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Improved Oscillation Theorems for Even-Order Quasi-Linear Neutral Differential Equations

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Abstract: In this study, our goal was to establish improved inequalities that enhance the asymptotic and oscillatory behaviors of solutions to even-order neutral differential equations. In the oscillation theory of neutral differential equations, the connection between the solution and its corresponding function plays a critical role. We refined these relationships by leveraging the modified monotonic properties of positive solutions and introduced new conditions that ensure the absence of positive solutions, confirming the oscillation of all solutions to the studied equation. Based on the concept of symmetry between the positive and negative solutions of the studied equation, we obtained criteria that guarantee the oscillation of all solutions by excluding positive solutions only. In order to demonstrate the significance of our findings, we examined certain instances of the studied equation and compared them with previous results in the literature.

Keywords: neutral differential equations; oscillatory properties; even-order differential equation

MSC: 34C10; 34K11

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1. Introduction

The objective of this research paper was to examine the oscillatory characteristics of solutions to an even-order quasi-linear neutral differential equation expressed as follows:

$$\left(a(s)\left(H^{(n-1)}(s)\right)^{\alpha}\right)' + q(s)h^{\alpha}(\sigma(s)) = 0,\ s \geq s_0, \tag{1}$$

where $H(s) = h(s) + \phi(s)h(\varrho(s))$. We assume throughout this paper that:

(H$_1$) $n \geq 4$, α is the ratio of two positive odd integers;
(H$_2$) $a, \varrho, \sigma \in C^1([s_0, \infty))$, and $q(s) \in C([s_0, \infty))$;
(H$_3$) $\varrho(s) \leq s, \sigma(s) \leq s, \sigma'(s) > 0$, and $\lim_{s\to\infty} \varrho(s) = \lim_{s\to\infty} \sigma(s) = \infty$;
(H$_4$) $a(s) > 0, a'(s) \geq 0, 0 \leq \phi(s) < \phi_0$ and $q(s) \geq 0$;
(H$_5$) $\pi_0(s_0) < \infty$, where

$$\pi_0(s) := \int_s^\infty \frac{1}{a^{1/\alpha}(v)} \mathrm{d}v,$$

and

$$\pi_i(s) := \int_s^\infty \pi_{i-1}(v) \mathrm{d}v,\ i = 1, 2, \ldots, n-2.$$

A function $h \in C^{n-1}([S_h, \infty))$, $S_h \geqslant s_0$, is said to be a solution of (1), which has the property $a(H^{(n-1)})^\alpha \in C^1[S_h, \infty)$ and satisfies Equation (1) for all $h \in [S_h, \infty)$. We consider only those solutions h of (1) that exist on some half-line $[S_h, \infty)$ and satisfy the condition

$$\sup\{|h(s)| : s \geqslant S\} > 0, \text{ for all } S \geq S_h.$$

Differential equations play a crucial role in solving real-world problems across many fields, including physics, engineering, biology, economics, and more. These equations help to model complex systems by describing how variables change over time based on their current values and rates of change. Through the use of mathematical tools and techniques, differential equations can be solved to provide insights into the behavior of the system being modeled and to make predictions about its future behavior. Applications of differential equations include modeling the spread of diseases, predicting weather patterns, analyzing the behavior of electrical circuits, designing control systems, and many more. In general, differential equations provide a powerful and versatile framework for understanding and solving real-world problems; see [1–3].

Neutral differential equations are an important type of differential equation that arises in many areas of science and engineering. They include a time delay in both the derivatives and the function itself and can be linear or nonlinear. Neutral differential equations have applications in control theory, neuroscience, chemical kinetics, population dynamics, and electrical engineering. They are used to model systems that have delayed feedback, such as control systems, neural networks, chemical reactions, populations, and electronic circuits; see [4–7].

The oscillation theory is one of many theories that fall under the qualitative theory. The qualitative theory is the theory concerned with studying the qualitative behavior of solutions to differential inequalities such as stability, periodicity, symmetry, oscillation, and others. The principle of symmetry between positive and negative solutions, which means that every negative value of a positive solution is also a solution and vice versa, is the main reason why the study focuses on excluding positive solutions only.

In general, neutral differential equations can have oscillatory solutions depending on the specific parameters and initial conditions of the equation. However, conditions for oscillatory behavior in neutral differential equations can be more complicated than in regular differential equations due to the presence of delayed and advanced terms.

In recent times, the field of oscillation theory has witnessed significant growth and advancement. It now encompasses the examination of oscillation for solutions of various types of differential equations, including ordinary, fractional, and partial differential equations with delay and neutral terms. Of these, the study of delay differential equations, particularly in noncanonical cases, has garnered the most attention, as evidenced by works such as [8–10] for delay differential equations and [11–15] for neutral differential equations. Moaaz et al. [16,17] contributed to this expansion by extending the analysis to even-order equations.

Numerous studies have delved into the topic of even-order NDE oscillation and proposed various techniques for determining oscillation standards for the analyzed equations. This has been extensively researched in the canonical case; that is,

$$\int_{s_0}^{\infty} \frac{1}{a^{1/\alpha}(v)} \mathrm{d}v = \infty, \tag{2}$$

see [18–21].

Below, we will highlight some of the findings from previous years papers that have played a critical role in the advancement of research on even-order differential equations.

Baculíková [9] investigated the monotonic characteristics of non-oscillatory solutions for the linear equation

$$\left(a(s)h'(s)\right)' + q(s)h(\varrho(s)) = 0,$$

in both delay and advanced cases. Additionally, Baculíková [22] enhanced the findings by providing criteria for oscillation in the NDE

$$\left(a(s)\left((h(s)+\phi(s)h(\varrho(s)))'\right)^{\alpha}\right)'+q(s)h^{\alpha}(\sigma(s))=0.$$

Muhib et al. [23] investigated the asymptotic properties of positive solutions to the fourth-order neutral differential equation

$$\left(a(s)(h(s)+\phi(s)h(\varrho(s)))'''\right)'+f(s,h(\sigma(s)))=0,$$

which involves the noncanonical operator given by

$$\int_{s_0}^{\infty}\frac{1}{a^{1/\alpha}(v)}\mathrm{d}v<\infty. \tag{3}$$

In [24], Almarri et al. established asymptotic properties of positive solutions to the even-order neutral differential equation

$$\left(a(s)(h(s)+\phi(s)h(\varrho(s)))^{(n-1)}\right)'+q(s)h(\sigma(s))=0,$$

under the condition (3).

Xing et al. [25] investigated oscillation theorems for the equation

$$\left(a(s)\left((h(s)+\phi(s)h(\varrho(s)))^{(n-1)}\right)^{\alpha}\right)'+q(s)h^{\alpha}(\sigma(s))=0,$$

under the condition (2).

The initial step of our investigation involved the classification of positive solutions to the studied equation according to the signs of their derivatives. Then, for some positive solutions, we obtained additional monotonic characteristics. We improved the relationship between the solution and the associated function of the studied equation based on these properties. We also utilized these new relationships to rule out the possibility of positive solutions. We also present an example to demonstrate the importance of our results.

2. Auxiliary Results

In this section, we will establish some important lemmas that we will use to prove the main results.

The study of the asymptotic and oscillatory behavior of solutions of neutral-type differential equations heavily relies on the connection between the solution h and its corresponding function H. Typically, the canonical case of second-order equations uses the traditional relationship

$$h(s)>(1-\phi(s))H(s), \tag{4}$$

whereas positive decreasing solutions in the non-canonical case often use the relationship

$$h(s)>\left(1-\phi(s)\frac{\pi_0(\varrho(s))}{\pi_0(s)}\right)H(s), \tag{5}$$

see [26,27].

Lemma 1 ([28])**.** *Let $f\in C^n([s_0,\infty),\mathbb{R}^+)$. If $f^{(n)}(s)$ is eventually of one sign for all large s, then there exist a $s_h\geq s_0$ and an integer l, $0\leq l\leq n$, with $n+l$ even for $f^{(n)}(s)\geq 0$, or $n+l$ odd for $f^{(n)}(s)\leq 0$ such that*

$$l>0 \text{ yields } f^{(k)}(s)>0 \text{ for } s\geq s_h,\ k=0,1,\dots,l-1,$$

and

$$l \le n-1 \text{ yields } (-1)^{l+k} f^{(k)}(s) > 0 \text{ for } s \ge s_h,\ k = l, l+1, \ldots, n-1.$$

Lemma 2 ([29]). *Let $w \in C^m([s_0,\infty),(0,\infty))$, $w^{(i)}(s) > 0$ for $i = 1,2,\ldots,m$, and $w^{(m+1)}(s) \le 0$, eventually. Then,*

$$\frac{w(s)}{w'(s)} \ge \frac{\epsilon}{m} s,$$

for every $\epsilon \in (0,1)$.

Lemma 3 ([30], Lemma 2.2.3). *Suppose that $f \in C^m([s_0,\infty),\mathbb{R}^+)$. Assume that $f^{(m)}(s)$ is of fixed sign and not identically zero on $[s_0,\infty)$ and that there exists $s_1 \ge s_0$ such that $f^{(m-1)}(s) f^{(m)}(s) \le 0$ for all $s_1 \ge s_0$. If $\lim_{s\to\infty} f(s) \ne 0$, then, for every $\epsilon \in (0,1)$, there exists $s_\delta \in [s_1,\infty)$ such that the inequality*

$$f(s) \ge \frac{\epsilon}{(m-1)!} s^{m-1} \left| f^{(m-1)}(s) \right|,$$

holds for all $s \in [s_\delta,\infty)$.

Lemma 4 ([31]). *Suppose that $h(s)$ is a positive solution to Equation (1). Then, $a(s)\left(H^{(n-1)}(s)\right)^{\alpha}$ is a decreasing function, and $H(s)$ satisfies one of the following cases:*

$$(\mathrm{N}_1)\ H^{(r)}(s) > 0 \text{ for } r = 0,1,n-1 \text{ and } H^{(n)}(s) < 0;$$
$$(\mathrm{N}_2)\ H^{(r)}(s) > 0 \text{ for } r = 0,1,n-2 \text{ and } H^{(n-1)}(s) < 0;$$
$$(\mathrm{N}_3)\ (-1)^r H^{(r)}(s) > 0 \text{ for } r = 0,1,2,\ldots,n-1,$$

eventually.

Proof. Using Equation (1) and Lemma 1 leads to the proof of this lemma. □

Notation 1. *For more details on determining the sign of derivatives—for example, in the case where $n = 4$—see [32].*

Notation 2. *The symbol Ω_i refers to the set of all solutions that are eventually positive and whose corresponding function satisfies (N_i) for $i = 1,2,3$. For convenience, we define*

$$F^{[0]}(s) = F(s) \text{ and } F^{[j]}(s) = F\left(F^{[j-1]}(s)\right), \text{ for } j = 1,2,\ldots,\kappa. \tag{6}$$

Notation 3. *In order to simplify, we define the functions for any positive integer κ*

$$\phi_1(s;\kappa) = \sum_{r=0}^{\kappa} \left(\prod_{\gamma=0}^{2r} \phi\left(\varrho^{[\gamma]}(s)\right) \right) \left[\frac{1}{\phi\left(\varrho^{[2r]}(s)\right)} - 1 \right] \left(\frac{\varrho^{[2r]}(s)}{s} \right)^{(n-2)/\epsilon}, \tag{7}$$

$$\phi_2(s;\kappa) = \sum_{r=0}^{\kappa} \left(\prod_{\gamma=0}^{2r} \phi\left(\varrho^{[\gamma]}(s)\right) \right) \left[\frac{1}{\phi\left(\varrho^{[2r]}(s)\right)} - \frac{\pi_{n-2}\left(\varrho^{[2r+1]}(s)\right)}{\pi_{n-2}\left(\varrho^{[2r]}(s)\right)} \right], \tag{8}$$

and

$$\widehat{\phi}_2(s;\kappa) = \sum_{r=0}^{\kappa} \left(\prod_{\gamma=0}^{2r} \phi\left(\varrho^{[\gamma]}(s)\right) \right) \left[\frac{1}{\phi\left(\varrho^{[2r]}(s)\right)} - \frac{\pi_{n-2}\left(\varrho^{[2r+1]}(s)\right)}{\pi_{n-2}\left(\varrho^{[2r]}(s)\right)} \right] \frac{\pi_{n-2}^{k_m}\left(\varrho^{[2r]}(s)\right)}{\pi_{n-2}^{k_m}(s)}. \tag{9}$$

Lemma 5 ([33], Lemma 1)**.** *Assume that h is an eventually positive solution of (1). Then, it follows that, eventually,*

$$h(s) > \sum_{r=0}^{\kappa} \left(\prod_{\gamma=0}^{2r} \phi\left(\varrho^{[\gamma]}(s)\right) \right) \left[\frac{H\left(\varrho^{[2r]}(s)\right)}{\phi\left(\varrho^{[2r]}(s)\right)} - H\left(\varrho^{[2r+1]}(s)\right) \right], \tag{10}$$

for any integer $\kappa \geq 0$.

