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Radius Results for Certain Strongly Starlike Functions

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Abstract: This article comprises the study of strongly starlike functions which are defined by using the concept of subordination. The function φ defined by $\varphi(\zeta) = (1+\zeta)^{\lambda}$, $0 < \lambda < 1$ maps the open unit disk in the complex plane to a domain symmetric with respect to the real axis in the right-half plane. Using this mapping, we obtain some radius results for a family of starlike functions. It is worth noting that all the presented results are sharp.

Keywords: univalent function; subordination; analytic function; lemniscate of Bernoulli; Schwarz function

MSC: 30C45; 30C50

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1. Introduction and Preliminaries

Mathematicians, especially analysts, keep working to improve on their own results that are ultimately beneficial for not only research and development in mathematical results but equally helpful for physical scientists or engineers using it. The people related to research in GFT have been exploring new dimensions of their field that study the problems related to geometrical structures of various symmetric domains. Working in a similar direction with the same zeal, Stankiewicz [1,2] was one of the two mathematicians who discussed the idea of the functions that are called strongly starlike, the second being was Brannan and Kirwan [3], but both of them were unaware of the discovery of the other. The former then discovered the external characterization, which was geometrical in nature, for these newly invented functions, see [2]. While on the other end, Brannan and Kirwan discussed the geometrical property, but that was a sufficient condition for any function to be the part of this class of functions. Ma and Minda [4] explored another direction and, unlike Stankiewicz, proved the results for internal characterization, but they based their results in k-starlike domains. Then there was no end to new ideas coming to this area of research. One can see [5], Chapter IV, [6,7]. After that, we see Mocanu [8] coming up with "some starlike conditions for analytic functions" in his article titled so. It was followed by another research work by him [9] in which he explored two conditions for this property of functions, but they were simpler than the ones stated before by him. The following year, Nunokawa [10] gave a remarkable result involving the order of the type of functions under discussion. The results of Brannan and Kirwan were generalized by Obradović [11], who stated more conditions, which can be regarded as sufficient conditions that a function needs to satisfy to be a strongly starlike function. As a matter of fact and a part of natural improvement that has to come under research with the passage of time, Tuneski [12] improved the results stated earlier and made these sufficiency conditions similar in his paper, which was published in 2001. Lecko [6] in 2005 aimed at discovering the results related to this group of functions more geometrically and then switch between the analytical nature and that of the geometrical one in an interesting manner. He built a relation between the geometry of our class with spiral likeness of the same order. He did it in a simpler mean by discussing these beautiful geometrical properties for planar domain which was enhanced by the analytical forms of these results in their related classes. His paper is worth reading. In the same year, Sugawa [7] proved the necessary and sufficient conditions for a simply connected domain to be SS but for a specific order. After proving this remarkable result, he also constructed the connection between already discovered properties of these functions with each other and some other related interesting results. Kwon et al. [13] mainly discussed the characterization of Carathéodory function, including the sufficiency conditions and its related results, but they also applied their results successfully to study those analytic functions whose geometry involves starlikeness and other shapes too. They inspired many to work in a similar direction. Sim [14] was one of those who was inspired enough to explore new dimensions. In his article, referred to earlier, he compared his new results related to sufficiency conditions of Carathéodory function with already known results in the literature. These results also include those related to the geometrical side of the analytic functions associated with his proven results. We see a number of articles by Nunokawa [15–17] in the following years who explored various dimensions of research addressing the geometrical and analytical properties of strongly starlike functions. For the latest work on strongly starlike functions, see [18,19]. The exploration on strongly starlike functions that started in the early 1960s and had been a matter of interest for many mathematicians over that period of time motivated us to explore new dimensions of strongly starlike functions, which are explained in detail below after the basic concepts and definitions. In particular, we aim to focus on the geometrical properties such as radius results related to a class of strongly starlike functions associated with the function $\varphi(\zeta) = (1+\zeta)^{\lambda}$, $0 < \lambda < 1$ and to build its connection with the renowned classes $C, \mathcal{S}^*_{\eta}, S^*_M, \gamma-US^*, \gamma-UC$ and $\mathcal{S}^*_L$ of analytic functions, which is the main motivation of this work. For more relevant work, one can see [20–24].

Let Ω denote the class of analytic functions κ whose series form is given as

$$\kappa(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n \zeta^n, \quad \zeta \in \mathbb{D} := \{\zeta \in \mathbb{C} : |\zeta| < 1\}. \tag{1}$$

We say $\kappa \in \Omega$ is subordinate to $g \in \Omega$ (written symbolically as $\kappa \prec g$ or $\kappa(\zeta) \prec g(\zeta)$) if there exists a Schwarz function ϑ such that $\kappa(\zeta) = g(\vartheta(\zeta))$ for all $\zeta \in \mathbb{D}$.

