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Existence of Monotone Positive Solutions for Caputo–Hadamard Nonlinear Fractional Differential Equation with Infinite-Point Boundary Value Conditions

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Abstract: Numerical solutions and approximate solutions of fractional differential equations have been studied by mathematicians recently and approximate solutions and exact solutions of fractional differential equations are obtained in many kinds of ways, such as Lie symmetry, variational method, the optimal ADM method, and so on. In this paper, we obtain the positive solutions by iterative methods for sum operators. Green's function and the properties of Green's function are deduced, then based on the properties of Green's function, the existence of iterative positive solutions for a nonlinear Caputo–Hadamard infinite-point fractional differential equation are obtained by iterative methods for sum operators; an example is proved to illustrate the main result.

Keywords: Caputo–Hadamard fractional differential equation; iterative positive solution; infinite-point; sum operator

MSC: 34B16; 34B18

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1. Introduction

Compared with the traditional integer differential model, the fractional-order differential model has more accurate performance, memory effect and genetic characteristics when describing the dynamic behavior of the system, so many systems are modeled by fractional-order differential equations. It is precisely because fractional differential models show some special properties and have relatively fine results when simulating systems that people have strong research interest in then. For example, Boulham [1] proposed a kind of adaptive control scheme based on fractional sliding surface and radial basis function (RBF) neural network, and numerical simulation verified the effectiveness and efficiency of the controller. Huang [2] established a fractional-order viscoelastic-plastic creep model that can accurately describe the strain–time relationship of damaged coal samples. Khajji [3] and Abdullahi [4] established infectious disease dynamics models of a fractional COVID-19 epidemic model and pine fusarium wilt infection that can provide dynamic complexity information, respectively.

Some mathematicians study numerical solutions and approximate solutions of fractional differential equations and they obtain approximate solutions and exact solutions of fractional differential equations in many kinds of ways, such as Lie symmetry [5], variational methods [6], the optimal ADM method [7], and so on. In [5], based on (2 + 1) independent variables and one dependent variable, the authors proposed Lie symmetry analysis for space-time convection-diffusion Riemann–Liouville fractional differential equations. A reduction form of our governed fractional differential equation is obtained, then, for solving the two-dimensional fractional-order heat equation with some initial value conditions, a computational method is yielded by the spectral analysis method based on Bernstein operational matrices. In [6], for solving partial and ordinary differential equations under the definition of fractional derivative, the optimal variational iteration method is used. First, a parameter is introduced to the standard variational iteration method and the new method accelerates convergence; under the L2 norm and based on calculating the residual of the parameter, the value of the convergence acceleration coefficient is determined; then, the numerical simulation was performed by author, and the numerical results showed that the proposed method with an introducing parameter gives much more and better accurate results than the standard VIM method, due to the effectiveness of the VIM method with adding the parameter and was clear in its calculation method, so the authors obtained the solution for ordinary and partial differential equations under the definition of the fractional derivative. Moreover, the authors solved the fractional Kawahara and foam drainage equations. In [7], for solving the present model, the authors used a novel semi-analytical technique that called fractional reduced differential transform method, whose characteristic was the time-fractional derivative. This is compared with the solution of other existing methods of obtained outcomes for a particular case. Moreover, the authors studied the convergence analysis of this fractional model. Even though the practicality and applicability have been studied by many scientists, the analytical approach of this fractional model is seldom studied in the existing literature.

There are also some scientists working on symmetric solutions nowadays, for example, in [8,9], nonlinear equations with the fractional p-Laplacian was considered, in order to continue with the method of moving planes, a maximum principle for anti-symmetric functions was proved, and other key ingredients were obtained, such as a variant for the Hopf Lemma (a Lemma on the boundary estimate), which played an important role in the narrow regional principle. Then, the radial symmetry and monotonicity for positive solutions were established for semilinear equations for the fractional p-Laplacian in the whole space and in a unit ball. In [9], equations involving fully nonlinear nonlocal operators are investigated, a maximum principle was proved and key ingredients were obtained for carrying on the method of moving planes, such as decayed at infinity and narrow region principle. Then, radial symmetry and monotonicity for positive solutions were obtained for Dirichlet problems associated with such fully nonlinear fractional order equations in a unit ball and in the whole space, the authors also obtained non-existence results on a half space. The authors claim that the developed methods can be applied to a variety of problems with many kinds of fully nonlinear nonlocal operators.