In the following section, we highlight the improved asymptotic and monotonic properties of the positive solutions for the studied equation. Additionally, we establish certain conditions that guarantee the absence of positive solutions satisfying (N_1), (N_2), and (N_3) within Category Ω_2, Ω_2, and Ω_3, respectively.

3. Asymptotic and Monotonic Properties

This section presents the improved asymptotic and monotonous properties of the positive solutions of the studied equation. It is divided into three subsections, which are as follows:

3.1. Category Ω_2

Lemma 6. *Assume that $h \in \Omega_2$. Then, eventually,*

$(\mathrm{S}_{1,1})$ $H(s) \geq \frac{\epsilon}{n-2} s H'(s)$;

$(\mathrm{S}_{1,2})$ $H(s) \geq \frac{\epsilon}{(n-2)!} s^{n-2} H^{(n-2)}(s)$ *for all* $\epsilon \in (0,1)$;

$(\mathrm{S}_{1,3})$ $H^{(n-2)}(s) \geq -a^{1/\alpha}(s)\pi_0(s)H^{(n-1)}(s)$;

$(\mathrm{S}_{1,4})$ $H^{(n-2)}(s)/\pi_0(s)$ *is increasing;*

$(\mathrm{S}_{1,5})$ $h(s) \geq \phi_1(s;\kappa)H(s)$;

$(\mathrm{S}_{1,6})$ $\left(a(s)\left(H^{(n-1)}(s)\right)^{\alpha}\right)' \leq -q(s)\phi_1^{\alpha}(\sigma(s);\kappa)H^{\alpha}(\sigma(s))$.

Proof. Assume that $h \in \Omega_2$.

$(\mathrm{S}_{1,1})$ Using Lemma 2 with $m = n-2$ and $w = H$, we have

$$H(s) \geq \frac{\epsilon}{n-2} s H'(s).$$

$(\mathrm{S}_{1,2})$ Using Lemma 3 with $m = n-1$ and $f = H$, we have

$$H(s) \geq \frac{\epsilon_0}{(n-2)!} s^{n-2} H^{(n-2)}(s),$$

for all $\epsilon_0 \in (0,1)$.

$(\mathrm{S}_{1,3})$ Since $a^{1/\alpha}(s)H^{(n-1)}(s)$ is decreasing, we obtain

$$H^{(n-2)}(s) \geq -\int_s^{\infty} H^{(n-1)}(v)\mathrm{d}v \geq -a^{1/\alpha}(s)\pi_0(s)H^{(n-1)}(s).$$

$(\mathrm{S}_{1,4})$ From $(\mathrm{S}_{1,3})$, we obtain

$$\left(\frac{H^{(n-2)}(s)}{\pi_0(s)}\right)' = \frac{1}{a^{1/\alpha}(s)\pi_0^2(s)} \left(a^{1/\alpha}(s)\pi_0(s)H^{(n-1)}(s) + H^{(n-2)}(s)\right) \geq 0.$$

$(S_{1,5})$ From Lemma 5, (10) holds. Based on the properties of solutions in the class Ω_2, we conclude that $H\left(\varrho^{[2r]}(s)\right) \geq H\left(\varrho^{[2r+1]}(s)\right)$ for $i = 1, 2, \ldots$. Thus, (10) becomes

$$h(s) > \sum_{r=0}^{\kappa}\left(\prod_{\gamma=0}^{2r}\phi\left(\varrho^{[\gamma]}(s)\right)\right)\left[\frac{1}{\phi\left(\varrho^{[2r]}(s)\right)} - 1\right]H\left(\varrho^{[2r]}(s)\right).$$

Using $(S_{1,1})$, we obtain

$$H\left(\varrho^{[2r]}(s)\right) \geq \left(\frac{\varrho^{[2r]}(s)}{s}\right)^{(n-2)/\epsilon} H(s),$$

which, with (11), gives

$$\begin{aligned} h(s) &> \sum_{r=0}^{\kappa}\left(\prod_{\gamma=0}^{2r}\phi\left(\varrho^{[\gamma]}(s)\right)\right)\left[\frac{1}{\phi\left(\varrho^{[2r]}(s)\right)} - 1\right]\left(\frac{\varrho^{[2r]}(s)}{s}\right)^{(n-2)/\epsilon} H(s) \\ &= \phi_1(s;\kappa)H(s). \end{aligned}$$

$(S_{1,6})$ Equation (1) with $(S_{1,5})$ becomes

$$\begin{aligned} \left(a(s)\left(H^{(n-1)}(s)\right)^{\alpha}\right)' &= -q(s)h^{\alpha}(\sigma(s)) \\ &\leq -q(s)\phi_1^{\alpha}(\sigma(s);\kappa)H^{\alpha}(\sigma(s)). \end{aligned}$$

Therefore, the proof of the Lemma is complete. □

Remark 1. *The verification of $\phi_1(s;0) = 1 - \phi(s)$ is straightforward. Substituting $\kappa = 0$ into $(S_{1,5})$ yields the classical relation (4).*

Lemma 7. *Assume that $h \in \Omega_2$ and that there are $\delta > 0$ and $s_1 \geq s_0$ such that*

$$\frac{1}{\alpha}a^{1/\alpha}(s)\pi_0^{1+\alpha}(s)\left(\sigma^{n-2}(s)\right)^{\alpha}q(s)\phi_1^{\alpha}(\sigma(s);\kappa) \geq ((n-2)!)^{\alpha}\delta, \tag{11}$$

We obtain, for $s \geq s_1$,

$(S_{2,1})$ $\lim_{s\to\infty} H^{(n-2)}(s) = 0$;

$(S_{2,2})$ $H^{(n-2)}(s)/\pi_0^{\beta_0}(s)$ *is decreasing;*

$(S_{2,3})$ $\lim_{s\to\infty} H^{(n-2)}(s)/\pi_0^{\beta_0}(s) = 0$;

$(S_{2,4})$ $H^{(n-2)}(s)/\pi_0^{1-\beta_0}(s)$ *is increasing;*

for $s \geq s_0$, where $\beta_0 = \epsilon\delta^{1/\alpha}$, $\epsilon \in (0,1)$ and $\alpha \leq 1$.

Proof. Assume that $h \in \Omega_2$ and that there are $\delta > 0$ and $s_1 \geq s_0$ such that (11) holds.

$(S_{2,1})$ Given that $h \in \Omega_2$, we can conclude that $(S_{1,1})$–$(S_{1,6})$ in Lemma 6 hold for all $s \geq s_1$, where s_1 is sufficiently large. Since $H^{(n-2)}(s)$ is a positive decreasing function, it follows that $\lim_{s\to\infty} H^{(n-2)}(s) = \ell_1 \geq 0$. We claim that $\ell_1 = 0$. If we suppose not, then $H^{(n-2)}(s) \geq \ell_1 > 0$ eventually, which, together with $(S_{1,2})$, yields

$$\begin{aligned} H(s) &\geq \frac{\epsilon}{(n-2)!}s^{n-2}H^{(n-2)}(s) \\ &\geq \frac{\epsilon\ell_1}{(n-2)!}s^{n-2}, \end{aligned}$$

for all $\epsilon \in (0,1)$. Thus, from $(S_{1,6})$, we obtain

$$\begin{aligned}\left(a(s)\left(H^{(n-1)}(s)\right)^{\alpha}\right)' &\leq -q(s)\phi_1^{\alpha}(\sigma(s);\kappa)H^{\alpha}(\sigma(s)) \\ &\leq -\left(\frac{\epsilon\ell_1}{(n-2)!}\sigma^{n-2}(s)\right)^{\alpha} q(s)\phi_1^{\alpha}(\sigma(s);\kappa) \\ &\leq -\epsilon^{\alpha}\ell_1^{\alpha}\frac{(\sigma^{n-2}(s))^{\alpha}}{((n-2)!)^{\alpha}}q(s)\phi_1^{\alpha}(\sigma(s);\kappa),\end{aligned}$$

which, with (11), gives

$$\begin{aligned}\left(a(s)\left(H^{(n-1)}(s)\right)^{\alpha}\right)' &\leq -\alpha\ell_1^{\alpha}\epsilon^{\alpha}\delta\frac{1}{a^{1/\alpha}(s)\pi_0^{1+\alpha}(s)} \\ &\leq -\alpha\ell_1^{\alpha}\beta_0^{\alpha}\frac{1}{a^{1/\alpha}(s)\pi_0^{1+\alpha}(s)}.\end{aligned}$$

Integrating the previous inequality from s_2 to s, we have

$$\begin{aligned}a(s)\left(H^{(n-1)}(s)\right)^{\alpha} &\leq a(s_2)\left(H^{(n-1)}(s_2)\right)^{\alpha} - \alpha\ell_1^{\alpha}\beta_0^{\alpha}\int_{s_2}^{s}\frac{1}{a^{1/\alpha}(v)\pi_0^{1+\alpha}(v)}\mathrm{d}v \\ &\leq \beta_0^{\alpha}\ell_1^{\alpha}\left(\frac{1}{\pi_0^{\alpha}(s_2)} - \frac{1}{\pi_0^{\alpha}(s)}\right).\end{aligned} \tag{12}$$

Since $\pi_0^{-\alpha}(s) \to \infty$ as $s \to \infty$, there is a $s_3 \geq s_2$ such that $\pi_0^{-\alpha}(s) - \pi_0^{-\alpha}(s_2) \geq \mu_0\pi_0^{-\alpha}(s)$ for all $\mu_0 \in (0,1)$. Hence, (12) becomes

$$H^{(n-1)}(s) \leq -\ell_1\mu_0^{1/\alpha}\beta_0\frac{1}{a^{1/\alpha}(s)\pi_0(s)},$$

for all $s \geq s_3$. Integrating the last inequality from s_3 to s, we find that

$$\begin{aligned}H^{(n-2)}(s) &\leq H^{(n-2)}(s_3) - \ell_1\mu_0^{1/\alpha}\beta_0\int_{s_3}^{s}\frac{1}{a^{1/\alpha}(v)\pi_0(v)}\mathrm{d}v \\ &\leq H^{(n-2)}(s_3) - \ell_1\mu_0^{1/\alpha}\beta_0\ln\frac{\pi_0(s_3)}{\pi_0(s)} \to -\infty \text{ as } s \to \infty,\end{aligned}$$

which is a contradiction. Then, $\ell_1 = 0$.