Shanmugam [25] started the study of the general classes of convex and starlike functions by using the convolution techniques to examine inclusion-related problems. Ma and Minda [26] further developed this theory and established some coefficient-related results. For this purpose, they considered analytic functions $\varphi(\zeta)$ with $\mathrm{Re}\varphi(\zeta) > 0$, $\varphi(0) = 1$ and $\varphi'(0) > 0$ in $\mathbb{D}$ such that $\varphi(\zeta)$ maps $\mathbb{D}$ onto regions that are starlike with respect to 1. They introduced the following classes of functions:

$$\mathcal{S}^*(\varphi) = \left\{ \kappa \in \Omega \colon \frac{\zeta \kappa'(\zeta)}{\kappa(\zeta)} \prec \varphi(\zeta) \right\}$$

and

$$\mathcal{C}(\varphi) = \left\{ \kappa \in \Omega \colon \frac{(\zeta \kappa'(\zeta))'}{\kappa'(\zeta)} \prec \varphi(\zeta) \right\}.$$

Specializing the choices of φ in the class, $\mathcal{S}^*(\varphi)$ reduces to subclasses of starlike univalent functions that are introduced in the literature. For example, $\mathcal{S}^*(\sqrt{1+\zeta}) := \mathcal{S}^*_L$ is the class of functions $\kappa(\zeta)$ that map $\mathbb{D}$ onto the region bounded by lemniscate of Bernoulli [27], which is symmetric around the real axis. For $\varphi(\zeta) = (1+s\zeta)^2$, $0 < s \leq \frac{1}{\sqrt{2}}$, the class $\mathcal{S}^*(\varphi)$

reduces to $ST_L(s)$, which consists of starlike functions associated with symmetric Limaçon domain. This Limaçon is also symmetric around the real axis, see [28]. Closely related to this class, Sokół [29] introduced the class $\mathcal{S}_L^*(c)$, defined as

$$\mathcal{S}_L^*(c) = \left\{ \kappa \in \Omega \colon \left| \left(\frac{\zeta\kappa'(\zeta)}{\kappa(\zeta)} \right)^2 - 1 \right| < c, \quad c \in (0,1],\, \zeta \in \mathbb{D} \right\}.$$

Similarly , for $\varphi(\zeta) = ((1+\zeta)/(1-\zeta))^\eta$, $0 < \eta \leq 1$, we have the class $\mathcal{S}_\eta^*$ of strongly starlike functions [2,3]. This class is equivalently defined as

$$\mathcal{S}_\eta^* = \left\{ \kappa \in \Omega \colon \left| \arg\left(\frac{\zeta\kappa'(\zeta)}{\kappa(\zeta)} \right) \right| < \frac{\eta\pi}{2}, \quad \zeta \in \mathbb{D} \right\}.$$

Additionally, with $\varphi(\zeta) := p_\gamma(\zeta)$, where $p_\gamma(\zeta)$ is defined in [30,31], we have the classes γ-UC and γ-US^* of γ-uniformly convex and corresponding γ-starlike functions, respectively, which are defined as

$$\gamma - UC = \left\{ \kappa \in \Omega \colon \mathrm{Re}\left(\frac{(\zeta\kappa'(\zeta))'}{\kappa'(\zeta)} \right) > \gamma \left| \frac{(\zeta\kappa'(\zeta))'}{\kappa'(\zeta)} - 1 \right|, \gamma \geq 0,\, \zeta \in \mathbb{D} \right\},$$

$$\gamma - US^* = \left\{ \kappa \in \Omega \colon \mathrm{Re}\left(\frac{\zeta\kappa'(\zeta)}{\kappa(\zeta)} \right) > \gamma \left| \frac{\zeta\kappa'(\zeta)}{\kappa(\zeta)} - 1 \right|, \gamma \geq 0,\, \zeta \in \mathbb{D} \right\}.$$

In particular for $\gamma = 0$, these classes reduce to the well-known classes C and S^* of convex and starlike functions, respectively. For $\varphi(\zeta) = (1+\zeta)/(1-\delta\zeta)$ with $\delta = 1 - 1/M$, $M > 1/2$, Janowski [32] introduced the class S_M^*, which is equivalently defined as

$$\mathcal{S}_M^* = \left\{ \kappa \in \Omega \colon \left| \frac{\zeta\kappa'(\zeta)}{\kappa(\zeta)} - M \right| < M, \quad \zeta \in \mathbb{D} \right\}.$$

Recently, Liu et al. [33] introduced and studied the class $\mathcal{S}_L^*(\lambda)$ for which $\varphi(\zeta) = (1+\zeta)^\lambda$, $0 < \lambda < 1$ whose geometric characterization was studied by Masih et al. [34]. For more information regarding other choices of φ, one may see [35–39]. For a detailed list of such functions φ that give symmetric geometrical structures, we refer to [40,41] and the references therein.