Some mathematicians study the existence and multiplicity of solutions for fractional differential equations by the symmetric mountain pass theorem [10,11]. In [10], the existence and multiplicity of solutions for fractional-Kirchhoff-type problems were investigated by the authors, based on the theory of the fractional Sobolev space with variable exponents combining symmetric mountain pass theorem, the case of solution was obtained by weakening the condition of this equation. In [11], the compact embedding of the space was established by the authors. Using this embedding result and some critical-points theorems, the existence and multiplicity results were obtained by the authors based on the critical-points theorems and embedding result for the following class of fractional p(x,.)-Kirchhoff-type problems in space.

Some mathematicians study the case of solutions for fractional differential equations qualitatively and they obtain results using different methods such as fixed point theory [12], topological degree theory [13], and so on. In recent years, the research on the structure of boundary value problems of fractional differential models has developed rapidly [14–39] and the references therein). In [23], the authors discuss the existence of monotone positive solutions for the following problem:

$$D^{\alpha}_{0+}x(t) + \hbar(t, x(t), x(t))) + \Im(t, x(t)) = 0, t \in (0,1), n-1 < \alpha \leq n,$$

with boundary value condition $x^{(i)}(0) = 0, i = 0,1,2,\ldots,n-2,$ $[D^{\beta}_{0+}x(t)]_{t=1} = 0$, where $2 \leq \beta \leq n-2, n-1 < \alpha \leq n, n > 3(n \in \mathbb{N}), D^{\alpha}_{0+}$ is the Riemann–Liouville differential fractional derivative of order, $\hbar : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ and $\Im : [0,1] \times [0,+\infty) \to [0,+\infty)$ are given continuous functions. The authors obtained a unique solution by a mixed monotone operator method. In [20], we consider the following fractional differential equation:

$$^{c}D_{0^+}^{\alpha}x(t) + \ell(t, x(t), x'(t)) = 0,\ 0 < t < 1,$$

with boundary condition $x(0) = x''(0) = 0, x'(1) = \sum_{i=1}^{\infty} \wp_j x(\xi_j)$, where $\wp_j \geq 0, 2 < \alpha \leq 3$, $\wp_j \geq 0, 0 < \xi_j < 1(j = 1, 2, \cdots)$, $\ell(t, x, y)$ may be singular at $t = 0$, we obtain the result of multiple positive solutions by Avery and Peterson fixed point theorem. Boutiara [40] studied the following Caputo–Hadamard fractional differential equation

$$^{cH}D_{1^+}^{\vartheta}x(t) = \omega(t, x(t)), t \in J = [1, T], 0 < \vartheta \leq 1,$$

with three-point boundary condition

$$\alpha x(1) + \beta x(T) = \mu I^{\kappa} x(\tau) + \chi, \kappa \in (0, 1]$$

where I^{κ} denotes the standard Hadamard fractional integral and $^{cH}D_{1^+}^{\vartheta}$ denotes the Caputo–Hadamard fractional derivative, $0 < \vartheta, \kappa \leq 1, \omega : [1, T] \times \mathbb{R} \to \mathbb{R}$ is continuous. $\alpha, \beta, \mu, \kappa$ are real constants, and $\tau \in (1, T)$. The uniqueness results were obtained using Boyd and Wong's and Banach's fixed point theorems. Ardjouni [41] discusses the existence case of positive solutions of the following nonlinear fractional differential $^{cH}D_{1^+}^{\sigma}y(t) = f(t, x(t)), t \in J$ with integral boundary conditions $y(1) = b\int_1^e y(s)ds + d$, where $J = [1, e]$, $^{cH}D_{1^+}^{\sigma}$ denote the Caputo–Hadamard fractional derivative, $0 < \sigma \leq 1$ and $f : J \times [0, \infty) \to [0, \infty)$ is a given continuous function. The authors discuss the existence and uniqueness of positive solutions by some methods. Motivated by the excellent results above, in this paper, we investigate the monotone positive solutions for fractional differential equation under Caputo–Hadamard fractional order differential with infinite-point boundary value conditions.