$(\mathrm{S}_{2,2})$ From (11), $(\mathrm{S}_{1,2})$, and $(\mathrm{S}_{1,6})$, we obtain

$$\left(a(s)\left(H^{(n-1)}(s)\right)^{\alpha}\right)' \leq -\frac{\alpha\beta_0^{\alpha}}{a^{1/\alpha}(s)\pi_0^{1+\alpha}(s)}\left(H^{(n-2)}(\sigma(s))\right)^{\alpha}.$$

By integrating the last inequality from s_1 to s and taking into account that $H^{(n-1)}(s) < 0$, we obtain

$$a(s)\left(H^{(n-1)}(s)\right)^{\alpha} \leq a(s_1)\left(H^{(n-1)}(s_1)\right)^{\alpha} + \frac{\beta_0^{\alpha}}{\pi_0^{\alpha}(s_1)}\left(H^{(n-2)}(s)\right)^{\alpha} - \frac{\beta_0^{\alpha}}{\pi_0^{\alpha}(s)}\left(H^{(n-2)}(s)\right)^{\alpha}.$$

Because $H^{(n-2)}(s) \to 0$ as $s \to \infty$, there is a $s_2 \geq s_1$ such that

$$a(s_1)\left(H^{(n-1)}(s_1)\right)^{\alpha} + \frac{\beta_0^{\alpha}}{\pi_0^{\alpha}(s_1)}\left(H^{(n-2)}(s)\right)^{\alpha} \leq 0,$$

for $s \geq s_2$. Therefore, we have

$$a(s)\left(H^{(n-1)}(s)\right)^{\alpha} \leq -\frac{\beta_0^{\alpha}}{\pi_0^{\alpha}(s)}\left(H^{(n-2)}(s)\right)^{\alpha},$$

or equivalent

$$a^{1/\alpha}(s)H^{(n-1)}(s)\pi_0(s)+\beta_0 H^{(n-2)}(s)\leq 0. \tag{13}$$

Thus,

$$\left(\frac{H^{(n-2)}(s)}{\pi_0^{\beta_0}(s)}\right)'=\frac{a^{1/\alpha}(s)H^{(n-1)}(s)\pi_0(s)+\beta_0 H^{(n-2)}(s)}{a^{1/\alpha}(s)\pi_0^{1+\beta_0}(s)}\leq 0.$$

$(S_{2,3})$ Since $H^{(n-2)}(s)/\pi_0^{\beta_0}(s)$ is a positive decreasing function, $\lim_{s\to\infty} H^{(n-2)}(s)/\pi_0^{\beta_0}(s)=\ell_2\geq 0$. We claim that $\ell_2=0$. If not, then $H^{(n-2)}(s)/\pi_0^{\beta_0}(s)\geq \ell_2>0$ eventually. Now, we introduce the function

$$w(s)=\frac{H^{(n-2)}(s)+\pi_0(s)a^{1/\alpha}(s)H^{(n-1)}(s)}{\pi_0^{\beta_0}(s)}.$$

In view of $(S_{1,3})$, we observe that $w(s)>0$ and

$$\begin{aligned} w'(s) &= \frac{H^{(n-1)}(s)+\pi_0(s)\left(a^{1/\alpha}(s)H^{(n-1)}(s)\right)'-H^{(n-1)}(s)}{\pi_0^{\beta_0}(s)} \\ &\quad +\beta_0\frac{H^{(n-2)}(s)+\pi_0(s)a^{1/\alpha}(s)H^{(n-1)}(s)}{a^{1/\alpha}(s)\pi_0^{1+\beta_0}(s)} \\ &= \frac{\left(a^{1/\alpha}(s)H^{(n-1)}(s)\right)'}{\pi_0^{\beta_0-1}(s)}+\beta_0\frac{H^{(n-2)}(s)}{a^{1/\alpha}(s)\pi_0^{1+\beta_0}(s)}+\beta_0\frac{H^{(n-1)}(s)}{\pi_0^{\beta_0}(s)} \\ &= \frac{1}{\alpha}\frac{\left(a(s)\left(H^{(n-1)}(s)\right)^{\alpha}\right)'\left(a^{1/\alpha}(s)H^{(n-1)}(s)\right)^{1-\alpha}}{\pi_0^{\beta_0-1}(s)} \\ &\quad +\beta_0\frac{H^{(n-2)}(s)}{a^{1/\alpha}(s)\pi_0^{1+\beta_0}(s)}+\beta_0\frac{H^{(n-1)}(s)}{\pi_0^{\beta_0}(s)}. \end{aligned}$$

Using $(S_{1,2})$, $(S_{1,6})$, and (11), we obtain

$$\begin{aligned} \left(a(s)\left(H^{(n-1)}(s)\right)^{\alpha}\right)' &\leq -\left(\frac{\epsilon}{(n-2)!}\sigma^{n-2}(s)\right)^{\alpha}q(s)\phi_1^{\alpha}(\sigma(s);\kappa)\left(H^{(n-2)}(\sigma(s))\right)^{\alpha} \\ &\leq -\alpha\beta_0^{\alpha}\frac{1}{a^{1/\alpha}(s)\pi_0^{1+\alpha}(s)}\left(H^{(n-2)}(\sigma(s))\right)^{\alpha}. \end{aligned} \tag{14}$$

From (13), we know that

$$a^{1/\alpha}(s)H^{(n-1)}(s)\leq -\beta_0\frac{H^{(n-2)}(s)}{\pi_0(s)},$$

and

$$\left(a^{1/\alpha}(s)H^{(n-1)}(s)\right)^{1-\alpha}\geq\left(\beta_0\frac{H^{(n-2)}(s)}{\pi_0(s)}\right)^{1-\alpha}. \tag{15}$$

Using (14) and (15), we obtain

$$\begin{aligned} w'(s) &\leq -\frac{\beta_0^{\alpha}}{\pi_0^{\beta_0-1}(s)}\frac{1}{a^{1/\alpha}(s)\pi_0^{1+\alpha}(s)}\left(H^{(n-2)}(\sigma(s))\right)^{\alpha}\left(\beta_0\frac{H^{(n-2)}(s)}{\pi_0(s)}\right)^{1-\alpha} \\ &\quad +\beta_0\frac{H^{(n-2)}(s)}{a^{1/\alpha}(s)\pi_0^{1+\beta_0}(s)}+\beta_0\frac{H^{(n-1)}(s)}{\pi_0^{\beta_0}(s)}. \end{aligned}$$

Since $H^{(n-1)}(s) < 0$ and $\sigma(s) \leq s$, we obtain $H^{(n-2)}(\sigma(s)) \geq H^{(n-2)}(s)$, and then

$$\begin{aligned} w'(s) &\leq -\frac{\beta_0^{\alpha}}{\pi_0^{\beta_0-1}(s)} \frac{1}{a^{1/\alpha}(s)\pi_0^{1+\alpha}(s)} \left(H^{(n-2)}(s)\right)^{\alpha} \left(\beta_0 \frac{H^{(n-2)}(s)}{\pi_0(s)}\right)^{1-\alpha} \\ &\quad +\beta_0 \frac{H^{(n-2)}(s)}{a^{1/\alpha}(s)\pi_0^{1+\beta_0}(s)} + \beta_0 \frac{H^{(n-1)}(s)}{\pi_0^{\beta_0}(s)} \\ &\leq -\beta_0 \frac{H^{(n-2)}(s)}{a^{1/\alpha}(s)\pi_0^{1+\beta_0}(s)} + \beta_0 \frac{H^{(n-2)}(s)}{a^{1/\alpha}(s)\pi_0^{1+\beta_0}(s)} + \beta_0 \frac{H^{(n-1)}(s)}{\pi_0^{\beta_0}(s)} \\ &\leq \beta_0 \frac{H^{(n-1)}(s)}{\pi_0^{\beta_0}(s)}. \end{aligned}$$

Using the fact that $H^{(n-2)}(s)/\pi_0^{\beta_0}(s) \geq \ell_2$ and (13), we obtain

$$\begin{aligned} w'(s) &\leq \beta_0 \frac{H^{(n-1)}(s)}{\pi_0^{\beta_0}(s)} \leq \beta_0 \frac{1}{\pi_0^{\beta_0}(s)} \left(\frac{-\beta_0 H^{(n-2)}(s)}{a^{1/\alpha}(s)\pi_0(s)}\right) \\ &\leq -\frac{H^{(n-2)}(s)}{\pi_0^{\beta_0}(s)} \frac{\beta_0^2}{a^{1/\alpha}(s)\pi_0(s)} \leq \frac{-\ell_2\beta_0^2}{a^{1/\alpha}(s)\pi_0(s)} < 0. \end{aligned}$$

We can conclude that the function $w(s)$ converges to a non-negative constant since it is a positive decreasing function. By integrating the previous inequality from s_3 to ∞, we obtain

$$-w(s_3) \leq -\beta_0^2\ell_2 \lim_{s\to\infty} \ln \frac{\pi_0(s_3)}{\pi_0(s)},$$

or, equivalently,

$$w(s_3) \geq \beta_0^2 c_2 \lim_{s\to\infty} \ln \frac{\pi_0(s_3)}{\pi_0(s)} \to \infty,$$

which is a contradiction, and we obtain that $\ell_2 = 0$.

$(S_{2,4})$ Now, we have

$$\begin{aligned} &\left(a^{1/\alpha}(s)H^{(n-1)}(s)\pi_0(s) + H^{(n-2)}(s)\right)' \\ &= \left(a^{1/\alpha}(s)H^{(n-1)}(s)\right)'\pi_0(s) - H^{(n-1)}(s) + H^{(n-1)}(s) \\ &= \left(a^{1/\alpha}(s)H^{(n-1)}(s)\right)'\pi_0(s) \\ &= \frac{1}{\alpha}\left(a(s)\left(H^{(n-1)}(s)\right)^{\alpha}\right)'\left(a^{1/\alpha}(s)H^{(n-1)}(s)\right)^{1-\alpha}\pi_0(s), \end{aligned}$$

which, with (14) and (15), we obtain

$$\begin{aligned} &\left(a^{1/\alpha}(s)H^{(n-1)}(s)\pi_0(s) + H^{(n-2)}(s)\right)' \\ &\leq -\beta_0^{\alpha} \frac{1}{a^{1/\alpha}(s)\pi_0^{1+\alpha}(s)} \left(H^{(n-2)}(\sigma(s))\right)^{\alpha} \left(\beta_0 \frac{H^{(n-2)}(s)}{\pi_0(s)}\right)^{1-\alpha} \pi_0(s) \\ &\leq -\beta_0^{\alpha} \frac{1}{a^{1/\alpha}(s)\pi_0^{\alpha}(s)} \left(H^{(n-2)}(s)\right)^{\alpha} \left(\beta_0 \frac{H^{(n-2)}(s)}{\pi_0(s)}\right)^{1-\alpha} \\ &\leq \frac{-\beta_0}{a^{1/\alpha}(s)\pi_0(s)} H^{(n-2)}(s). \end{aligned}$$

By integrating the previous inequality from s to ∞, we derive

$$-a^{1/\alpha}(s)H^{(n-1)}(s)\pi_0(s)-H^{(n-2)}(s)\leq-\beta_0\int_s^\infty\frac{1}{a^{1/\alpha}(v)\pi_0(v)}H^{(n-2)}(v)\mathrm{d}v,$$

or, equivalently,

$$\begin{aligned}a^{1/\alpha}(s)H^{(n-1)}(s)\pi_0(s)+H^{(n-2)}(s) &\geq \beta_0\int_s^\infty\frac{1}{a^{1/\alpha}(v)\pi_0(v)}H^{(n-2)}(v)\mathrm{d}v\\ &\geq \beta_0\frac{H^{(n-2)}(s)}{\pi_0(s)}\int_s^\infty\frac{1}{a^{1/\alpha}(v)}\mathrm{d}v\geq\beta_0H^{(n-2)}(s);\end{aligned}$$

that is,

$$a^{1/\alpha}(s)H^{(n-1)}(s)\pi_0(s)+(1-\beta_0)H^{(n-2)}(s)\geq 0.$$

Thus,

$$\left(\frac{H^{(n-2)}(s)}{\pi_0^{1-\beta_0}(s)}\right)'=\frac{\pi_0(s)a^{1/\alpha}(s)H^{(n-1)}(s)+(1-\beta_0)H^{(n-2)}(s)}{a^{1/\alpha}(s)\pi_0^{2-\beta_0}(s)}\geq 0. \tag{16}$$

Therefore, the proof of the Lemma is complete. □

Assuming that $\beta_0\leq 1/2$, the properties stated in Lemma 7 can be further improved as demonstrated in the following lemma.

