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Time Optimal Feedback Control for 3D Navier–Stokes–Voigt Equations

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Abstract: In this article, we discuss a time optimal feedback control for asymmetrical 3D Navier–Stokes–Voigt equations. Firstly, we consider the existence of the admissible trajectories for the asymmetrical 3D Navier–Stokes–Voigt equations by using the well-known Cesari property and the Fillippove's theorem. Secondly, we study the existence result of a time optimal control for the feedback control systems. Lastly, asymmetrical Clarke's subdifferential inclusions and asymmetrical 3D Navier–Stokes–Voigt differential variational inequalities are given to explain our main results.

Keywords: 3D Navier–Stokes–Voigt equations; admissible trajectories set; admissible control set; feedback control; time optimal control

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1. Introduction

Let Ω be an open bounded domain in $\mathbb{R}^3$ with C^1 boundary $\partial\Omega$. For $T > 0$, let $J = [0, T], Q = J \times \Omega$, we consider the following 3D Navier–Stokes–Voigt equations:

$$\begin{cases} z'_t - \mu\Delta z - \gamma^2\Delta z' + (z \cdot \nabla)z + \nabla p = f, & \text{in } Q, \\ z(t, y) = 0, & \text{on } (0, T) \times \partial Q, \\ z(0, y) = z_0(y), & \text{in } Q, \end{cases} \tag{1}$$

which is called the Navier–Stokes–Voigt equation. A model of motion of linear viscoelastic fluids was presented by Oskolkov in 1973 [1]. Furthermore, Oskolkov studied the existence of time periodic solutions and no-slip Dirichlet boundary conditions for the Navier–Stokes–Voigt equation. From then on, the existence results and optimal control problem for the Navier–Stokes–Voigt equation have drawn great attention, for example, Sviridyuk [2] discussed the weakly compressible for the Navier–Stokes–Voigt equation. The long time dynamics and attractors were researched by [3,4]. Anh-Nguyet [5] focused on an optimal control problem with quadratic objective functional for the Navier–Stokes–Voigt equation.

Control theory has become a very popular research field and has seen wide use in science and engineering. Many control systems are usually established upon feedback principles [6–15]. Many modern conveniences, including car cruise-control systems and thermostats, rely heavily on feedback control. These problems naturally promote the development of feedback control theory. In recent years, the optimal feedback of evolution systems were considered in the works of [16–22]. Zhang and Jia focused on the multiple inflows feedback control system in their paper [23]. For more details, we refer readers to the papers by Refs. [5,13–15,24–35].

Over the past few decades, the optimal control of the Navier–Stokes equation has been extensively researched by a large number of authors. For example, the absence of state constraint for Navier–Stokes control systems has been discussed by [36–38]. The optimal feedback control of Kelvin–Voigt fluid flows is presented in [39]. The presence of state constraint for the control systems were investigated [40]. Recently, Zeng [41] studied the feedback control for non-stationary 3D Navier–Stokes–Voigt Equations (3DNSVEs for short) by using monotone theory.

Since the concept of time optimal control was introduced by LaSalle [42] in 1960, the theories of time optimal control problems has caused widespread concern by many mathematics. For example, Berkovitz [7] and Warga [43] considered the time optimal control for functional equations, Barbu [6] studied the parabolic variational inequalities by using monotone theory. Fattorini [16,38] discussed the operational differential equation and viscous flows problem. Yong-Li gave the necessary and sufficient conditions for the time optimal control of distributed parameter equation [22], for more detail see the references therein.

In this paper, we consider the following 3DNSVEs:

$$\begin{cases} z_t' - \mu\Delta z - \gamma^2\Delta z' + (z\cdot\nabla)z + \nabla p = f + v, \quad \text{in } Q, \\ v(t) \in \Phi(t, y, z(t, y)), \quad \text{in } L^2(\Omega), \\ \Delta\cdot z = 0, \quad \text{in } Q, \\ z(t, y) = 0, \quad \text{on } (0, T)\times\partial Q, \\ z(0, y) = z_0(y), \quad \text{in } Q, \end{cases} \tag{2}$$

where z is a state function, Φ is a feedback multi-map, the control function $v \in \Phi$.

The aim of this article is to consider the existence results of admissible trajectories and a time optimal control for 3D Navier–Stokes–Voigt systems. To achieve this aim, the existence result of admissible trajectories is discussed using the help of monotone theory and the well-known Fillippove's theorem. Furthermore, we investigate the existence of time optimal control for the 3D Navier–Stokes–Voigt systems by using optimal control theory. We note that our theory obtained in this article could be widely applied across numerous practical problems, such as static, quasistatic and dynamic frictional and frictionless contact problems.

The rest of this paper is structured as follows. In Section 2, some useful preliminaries and notations on the data are introduced. In Section 3, the existence of admissible trajectories is discussed. In Section 4, a time optimal control of feedback control for 3DNSVs is studied. Lastly, we use two examples to demonstrate our main theory.