The concept of radius results or problems is one of the most fascinating geometric properties of the subclasses of the Shanmugam or Ma and Minda classes. Let $\mathcal{Q}$ be a set of functions and $\mathcal{P}$ be a property that $\mathcal{Q}$ may or may not have in $\mathbb{D}_r := \{\zeta \in \mathbb{C} : |\zeta| < r, 0 < r < 1\}$. Denoted by $R_{\mathcal{P}}(\mathcal{Q})$ is the radius for the property $\mathcal{P}$ in the set $\mathcal{Q}$, and is the largest R such that every function in $\mathcal{Q}$ has the property $\mathcal{P}$ in each $\mathbb{D}_r$ for every $r < R$. In this direction, Sokół [42] obtained a sharp radius relationship between the classes $C, \mathcal{S}_\eta^*, S_M^*$, γ-US^* and $\mathcal{S}_L^*$. Further, Cang and Liu [43] established the radius of inclusion for a certain geometric expression associated with the class $\mathcal{S}_L^*$. Recently, Saliu et al. [38] also obtained the radius relationship of the ratio of analytic functions related with limacon functions. On this note, Bano and Raza [44] obtained several radius results for the subclasses of starlike functions associated with the limacon class.

Motivated principally by the works of Sokół [42], Cang and Liu [43] and Masih et al. [34], we obtained the sharp radius of inclusions for the classes $C, \mathcal{S}_\eta^*, S_M^*$, γ-US^*, γ-UC and $\mathcal{S}_L^*$ associated with the class $\mathcal{S}_L^*(\lambda)$. Additionally, some sharp radius results are derived for certain geometric expressions related with $\mathcal{S}_L^*(\lambda)$.

2. Main Results

Theorem 1. *Let* $\kappa \in \mathcal{S}^*_L(\lambda)$*. Then,*

(a) $\kappa \in S^*_M$ *in the disc*

$$|\zeta| < r_\lambda(M) := \begin{cases} (2M)^{\frac{1}{\lambda}} - 1, & \frac{1}{2} < M < 2^{\lambda-1} \\ 1, & M \geq 2^{\lambda-1}, \end{cases} \tag{2}$$

(b) $\kappa \in \mathcal{S}^*_\eta$ *in the disc*

$$|\zeta| < \sin\left(\frac{\eta\pi}{2\lambda}\right),$$

(c) $\kappa \in \gamma - US^*$ *in the disc*

$$|\zeta| < r_\lambda(\gamma) := 1 - \left(\frac{\gamma}{1+\gamma}\right)^{\frac{1}{\lambda}},$$

(d) $\kappa \in \gamma - UC$ *in the disc* $|\zeta| < r_\lambda(\gamma)$*, where* $r_\lambda(\gamma)$ *is the smallest positive root of the equation*

$$(1-x)^\lambda - \frac{\lambda x}{1-x} - \frac{\gamma}{1+\gamma} = 0,$$

(e) $\kappa \in \mathcal{S}^*_L(c)$ *in the disc*

$$|\zeta| < r_\lambda(c) := (1+c)^{\frac{1}{2\lambda}} - 1.$$

All these radii cannot be improved since the function

$$\begin{aligned} \kappa_0(\zeta) =& \zeta \exp\left(\int_0^\zeta \frac{(1+t)^\lambda - 1}{t}\, dt\right) \\ =& \zeta + \lambda\zeta^2 + \left(\frac{3\lambda^2 - \lambda}{4}\right)\zeta^3 + \left(\frac{17\lambda^3 - 15\lambda^2 + 4\lambda}{36}\right)\zeta^4 + \cdots \end{aligned} \tag{3}$$

plays the role of an extremal function.

Proof. Consider

$$\frac{\zeta\kappa'(\zeta)}{\kappa(\zeta)} = \left(1 + \vartheta(re^{i\theta})\right)^\lambda, \quad 0 \leq \theta \leq 2\pi, \tag{4}$$

where ϑ is analytic in $\mathbb{D}$ with $\vartheta(0) = 0$ and $|\vartheta(\zeta)| < 1$. Then,

(a) for κ to be in S^*_M, we must have

$$\left|\left(1 + \vartheta(re^{i\theta})\right)^\lambda - M\right| < M. \tag{5}$$

An obvious geometric observation shows that $(1+r)^\lambda < 2M$ is sufficient for (5). Thus, we obtain (2). For the sharpness, consider the function $\kappa_0(\zeta)$ in (3). Then, at $\zeta = r_\lambda(M)$,

$$\left|\frac{\zeta\kappa_0'(\zeta)}{\kappa_0(\zeta)} - M\right| = \left|(1+\zeta)^\lambda - M\right| = M. \tag{6}$$