Up until now, for the fractional differential equation boundary value problem, most of the results are obtained in the sense of fractional derivatives such as Caputo and Riemann–Liouville, and there are few models under the Caputo–Hadamard fractional derivatives. Compared with the Caputo and Riemann–Liouville fractional derivative, the Caputo–Hadamard fractional order derivative contained logarithmic function of arbitrary order, which is invariant to dilation on the half-axis. As far as we are aware, there are few results on solutions of Caputo–Hadamard fractional differential equations so far. In this paper, we consider the following infinite-point Caputo–Hadamard fractional differential equation

$$^{cH}D^{\sigma}u(t)) + \aleph(t, u(t), u(t)) + \Re(t, u(t)) = 0,\ 1 < t < e, \tag{1}$$

with boundary value condition

$$u^{(j)}(1) = 0, j = 0, 1, 2, \ldots, n-1, j \neq i, u^{(i)}(e) = \sum_{j=1}^{\infty} \eta_j u(\xi_j), \tag{2}$$

where $n - 1 < \sigma \leq n$, $\sigma > i + 1$, $\eta_j \geq 0$, $1 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < e(j = 1, 2 \cdots)$, $\Sigma_{j=1}^{\infty}\eta_j(\ln \xi_j)^i < i!, i$ is a fixed constant, $\aleph : [1, e] \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ and $\Re : [1, e] \times [0, +\infty) \to \times[0, +\infty)$ are given continuous functions, $\aleph(t, x, y)$ may be singular at $t = 1$ and $^{cH}D_{1^+}^{\sigma}u$ is the standard Caputo–Hadamard derivative. The existence of multiple positive solutions is obtained for the boundary value problem under sufficient conditions.

Compared with [29], the derivative of this paper is a Caputo–Hadamard fractional derivative and the solutions we obtained are iterative solutions. Compared with [23], the derivative of this paper is a Caputo–Hadamard fractional derivative and infinite point is involved in boundary value conditions in BVP (1–2), and an iterative solution is obtained by a sequence of iterations. As far as we are aware, this is the first paper to investigate this type of infinite-point Caputo–Hadamard fractional differential equation.

2. Preliminaries and Methods

We present some basic definitions, methods and lemmas which will be used in the proof of our results and can also be found in the recent literature such as [42].

Definition 1 ([42])**.** *The Hadamard fractional integral of order* $\alpha > 0$ *of a function* $\delta : (0, \infty) \to R$ *is given by*

$$^{H}I^{\sigma}_{1^+}\delta(t) = \frac{1}{\Gamma(\sigma)} \int_1^t (\ln \frac{t}{s})^{\sigma-1} \delta(s) \frac{ds}{s},$$

provided the right-hand side is pointwise defined on $(0, \infty)$*.*

Definition 2 ([42])**.** *The Caputo–Hadamard fractional derivative of order* $\sigma > 0$ *of a continuous function* $\delta : (0, \infty) \to R$ *is given by*

$$^{cH}D^{\sigma}_{1^+}\delta(t) = \frac{1}{\Gamma(n-\sigma)} \int_0^t (t\frac{d}{dt})^n \frac{\delta(s)}{(t-s)^{\sigma-n+1}} \frac{ds}{s},$$

where $n = [\sigma] + 1$*,* $[\sigma]$ *denotes the integer part of the number* σ*, provided that the right-hand side is pointwise defined on* $(0, \infty)$*.*

Definition 3 ([42])**.** *The Caputo fractional derivative of order* $\sigma > 0$ *of a function* $\delta : (0, \infty) \to R$ *is given by*

$$^{c}D^{\sigma}_{0^+}\delta(t) = \frac{1}{\Gamma(n-\sigma)} \int_0^t \frac{\delta^{(n)}(s)}{(t-s)^{\sigma-n+1}} ds,$$

where σ *is a fractional number,* $n = [\sigma] + 1$*, provided that the right-hand side is pointwise defined on* $(0, \infty)$*.*