Lemma 8. *Suppose that $h\in\Omega_2$ and (11) holds. If condition*

$$\lim_{s\to\infty}\frac{\pi_0(\sigma(s))}{\pi_0(s)}=\lambda<\infty, \tag{17}$$

and there exists an increasing sequence $\{\beta_j\}_{j=1}^m$ defined as

$$\beta_j:=\beta_0\frac{\lambda^{\beta_{j-1}}}{(1-\beta_{j-1})^{1/\alpha}},$$

with $\alpha\leq 1$, $\beta_0=\epsilon\delta^{1/\alpha}$, $\beta_{m-1}\leq 1/2$, and $\beta_m,\epsilon\in(0,1)$. Then, eventually,

($S_{3,1}$) *$H^{(n-2)}(s)/\pi_0^{\beta_m}(s)$ is decreasing;*

($S_{3,2}$) $\lim_{s\to\infty}H^{(n-2)}(s)/\pi_0^{\beta m}(s)=0$.

Proof. Since $h\in\Omega_2$, we can conclude that ($S_{1,1}$)–($S_{1,5}$) in Lemma 6 are satisfied for all $s\geq s_1$, s_1 large enough. Furthermore, from Lemma 7, we have that ($S_{2,1}$)–($S_{2,4}$) hold.

Now, assume that $\beta_0\leq 1/2$ and

$$\beta_1:=\beta_0\frac{\lambda^{\beta_0}}{(1-\beta_0)^{1/\alpha}}.$$

Next, we will prove that ($S_{3,1}$) and ($S_{3,2}$) at $m=1$. As in the proof of Lemma 7, we arrive at

$$\left(a(s)\left(H^{(n-1)}(s)\right)^\alpha\right)'\leq-\alpha\beta_0^\alpha\frac{1}{a^{1/\alpha}(s)\pi_0^{1+\alpha}(s)}\left(H^{(n-2)}(\sigma(s))\right)^\alpha.$$

Integrating the last inequality from s_1 to s, and using ($S_{2,2}$) and (17), we obtain

$$
\begin{aligned}
& a(s)\left(H^{(n-1)}(s)\right)^{\alpha} \\
\leq\ & a(s_1)\left(H^{(n-1)}(s_1)\right)^{\alpha} - \alpha\beta_0^{\alpha}\int_{s_1}^{s}\frac{1}{a^{1/\alpha}(v)\pi_0^{1+\alpha}(v)}\left(H^{(n-2)}(\sigma(v))\right)^{\alpha}\mathrm{d}v \\
\leq\ & a(s_1)\left(H^{(n-1)}(s_1)\right)^{\alpha} - \alpha\beta_0^{\alpha}\int_{s_1}^{s}\frac{1}{a^{1/\alpha}(v)\pi_0^{1+\alpha}(v)}\pi_0^{\alpha\beta_0}(\sigma(v))\left(\frac{H^{(n-2)}(v)}{\pi_0^{\beta_0}(v)}\right)^{\alpha}\mathrm{d}v \\
\leq\ & a(s_1)\left(H^{(n-1)}(s_1)\right)^{\alpha} - \alpha\beta_0^{\alpha}\left(\frac{H^{(n-2)}(s)}{\pi_0^{\beta_0}(s)}\right)^{\alpha}\int_{s_1}^{s}\frac{\pi_0^{-1-\alpha+\alpha\beta_0}(v)}{a^{1/\alpha}(v)}\frac{\pi_0^{\alpha\beta_0}(\sigma(v))}{\pi_0^{\alpha\beta_0}(v)}\mathrm{d}v \\
\leq\ & a(s_1)\left(H^{(n-1)}(s_1)\right)^{\alpha} - \alpha\beta_0^{\alpha}\lambda^{\alpha\beta_0}\left(\frac{H^{(n-2)}(s)}{\pi_0^{\beta_0}(s)}\right)^{\alpha}\int_{s_1}^{s}\frac{\pi_0^{-1-\alpha+\alpha\beta_0}(v)}{a^{1/\alpha}(v)}\mathrm{d}v \\
\leq\ & a(s_1)\left(H^{(n-1)}(s_1)\right)^{\alpha} - \frac{\beta_0^{\alpha}\lambda^{\alpha\beta_0}}{1-\beta_0}\left(\frac{H^{(n-2)}(s)}{\pi_0^{\beta_0}(s)}\right)^{\alpha}\left(\frac{1}{\pi_0^{\alpha(1-\beta_0)}(s)} - \frac{1}{\pi_0^{\alpha(1-\beta_0)}(s_1)}\right) \\
\leq\ & a(s_1)\left(H^{(n-1)}(s_1)\right)^{\alpha} + \beta_1^{\alpha}\frac{1}{\pi_0^{\alpha(1-\beta_0)}(s_1)}\left(\frac{H^{(n-2)}(s)}{\pi_0^{\beta_0}(s)}\right)^{\alpha} - \beta_1^{\alpha}\left(\frac{H^{(n-2)}(s)}{\pi_0(s)}\right)^{\alpha}.
\end{aligned}
$$

Using the fact that $H^{(n-2)}(s)/\pi_0^{\beta_0}(s) \to 0$ as $s \to \infty$, we have that

$$a(s_1)\left(H^{(n-1)}(s_1)\right)^{\alpha} + \beta_1^{\alpha}\frac{1}{\pi_0^{\alpha(1-\beta_0)}(s_1)}\left(\frac{H^{(n-2)}(s)}{\pi_0^{\beta_0}(s)}\right)^{\alpha} \leq 0.$$

Therefore,

$$a(s)\left(H^{(n-1)}(s)\right)^{\alpha} \leq -\beta_1^{\alpha}\left(\frac{H^{(n-2)}(s)}{\pi_0(s)}\right)^{\alpha},$$

or, equivalently,

$$a^{1/\alpha}(s)H^{(n-1)}(s)\pi_0(s) + \beta_1 H^{(n-2)}(s) \leq 0;$$

then,

$$\left(\frac{H^{(n-2)}(s)}{\pi_0^{\beta_1}(s)}\right)' = \frac{\pi_0(s)a^{1/\alpha}(s)H^{(n-1)}(s) + \beta_1 H^{(n-2)}(s)}{a^{1/\alpha}(s)\pi_0^{1+\beta_1}(s)} \leq 0.$$

Using the same method as before, we can show that

$$\lim_{s\to\infty}\frac{H^{(n-2)}(s)}{\pi_0^{\beta_1}(s)} = 0,$$

and

$$\left(\frac{H^{(n-2)}(s)}{\pi_0^{1-\beta_1}(s)}\right)' \geq 0.$$

In a similar manner, for $\beta_{k-1} < \beta_k \leq 1/2$, we can demonstrate that

$$a^{1/\alpha}(s)H^{(n-1)}(s)\pi_0(s) + \beta_k H^{(n-2)}(s) \leq 0, \tag{18}$$

and

$$\lim_{s\to\infty}\frac{H^{(n-2)}(s)}{\pi_0^{\beta_k}(s)} = 0,$$

for $k = 2, 3, \ldots, m$. Hence, we have completed the proof of the lemma. □

Theorem 1. *If (11) holds and*

$$\beta_0 > 1/2, \tag{19}$$

for some $\mu_0 \in (0,1)$, then we have $\Omega_2 = \varnothing$, where β_m is defined as in Lemma 7.

Proof. Assume for contradiction that $h \in \Omega_2$. By Lemma 7, we know that $H^{(n-2)}(s)/\pi_0^{\beta_0}(s)$ is decreasing for $s \geq s_1$and $H^{(n-2)}(s)/\pi_0^{1-\beta_0}(s)$ is increasing for $s \geq s_1$. This implies that

$$\beta_0 \leq 1/2,$$

which is a contradiction. The proof is complete. □

Theorem 2. *Assume that (11) and (17) hold. If there is a positive integer m such that*

$$\liminf_{s\to\infty} \int_{\sigma(s)}^{s} \pi_0(v)\pi_0^{\alpha-1}(\sigma(v))\left(\sigma^{n-2}(v)\right)^{\alpha} q(v)\phi_1^{\alpha}(\sigma(v);\kappa)\mathrm{d}v > \frac{\alpha\beta_m^{\alpha-1}(1-\beta_m)((n-2)!)^{\alpha}}{\mathrm{e}}, \tag{20}$$

then we can conclude that $\Omega_2 = \varnothing$, where $\alpha \leq 1$.

Proof. Suppose the opposite: that $h \in \Omega_2$. Then, based on Lemma 8, we know that $(\mathrm{S}_{3,1})$ and $(\mathrm{S}_{3,2})$ hold.

Now, we define the function

$$w(s) = a^{1/\alpha}(s)H^{(n-1)}(s)\pi_0(s) + H^{(n-2)}(s).$$

It follows from $(\mathrm{S}_{1,3})$ that $w(s) > 0$ for $s \geq s_1$. From $(\mathrm{S}_{3,1})$, we obtain

$$a^{1/\alpha}(s)H^{(n-1)}(s)\pi_0(s) \leq -\beta_m H^{(n-2)}(s).$$

Then, from the definition of $w(s)$, we have

$$\begin{aligned} w(s) &= a^{1/\alpha}(s)H^{(n-1)}(s)\pi_0(s) + \beta_m H^{(n-2)}(s) - \beta_m H^{(n-2)}(s) + H^{(n-2)}(s) \\ &\leq (1-\beta_m)H^{(n-2)}(s). \end{aligned} \tag{21}$$

Using Lemma 6, we find that $(\mathrm{S}_{1,1}) - (\mathrm{S}_{1,5})$ hold. From $(\mathrm{S}_{1,2})$ and $(\mathrm{S}_{1,6})$, we obtain

$$\begin{aligned} w'(s) &= \left(a^{1/\alpha}(s)H^{(n-1)}(s)\right)'\pi_0(s) \\ &\leq \frac{1}{\alpha}\left(a(s)\left(H^{(n-1)}(s)\right)^{\alpha}\right)'\left(a^{1/\alpha}(s)H^{(n-1)}(s)\right)^{1-\alpha}\pi_0(s) \\ &\leq -\frac{1}{\alpha}q(s)\phi_1^{\alpha}(\sigma(s);\kappa)H^{\alpha}(\sigma(s))\left(a^{1/\alpha}(s)H^{(n-1)}(s)\right)^{1-\alpha}\pi_0(s) \\ &\leq -\frac{1}{\alpha}q(s)\phi_1^{\alpha}(\sigma(s);\kappa)H^{\alpha}(\sigma(s))\left(\beta_m\frac{H^{(n-2)}(s)}{\pi_0(s)}\right)^{1-\alpha}\pi_0(s) \\ &\leq -\frac{1}{\alpha}\beta_m^{1-\alpha}q(s)\phi_1^{\alpha}(\sigma(s);\kappa)\pi_0(s)H^{\alpha}(\sigma(s))\left(\frac{H^{(n-2)}(s)}{\pi_0(s)}\right)^{1-\alpha} \\ &\leq -\frac{1}{\alpha}\beta_m^{1-\alpha}q(s)\phi_1^{\alpha}(\sigma(s);\kappa)\pi_0(s)\left(\frac{\epsilon}{(n-2)!}\sigma^{n-2}(s)\right)^{\alpha}\left(H^{(n-2)}(\sigma(s))\right)^{\alpha}\left(\frac{H^{(n-2)}(s)}{\pi_0(s)}\right)^{1-\alpha}. \end{aligned}$$