(b) $\kappa \in \mathcal{S}^*_\eta$ if

$$\left|\arg\left(1 + \vartheta(re^{i\theta})\right)^\lambda\right| < \frac{\eta\pi}{2}. \tag{7}$$

Recall that $|\arg(1+\zeta)| \leq \arcsin r$, see [45]. Thus, the condition $\arcsin r < \eta\pi/2\lambda$ or $r < \sin(\eta\pi/2\lambda)$ is sufficient for (7). For the sharpness, let $\eta < \lambda$ and consider

$$\zeta = \sin\frac{\eta\pi}{2\lambda}\left[\cos\left(\frac{\eta\pi}{2\lambda}+\frac{\pi}{2}\right)+i\sin\left(\frac{\eta\pi}{2\lambda}+\frac{\pi}{2}\right)\right]$$

with $|\zeta| = \sin\frac{\eta\pi}{2\lambda}$ for $\kappa_0(\zeta)$ such that

$$\begin{aligned}\left|\arg\left(\frac{\zeta\kappa_0'(\zeta)}{\kappa_0(\zeta)}\right)\right| &= \left|\arg\left(1+\zeta\right)^{\lambda}\right| \\ &= \left|\arg\left[\cos^2\frac{\eta\pi}{2\lambda}+i\sin\frac{\eta\pi}{2\lambda}\cos\frac{\eta\pi}{2\lambda}\right]^{\lambda}\right| \\ &= \left|\arg\left(\cos\frac{\eta\pi}{2\lambda}\right)^{\lambda}e^{i\frac{\eta\pi}{2}}\right| \\ &= \frac{\eta\pi}{2}.\end{aligned}$$

(c) $\kappa \in \gamma - US^*$ if

$$\begin{aligned}\text{Re}\left(1+\vartheta(re^{i\theta})\right)^{\lambda} &> \gamma\left|\left(1+\vartheta(re^{i\theta})\right)^{\lambda}-1\right| \\ &\geq \gamma - \gamma\text{Re}\left(1+\vartheta(re^{i\theta})\right)^{\lambda},\end{aligned}$$

that is

$$\text{Re}\left(1+\vartheta(re^{i\theta})\right)^{\lambda} > \frac{\gamma}{\gamma+1}. \tag{8}$$

Since $\text{Re}\left(1+\vartheta(re^{i\theta})\right)^{\lambda} \geq (1-r)^{\lambda}$, we see that condition (8) will be satisfied if

$$(1-r)^{\lambda} > \frac{\gamma}{\gamma+1}.$$

Hence, $r < 1-(\gamma/(\gamma+1))^{\frac{1}{\lambda}}$. To establish the sharpness, we consider κ_0 at $\zeta = -r_{\lambda}(\gamma)$, and have

$$\text{Re}\left(\frac{\zeta\kappa_0'(\zeta)}{\kappa_0(\zeta)}\right) = \text{Re}\left[1-\left(1-\left(\frac{\gamma}{\gamma+1}\right)^{\frac{1}{\lambda}}\right)\right]^{\lambda} = \frac{\gamma}{\gamma+1}$$

and

$$\gamma\left|1-\frac{\zeta\kappa_0'(\zeta)}{\kappa_0(\zeta)}\right| = \gamma\left|1-\left[1-\left(1-\left(\frac{\gamma}{\gamma+1}\right)^{\frac{1}{\lambda}}\right)\right]^{\lambda}\right| = \frac{\gamma}{\gamma+1}.$$

Thus,

$$\text{Re}\left(\frac{\zeta\kappa_0'(\zeta)}{\kappa_0(\zeta)}\right) = \gamma\left|\frac{\zeta\kappa_0'(\zeta)}{\kappa_0(\zeta)}-1\right|.$$

(d) From (4), a computation gives

$$\frac{(\zeta\kappa'(\zeta))'}{\kappa'(\zeta)} = (1+\vartheta(\zeta))^{\lambda} + \frac{\lambda\zeta\vartheta'(\zeta)}{1+\vartheta(\zeta)}.$$

Therefore, $\kappa \in \gamma - UC$ if

$$\operatorname{Re}\left((1+\vartheta(\zeta))^{\lambda} + \frac{\lambda\zeta\vartheta'(\zeta)}{1+\vartheta(\zeta)}\right) > \gamma\left|(1+\vartheta(\zeta))^{\lambda} + \frac{\lambda\zeta\vartheta'(\zeta)}{1+\vartheta(\zeta)} - 1\right|$$
$$\geq \gamma - \gamma\operatorname{Re}\left((1+\vartheta(\zeta))^{\lambda} + \frac{\lambda\zeta\vartheta'(\zeta)}{1+\vartheta(\zeta)}\right),$$

that is

$$\operatorname{Re}\left((1+\vartheta(\zeta))^{\lambda} + \frac{\lambda\zeta\vartheta'(\zeta)}{1+\vartheta(\zeta)}\right) > \frac{\gamma}{\gamma+1}. \tag{9}$$