2. Some Notations, Definitions and Preliminaries

We set $(X, \|\cdot\|_X)$ as a Banach space and denote its dual space as X^*. Furthermore, we us $\langle\cdot,\cdot\rangle_X$ to denote duality pair between X and X^*. For any $T > 0$, Let $C(J, X)$ denote the Banach space of all continuous functions from $J = [0, T]$ into X with the norm $\|x\|_{C(J,X)} = \sup_{t\in J}\|x(t)\|$ and $L^2([0, T]; X)$ denote the Banach space of all square integrable functions from $[0, T]$ into X with the norm $\|x\|_{L^2} = \left(\int_0^T \|x(t)\|_X^2 dt\right)^{\frac{1}{2}}$. We denote the strong convergence as "$\rightarrow$" and "$\rightharpoonup$" as the weak convergence.

Now, we give the abstract framework for our main work. Defining two inner products as

$$(x, v)_1 := \int_\Omega \sum_{k=1}^{3} x_k v_k ds, \quad x = (x_1, x_2, x_3), v = (v_1, v_2, v_3) \in (L^2(\Omega))^3$$

and

$$(x, v)_2 := \int_\Omega \sum_{k=1}^{3} \nabla x_k\cdot\nabla v_k ds, \quad x = (x_1, x_2, x_3), v = (v_1, v_2, v_3) \in (H_0^1(\Omega))^3,$$

also, with the norms $\|x\|_1 := \sqrt{(x, x)_1}$ and $\|x\|_2 := \sqrt{(x, x)_2}$. We set

$$V_0 = \{x \in (C_0^\infty(\Omega))^3 : \nabla x = 0\}.$$

The closure of V_0 is denoted in $(L^2(\Omega))^3$ (resp. in $(H_0^1(\Omega))^3$) as H (resp. V). One can easily know that H (resp. V) is a Hilbert space with scalar products $(\cdot,\cdot)_1$ (resp. $(\cdot,\cdot)_2$). It follows that $V \subset H \equiv H^* \subset V^*$, here the embeddings are dense, continuous, and compact. Furthermore, we give the pairing between $L^2(J,V^*)$ and $L^2(J,V)$ as

$$\langle x^*, x\rangle_{L^2(J,V)} := \int_0^T \langle x^*(t), x(t)\rangle_V dt, \quad x \in L^2(J,V), x^* \in L^2(J,V^*).$$

We define the Sobolev space as

$$H^1(J,V) := \{x \in L^2(J,V) : x^* \in L^2(J,V^*)\}$$

endow with the norm

$$\|x\|_{H^1(J,V)} := \|x\|_{L^2(J,V)} + \|x^*\|_{L^2(J,V^*)}.$$

Clearly, $H^1(J,V) \hookrightarrow C(J,V)$ is continuous and $H^1(J,V) \hookrightarrow (L^2(\Omega))^3$ is compact.

Now, recalling some basic definitions and properties of multi-valued maps, which can be seen in the monograph [44].

Let $P(X)$ be the set of all nonempty subsets of X, $P_f(X)$ is the set of all nonempty closed subsets of X. Defining the Hausdorff metric as follows

$$h(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\},$$

where $d(x,D)$ is the distance from a point x to D. A multi-map is said to be h-continuous if it is continuous in the Hausdorff metric $h(\cdot,\cdot)$.

Setting (Ω,Σ) be a measurable space and Y be a separable Banach space. A multifunction $\Gamma : \Omega \to P_f(Y)$ is called to be measurable if $\Gamma^{-1}(E) = \{t \in \Omega : \Gamma(t) \cap E \neq \emptyset\} \in \Sigma$ for every closed set $E \subseteq Y$.

Let G, D be two Hausdorff topological spaces and $\Gamma : G \to P(D)$. Γ is said to be lower semicontinuous (l.s.c for short) at $x_0 \in G$, if for any open set $O \subseteq D$, $\Gamma(x_0) \cap O \neq \emptyset$, there exists a neighborhood $\Theta(x_0)$ of x_0 such that $\Gamma(x) \cap O \neq \emptyset$ for all $x \in \Theta(x_0)$. Γ is called to be upper semicontinuous at $x_0 \in G$, if for any open set $O \subseteq D$, $\Gamma(x_0) \subseteq O$, there is a neighborhood $\Theta(x_0)$ of x_0 such that $\Gamma(x) \subseteq O$ for all $x \in \Theta(x_0)$. For more details, one can see the monograph [44].

Let $F : \Omega \to P(X)$ be a multifunction. For $1 \leq p \leq +\infty$, we can define $S_F^p = \{f \in L^p(\Omega, X) : f(t) \in F(t) \text{ a.e. on } \Omega\}$.

Besides the standard norm on $L^q(J,X)$ (here X is a separable, reflexive Banach space) for $1 < q < \infty$, we also consider the so called weak norm

$$\|u(\cdot)\|_\omega = \sup_{0\leq t_1\leq t_2\leq b} \left\| \int_{t_1}^{t_2} u(s)ds \right\|_X \text{ for } u \in L^q(J,X). \tag{3}$$

Space $L^q(J,X)$ furnished with this norm will be denoted by $L^q_\omega(J,X)$. The following result establishes a relation between convergence in ω-$L^q(J,X)$ and convergence in $L^q_\omega(J,X)$.