Applying $(\mathrm{S}_{1,4})$ from Lemma 6, we can observe that $H^{(n-2)}(s)/\pi_0(s)$ is increasing. Therefore, we have

$$\frac{H^{(n-2)}(\sigma(s))}{\pi_0(\sigma(s))} \leq \frac{H^{(n-2)}(s)}{\pi_0(s)},$$

and
$$\left(\frac{H^{(n-2)}(\sigma(s))}{\pi_0(\sigma(s))}\right)^{1-\alpha} \le \left(\frac{H^{(n-2)}(s)}{\pi_0(s)}\right)^{1-\alpha}.$$

Therefore,
$$\begin{aligned} w'(s) &\le -\frac{1}{\alpha}\beta_m^{1-\alpha} q(s)\phi_1^\alpha(\sigma(s);\kappa)\pi_0(s)\left(\frac{\epsilon}{(n-2)!}\sigma^{n-2}(s)\right)^\alpha \\ &\quad \times\left(H^{(n-2)}(\sigma(s))\right)^\alpha \left(\frac{H^{(n-2)}(\sigma(s))}{\pi_0(\sigma(s))}\right)^{1-\alpha} \\ &\le -\frac{1}{\alpha}\frac{\beta_m^{1-\alpha}\epsilon^\alpha}{((n-2)!)^\alpha} q(s)\phi_1^\alpha(\sigma(s);\kappa)\frac{\pi_0(s)}{\pi_0^{1-\alpha}(\sigma(s))}\left(\sigma^{n-2}(s)\right)^\alpha H^{(n-2)}(\sigma(s)), \end{aligned}$$

which, from (21), gives
$$w'(s) + \frac{1}{\alpha}\frac{\epsilon^\alpha\beta_m^{1-\alpha}}{((n-2)!)^\alpha(1-\beta_m)}\frac{\pi_0(s)}{\pi_0^{1-\alpha}(\sigma(s))}\left(\sigma^{n-2}(s)\right)^\alpha q(s)\phi_1^\alpha(\sigma(s);\kappa)w(\sigma(s)) \le 0. \quad (22)$$

Therefore, we can conclude that $w(s)$ satisfies the differential inequality (22) with positive values. However, according to Theorem 2.1.1 in [5], condition (20) ensures that (22) is oscillatory. This leads to a contradiction, completing the proof of the theorem. □

3.2. Category Ω_3

Lemma 9. *Assume that $h \in \Omega_3$. Then, eventually,*

($S_{4,1}$) $H(s)/\pi_{n-2}(s)$ *is increasing;*

($S_{4,1}$) $(-1)^{i+1}H^{(n-i-2)}(s) \le a^{1/\alpha}(s)H^{(n-1)}(s)\pi_i(s)$, *for* $i = 0, 1, 2, \ldots, n-2$.

Proof. Assume that $h \in \Omega_3$.

($S_{4,1}$) From (1), we have that $a(s)\left(H^{(n-1)}(s)\right)^\alpha$ is decreasing, and hence
$$\begin{aligned} a^{1/\alpha}(s)H^{(n-1)}(s)\int_s^\infty \frac{1}{a^{1/\alpha}(v)}\mathrm{d}v &\ge \int_s^\infty \frac{1}{a^{1/\alpha}(v)}a^{1/\alpha}(v)H^{(n-1)}(v)\mathrm{d}v \\ &= \lim_{s\to\infty} H^{(n-2)}(s) - H^{(n-2)}(s). \end{aligned} \quad (23)$$

Since $H^{(n-2)}(s)$ is a positive decreasing function, we have that $H^{(n-2)}(s)$ converges to a non-negative constant when $s \to \infty$. Thus, (23) becomes
$$-H^{(n-2)}(s) \le a^{1/\alpha}(s)H^{(n-1)}(s)\pi_0(s),$$

which implies that
$$\left(\frac{H^{(n-2)}(s)}{\pi_0(s)}\right)' = \frac{a^{1/\alpha}(s)\pi_0(s)H^{(n-1)}(s) + H^{(n-2)}(s)}{a^{1/\alpha}(s)\pi_0^2(s)} \ge 0,$$

which leads to
$$-H^{(n-3)}(s) = \int_s^\infty \frac{H^{(n-2)}(v)}{\pi_0(v)}\pi_0(v)\mathrm{d}v \ge \frac{H^{(n-2)}(s)}{\pi_0(s)}\pi_1(s).$$

This implies that

$$\left(\frac{H^{(n-3)}(s)}{\pi_1(s)}\right)' = \frac{\pi_1(s)H^{(n-2)}(s) + H^{(n-3)}(s)\pi_0(s)}{\pi_1^2(s)} \leq 0.$$

Similarly, we repeat the same previous process $(n-4)$ times and obtain

$$\left(\frac{H'(s)}{\pi_{n-3}(s)}\right)' \leq 0.$$

Now,

$$-H(s) = \int_s^\infty \frac{H'(v)}{\pi_{n-3}(v)}\pi_{n-3}(v)\mathrm{d}v \leq \frac{H'(s)}{\pi_{n-3}(s)}\pi_{n-2}(s).$$

This implies that

$$\left(\frac{H(s)}{\pi_{n-2}(s)}\right)' = \frac{\pi_{n-2}(s)H'(s) + H(s)\pi_{n-3}(s)}{\pi_{n-2}^2(s)} \geq 0.$$

$(\mathrm{S}_{4,2})$ Assume that $h \in \Omega_3$. Then, we obtain

$$a^{1/\alpha}(s)H^{(n-1)}(s)\pi_0(s) \geq \int_s^\infty \frac{a^{1/\alpha}(v)H^{(n-1)}(v)}{a^{1/\alpha}(v)}\mathrm{d}v \geq -H^{(n-2)}(s),$$

or, equivalently,

$$H^{(n-2)}(s) \geq -a^{1/\alpha}(s)H^{(n-1)}(s)\pi_0(s).$$

Integrating the last inequality from s to ∞, we obtain

$$\begin{aligned} -H^{(n-3)}(s) &\geq -\int_s^\infty a^{1/\alpha}(v)H^{(n-1)}(v)\pi_0(v)\mathrm{d}v \\ &\geq -a^{1/\alpha}(s)H^{(n-1)}(s)\int_s^\infty \pi_0(v)\mathrm{d}v \\ &\geq -a^{1/\alpha}(s)H^{(n-1)}(s)\pi_1(s), \end{aligned}$$

or, equivalently,

$$H^{(n-3)}(s) \leq a^{1/\alpha}(s)H^{(n-1)}(s)\pi_1(s).$$

Integrating the last inequality from s to ∞, we have

$$\begin{aligned} -H^{(n-4)}(s) &\leq \int_s^\infty a^{1/\alpha}(v)H^{(n-1)}(v)\pi_1(v)\mathrm{d}v \\ &\leq a^{1/\alpha}(s)H^{(n-1)}(s)\int_s^\infty \pi_1(v)\mathrm{d}v \\ &\leq a^{1/\alpha}(s)H^{(n-1)}(s)\pi_2(s), \end{aligned}$$

or, equivalently,

$$H^{(n-4)}(s) \geq -a^{1/\alpha}(s)H^{(n-1)}(s)\pi_2(s).$$

Through the repeated integration of the previous inequalities from s to ∞, we obtain

$$(-1)^{i+1}H^{(n-i-2)}(s) \leq a^{1/\alpha}(s)H^{(n-1)}(s)\pi_i(s),$$

for $i = 0, 1, 2, \ldots, n-2$.

Hence, we have completed the proof of the lemma. □

Lemma 10. *Suppose that $h \in \Omega_3$. Then, eventually,*

($S_{5,1}$) $h(s) > \phi_2(s,\kappa)H(s)$;
($S_{5,2}$) $\left(a(s)\left(H^{(n-1)}(s)\right)^{\alpha}\right)' \leq -q(s)\phi_2^{\alpha}(\sigma(s);\kappa)H^{\alpha}(\sigma(s))$.

Proof. Suppose that $h \in \Omega_3$.
($S_{5,1}$) From Lemma 5, we have that (10) holds. From ($S_{4,1}$), we conclude that

$$H\left(\varrho^{[2r+1]}(s)\right) \leq \frac{\pi_{n-2}\left(\varrho^{[2r+1]}(s)\right)}{\pi_{n-2}\left(\varrho^{[2r]}(s)\right)} H\left(\varrho^{[2r]}(s)\right),$$

which, with (10), gives

$$h(s) > \sum_{r=0}^{\kappa}\left(\prod_{\gamma=0}^{2r}\phi\left(\varrho^{[\gamma]}(s)\right)\right)\left[\frac{1}{\phi\left(\varrho^{[2r]}(s)\right)} - \frac{\pi_{n-2}\left(\varrho^{[2r+1]}(s)\right)}{\pi_{n-2}\left(\varrho^{[2r]}(s)\right)}\right] H\left(\varrho^{[2r]}(s)\right). \tag{24}$$

Since H is decreasing, then (24) becomes

$$\begin{aligned} h(s) &> \sum_{r=0}^{\kappa}\left(\prod_{\gamma=0}^{2r}\phi\left(\varrho^{[\gamma]}(s)\right)\right)\left[\frac{1}{\phi\left(\varrho^{[2r]}(s)\right)} - \frac{\pi_{n-2}\left(\varrho^{[2r+1]}(s)\right)}{\pi_{n-2}\left(\varrho^{[2r]}(s)\right)}\right] H(s) \\ &= \phi_2(s,\kappa)H(s). \end{aligned}$$

($S_{5,2}$) Equation (1) with ($S_{5,1}$) becomes

$$\begin{aligned} \left(r(s)\left(H^{(n-1)}(s)\right)^{\alpha}\right)' &= -q(s)h^{\alpha}(\sigma(s)) \\ &\leq -q(s)\phi_1^{\alpha}(\sigma(s);\kappa)H^{\alpha}(\sigma(s)). \end{aligned}$$

Therefore, the proof of the Lemma is complete. □

Remark 2. *The verification of*

$$\phi_2(s;0) = 1 - \phi(s)\frac{\pi_{n-2}(\varrho(s))}{\pi_{n-2}(s)}$$

is straightforward. Substituting $\kappa = 0$ and $n = 2$ into ($S_{5,1}$) yields the classical relation (5).

Lemma 11. *Assume that $h \in \Omega_3$. If*

$$\int_{s_0}^{\infty}\left(\frac{1}{a(w)}\int_{s_0}^{w} q(v)\phi_2^{\alpha}(\sigma(v);\kappa)\mathrm{d}v\right)^{1/\alpha}\mathrm{d}w = \infty, \tag{25}$$

and there exists a $k_0 \in (0,1)$ such that

$$\frac{1}{\alpha}\pi_{n-2}^{\alpha+1}(s)\pi_{n-3}^{-1}(s)q(s)\phi_2^{\alpha}(\sigma(s);\kappa) \geq k_0^{\alpha}, \tag{26}$$

then
($S_{6,1}$) $\lim_{s\to\infty} H(s) = 0$;
($S_{6,2}$) $H(s)/\pi_{n-2}^{k_0}(s)$ *is decreasing;*
($S_{6,3}$) $\lim_{s\to\infty} H(s)/\pi_{n-2}^{\beta_0}(s) = 0$;

Proof. Assume that $h \in \Omega_3$.