By the Schwarz Pick lemma [46] (Vol. I, p. 84), we have

$$|\vartheta'(\zeta)| \leq \frac{1-|\vartheta(\zeta)|^2}{1-|\zeta|^2}, \tag{10}$$

we have

$$\operatorname{Re}\left((1+\vartheta(\zeta))^{\lambda} + \frac{\lambda\zeta\vartheta'(\zeta)}{1+\vartheta(\zeta)}\right) \geq \operatorname{Re}(1+\vartheta(\zeta))^{\lambda} - \frac{\lambda|\zeta||\vartheta'(\zeta)|}{1-|\vartheta(\zeta)|}$$
$$\geq (1-r)^{\lambda} - \frac{\lambda r(1-|\vartheta(\zeta)|^2)}{(1-|\vartheta(\zeta)|)(1-|\zeta|^2)}$$
$$\geq (1-r)^{\lambda} - \frac{\lambda r(1+|\zeta|)}{1-|\zeta|^2},$$

that is

$$\operatorname{Re}\left((1+\vartheta(\zeta))^{\lambda} + \frac{\lambda\zeta\vartheta'(\zeta)}{1+\vartheta(\zeta)}\right) > (1-r)^{\lambda} - \frac{\lambda r}{1-r}.$$

Thus, we see that condition (9) will be satisfied if

$$(1-r)^{\lambda} - \frac{\lambda r}{1-r} > \frac{\gamma}{\gamma+1}.$$

So, the function

$$g(r) := (1-r)^{\lambda} - \frac{\lambda r}{1-r} - \frac{\gamma}{\gamma+1}$$

is decreasing in $(0,1)$ and $g(0) = 1/(\gamma+1)$. Hence, $\kappa \in \gamma - UC$ in the disc $|\zeta| < r_{\lambda}(\gamma)$. Consider the function $\kappa_0(\zeta)$ at $\zeta = -r_{\lambda}(\gamma)$, we have

$$\operatorname{Re}\frac{(\zeta\kappa_0'(\zeta))'}{\kappa_0'(\zeta)} = (1+\zeta)^{\lambda} + \frac{\lambda\zeta}{1+\zeta} = \frac{\gamma}{\gamma+1} = \gamma\left|1 - \frac{(\zeta\kappa_0'(\zeta))'}{\kappa_0'(\zeta)}\right|,$$

where

$$\gamma\left|1 - \frac{(\zeta\kappa_0'(\zeta))'}{\kappa_0'(\zeta)}\right| = \gamma\left|1 - \left((1-r_{\lambda}(\gamma))^{\lambda} - \frac{\lambda r_{\lambda}(\gamma)}{1-r_{\lambda}(\gamma)}\right)\right| = \gamma\left|1 - \frac{\gamma}{\gamma+1}\right| = \frac{\gamma}{\gamma+1}.$$

This shows that the radius is sharp.

(e) $\kappa \in \mathcal{S}_L^*(c)$ if

$$\left|\left(1+\vartheta(re^{i\theta})\right)^{2\lambda} - 1\right| < c,$$

which implies

$$\left|\left(1+\vartheta(re^{i\theta})\right)^{\lambda}\right| < \sqrt{1+c}, \tag{11}$$

It is easy to see that the inequality

$$(1+r)^{\lambda} < \sqrt{1+c}$$

is sufficient for (11). Thus, we have the required result. To establish the sharpness, we consider κ_0 at $\zeta = r_\lambda(c)$ such that

$$\left|\left(\frac{\zeta\kappa_0'(\zeta)}{\kappa_0(\zeta)}\right)^{2\lambda} - 1\right| = \left|\Big(1 + r_\lambda(c)\Big)^{2\lambda} - 1\right| = c.$$

□

Remark 1. *When we choose $\lambda = 1/2$ in Theorem 1, results $(a) - (c)$ reduce to the one obtained by Sokół [42]. For $\gamma = 0$, (d) becomes the radius of convexity for the class $\mathcal{S}_L^*(\lambda)$. Furthermore, for $\lambda = 1/2$, (d) gives the radius of convexity for the class $\mathcal{S}_L^*$ of the lemniscate of Bernoulli [27]. Additionally, for $\lambda = 1/2$, (e) illustrates the radius of inclusion between the classes $\mathcal{S}_L^*$ and $\mathcal{S}_L^*(c)$.*