Definition 1 ([45])**.** *Let E and T be two metric spaces. A multifunction $\Gamma : T \to 2^E$ is called to be pseudo-continuous at point $t \in T$ if*

$$\bigcap_{\epsilon>0} \overline{\Gamma(O_\epsilon(t))} = \Gamma(t),$$

where $O_\epsilon(t) = \{\tau \in J : \|\tau - t\| < \epsilon\}$. The multifunction Γ is said to be pseudo-continuous on T if Γ is pseudo-continuous at every point $t \in T$.

Definition 2 ([45]). *Let X be a metric space and Y be a Banach space. Let $\Gamma : X \to P(Y)$ be a multifunction. Γ is said to possess the Cesari property at point $x_0 \in X$, if*

$$\bigcap_{\delta>0} \overline{co}\Gamma(O_\delta(x_0)) = \Gamma(x_0),$$

where $\overline{co}D$ is the closed convex hull of D, $O_\delta(x)$ is the δ-neighborhood of x. If Γ has the Cesari property at every point $x \in Q \subset X$, then Γ has the Cesari property on Q.

Example 1. *Let $E = [0,1] \times \mathbb{R}$, let Z be a closed Cantor subset of $[0,1]$ whose measure is positive, and let $Z' = [0,1]\backslash Z$. Then Z' is the countable union of disjoint subintervals of $[0,1]$. Let $\sigma(t)$ be a continuous function on Z'. which tends to $+\infty$ whenever t tends to an end of any interval component of Z'. We define a multifunction as follows*

$$U(t) = \begin{cases} \{-1\}, & if\, t \in Z, \\ \{u \in \mathbb{R} | u \geq \sigma(t)\}, if\, t \in Z'. \end{cases}$$

Then, $U(t)$ has the Cesari property.

Now, let us denote the trilinear form $g : V \times V \times V \to \mathbb{R}$ as

$$g(x,y,z) := \sum_{i,j=1}^{3} \int_\Omega x_i(D_i y_j) z_j ds$$

whenever the integrals make sense. Let us give the property of g.

Lemma 1 ([46]). *The following properties hold:*

(i) *for any $x, y, z \in V$,*
$$g(x,y,z) = -g(x,z,y).$$
(ii) *for any $x, y \in V$,*
$$g(x,y,y) = 0.$$
(iii) *for any $x, y, z \in V$,*
$$|g(x,y,z)| \leq \begin{cases} C\|x\|_1^{\frac{1}{4}}\|x\|_2^{\frac{3}{4}}\|y\|_1^{\frac{1}{4}}\|z\|_2^{\frac{3}{4}} \\ C\|x\|_2\|y\|_2\|z\|_2, \end{cases}$$
in particular, $|g(x,y,x)| \leq C\|x\|_1^{\frac{1}{2}}\|x\|_2^{\frac{3}{2}}\|y\|_2$.

Now, let us give the definition of a weak solution for the problem (2).

Definition 3. *For each $f \in L^2(J, V^*), z(0) = z_0 \in V$, $z(\cdot)$ is said to be a weak solution of the problem (1), if*

$$\begin{cases} z \in C(J,V), \qquad z' \in L^2(J,V), \\ (z'(t),w)_1 + \mu(z(t),w)_2 + \gamma^2(z'(t),w)_2 + g(z(t),z(t),w) = \langle f(t),w\rangle_V, \quad \forall w \in V, \text{ a.e. } t \in J, \\ z(0) = z_0. \end{cases}$$

Now, we define a linear and continuous operator $A : L^2(J,V) \to L^2(J,V^*)$ as follows

$$\langle Az, w\rangle_{L^2(J,V)} := \int_0^T \langle (Az)(t), w(t)\rangle_V dt := \int_0^T (z(t),w(t))_2 dt, \quad \forall z,w \in L^2(J,V),$$

also, a nonlinear operator $B : H^1(J,V) \to L^2(J,V)$ is defined as

$$\langle Bz, w\rangle_{L^2(J,V)} := \int_0^T \langle (Bz)(t), w(t)\rangle_2 dt := \int_0^T g(z(t),z(t),w(t)) dt, \quad \forall z,w \in H^1(J,V).$$

By Lemma 1, one can know that B is a bounded mapping from $H^1(J,V)$ to $L^2(J,V^*)$, i.e., there is a positive constant $c>0$ such that

$$\|Bz\|_{L^2(J,V^*)} \leq c\|z\|_{H^1(J,V)}.$$

According to the work of [5], we have the following results for operators A and B.