Lemma 1 ([42])**.** *Let* $n - 1 < \sigma \le n, n \in \mathbb{N}$ *and* $x \in C^n([1, T])$*. Then,*

$$({}^{cH}D^{\sigma}_{1^+}\,{}^{H}I_{1^+}{}^{\sigma}x)(t) = x(t),$$

$$({}^{H}I_{1^+}{}^{\sigma cH}D^{\alpha}_{1^+}x)(t) = x(t) + \Sigma_{k=0}^{n-1} c_k (\ln t)^k,$$

where $\sigma,\ c_k \in R\ (k = 1, 2, \ldots, n-1)$*.*

Lemma 2. *Given* $h \in L^1[1, e]$*, then the Caputo–Hadamard linear fractional order differential equation*

$$^{cH}D^{\sigma}_{1^+}u(t)) + h(t) = 0,\ 1 < t < e, \tag{3}$$

with boundary condition (2) can be expressed by

$$u(t) = \int_1^e \Xi(t, s) h(s) \frac{ds}{s},\ t \in [1, e], \tag{4}$$

where

$$\Xi(t, s) = \frac{1}{\Delta\Gamma(\sigma)} \begin{cases} (\ln t)^i \vartheta(s) (\ln \frac{e}{s})^{\sigma-i-1} - \Delta(\ln \frac{t}{s})^{\sigma-1},\ 1 \le s \le t \le e, \\ (\ln t)^i P(s) (\ln \frac{e}{s})^{\sigma-i-1},\ 1 \le t \le s \le e, \end{cases} \tag{5}$$

in which $\vartheta(s) = (\sigma-1)(\sigma-2) \times (\sigma-i) - \Sigma_{s \le \xi_j} \eta_j (\frac{\ln \frac{\xi_j}{s}}{\ln \frac{e}{s}})^{\sigma-1} (\ln \frac{e}{s})^i$ *and* $\Delta = i! - \Sigma_{j=1}^{\infty} \eta_j (\ln \xi_j)^i$*.*

Proof. By Lemma 1, we can deduce Equation (3) to the following equivalent equation

$$u(t) = -{}^{cH}I^{\sigma}_{1^+}h(t) + r_0 + r_1(\ln t) + \cdots + r_{i-1}(\ln t)^{i-1} + r_{i+1}(\ln t)^{i+1} + \ldots + r_n(\ln t)^{n-1},$$

for $r_i(i = 1, 2, \ldots, n-1) \in \mathbb{R}$. From $x^{(j)}(1) = 0, j = 0, 1, 2, \ldots, n-1, j \neq i$, we have $r_j = 0, j \neq i$. Consequently, we obtain

$$u(t) = r_i(\ln t)^i -^{cH} I^{\sigma}_{1^+} h(t),$$

$$u^{(i)}(t) = -^{cH} I^{\sigma-i}_{1^+} h(t) + i! r_i.$$

On the other hand, $u^{(i)}(1) = \sum_{j=1}^{\infty} \eta_j u(\xi_j)$, combining with

$$u^{(i)}(1) = i! r_i -^{cH} I^{\sigma-i}_{1^+} h(t),$$

we obtain

$$\begin{aligned} r_i &= \int_1^e \frac{(\ln \frac{e}{s})^{\sigma-i-1}}{\Gamma(\sigma-i)(i! - \sum_{j=1}^{\infty} \eta_j (\ln \xi_j)^i)} h(s) \frac{ds}{s} \\ &\quad - \sum_{j=1}^{\infty} \eta_j \int_1^{\xi_j} \frac{(\xi_j - s)^{\sigma-1}}{\Gamma(\sigma)(i! - \sum_{j=1}^{\infty} \eta_j (\ln \xi_j)^i)} h(s) \frac{ds}{s} \\ &= \int_1^e \frac{(\ln \frac{e}{s})^{\sigma-i-1} \vartheta(s)}{\Gamma(\sigma)\Delta} h(s) \frac{ds}{s}, \end{aligned}$$