Theorem 2. *Let p be analytic in $\mathbb{D}$ with $p(0) = 1$. Let $0 \le \alpha \le 1$, $0 \le \beta \le 1$ such that $\alpha\lambda + (1-\alpha) + (1-\beta) > 2\sqrt{(1-\alpha)(1-\beta)}, 0 < \lambda < 1$. If $p(\zeta) \prec (1+\zeta)^{\lambda}$, then*

$$\operatorname{Re}\left((1-\alpha)p^{\frac{1}{\lambda}}(\zeta) + \alpha\left(1 + \frac{\zeta p'(\zeta)}{p(\zeta)}\right)\right) > \beta \tag{12}$$

in the disc $|\zeta| < r_\lambda(\alpha,\beta)$, where

$$r_\lambda(\alpha,\beta) = \frac{2(1-\beta)}{\alpha\lambda + (1-\alpha) + (1-\beta) + \sqrt{(\alpha\lambda + (1-\alpha) + (1-\beta))^2 - 4(1-\alpha)(1-\beta)}}. \tag{13}$$

This result cannot be improved.

Proof. Let $p(\zeta) = (1 + \zeta\vartheta(\zeta))^{\lambda}$, where $\zeta\vartheta(\zeta)$ is a Schwarz function with $|\vartheta(\zeta)| < 1$ for all $\zeta \in \mathbb{D}$. Then, $p^{\frac{1}{\lambda}} = 1 + \zeta\vartheta(\zeta) := u + iv$. We have that $\zeta\vartheta(\zeta) = u - 1 + iv$, and so

$$1 - r \le u \le 1 + r, \quad r = |\zeta|. \tag{14}$$

A computation gives

$$\frac{\zeta p'(\zeta)}{p(\zeta)} = \frac{\lambda\zeta(\vartheta(\zeta))}{1+\zeta\vartheta(\zeta)} + \frac{\lambda\zeta^2\vartheta'(\zeta)}{1+\zeta\vartheta(\zeta)}.$$

Therefore,

$$\begin{aligned}
\operatorname{Re}\left((1-\alpha)p^{\frac{1}{\lambda}}(\zeta) + \alpha\left(1 + \frac{\zeta p'(\zeta)}{p(\zeta)}\right)\right) &= \operatorname{Re}\left[(1-\alpha)(1+\zeta\vartheta(\zeta)) + \alpha\left(1 + \frac{\lambda\zeta\vartheta(\zeta)}{1+\zeta\vartheta(\zeta)} + \frac{\lambda\zeta^2\vartheta'(\zeta)}{1+\zeta\vartheta(\zeta)}\right)\right] \\
&\ge (1-\alpha)u + \alpha(1+\lambda) - \alpha\lambda\operatorname{Re}\left(\frac{1}{u+iv}\right) - \frac{\alpha\lambda r^2|\vartheta'(\zeta)|}{|u+iv|} \\
&\ge (1-\alpha)u + \alpha(1+\lambda) - \frac{\alpha\lambda u}{u^2+v^2} - \frac{\alpha\lambda r^2(1-|\vartheta(\zeta)|^2)}{(1-r^2)(u^2+v^2)^{\frac{1}{2}}} \\
&= (1-\alpha)u + \alpha(1+\lambda) - \frac{\alpha\lambda u}{u^2+v^2} + \frac{\alpha\lambda\big(v^2 + (u-1)^2 - r^2\big)}{(1-r^2)(u^2+v^2)^{\frac{1}{2}}} \\
&:= G(u,v),
\end{aligned}$$

where we have used (10). Thus,

$$\frac{\partial G}{\partial v} = \frac{2\alpha\lambda uv}{(u^2+v^2)^2} + \frac{2\alpha\lambda v}{(1-r^2)(u^2+v^2)^{\frac{1}{2}}} - \frac{\alpha\lambda\big(v^2+(u-1)^2-r^2\big)v}{(1-r^2)(u^2+v^2)^{\frac{3}{2}}}$$

and clearly $\frac{\partial G}{\partial v} = 0$ at $v = 0$. So, at $v = 0$, we have

$$\frac{\partial^2 G}{\partial v^2} = \frac{2\alpha\lambda}{u^3} + \frac{2\alpha\lambda}{(1-r^2)u} - \frac{\alpha\lambda\Big(u-(1+r)\Big)\Big(u-(1-r)\Big)}{(1-r^2)u^3} > 0\,,$$

where we have used (14). This shows that $G(u,v)$ experiences its minimum value at $v = 0$. Thus,

$$\begin{aligned} G(u,v) \geq G(u,0) &= (1-\alpha)u + \alpha(1+\lambda) - \frac{\alpha\lambda}{u} + \frac{\alpha\lambda\big((u-1)^2-r^2\big)}{(1-r^2)u} \\ &= \frac{\big[(1-r^2)(1-\alpha)+\lambda\alpha\big]u - \big[(1+\lambda)r^2-(1-\lambda)\big]\alpha}{1-r^2} \\ &:= H(u) \end{aligned}$$

and

$$\frac{dH}{du} = 1-\alpha+\frac{\alpha\lambda}{1-r^2} > 0,$$

which indicates that $H(u)$ is increasing on the close interval $[1-r, 1+r]$. Thus,

$$H(u) \geq H(1-r) = \frac{(1-\alpha)(1-r)^2+\alpha(1+\lambda)(1-r)-\alpha\lambda}{1-r}.$$