Lemma 2. *Let $z_n \rightharpoonup z$ in $H^1(J,V)$ as $n\to\infty$, then*

$$\begin{aligned} Az_n &\rightharpoonup Az \quad \text{in } L^2(J,V^*),\\ Az_n' &\rightharpoonup Az' \quad \text{in } L^2(J,V^*),\\ Bz_n &\rightharpoonup Bz \quad \text{in } L^2(J,V^*). \end{aligned}$$

Thanks to the properties of operators A and B, we can now give an equivalent formulate with Definition 3.

Definition 4. *The function $z\in H^1(J,V)$ is said to be a weak solution of the problem (2), if*

$$\begin{cases} z' + \mu Az + \gamma^2 Az' + Bz = f, & \text{in } L^2(J,V^*),\\ z(0) = z_0 & \text{in } V. \end{cases}$$

At the end of this section, we give a well-known priori estimate for the system (1) (see [41,47]).

Theorem 1. *For every $f\in L^2(J,V^*)$, problem (1) has a unique weak solution $x\in H^1(J,V)$. Furthermore, there is a positive constant M such that*

$$\|z\|_{H^1(J,V)} \leq M(1+\|f\|_{L^2(J,V^*)}). \tag{4}$$

To discuss the main results, we need the following definitions.

Definition 5. *Assume that $z\in V$, control pair $(z(\cdot),v(\cdot))\in H^1(J,V)\times(L^2(\Omega))^3$ is called to be an admissible pair with $z_0\in H^1(J,V)$ on J if $z(\cdot)$ is the weak solution for control system (2) and*

$$v(t)\in\Phi(t,z(t)) \quad a.e.\ t\in J.$$

where $z(\cdot)$ is the admissible trajectory and $v(\cdot)$ is the admissible control.

We set that $\Pi: J\to P(V)$ is a target trajectories set. Denoting the admissible control pair set as

$$\begin{aligned} \mathscr{H}(J,z_0) = &\{(z(\cdot),v(\cdot))\in H^1(J,V)\times(L^2(\Omega))^3 : (z(\cdot),v(\cdot))\\ &\text{is an admissible pair with } z_0\in V\}, \end{aligned}$$

and denoting the admissible trajectories set as

$$\mathscr{G}(J,z_0) = \{z(\cdot)\in H^1(J,V) : (z(\cdot),v(\cdot))\in\mathscr{H}(J,z_0),\ \text{for some } v(\cdot)\in(L^2(\Omega))^3\},$$

Moreover, we denote the reachable set as

$$\mathscr{K}(r;0,x_0) = \{x(r) : x(\cdot)\in\mathscr{G}([0,r],x_0)\}.$$

Furthermore, set

$$\mathscr{C}(0,z_0) = \left\{(z(\cdot),v(\cdot))\in\bigcup_{r>0}\mathscr{H}([0,\tau],z_0) : z(\bar{t})\in\Pi(\bar{t}) \text{ for some } \bar{t}\geq 0\right\}$$

to be the target admissible control pair set, and denote the target of time set by

$$\mathscr{J}(z(\cdot)) = \{\bar{t} \in J : z(\bar{t}) \in \Pi(\bar{t})\}, \qquad \forall z(\cdot) \in \bigcup_{0<r<T} H^1([0,\tau],V).$$

Now, we give our main problem as follows:

Problem (T): Assume $\mathscr{C}(0,z_0) \neq \emptyset$. Find control pair $(z^*(\cdot),v^*(\cdot)) \in \mathscr{C}(0,z_0)$ and $t^* \in \mathscr{J}(x^*(\cdot))$, such that

$$t^* = \min_{(z(\cdot),v(\cdot))\in\mathscr{C}(0,z_0)} \inf \mathscr{J}(z(\cdot)). \tag{5}$$

3. Existence Results for Admissible Trajectories

The aim of this section is to study the existence results for admissible trajectories of the system (2). To achieve this aim, we need the conditions as follows.

H(Φ): $\Phi : J \times (L^2(\Omega))^3 \to P((L^2(\Omega))^3)$ is pseudo-continuous and

(i) $\forall (t,z) \in J \times V$, there are $\theta \in L^2(J,\mathbb{R}^+)$ and $L_\Phi > 0$, such that

$$\|\Phi(t,z)\| \leq \sup_{v\in\Phi(t,z)} \|v\|_{(L^2(\Omega))^3} \leq \theta(t) + L_\Phi \|z\|_{(L^2(\Omega))^3}, \text{for all } (t,z) \in J \times (L^2(\Omega))^3,$$

(ii) for a.e. $t \in J, z \in (L^2(\Omega))^3$, the set $\Phi(t,x)$ satisfies

$$\bigcap_{\delta>0} \overline{\text{co}}\Phi(O_\delta(t,z)) = \Phi(t,z).$$

H(Π): $\Pi : J \to P((L^2(\Omega))^3)$ is pseudo-continuous.