$\vartheta(s) = (\sigma-1)(\sigma-2) \times (\sigma-i) - \Sigma_{s \leq \xi_j} \eta_j (\frac{\ln \frac{\xi_j}{s}}{\ln \frac{e}{s}})^{\sigma-1} (\ln \frac{e}{s})^i$ and $\Delta = i! - \Sigma_{j=1}^{\infty} \eta_j (\ln \xi_j)^i$. Hence

$$\begin{aligned} u(t) =& r_i(\ln t)^i -^{cH} I^{\sigma}_{1^+} h(t) \\ =& - \int_1^t \frac{\Delta(\ln \frac{t}{s})^{\sigma-1}}{\Gamma(\sigma)\Delta} h(s) \frac{ds}{s} + \int_1^e \frac{(\ln \frac{e}{s})^{\sigma-i-1} t^i \vartheta(s)}{\Gamma(\sigma)\Delta} h(s) \frac{ds}{s} \\ =& \int_1^e \Xi(t,s) h(s) \frac{ds}{s}. \end{aligned}$$

Therefore, expression (4) holds. □

Lemma 3. *The Green function (5) has the following properties:*

$$\Delta(\ln t)^i (\ln \frac{e}{s})^{\sigma-i-1}[1 - (\ln \frac{e}{s})^i] \leq \Delta\Gamma(\sigma)\Xi(t,s) \leq (\ln t)^i \Gamma(\sigma)\vartheta(s)(\ln \frac{e}{s})^{\alpha-i-1}.$$

Proof. By direct calculation, we obtain $\vartheta'(s) \geq 0,\ s \in [1, e]$, and so $\vartheta(s)$ is nondecreasing with respect to s. For $s \in [1, e], \sigma - 1 > i$, we obtain

$$\begin{aligned} \vartheta(s) =& (\sigma-1)(\sigma-2) \times (\sigma-i) - \sum_{s \leq \xi_j} \eta_j \left(\frac{\ln \frac{\xi_j}{s}}{\ln \frac{e}{s}} \right)^{\sigma-1} (\ln \frac{e}{s})^i \\ \geq& \vartheta(1) = (\sigma-1)(\sigma-2) \times (\sigma-i) - \sum_{j=1}^{\infty} \eta_j (\ln \xi_j)^{\sigma-1} \\ \geq& i! - \sum_{j=1}^{\infty} \eta_j (\ln \xi_j)^i = \Delta. \end{aligned}$$

Then, we prove Lemma 3. The right inequality of Lemma 3 is trivial. We have only to prove the left inequality. If $1 \leq s \leq t \leq e$, we have $\ln t - \ln s \leq \ln t - \ln t \ln s = (1 - \ln s) \ln t$, which implies that

$$(\ln \frac{t}{s})^{\sigma-1} \leq (\ln \frac{e}{s})^{\sigma-1} (\ln t)^{\sigma-1}.$$

Then,

$$\begin{aligned}\Delta\Gamma(\sigma)\Xi(t,s) =&(\ln t)^i\vartheta(s)(\ln\frac{e}{s})^{\sigma-i-1}-\Delta(\ln\frac{t}{s})^{\sigma-1}\\ \geq&\Delta[(\ln t)^i(\ln\frac{e}{s})^{\sigma-i-1}-\Delta(\ln\frac{t}{s})^{\sigma-1}]\\ \geq&\Delta[(\ln t)^i(\ln\frac{e}{s})^{\sigma-i-1}-(\ln\frac{e}{s})^{\sigma-1}(\ln t)^{\sigma-1}]\\ \geq&\Delta(\ln t)^i(\ln\frac{e}{s})^{\sigma-i-1}[1-(\ln\frac{e}{s})^i]\end{aligned}$$

If $1\leq t\leq s\leq e$, then we have

$$\begin{aligned}\Delta\Gamma(\sigma)\Xi(t,s) =&(\ln t)^i\Gamma(\sigma)\vartheta(s)(\ln\frac{e}{s})^{\sigma-i-1}\\ \geq&\Delta[t^i(\ln\frac{e}{s})^{\sigma-i-1}-\Delta(\ln\frac{t}{s})^{\sigma-1}]\\ \geq&\Delta[t^i(\ln\frac{e}{s})^{\sigma-i-1}-(\ln\frac{e}{s})^{\sigma-1}(\ln t)^{\sigma-1}]\\ \geq&\Delta t^i(\ln\frac{e}{s})^{\sigma-i-1}[1-(\ln\frac{e}{s})^i].\end{aligned}$$

Thus, the left inequality is proved. Next, we will present some concepts of complete spaces and a fixed point theorem, which will be used later. □

Suppose that $(E,\|\cdot\|)$ is a real Banach space, which is partially ordered by a cone $K\subset E$, i.e.,

$$u,v\in E, u\preceq v\Leftrightarrow v-u\in K.$$

If $u\preceq v$ and $u\neq v$, then we denote $u\prec v$ or $v\succ u$. The zero element in E is denoted by θ_E.