We need to show that

$$\frac{(1-\alpha)(1-r)^2+\alpha(1+\lambda)(1-r)-\alpha\lambda}{1-r} - \beta > 0,$$

that is

$$\frac{(1-\alpha)r^2-((\lambda-1)\alpha+2-\beta)r+1-\beta}{1-r} > 0,$$

which is possible if

$$(1-\alpha)r^2-((\lambda-1)\alpha+2-\beta)r+1-\beta > 0. \tag{15}$$

Let $T(r)$ be the left side of (15), then $T(0)T(1) < 0$. By the intermediate value theorem, there exists $r = r_\lambda(\alpha,\beta)$ such that

$$(1-\alpha)r^2-((\lambda-1)\alpha+2-\beta)r+1-\beta = 0,$$

which gives the required result. For the sharpness, we let $p(\zeta) = (1+\zeta)^\lambda$ and at $\zeta = -r_\lambda(\alpha,\beta)$,

$$(1-\alpha)p^{\frac{1}{\lambda}}(\zeta)+\alpha\left(1+\frac{\zeta p'(\zeta)}{p(\zeta)}\right) = (1-\alpha)(1-r_\lambda(\alpha,\beta))+\alpha\left(1-\frac{\lambda r_\lambda(\alpha,\beta)}{1-r_\lambda(\alpha,\beta)}\right) = \beta.$$

□

Setting $p(\zeta) = \zeta\kappa'(\zeta)/\kappa(\zeta)$ for $\kappa \in \Omega$, we have the following result.

Corollary 1. *Let $\kappa \in \mathcal{S}^*_L$. Then,*

$$\mathrm{Re}\left[(1-\alpha)\left(\frac{\zeta\kappa'(\zeta)}{\kappa(\zeta)}\right)^{\frac{1}{\lambda}} + \alpha\left(1 + \frac{(\zeta\kappa'(\zeta))'}{\kappa'(\zeta)} - \frac{\zeta\kappa'(\zeta)}{\kappa(\zeta)}\right)\right] > \beta$$

in the disc given in Theorem 2.

Remark 2. *For $\alpha = 1, \beta = 0$ in Corollary 1, we obtain the radius of convexity for the class $\kappa \in \mathcal{S}^*_L$ given as*

$$r_\lambda = \frac{1}{\lambda+1},$$

and with $\lambda = 1/2, r = 2/3 \approx 0.66667$ is the radius of convexity for the class of lemniscate of Bernoulli introduced by Sokół [27]. However, this result is slightly greater than the one Sokół obtained.

Theorem 3. *Let p be analytic in $\mathbb{D}$ with $p(0) = 1$. Let $\alpha > 0$, $0 < \beta \leq 1$ and $0 < \lambda < 1$. If $p(\zeta) \prec (1+\zeta)^\lambda$, then*

$$\mathrm{Re}\left(p^{\frac{1}{\lambda}}(\zeta) + \alpha\zeta p'(\zeta)\right) > \beta \tag{16}$$

in the disc $|\zeta| < r_\lambda(\alpha,\beta)$, where $r_\lambda(\alpha,\beta)$ is the smallest positive root of the equation

$$(1-x)^2 - \alpha\lambda x(1-x)^\lambda - (1-x)\beta = 0. \tag{17}$$

This result is the best possible.

Proof. Following the initial procedure of Theorem 2, we arrive at

$$\begin{aligned}
\mathrm{Re}\left(p^{\frac{1}{\lambda}}(\zeta) + \alpha\zeta p'(\zeta)\right) &= \mathrm{Re}\left[1 + \zeta\vartheta(\zeta) + \alpha\lambda\zeta\big(\zeta\vartheta'(\zeta) + \vartheta(\zeta)\big)(1+\zeta\vartheta(\zeta))^{\lambda-1}\right] \\
&= \mathrm{Re}\left[1 + \zeta\vartheta(\zeta) + \alpha\lambda(1+\zeta\vartheta(\zeta))^\lambda + \alpha\lambda\left(\zeta^2\vartheta'(\zeta) - 1\right)(1+\zeta\vartheta(\zeta))^{\lambda-1}\right] \\
&\geq u + \alpha\lambda u^\lambda - \alpha\lambda\left|\zeta^2\vartheta'(\zeta) - 1\right| |1+\zeta\vartheta(\zeta)|^{\lambda-1} \\
&\geq u + \alpha\lambda u^\lambda - \alpha\lambda\left(\frac{r^2 - ((u-1)^2 + v^2)}{1-r^2} + 1\right)\left(u^2+v^2\right)^{\frac{\lambda-1}{2}} \\
&\geq u + \alpha\lambda u^\lambda + \alpha\lambda\left(\frac{(u-1)^2 + v^2 - 1}{1-r^2}\right)\left(u^2+v^2\right)^{\frac{\lambda-1}{2}} \\
&:= G(u,v),
\end{aligned}$$