Theorem 2. *Assume $H(\Phi)$ be satisfied, then for each $v \in (L^2(\Omega))^3$, the set $\mathscr{G}(J,z_0)$ is nonempty. Furthermore, for any $v \in (L^2(\Omega))^3$, $\mathscr{G}(J,z_0)$ is compact in $C(J,V)$.*

Proof. For each $l > 0$, we assume that $\tau_i = \frac{i}{k}T, 0 \leq i \leq l-1$, then

$$v_l(t) = \sum_{i=0}^{l-1} v^i \chi_{[t_i,t_{i+1})}(t),\ t \in J,$$

here $\chi_{[t_i,t_{i+1})} = \begin{cases} 1, t \in [t_i,t_{i+1}) \\ 0, t \notin [t_i,t_{i+1}) \end{cases}$. The sequence $\{v^i\}$ is constructed as following.

To begin, we take $v^0 \in \Phi(0,x_0)$, using Theorem 1, that has a unique $z_l \in H^1(J,V)$ given by $z_l(0) = z_0$ and

$$z_l' + \mu A z_l + \gamma^2 A z_l' + B z_l = f + v^0, \quad \text{in } L^2(J,V^*).$$

Then taking $v^1 \in \Phi(\frac{T}{l}, z_l(\frac{T}{l}))$, repeating this process to obtain z_l on $[\frac{T}{l},\frac{2T}{l}]$, etc. Using a similar approach, we denote as follows:

$$\begin{cases} z_l' + \mu A z_l + \gamma^2 A z_l' + B z_l = f + v_l,\ t \in [\frac{iT}{l}, \frac{(i+1)T}{l}), \\ v_l(t) \in \Phi(\frac{iT}{l}, z_l(\frac{iT}{l})), \end{cases}$$

Similar to the Theorem 1 and applying $H(U)(i)$, we can find that there is a constant $\vartheta_0 > 0$ such that

$$\|z_l\|_{H^1(J,V)} \leq \vartheta_0.$$

Hence $\{z_l\}$ is boundness in $H^1(J,V)$. There then exists a subsequence of $\{z_l\}$, denoting as $\{z_l\}$ again, such that

$$z_l \rightharpoonup \bar{z} \quad \text{in } H^1(J,V),$$

and

$$z_l \rightharpoonup \bar{z} \quad \text{in } L^2(J,V),$$
$$z_l' \rightharpoonup \bar{z}' \quad \text{in } L^2(J,V^*)$$

Since $H^1(J,V) \hookrightarrow C(J,V)$ is continuous, then $z_l(0) \rightharpoonup \bar{z}(0)$ in $(L^2(\Omega))^3$, which implies that $\bar{x}(0) = x_0$. Given the fact that the embedding $H^1(J,V) \subset (L^2(\Omega))^3$ is compact, then

$$z_l \to \bar{z} \quad \text{in } (L^2(\Omega))^3.$$

According to the fact that $z_k \in C(J,V)$ and $V \in L^2(\Omega)$, for each $\delta > 0$ there are positive constants l, k_0 such that

$$\begin{aligned} \|z_l(t_i) - \bar{z}(t)\|_{L^2(\Omega)} &\le \|z_l(t_i) - z_l(t)\|_{L^2(\Omega)} + \|z_l(t) - \bar{z}(t)\|_{L^2(\Omega)} \\ &\le \tau\|z_l(t_i) - z_l(t)\|_V + \|z_l(t) - \bar{z}(t)\|_{L^2(\Omega)} \\ &< \frac{\delta}{2} + \frac{\delta}{2} = \delta \end{aligned} \tag{6}$$

By (6), for each $\delta > 0$, there is a constant $\zeta_0 > 0$ such that

$$z_l(t) \in O_\delta(z_l(t_i)), t \in J, k \ge k_0. \tag{7}$$

Combining Lemma 2, we have

$$Az_n \rightharpoonup Az \quad \text{in } L^2(J,V^*),$$
$$Az_n' \rightharpoonup Az' \quad \text{in } L^2(J,V^*),$$
$$Bz_n \rightharpoonup Bz \quad \text{in } L^2(J,V^*).$$

Moreover, by $H(\Phi)(i)$, there is η_1, such that

$$\|v_l\|_{(L^2(Q))^3} = \int_0^T \|v_l(s)\|_{(L^2(Q))^3} ds \le \int_0^T \Big(\theta(s) + L_\Phi \|z(s)\|_{(L^2(\Omega))^3}\Big) ds < \eta_1.$$

This means that the sequence $\{v_l\}_{l\ge1}$ is bounded in $(L^2(Q))^3$. Then, there exists a subsequence of $\{v_l\}_{l\ge1}$, that we also denote as $\{v_l\}_{l\ge1}$, such that

$$v_k \rightharpoonup \bar{v} \quad \text{in } (L^2(Q))^3. \tag{8}$$

Hence,

$$\bar{z}' + \mu A\bar{z} + \gamma^2 A\bar{z}' + B\bar{z} = f + \bar{v}, \quad \text{in } L^2(J,V^*),$$

Furthermore, for l that is large enough, from the definition of $u_l(\cdot)$, we have

$$v_l(t) \in \Phi(t_i, z_l(t_i)) \subset \Phi(O_\delta(t_i, z_l(t_i))). \tag{9}$$

