20,21]. In this part, we show that there is a unique solution to the Atangana and Baleanu fractional integral.

Let $\Theta = C([0,1], \mathbb{R})$, and define $S : \Theta \times \Theta \times \Theta \to [0, \infty)$ by $S(f, g, h) = d(f, h) + d(g, h)$ for all $f(x), g(x), h(x) \in \Theta$ and $x \in [0,1]$, where $d : \Theta \times \Theta \to [0, \infty)$ defined by

$$d(f, g) = |f(x) - g(x)|^2$$

is a controlled metric type. Then, it is obvious that S is a controlled S-metric type.

On the other hand, the general form of the Atangana and Baleanu fractional integral is

$$^{AB}_{\ a}I^{\nu}_{x} f(x) = \frac{1-\nu}{\beta(\nu)} f(x) + \frac{\nu}{\beta(\nu)\Gamma(\nu)} \int_a^x f(t)(x-t)^{\nu-1} dt, \tag{16}$$

where $\nu \in (0,1]$, $f(t) \in \Theta$ (Θ is the set of all continuous functions from $[0,1]$ onto $\mathbb{R}$), and $x, t \in [0,1]$. Note that β is the normalization function such that $\beta(0) = \beta(1) = 1$, that is, it is a function in which the integral is equal to 1 in the interval of domain f.

Now, we prove that the integral in (16) has a unique solution if the following condition holds:

$$\frac{1-\nu}{\beta(\nu)} + \frac{a^{\nu}}{\beta(\nu)\Gamma(\nu)} < L, \text{ where } L \in (0,1). \tag{17}$$

Define an operator $T_{AB} : \Theta \to \Theta$ by

$$T_{AB} f(x) = \frac{1-\nu}{\beta(\nu)} f(x) + \frac{\nu}{\beta(\nu)\Gamma(\nu)} \int_a^x f(t)(x-t)^{\nu-1} dt. \tag{18}$$

Thus, the existence of a unique solution (16) is equivalent to finding a unique fixed point of the integral operator (18). Consider

$$\begin{aligned}
S(Tf, Tf, Tg) &= d(Tf, Tg) + d(Tf, Tg) \\
&= 2d(Tf, Tg) \\
&= 2|Tf(x) - Tg(x)|^2 \\
&= 2\Bigg|\left(\frac{1-\nu}{\beta(\nu)} f(x) + \frac{\nu}{\beta(\nu)\Gamma(\nu)} \int_a^x f(t)(x-t)^{\nu-1} dt\right) \\
&- \left(\frac{1-\nu}{\beta(\nu)} g(x) + \frac{\nu}{\beta(\nu)\Gamma(\nu)} \int_a^x g(t)(x-t)^{\nu-1} dt\right)\Bigg|^2 \\
&= 2\left|\left(\frac{1-\nu}{\beta(\nu)} [f(x) - g(x)] + \frac{\nu}{\beta(\nu)\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} dt [f(t) - g(t)]\right)\right|^2 \\
&\leq 2\left|\left(\frac{1-\nu}{\beta(\nu)} |f(x) - g(x)| + \frac{\nu}{\beta(\nu)\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} dt |f(t) - g(t)|\right)\right|^2 \\
&= 2\left|\left(\frac{1-\nu}{\beta(\nu)} |f(x) - g(x)| - \frac{\nu}{\beta(\nu)\Gamma(\nu)} \left[\frac{(x-t)^{\nu}}{\nu}\Big|_a^x\right] |f(t) - g(t)|\right)\right|^2 \\
&= 2\left|\left(\frac{1-\nu}{\beta(\nu)} |f(x) - g(x)| - \frac{\nu}{\beta(\nu)\Gamma(\nu)} \frac{(x-a)^{\nu}}{\nu} |f(t) - g(t)|\right)\right|^2 \\
&\leq 2\left|\left(\frac{1-\nu}{\beta(\nu)} + \frac{\nu}{\beta(\nu)\Gamma(\nu)} \frac{(x-a)^{\nu}}{\nu} |f(x) - g(x)|\right)\right|^2 \\
&\leq 2L|f(x) - g(x)|^2 \\
&\leq 2Ld(f, g) \\
&= LS(f, f, g).
\end{aligned}$$

Thus, all the hypotheses of Theorem 1 are verified, and T has a unique fixed point. As a result, the Atangana–Baleanu fractional integral equation has a unique solution.

5. Conclusions

In this article, we introduced the notion of a controlled S-metric-type space. Some examples and properties were given in this new space. Then, we proved the Banach contraction principle in controlled S-metric-type spaces and a more general fixed point theorem. As future work, researchers should prove new fixed point theorems in this new space. Moreover, the fixed point approach used in this paper for the Riemann–Liouville and Atangana–Baleanu fractional integral equations can be applied to other differential and integral equations.

Author Contributions: Conceptualization, N.E.Y. and O.E.; methodology, N.E.Y. and O.E.; validation, N.E.Y. and O.E.; formal analysis, N.E.Y., A.M. and O.E.; investigation, N.E.Y. and O.E.; writing—original draft preparation, N.E.Y., A.M. and O.E.; writing—review and editing, N.E.Y., O.E. and N.M.; supervision, O.E. and N.M.; project administration, O.E., N.M. and A.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research did not receive any external funding.

Data Availability Statement: No data were used to support this work.

Acknowledgments: The authors N.M. and A.M. thank Prince Sultan University for paying the APC and for their support through the TAS research lab. This study is a part of Master thesis of the first author.

Conflicts of Interest: The authors declare that they have no competing interests.

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