($S_{6,1}$) Since H is positive decreasing, we have that $\lim_{s\to\infty} H(s) = \ell_3 \geq 0$. Assume the contrary, that $\ell_3 > 0$. Then, there exists a $s_2 \geq s_1$ with $H(s) \geq \ell_3$ for $s \geq s_2$. Then, from ($S_{5,2}$), we obtain

$$\left(a(s)\left(H^{(n-1)}(s)\right)^{\alpha}\right)' \leq -q(s)\phi_2^{\alpha}(\sigma(s);\kappa)H^{\alpha}(\sigma(s)) \leq -\ell_3^{\alpha} q(s)\phi_2^{\alpha}(\sigma(s);\kappa).$$

Integrating the inequality twice from s_2 to s, we obtain

$$a(s)\left(H^{(n-1)}(s)\right)^{\alpha} - a(s_2)\left(H^{(n-1)}(s_2)\right)^{\alpha} \leq -\ell_3^{\alpha}\int_{s_2}^{s} q(v)\phi_2^{\alpha}(\sigma(v);\kappa)\mathrm{d}v.$$

Using case (N_3), we have $H^{(n-1)}(s) < 0$ for $s \geq s_1$. Then, $a(s_2)\left(H^{(n-1)}(s_2)\right)^{\alpha} < 0$, and so

$$H^{(n-1)}(s) \leq -\frac{\ell_3}{a^{1/\alpha}(s)}\int_{s_2}^{s} q(v)\phi_2^{\alpha}(\sigma(v);\kappa)\mathrm{d}v.$$

Then,

$$H^{(n-2)}(s) \leq H^{(n-2)}(s_2) - \ell_3\int_{s_2}^{s}\left(\frac{1}{a(w)}\int_{s_2}^{w} q(v)\phi_2^{\alpha}(\sigma(v);\kappa)\mathrm{d}v\right)^{1/\alpha}\mathrm{d}w \to -\infty \text{ as } s\to\infty,$$

a contradiction with the positivity of $h^{(n-2)}(s)$. Therefore, $\ell_3 = 0$.

($S_{6,2}$) Integrating ($S_{5,2}$) from s_2 to s and using (26), we obtain

$$\begin{aligned}
a(s)\left(H^{(n-1)}(s)\right)^{\alpha} &\leq a(s_2)\left(H^{(n-1)}(s_2)\right)^{\alpha} - \int_{s_2}^{s} q(v)\phi_2^{\alpha}(\sigma(v);\kappa)H^{\alpha}(\sigma(v))\mathrm{d}v \\
&\leq a(s_2)\left(H^{(n-1)}(s_2)\right)^{\alpha} - H^{\alpha}(s)\int_{s_2}^{s} q(v)\phi_2^{\alpha}(\sigma(v);\kappa)\mathrm{d}v \\
&\leq a(s_2)\left(H^{(n-1)}(s_2)\right)^{\alpha} - H^{\alpha}(s)\int_{s_2}^{s} \alpha k_0^{\alpha}\frac{\pi_{n-3}(v)}{\pi_{n-2}^{\alpha+1}(v)}\mathrm{d}v \\
&\leq a(s_2)\left(H^{(n-1)}(s_2)\right)^{\alpha} + k_0^{\alpha}\frac{H^{\alpha}(s)}{\pi_{n-2}^{\alpha}(s_2)} - k_0^{\alpha}\frac{H^{\alpha}(s)}{\pi_{n-2}^{\alpha}(s)},
\end{aligned}$$

which, with ($S_{6,1}$), gives

$$a(s)\left(H^{(n-1)}(s)\right)^{\alpha} \leq -k_0^{\alpha}\frac{H^{\alpha}(s)}{\pi_{n-2}^{\alpha}(s)},$$

or, equivalently,

$$a^{1/\alpha}(s)H^{(n-1)}(s) \leq -k_0\frac{H(s)}{\pi_{n-2}(s)}. \tag{27}$$

Thus, from ($S_{4,2}$) at $i = n-3$, we have

$$\frac{H'(s)}{\pi_{n-3}(s)} \leq -k_0\frac{H(s)}{\pi_{n-2}(s)},$$

or, equivalently,

$$\pi_{n-2}(s)H'(s) + k_0\pi_{n-3}(s)H(s) \leq 0. \tag{28}$$

Consequently,

$$\left(\frac{H(s)}{\pi_{n-2}^{k_0}(s)}\right)' = \frac{\pi_{n-2}(s)H'(s) + k_0\pi_{n-3}(s)H(s)}{\pi_{n-2}^{k_0+1}(s)} \leq 0.$$

($S_{6,3}$) Given that $H(s)/\pi_{n-2}^{k_0}(s)$ is a positive decreasing function, it follows that

$$\lim_{s\to\infty} H(s)/\pi_{n-2}^{k_0}(s) = \ell_4 \geq 0.$$

Assume the contrary, that $\ell_4 > 0$. Then, there exists a $s_2 \geq s_1$ with $H(s)/\pi_{n-2}^{k_0}(s) \geq \ell_4$ for $s \geq s_2$. Next, we define

$$\varphi(s) := \frac{H(s) + a^{1/\alpha}(s)H^{(n-1)}(s)\pi_{n-2}(s)}{\pi_{n-2}^{k_0}(s)}.$$

Then, from ($S_{4,2}$), $\varphi(s) \geq 0$ for $s \geq s_2$. Differentiating $\varphi(s)$ and ($S_{4,2}$), we find

$$\begin{aligned} &\varphi'(s) \\ =\ & \frac{1}{\pi_{n-2}^{2k_0}(s)}\Big[\pi_{n-2}^{k_0}(s)\Big(H'(s) - a^{1/\alpha}(s)H^{(n-1)}(s)\pi_{n-3}(s) + \Big(a^{1/\alpha}(s)h^{(n-1)}(s)\Big)'\pi_{n-2}(s)\Big) \\ & +k_0\pi_{n-2}^{k_0-1}(s)\pi_{n-3}(s)\Big(H(s) + a^{1/\alpha}(s)H^{(n-1)}(s)\pi_{n-2}(s)\Big)\Big] \\ \leq\ & \frac{1}{\pi_{n-2}^{k_0+1}(s)}\Big[\Big(a^{1/\alpha}(s)H^{(n-1)}(s)\Big)'\pi_{n-2}^2(s) + k_0\pi_{n-3}(s)\Big(H(s) + a^{1/\alpha}(s)H^{(n-1)}(s)\pi_{n-2}(s)\Big)\Big] \\ \leq\ & \frac{1}{\pi_{n-2}^{k_0+1}(s)}\Big[\frac{1}{\alpha}\Big(a(s)\Big(H^{(n-1)}(s)\Big)^{\alpha}\Big)'\Big(a^{1/\alpha}(s)H^{(n-1)}(s)\Big)^{1-\alpha}\pi_{n-2}^2(s) \\ & +k_0\pi_{n-3}(s)\Big(H(s) + a^{1/\alpha}(s)H^{(n-1)}(s)\pi_{n-2}(s)\Big)\Big]. \end{aligned}$$

Using ($S_{5,2}$), we find

$$\begin{aligned} \varphi'(s) \leq\ & \frac{1}{\pi_{n-2}^{k_0+1}(s)}\Big[\frac{-1}{\alpha}q(s)\phi_2^{\alpha}(\sigma(s);\kappa)H^{\alpha}(\sigma(s))\Big(a^{1/\alpha}(s)H^{(n-1)}(s)\Big)^{1-\alpha}\pi_{n-2}^2(s) \\ & +k_0\pi_{n-3}(s)H(s) + k_0\pi_{n-3}(s)a^{1/\alpha}(s)H^{(n-1)}(s)\pi_{n-2}(s)\Big]. \end{aligned}$$

Since $\alpha \leq 1$, $H^{(n-1)}(s) \leq 0$, and

$$a^{1/\alpha}(s)H^{(n-1)}(s) \leq -k_0\frac{H(s)}{\pi_{n-2}(s)},$$

also

$$-a^{1/\alpha}(s)H^{(n-1)}(s) \geq k_0\frac{H(s)}{\pi_{n-2}(s)},$$

which implies that

$$\Big(a^{1/\alpha}(s)H^{(n-1)}(s)\Big)^{1-\alpha} \geq \left(k_0\frac{H(s)}{\pi_{n-2}(s)}\right)^{1-\alpha}.$$

Then,

$$\begin{aligned}\varphi'(s) &\leq \frac{1}{\pi_{n-2}^{k_0+1}(s)}\left[\frac{-1}{\alpha}q(s)\phi_2^{\alpha}(\sigma(s);\kappa)H^{\alpha}(s)\left(k_0\frac{H(s)}{\pi_{n-2}(s)}\right)^{1-\alpha}\pi_{n-2}^2(s)\right.\\ &\left.\quad+k_0\pi_{n-3}(s)H(s)+k_0\pi_{n-3}(s)a^{1/\alpha}(s)H^{(n-1)}(s)\pi_{n-2}(s)\right]\\ &\leq \frac{1}{\pi_{n-2}^{k_0+1}(s)}\left[\frac{-k_0^{1-\alpha}}{\alpha}q(s)\phi_2^{\alpha}(\sigma(s);\kappa)\pi_{n-2}^{\alpha+1}(s)H(s)\right.\\ &\left.\quad+k_0\pi_{n-3}(s)H(s)+k_0\pi_{n-3}(s)a^{1/\alpha}(s)H^{(n-1)}(s)\pi_{n-2}(s)\right].\end{aligned}$$

Using (26), we obtain

$$\begin{aligned}\varphi'(s) &\leq \frac{1}{\pi_{n-2}^{k_0+1}(s)}\left[-k_0\pi_{n-3}(s)H(s)+k_0\pi_{n-3}(s)H(s)+k_0\pi_{n-3}(s)a^{1/\alpha}(s)H^{(n-1)}(s)\pi_{n-2}(s)\right]\\ &= \frac{1}{\pi_{n-2}^{k_0}(s)}k_0\pi_{n-3}(s)a^{1/\alpha}(s)H^{(n-1)}(s).\end{aligned} \tag{29}$$

Using the fact that $H(s)/\pi_{n-2}^{k_0}(s) \geq \ell_4$ with (27), we obtain

$$a^{1/\alpha}(s)H^{(n-1)}(s) \leq -k_0\frac{H(s)}{\pi_{n-2}(s)} \leq -k_0\ell_4\pi_{n-2}^{k_0-1}(s). \tag{30}$$

Combining (29) and (30), we obtain

$$\varphi'(s) \leq -k_0^2\ell_4\frac{\pi_{n-3}(s)}{\pi_{n-2}(s)} < 0.$$

By integrating the last inequality from s_2 to s, we find

$$\varphi(s_2) \geq k_0^2\ell_4 \ln\frac{\pi_{n-2}(s_2)}{\pi_{n-2}(s)} \to \infty \text{ as } s \to \infty,$$

a contradiction, and so $\ell_4 = 0$.

Therefore, the proof of the Lemma is complete. □

Lemma 12. *Assume that $h(s) \in \Omega_3$ and (25) and (26) hold for some $k_0 \in (0,1)$. If $k_{i-1} \leq k_i < 1$ for all $i = 1, 2, \ldots, m-1$, then*

$(S_{7,1,m})$ *$H(s)/\pi_{n-2}^{k_m}(s)$ is decreasing;*
$(S_{7,2,m})$ $\lim_{s\to\infty} H(s)/\pi_{n-2}^{k_m}(s) = 0$;
where

$$k_j = k_0\frac{\lambda_1^{k_{j-1}}}{(1-k_{j-1})^{1/\alpha}},\ j = 1, 2, \ldots, m, \tag{31}$$

and

$$\frac{\pi_{n-2}(\sigma(s))}{\pi_{n-2}(s)} \geq \lambda_1, \text{ for all } s \geq s_1, \tag{32}$$

for some $\lambda \geq 1$.