Secondly, applying (8) and Mazur Lemma, we set $\alpha_{il} \ge 0$ and $\sum_{i\ge1} \alpha_{i\tau} = 1$, such that

$$\lambda_\tau(\cdot) = \sum_{i\ge1} \alpha_{i\tau} v_{i+\tau}(\cdot) \to \bar{v}(\cdot) \text{ in } (L^2(Q))^3.$$

Then there is a subsequence of $\{\lambda_l\}$, without loss of generality, such that

$$\lambda_\tau(t) \to \bar{v}(t) \text{ in } (L^2(Q))^3 \text{ a.e. } t \in J.$$

Hence, from (7) and (9), for l that is large enough, we get

$$\lambda_\tau(t) \in \text{co}\Phi(O_\delta(t, \bar{z}(t)), \text{ a.e. } t \in J.$$

Therefor, for any $\delta > 0$, one can have

$$\overline{v}(t) \in \overline{\text{co}}\Phi(O_\delta(t, \overline{z}(t)), \text{ a.e. } t \in J.$$

Using $H(\Phi)ii)$, we get

$$\overline{v}(t) \in \overline{\text{co}}\Phi(t, \overline{z}(t)), \text{ a.e. } t \in J. \tag{10}$$

From the above work, we have

$$(\overline{z}, \overline{v}) \in \mathscr{A}(J, z_0).$$

In the end, setting $\{z^l(\cdot)\}_{l\geq 1} \subset \mathscr{G}(J, x_0)$ and

$$\|z^l\|_{C(J,V)} \leq \gamma_0, \quad \text{in } C(J, V).$$

Then, by the same way, we can get that $\{z^l(\cdot)\}_{l\geq 1}$ is relatively compact in space $C(J, V)$. Furthermore, there is a subsequence of $\{z^l(\cdot)\}_{l\geq 1} \subset \mathscr{G}(J, z_0)$, denoted by $\{z^l(\cdot)\}_{l\geq 1}$ again, such that

$$z^l(\cdot) \to \widehat{z}(\cdot), \quad \text{in } C(J, V).$$

By hypothesis $H(\Phi)(ii)$, one can obtain $\widehat{z}(\cdot) \in \mathscr{G}(J, z_0)$. Hence, the admissible set $\mathscr{G}(J, z_0)$ is compact in space $C(J, V)$. □

Corollary 2. *Let hypothesis $H(\Phi)$ hold. Then for each $z_0 \in V$ and $0 \leq t \leq T$, the reachable set $\mathscr{K}(t; 0, z_0)$ is nonempty and compact in V.*

Theorem 3. *Assume the condition $H(\Phi)$ hold. Then for every $z_0 \in V$, $\mathscr{K}(\cdot; 0, z_0) : J \to P(V)$ is h-continuous.*

Proof. According to the Theorem 2, we can infer that for any $z_0 \in V, t \in J$, and any $z(\cdot) \in \mathscr{G}(J, z_0)$, there is a continuous, nondecreasing function $\beta : [0, \infty) \to [0, \infty)$ with $\beta(0) = 0$, such that

$$\|z(\iota_1) - z(\iota_2)\| \leq \beta(|\iota_1 - \iota_2|), \quad \text{for } \iota_1, \iota_2 \in J.$$

Then,

$$h(\mathscr{K}(\iota_1; 0, z_0), \mathscr{K}(\iota_2; 0, z_0)) \leq \beta(|\iota_1 - \iota_2|), \quad \forall \iota_1, \iota_2 \in J.$$

Thus, the multifunction $\mathscr{K}(\cdot; 0, z_0)$ is continuous. □

4. Existence Results for Time Optimal Control

In the following section, we will study the existence of time optimal control for the 3DNSVs.

Theorem 4. *Let the hypotheses $H(\Phi), H(\Pi)$ hold, then Problem (T) has at least one optimal solution.*

Proof. Set $z^0 \in V, (z_n(\cdot), v_n(\cdot)) \in \mathscr{C}(0, x^0), \bar{t}_n \in \mathscr{J}(z_n(\cdot))$ and

$$\lim_{n\to\infty} \bar{t}_n = t^* = \inf_{(z(\cdot),v(\cdot))\in\mathscr{C}(0,z^0)} \inf \mathscr{J}(z(\cdot)). \tag{11}$$

By the notation of $\mathscr{J}(z_n(\cdot))$, we get

$$z_n(\bar{t}_n) \in \Pi(\bar{t}_n) \bigcap \mathscr{K}(\bar{t}_n; 0, z^0), \quad \forall n \geq 1. \tag{12}$$