where we have used (10). Then,

$$\frac{\partial G}{\partial v} = \alpha\lambda v\left((\lambda-1)\left(\frac{(u-1)^2+v^2-1}{1-r^2}\right)\left(u^2+v^2\right)^{\frac{\lambda-3}{2}} + \frac{2\left(u^2+v^2\right)^{\frac{\lambda-1}{2}}}{1-r^2}\right) = 0$$

for $v = 0$. Thus,

$$\frac{\partial^2 G(u,0)}{\partial v^2} = \alpha\lambda\left(\frac{(\lambda-1)u^{\lambda-2}(u-2) + 2u^{\lambda-1}}{1-r^2}\right) > 0.$$

This shows that $G(u,v)$ assumes its minimum value at $v = 0$. Therefore,

$$G(u,v) \geq G(u,0) = u + \alpha\lambda u^\lambda\left(1 + \frac{u-2}{1-r^2}\right) := H(u),$$

and

$$\begin{aligned}\frac{dH}{du} &= 1+\alpha\lambda^2 u^{\lambda-1}\left(\frac{u-(1+r^2)}{1-r^2}\right)+\frac{\alpha\lambda u^{\lambda}}{1-r^2}\\ &\geq 1+\alpha\lambda^2 u^{\lambda}\left(\frac{u-(1+r)}{u(1-r^2)}\right)+\frac{\alpha\lambda u^{\lambda}}{1-r^2}\\ &\geq 1-\frac{2r\alpha\lambda^2(1-r)^{\lambda-1}}{(1+r)^2}+\frac{\alpha\lambda(1-r)^{\lambda-1}}{(1+r)}\\ &= 1+\frac{\alpha\lambda(1-r)^{\lambda-1}}{1+r}\left(1-\frac{2r\lambda}{1+r}\right)\\ &> 1+\frac{\alpha\lambda(1-r)^{\lambda-1}}{1+r}(1-\lambda)>0.\end{aligned}$$

This shows that $H(u)$ is an increasing function of u in $[1-r,1+r]$, and so

$$H(u)\geq H(1-r)=\frac{(1-r)^2-\alpha\lambda r(1-r)^{\lambda}}{1-r}.$$

To prove our result, it is enough to show that

$$\frac{(1-r)^2-\alpha\lambda r(1-r)^{\lambda}}{1-r}>\beta,$$

that is

$$(1-r)^2-\alpha\lambda r(1-r)^{\lambda}-(1-r)\beta>0, \tag{18}$$

Let $T(r)$ be left side of (18) and consider $\epsilon>0$ with $\epsilon<\beta$. Then $(0,1-\epsilon]\subset(0,1)$. Therefore, $T(0)=1-\beta\geq 0$ and $T(1-\epsilon)=\epsilon(\epsilon-\beta)-\alpha\lambda(1-\epsilon)\epsilon^{\lambda}<0$ such that $T(0)T(1-\epsilon)\leq 0$. Thus, there exists $r\in(0,1-\epsilon]$ such that

$$(1-r)^2-\alpha\lambda r(1-r)^{\lambda}-(1-r)\beta=0,$$

and hence, we have the desired result. For the sharpness, consider the function $p(\zeta)=(1+\zeta)^{\lambda}$. Then, at $\zeta=-r_{\lambda}(\alpha,\beta)$, we have

$$p^{\frac{1}{\lambda}}(\zeta)+\alpha\zeta p'(\zeta)=1+\zeta+\alpha\lambda z(1+\zeta)^{\lambda-1}\Big|_{\zeta=-r_{\lambda}(\alpha,\beta)}=\beta.$$

□

When we set $p(\zeta)=\kappa',\alpha=1,\beta=1$ and $\lambda=1/2$ in Theorem 3, we arrive at the following result.

Corollary 2. *Let $\kappa'(\zeta)\prec(1+\zeta)^{\lambda}$. Then*

$$\mathrm{Re}\left(\left(\kappa'(\zeta)\right)^{\frac{1}{2}}-1+\zeta\kappa''(\zeta)\right)>0$$

in the disc $|\zeta|<3/4$.

3. Conclusions

We have considered the function φ defined by $\varphi(\zeta)=(1+\zeta)^{\lambda}$, $0<\lambda<1$, which maps the open unit disk to a symmetric domain and by using this function, we have studied the class of strongly starlike functions. Hence, certain sharp radius results for a family of starlike functions were found.

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