Proof. Assuming that $h(s) \in \Omega_3$, we can use Theorem 11 to conclude that $(S_{6,1})$–$(S_{6,3})$ are satisfied. Furthermore, by applying induction and Lemma 12, we can establish that $(S_{7,1,0})$–$(S_{7,3,0})$ hold.

Assuming that $(\mathrm{S}_{7,1,m-1})$–$(\mathrm{S}_{7,3,m-1})$ hold, we can integrate $(\mathrm{S}_{5,2})$ from s_2 to s, resulting in

$$a(s)\left(H^{(n-1)}(s)\right)^{\alpha} \leq a(s_2)\left(H^{(n-1)}(s_2)\right)^{\alpha} - \int_{s_2}^{s} q(v)\phi_2^{\alpha}(\sigma(v);\kappa)H^{\alpha}(\sigma(v))\mathrm{d}v. \tag{33}$$

Using $(\mathrm{S}_{7,1,m-1})$, we obtain that

$$H(\sigma(s)) \geq \pi_{n-2}^{k_{m-1}}(\sigma(s))\frac{H(s)}{\pi_{n-2}^{k_{m-1}}(s)}.$$

Then, (33) becomes

$$\begin{aligned} a(s)\left(H^{(n-1)}(s)\right)^{\alpha} \leq\ & a(s_2)\left(H^{(n-1)}(s_2)\right)^{\alpha} \\ & - \int_{s_2}^{s} q(v)\phi_2^{\alpha}(\sigma(v);\kappa)\pi_{n-2}^{\alpha k_{m-1}}(\sigma(v))\frac{H^{\alpha}(v)}{\pi_{n-2}^{\alpha k_{m-1}}(v)}\mathrm{d}v. \end{aligned}$$

which, with the fact that $H(s)/\pi_{n-2}^{k_{m-1}}(s)$ is a decreasing function, gives

$$\begin{aligned} a(s)\left(H^{(n-1)}(s)\right)^{\alpha} \leq\ & a(s_2)\left(H^{(n-1)}(s_2)\right)^{\alpha} \\ & - \frac{H^{\alpha}(s)}{\pi_{n-2}^{\alpha k_{m-1}}(s)}\int_{s_2}^{s} q(v)\phi_2^{\alpha}(\sigma(v);\kappa)\pi_{n-2}^{\alpha k_{m-1}}(v)\frac{\pi_{n-2}^{\alpha k_{m-1}}(\sigma(v))}{\pi_{n-2}^{\alpha k_{m-1}}(v)}\mathrm{d}v. \end{aligned}$$

Hence, from (26) and (32), we obtain

$$\begin{aligned} & a(s)\left(H^{(n-1)}(s)\right)^{\alpha} \\ \leq\ & a(s_2)\left(H^{(n-1)}(s_2)\right)^{\alpha} - \lambda_1^{\alpha k_{m-1}}\frac{H^{\alpha}(s)}{\pi_{n-2}^{\alpha k_{m-1}}(s)}\int_{s_2}^{s} q(v)\phi_2^{\alpha}(\sigma(v);\kappa)\pi_{n-2}^{\alpha k_{m-1}}(v)\mathrm{d}v \\ \leq\ & a(s_2)\left(H^{(n-1)}(s_2)\right)^{\alpha} - \alpha k_0^{\alpha}\lambda_1^{\alpha k_{m-1}}\frac{H^{\alpha}(s)}{\pi_{n-2}^{\alpha k_{m-1}}(s)}\int_{s_2}^{s}\frac{\pi_{n-3}(v)}{\pi_{n-2}^{\alpha(1-k_{m-1})+1}(v)}\mathrm{d}v \\ =\ & a(s_2)\left(H^{(n-1)}(s_2)\right)^{\alpha} - k_0^{\alpha}\frac{\lambda_1^{\alpha k_{m-1}}}{1-k_{m-1}}\frac{H^{\alpha}(s)}{\pi_{n-2}^{\alpha k_{m-1}}(s)}\left(\frac{1}{\pi_{n-2}^{\alpha(1-k_{m-1})}(s)} - \frac{1}{\pi_{n-2}^{\alpha(1-k_{m-1})}(s_2)}\right) \\ =\ & a(s_2)\left(H^{(n-1)}(s_2)\right)^{\alpha} + k_m^{\alpha}\frac{H^{\alpha}(s)}{\pi_{n-2}^{\alpha k_{m-1}}(s)}\frac{1}{\pi_{n-2}^{\alpha(1-k_{m-1})}(s_2)} - k_m\frac{H^{\alpha}(s)}{\pi_{n-2}^{\alpha}(s)}, \end{aligned}$$

which, with the fact that $\lim_{s\to\infty} H(s)/\pi_{n-2}^{k_{m-1}}(s) = 0$, gives

$$a(s)\left(H^{(n-1)}(s)\right)^{\alpha} \leq -k_m^{\alpha}\frac{H^{\alpha}(s)}{\pi_{n-2}^{\alpha}(s)},$$

or, equivalently,

$$a^{1/\alpha}(s)H^{(n-1)}(s) \leq -k_m\frac{H(s)}{\pi_{n-2}(s)}. \tag{34}$$

Thus, from $(\mathrm{S}_{4,2})$ at $i = n-3$, we have

$$\frac{H'(s)}{\pi_{n-3}(s)} \leq -k_m\frac{H(s)}{\pi_{n-2}(s)},$$

or, equivalently,

$$\pi_{n-2}(s)H'(s) + k_m\pi_{n-3}(s)H(s) \leq 0. \tag{35}$$

Consequently,

$$\left(\frac{H(s)}{\pi_{n-2}^{k_m}(s)}\right)' = \frac{1}{\pi_{n-2}^{k_m+1}(s)}\left(\pi_{n-2}(s)H'(s) + k_m(s)\pi_{n-3}(s)H(s)\right) \le 0.$$

By following the same method used to prove ($S_{6,2}$) in Lemma 11, we can demonstrate that $\lim_{s\to\infty} H(s)/\pi_{n-2}^{k_m}(s) = 0$.

Hence, we have successfully completed the proof. □

Lemma 13. *Suppose that $h(s) \in \Omega_3$ and (25) and (26) hold for some $k_0 \in (0,1)$. If $k_{i-1} \le k_i < 1$ for all $i = 1, 2, \ldots, m-1$, then*

$$h(s) > \widehat{\phi}_2(s;\kappa)H(s).$$

Proof. Using the same method employed in the proof of Lemma 10, we can derive (24). Furthermore, from ($S_{7,1,m}$), we can infer that

$$H\left(\varrho^{[2r]}(s)\right) \ge \frac{\pi_{n-2}^{k_m}\left(\varrho^{[2r]}(s)\right)}{\pi_{n-2}^{k_m}(s)}H(s),$$

which, with (24), gives

$$\begin{aligned} h(s) &> \sum_{r=0}^{\kappa}\left(\prod_{s=0}^{2r}\phi\left(\varrho^{[s]}(s)\right)\right)\left[\frac{1}{\phi\left(\varrho^{[2r]}(s)\right)} - \frac{\pi_{n-2}\left(\varrho^{[2r+1]}(s)\right)}{\pi_{n-2}\left(\varrho^{[2r]}(s)\right)}\right]\frac{\pi_{n-2}^{k_m}\left(\varrho^{[2r]}(s)\right)}{\pi_{n-2}^{k_m}(s)}H(s) \\ &= \widehat{\phi}_2(s;\kappa)H(s). \end{aligned}$$

□

Theorem 3. *Assume that (25) and (26) hold. If there is a positive integer m such that*

$$\liminf_{s\to\infty}\int_{\sigma(s)}^{s}\pi_{n-2}(v)\pi_{n-2}^{\alpha-1}(\sigma(v))q(v)\phi_2^{\alpha}(\sigma(v);\kappa)\mathrm{d}v > \frac{\alpha k_m^{\alpha-1}(1-k_m)}{\mathrm{e}}, \tag{36}$$

then $\Omega_3 = \varnothing$, where $\alpha \le 1$ and k_m is defined as in Lemma 12.

Proof. Suppose that the opposite is true: that $h \in \Omega_3$. According to Lemma 12, we know that both ($S_{7,1,m}$) and ($S_{7,2,m}$) hold.

We can now introduce the function

$$w(s) = a^{1/\alpha}(s)H^{(n-1)}(s)\pi_{n-2}(s) + H(s).$$

From ($S_{4,2}$) at $i = n-2$, $w(s) \ge 0$ for $s \ge s_2$, and from (34), we obtain

$$a^{1/\alpha}(s)H^{(n-1)}(s)\pi_{n-2}(s) \le -k_m H(s).$$

Then, from the definition of $w(s)$, we have

$$\begin{aligned} w(s) &= a^{1/\alpha}(s)H^{(n-1)}(s)\pi_{n-2}(s) + k_m H^{(n-2)}(s) - k_m H^{(n-2)}(s) + H^{(n-3)}(s) \\ &\le (1-k_m)H^{(n-2)}(s). \end{aligned} \tag{37}$$

Thus,

$$w'(s) = \left(a^{1/\alpha}(s)H^{(n-1)}(s)\right)'\pi_{n-2}(s) - a^{1/\alpha}(s)H^{(n-1)}(s)\pi_{n-3}(s) + H'(s).$$

From $(S_{4,2})$ at $i = n-3$, we obtain

$$\begin{aligned} w'(s) &\leq \left(a^{1/\alpha}(s)H^{(n-1)}(s)\right)'\pi_{n-2}(s) \\ &= \frac{1}{\alpha}\left(a(s)\left(H^{(n-1)}(s)\right)^{\alpha}\right)'\left(a^{1/\alpha}(s)H^{(n-1)}(s)\right)^{1-\alpha}\pi_{n-2}(s). \end{aligned}$$

Using $(S_{5,2})$ and $(S_{4,2})$ at $i = n-2$, we have

$$\begin{aligned} w'(s) &\leq \frac{-1}{\alpha}q(s)\phi_2^{\alpha}(\sigma(s);\kappa)H^{\alpha}(\sigma(s))\left(a^{1/\alpha}(s)H^{(n-1)}(s)\right)^{1-\alpha}\pi_{n-2}(s) \\ &\leq \frac{-1}{\alpha}q(s)\phi_2^{\alpha}(\sigma(s);\kappa)H^{\alpha}(\sigma(s))\left(-k_m\frac{H(s)}{\pi_{n-2}(s)}\right)^{1-\alpha}\pi_{n-2}(s) \\ &= \frac{-k_m^{1-\alpha}}{\alpha}q(s)\phi_2^{\alpha}(\sigma(s);\kappa)H^{\alpha}(\sigma(s))\left(\frac{H(s)}{\pi_{n-2}(s)}\right)^{1-\alpha}\pi_{n-2}(s). \end{aligned}$$

Using $(S_{4,1})$ in Lemma 9, we note that $H(s)/\pi_{n-2}(s)$ is increasing; then,

$$\frac{H(\sigma(s))}{\pi_{n-2}(\sigma(s))} \leq \frac{H(s)}{\pi_{n-2}(s)},$$

and

$$\left(\frac{H(\sigma(s))}{\pi_{n-2}(\sigma(s))}\right)^{1-\alpha} \leq \left(\frac{H(s)}{\pi_{n-2}(s)}\right)^{1-\alpha}.$$