Thanks to Theorem 2, for each $t \in J$ and $\overline{z}(\cdot) \in \mathscr{G}(J, z^0)$, there has $\{z_n(\cdot)\}_{n\geq 1}$ such that

$$z_n(\cdot) \to \overline{z}(\cdot) \quad \text{in } C(J, V). \tag{13}$$

From (13), we can easily calculate that

$$z_n(\bar{t}_n) \to \bar{z}(t^*) \quad \text{in } V.$$

Combing with Theorem 3 and (12), one can infer that

$$\bar{z}(t^*) \in \mathscr{K}(t^*; 0, z^0). \tag{14}$$

Applying (11), for any $\delta > 0$, and n large enough, we get

$$z_n(\bar{t}_n) \in \Pi(\bar{t}_n) \subset \Pi(O_\delta(t^*)).$$

Since $\Pi(\cdot)$ is pseudo-continuous, we know

$$\bar{z}(t^*) \in \bigcap_{\delta>0} \overline{\Pi(O_\delta(t^*))} = \Pi(t^*). \tag{15}$$

From (14) and (15), one can get

$$\bar{z}(t^*) \in \Pi(t^*) \bigcap \mathscr{K}(t^*; 0, z^0).$$

Hence, the Problem (T) have at least one optimal solution. The proof is finished. □

5. Application

In this section, we apply our main results to existence results for Clarke's subdifferential inclusions and a class of differential hemivariational inequalities.

5.1. Clarke's Subdifferential Systems

Let us recall the definition of the Clarke's subdifferential for a locally Lipschitz function $j : K \subset L^2(\Omega) \to \mathbb{R}$, where K is a nonempty subset of a Banach space $L^2(\Omega)$ (one can see [48–50]). We denote by $j^0(x; y)$ the Clarke's generalized directional derivative of j at the point $x \in K$ in the direction $y \in L^2(\Omega)$, that is

$$j^0(x; y) := \limsup_{\lambda \to 0^+,\, \zeta \to x} \frac{j(\zeta + \lambda y) - j(\zeta)}{\lambda}.$$

Recall also that the Clarke's subdifferential or generalized gradient of j at $x \in K$, denoted by $\partial j(x)$, is a subset of $L^2(\Omega)^*$ given by

$$\partial j(x) := \{x^* \in X^* : j^0(x; y) \geq \langle x^*, y \rangle,\ \forall y \in L^2(\Omega)\}.$$

Lemma 3 ([50], Proposition 3.23)**.** *If $j : K \to \mathbb{R}$ is locally Lipschitz function, then*
(i) the function $(x, y) \mapsto j^0(x; y)$ is u.s.c. from $K \times L^2(\Omega)$ into $\mathbb{R}$;
(ii) for every $x \in K$ the gradient $\partial j(x)$ is a nonempty, convex and weakly compact subset of $L^2(\Omega)^*$ which is bounded by the Lipschitz constant $L_x > 0$ of j near x;*
(iii) the graph of ∂j is closed in $L^2(\Omega) \times L^2(\Omega)^*_{w^*}$;
(iv) the multifunction ∂j is u.s.c. from K into $L^2(\Omega)^*_{w^*}$.

Consider the following Clarke's subdifferential inclusion:

$$\begin{cases} z'_t - \mu\Delta z' - \gamma^2 \Delta z' + (z \cdot \nabla)z + \nabla p = f + u, & t \in (0, T], \\ u(t) \in \partial j(t, \gamma z(t, y)), \quad \text{a.e. } t \in [0, T], \\ \Delta \cdot z = 0, \quad \text{in } Q, \\ z(t, y) = 0, \quad \text{on } (0, T) \times \partial Q, \\ z(0, y) = z_0(y), \quad \text{in } Q, \end{cases} \tag{16}$$

where $j : [0,T] \times L^2(\Omega) \to \mathbb{R}$ is a locally Lipschitz function with respect to the second variable with Y being a separable reflexive Banach space, $\partial j(t,\cdot)$ denotes the Clarke's subdifferential of $j(t,\cdot)$ for $t \in [0,T]$ and $\gamma : L^2(\Omega) \to L^2(\Omega)$ is a linear, continuous and compact operator.

We need the following hypothesis.

(H_j) $j : [0,T] \times Y \to \mathbb{R}$ is continuous on $[0,T]$ and locally Lipschitz continuous on Y, and there exist a function $\phi_5 \in L^2([0,T];\mathbb{R}_+)$ and constants $L_5 > 0$ such that

$$\|\partial j(t,y)\| \leq \phi_5(t) + L_5\|y\|_{L^2(\Omega)}$$

for all $y \in Y$, a.e. $t \in [0,T]$.

We have the following result.

Theorem 5. *If (H_j) hold, then the system (16) has a solution.*

Proof. Thanks to the properties of ∂j in Lemma 3 and the compactness of γ, we infer that the multifunction $\Phi : [0,T] \times L^2(\Omega) \to 2^{L^2(\Omega)^*}$, defined by $\Phi(t,y) = \partial j(t,\gamma y)$ for $t \in [0,T], y \in L^2(\Omega)$, satisfies the condition $H(\Phi)$. The result of this theorem is a consequence of Theorem 2. □

5.2. Time Optimal Control for Differential Hemivariational Inequalities

In this section, we apply our previous results to the following differential variational inequalities:

$$\begin{cases} x_t' - \mu\Delta x' - \gamma^2\Delta x' + (x\cdot\nabla)x + \nabla p = f + \eta, & \text{in } Q, \\ \eta(t,y) \in SOL(K,\rho(t,x(t),\cdot),B,\phi,J), & \text{in } Q, \\ \Delta\cdot x = 0, & \text{in } Q, \\ x(t,y) = 0, & \text{on } (0,T)\times\partial Q, \\ x(0,y) = x_0(y), & \text{in } Q, \end{cases} \tag{17}$$

where $SOL(K,\rho(t,x(t),\cdot),\phi,J)$ denotes the solution set of the following mixed variational inequality in V: find $\eta : [0,T] \to K \subset L^2(\Omega)$ and $\eta^* \in \rho(t,x(t),\eta(t))$ such that $\langle \eta^* - B(\eta(t)), v - \eta(t)\rangle_Q + \phi(v) - \phi(\eta(t)) + J^0(x(t),\eta(t);v-\eta(t)) \geq 0, \forall v \in K,\ t \in [0,T]$. The notation of $J^0(x(t),\cdot;\cdot)$ means the generalized directional derivative of the function $J(x(t),\cdot)$. Then, we obtain

Lemma 4 ([25])**.** *Let K be a nonempty compact and convex subset of $(L^2(\Omega))^3$. Assume that:*

- **(i)** *$\phi : (L^2(\Omega))^3 \to \mathbb{R}^3$ is a proper convex l.s.c. functional, $B : (L^2(\Omega))^3 \to (L^2(\Omega))^3$ is a linear continuous operator, $g : [0,T] \times (L^2(\Omega))^3 \times K \to P((L^2(\Omega))^3)$ is a l.s.c. set-valued mapping, and $J : (L^2(\Omega))^3 \to \mathbb{R}^3$ is a locally Lipschitz function such that $g(t,x,\cdot) - B + \partial J(\cdot) : K \to P((L^2(\Omega))^3)$ is monotone for all $(t,x) \in [0,T] \times (L^2(\Omega))^3$.*
- **(ii)** *For each $(t,x) \in [0,T] \times (L^2(\Omega))^3$ there holds $\Phi(t,x,\varepsilon) \neq 0$ for all $\varepsilon > 0$, and one has $\lim_{\varepsilon\to 0} \chi_{(L^2(\Omega))^3}(\Phi(t,x,\varepsilon)) = 0$, where $\Phi(t,x,\varepsilon) := \{\eta \in K :$ there exists $\eta^* \in \rho(t,x,\cdot)$ such that $\langle \eta^* - B(\eta), v - \eta\rangle + \phi(v) - \phi(\eta(t)) + J^0(x(t),\eta(t);v-\eta(t)) \geq -\varepsilon\|v-\eta\|_{(L^2(\Omega))^3}$ for all $v \in K\}$, $\chi_Q(\Phi)$ means the Hausdorff measure of noncompactness.*
- **(iii)** *There is a continuous function $\kappa : [0,t] \times \mathbb{R}^3 \to \mathbb{R}_+$ with $\eta(0,0) = 0$ such that for all $t_1,t_2 \in [0,T], x_1,x_2 \in (L^2(\Omega))^3, \eta \in K$ and $\eta_1^* \in \rho(t_1,x_1,\eta)$ we can find $\eta_2^* \in g(t_2,x_2,\eta)$ such that*

$$\|\eta_1^* - \eta_2^*\|_{(L^2(\Omega))^3} \leq \kappa(|t_1-t_2|, \|x_1-x_2\|_{(L^2(\Omega))^3}).$$

Then $\Psi : [0,T] \times (L^2(\Omega))^3 \to P(K)$ *defined by*

$$\begin{aligned}\Psi(t,x) : &= \{\eta \in K : \langle \rho(t,x,\eta) - B(\eta(t)), v - \eta(t)\rangle_Q + \phi(v) - \phi(\eta(t)) + J^0(x(t),\eta(t); v - \eta(t) \\ &\geq 0, \forall vs. \in K\}\end{aligned}$$

is u.s.c. with compact values.

By virtue of this lemma, we can easily know that the feedback multimap $\Psi : [0,T] \times Q \to P(Q)$ is pseudo-continuous. If we assume that Ψ satisfies

(i) for all $(t,x) \in [0,T] \times (L^2(\Omega))^3$, there are a function $\theta \in L^2(J,\mathbb{R}^+)$ and a positive constant L_w, such that

$$\|\Psi(t,x)\| \leq \theta(t) + L_w\|x\|_{(L^2(\Omega))^3};$$

(ii) for a.e. $t \in [0,T], x \in (L^2(\Omega))^3$, the set $\Psi(t,x)$ satisfies

$$\bigcap_{\delta>0} \overline{\text{co}}\Psi(O_\delta(t,x)) = \Psi(t,x).$$

Our main results can then be applied to problem (17).

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