Therefore,

$$\begin{aligned} w'(s) &\leq \frac{-k_m^{1-\alpha}}{\alpha}q(s)\phi_2^{\alpha}(\sigma(s);\kappa)H^{\alpha}(\sigma(s))\left(\frac{H(\sigma(s))}{\pi_{n-2}(\sigma(s))}\right)^{1-\alpha}\pi_{n-2}(s) \\ &= \frac{-k_m^{1-\alpha}}{\alpha}\pi_{n-2}^{\alpha-1}(\sigma(s))\pi_{n-2}(s)q(s)\phi_2^{\alpha}(\sigma(s);\kappa)H(\sigma(s)), \end{aligned}$$

which, from (37), gives

$$w'(s) + \frac{1}{\alpha}\frac{k_m^{1-\alpha}}{1-k_m}\pi_{n-2}^{\alpha-1}(\sigma(s))\pi_{n-2}(s)q(s)\phi_2^{\alpha}(\sigma(s);\kappa)w(\sigma(s)) \leq 0. \tag{38}$$

Therefore, $w(s)$ satisfies the differential inequality (38) and is positive. However, according to Theorem 2.1.1 in [5], condition (36) ensures that (38) is oscillatory. Thus, this contradiction concludes the proof of the theorem. □

Theorem 4. *Suppose that (25) and (26) hold. If there is a positive integer m such that*

$$\liminf_{s\to\infty}\int_{\sigma(s)}^{s}\pi_{n-2}(v)\pi_{n-2}^{\alpha-1}(\sigma(v))q(v)\widehat{\phi}_2^{\alpha}(\sigma(v);\kappa)\mathrm{d}v > \frac{\alpha k_m^{\alpha-1}(1-k_m)}{\mathrm{e}}, \tag{39}$$

then $\Omega_3 = \varnothing$, where $\alpha \leq 1$ and k_m is defined as in Lemma 12.

Proof. Apply the relation

$$h(s) > \widehat{\phi}_2(s;\kappa)H(s),$$

to (1) and utilize the same proof technique employed in the preceding theorem. □

3.3. Category Ω_1

Lemma 14. *If*

$$\liminf_{s\to\infty}\int_{\sigma(s)}^{s} q(v)(1-\phi(\sigma(v)))^{\alpha}\frac{\left(\sigma^{n-1}(v)\right)^{\alpha}}{a(\sigma(v))}\mathrm{d}v > \frac{((n-1)!)^{\alpha}}{\mathrm{e}}, \tag{40}$$

then $\Omega_1 = \varnothing$.

Proof. Assume the contrary: that $h \in \Omega_1$. Then, it is clear from (N_1) that

$$\lim_{s\to\infty} H(s) \neq 0.$$

Thus, it follows from Lemma 3 that, for every $\epsilon \in (0,1)$,

$$H(\sigma(s)) \geq \frac{\epsilon}{(n-1)!}\frac{\sigma^{n-1}(s)}{a(\sigma(s))}\Big(a(\sigma(s))H^{(n-1)}(\sigma(s))\Big), \tag{41}$$

eventually. Using (41) in Equation (1), we see that

$$\begin{aligned}
\left(a(s)\left(h^{(n-1)}(s)\right)^{\alpha}\right)' &= -q(s)h^{\alpha}(\sigma(s)) \\
&\leq -q(s)(1-\phi(\sigma(s)))^{\alpha}H^{\alpha}(\sigma(s)) \\
&\leq -q(s)(1-\phi(\sigma(s)))^{\alpha}\left(\frac{\epsilon}{(n-1)!}\frac{\sigma^{n-1}(s)}{a(\sigma(s))}\right)^{\alpha}\left(a(\sigma(s))\left(H^{(n-1)}(\sigma(s))\right)^{\alpha}\right).
\end{aligned}$$

Let $\theta(s) = a(s)\left(H^{(n-1)}(s)\right)^{\alpha}$ in the last inequality. We see that $\theta(s)$ is a positive solution of the delay differential inequality

$$\theta'(s) + \frac{\epsilon^{\alpha}}{((n-1)!)^{\alpha}}q(s)(1-\phi(\sigma(s)))^{\alpha}\frac{\left(\sigma^{n-1}(s)\right)^{\alpha}}{a(\sigma(s))}\theta(\sigma(s)) \leq 0. \tag{42}$$

Therefore, $w(s)$ satisfies the differential inequality (42) and is positive. However, according to Theorem 2.1.1 in [5], condition (40) ensures that (42) is oscillatory. Thus, this contradiction concludes the proof of the Theorem. □

4. Oscillation Criteria

In this section, we use the results of the previous section to obtain new criteria for checking the oscillation of all solutions of (1).

We now have conditions that exclude positive solutions for all (N_1), (N_2), and (N_3) cases. By combining these conditions, as outlined in the following theorem, we can derive criteria for oscillation.

Theorem 5. *Assume that (19), (36) and (40) hold. Then, (1) is oscillatory.*

Theorem 6. *Assume that (20), (36) and (40) hold. Then, (1) is oscillatory.*

Theorem 7. *Assume that (19), (39) and (40) hold. Then, (1) is oscillatory.*

Theorem 8. *Assume that (20), (39) and (40) hold. Then, (1) is oscillatory.*

Example 1. *Consider the NDE*

$$\left(s^4(h(s)+\phi_0 h(\varrho_0 s))'''\right)' + q_0 h(\sigma_0 s) = 0, \tag{43}$$

where $s>0$, $\phi_0\in(0,1)$, $\varrho_0,\sigma_0\in(0,1)$, *and* $q_0>0$. *By comparing (1) and (43), we can conclude that* $n=4$, $a(s)=s^4$, $\phi(s)=\phi_0$, $q(s)=q_0$, $\varrho(s)=\varrho_0 s$, *and* $\sigma(s)=\sigma_0 s$. *It is easy to verify that*

$$\pi_0(s)=\frac{1}{3s^3},\ \pi_1(s)=\frac{1}{6s^2},\ \pi_2(s)=\frac{1}{6s},$$

$$\phi_1(s;\kappa)=[1-\phi_0]\sum_{r=0}^{\kappa}\phi_0^{2r}\varrho^{4r/\epsilon},\ \phi_2(s;\kappa)=\left[\frac{1}{\phi_0}-\frac{1}{\varrho_0}\right]\sum_{r=0}^{\kappa}\phi_0^{2r+1},$$

$$\widehat{\phi}_2(s;\kappa)=\left[\frac{1}{\phi_0}-\frac{1}{\varrho_0}\right]\sum_{r=0}^{\kappa}\phi_0^{2r+1}\frac{1}{\varrho_0^{2rk_m}},$$

$$\delta=\frac{1}{18}\sigma_0^2q_0(1-\phi_0)\sum_{r=0}^{\kappa}\phi_0^{2r}\varrho^{4r/\epsilon},\ \beta_0=\frac{\epsilon}{18}\sigma_0^2q_0(1-\phi_0)\sum_{r=0}^{\kappa}\phi_0^{2r}\varrho^{4r/\epsilon},$$

$$k_0=\frac{1}{6}q_0\left[\frac{1}{\phi_0}-\frac{1}{\varrho_0}\right]\sum_{r=0}^{\kappa}\phi^{2r+1}.$$

Condition (19) results in

$$q_0>\frac{9}{\epsilon\sigma_0^2(1-\phi_0)\sum_{r=0}^{\kappa}\phi_0^{2r}\varrho^{4r/\epsilon}}, \tag{44}$$

condition (20) yields

$$q_0>\frac{18(1-\beta_m)}{\sigma_0^2(1-\phi_0)\sum_{r=0}^{\kappa}\phi_0^{2r}\varrho^{4r/\epsilon}\ln\frac{1}{\sigma_0}}\frac{1}{\mathrm{e}}, \tag{45}$$

condition (36) leads to

$$q_0>\frac{6(1-k_m)}{\left[\frac{1}{\phi_0}-\frac{1}{\varrho_0}\right]\sum_{r=0}^{\kappa}\phi^{2r+1}\ln\frac{1}{\sigma_0}}\frac{1}{\mathrm{e}}, \tag{46}$$

condition (39) produces

$$q_0>\frac{6(1-k_m)}{\left[\frac{1}{\phi_0}-\frac{1}{\varrho_0}\right]\sum_{r=0}^{\kappa}\phi_0^{2r+1}\frac{1}{\varrho_0^{2rk_m}}\ln\frac{1}{\sigma_0}}\frac{1}{\mathrm{e}}, \tag{47}$$

and condition (40) leads to

$$q_0>\frac{\sigma_0}{(1-\phi_0)\ln\frac{1}{\sigma_0}}\frac{(n-1)!}{\mathrm{e}}. \tag{48}$$

The oscillatory of Equation (43) can be determined by applying different theorems. Theorem 5 indicates that if (44), (46) and (48) are satisfied, then Equation (43) is oscillatory. Similarly, Theorem 6 shows that if (45), (46) and (48) are satisfied, then Equation (43) is oscillatory. Theorem 7 establishes that when (44), (47) and (48) are satisfied, Equation (43) is oscillatory. Finally, Theorem 8 states that if (45), (47) and (48) are satisfied, then Equation (43) is oscillatory.

Example 2. *Consider the NDE (43), where* $\phi_0=1/2$, $\varrho_0=0.9$ *and* $\rho_0=1/3$; *then, (43) becomes*

$$\left(s^4\left(h(s)+\frac{1}{2}h(0.9s)\right)'''\right)'+q_0h\left(\frac{1}{3}s\right)=0,\ s\geq 1. \tag{49}$$

Clearly,

$$\lambda=\frac{\pi_0(\sigma(s))}{\pi_0(s)}=27,\ \lambda_1=\frac{\pi_2(\sigma(s))}{\pi_2(s)}=3,$$

$$\phi_1(s;10)=\sum_{r=0}^{10}\left(\frac{1}{2}\right)^{2r+1}(0.9)^{4r/\epsilon}\cong 0.579,\ \textit{where}\ \epsilon=0.7,$$

$$\phi_2(s;10) = \left[2 - \frac{1}{0.9}\right] \sum_{r=0}^{10} \left(\frac{1}{2}\right)^{2r+1} \cong 0.67,$$

$$\delta = 0.004 q_0,\ \beta_0 = 0.003 q_0,\ \textit{where } \epsilon = 0.7,$$

$$\beta_j := 0.003 q_0 \frac{27^{\beta_{j-1}}}{1 - \beta_{j-1}},\ j = 1, 2, \ldots, m,$$

$$k_0 = 0.112 q_0,\ k_j = 0.112 q_0 \frac{3^{k_{j-1}}}{1 - k_{j-1}}\ \textit{for } j = 1, 2, \ldots, m.$$

The conditions (19), (20), (36) and (40) are satisfied when

$$q_0 > 166.667,$$

$$q_0 > 93.573(1 - \beta_m),$$

$$q_0 > 2.995(1 - k_m),$$

$$q_0 > 1.34,$$

respectively. Thus, from Theoerms 5 and 6, we conclude that (49) is oscillatory.

5. Conclusions

This research investigated the oscillatory behavior and monotonic properties of a class of even-order quasi-linear neutral differential equations. We introduced several enhanced relationships connecting the solution and its corresponding function in two out of the three cases of positive solutions for the examined equation. Utilizing these relationships, we established criteria verifying that categories Ω_2 and Ω_3 have no positive solutions. Furthermore, we demonstrated through comparisons and examples that the new relationships improved the criteria, ensuring that Ω_2 and Ω_3 were empty sets. Finally, we developed a new criterion to check the oscillation of Equation (1).

The theorems that we obtained not only extend current findings in the literature but also provide a basis for future research in different directions. For example, it would be of interest to extend the results of this paper to higher-order equations of type (1), where $n \geq 3$ is an odd natural number.

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