Putting $int(K)=\{u\in K|\ u \text{ is an interior point of } K\}$, if the interior $int(K)$ is nonempty, then the cone K is said to be solid. Moreover, K is called normal if there exists a constants $N>0$ such that, for all $u,v\in E$, $\theta_E\preceq u\preceq v$ implies $\|u\|\leq N\|v\|$. In this case, the smallest constant satisfying the above inequality is called the normality constant of K. For all $u,v\in E$, the notation $u\sim v$ means that there exists $\lambda_1>0$ and $\lambda_2>0$ such that

$$\lambda_1 v\preceq u\preceq\lambda_2 v.$$

Obviously, $\sim$ is an equivalence relation. Given $h\succ\theta_E$, we denote K_h by

$$K_h=\{u\in E|u\sim h\}.$$

It is obvious that $K_h\subset K$.

Definition 4. *An operator $\Theta:E\to E$ is said to be increasing (respectively, decreasing) if for all $u,v\in E, u\preceq v$ implies $\Theta u\preceq\Theta v$(respectively, $\Theta u\succeq\Theta v$).*

Definition 5. *If $\Theta(u,v)$ is increasing in first component and decreasing in second component, then operator $\Theta:K\times K\to K$ is said to be a mixed monotone operator, i.e.,*

$$(u_1,v_1),(u_2,v_2)\in K\times K, u_1\preceq u_2, v_1\succeq v_2\Rightarrow\Theta(u_1,v_1)\preceq\Theta(u_2,v_2).$$

Definition 6. *An operator $\Theta:K\to K$ is said to be sub-homogeneous if it satisfies*

$$\Theta(tu)\succeq t\Theta u,\forall t\in(1,e), u\in K.$$

Lemma 4 (See [43])**.** *Let $\gamma\in(0,1)$. Let $\Theta:K\to K$ be a mixed monotone operator that satisfies*

$$\Theta(tu,t^{-1}v)\succeq t^{\gamma}\Theta(u,v), t\in(1,e), u,v\in K.$$

Let $B : K \to K$ be an increasing sub-homogeneous operator. Assume the following:

(i) *There is $h_0 \in K_h$ such that $\Theta(h_0, h_0) \in K_h$ and $Bh_0 \in K_h$;*

(ii) *There exists a constant $\delta_0 > 0$ such that $\Theta(u, v) \succeq \delta_0 Bu$ for all $u, v \in K$, then:*

   (a) $\Theta : K_h \times K_h \to K_h, B : K_h \to K_h$;

   (b) *There exist $x_0, y_0 \in P_h$ and $r \in (0, 1)$ such that*

$$ry_0 \preceq x_0 \prec y_0, x_0 \preceq \Theta(x_0, y_0) + Bx_0 \preceq \Theta(y_0, x_0) + By_0 \preceq y_0;$$

   (c) *There exists a unique $u^\star \in K_h$ such that $u^\star = \Theta(u^\star, u^\star) + Bu^\star$;*

   (d) *For any initial values $u_0, u_0 \in K_h$, by constructing successively the sequences*

$$u_n = \Theta(u_{n-1}, u_{n-1}) + Bu_{n-1}, v_n = \Theta(v_{n-1}, u_{n-1}) + Bv_{n-1}, n = 1, 2, \ldots,$$

we have

$$\lim_{n\to\infty} \|u_n - u^\star\| = \lim_{n\to\infty} \|v_n - v^\star\| = 0.$$

3. Result

The main result of this paper is Theorem 1.

Let $E = C([1, e])$ be the Banach space of real continuous functions defined on $[1, e]$ endowed with the norm

$$\|u\| = \max\{|u(t)| : t \in [1, e]\},$$

and $K \subset E$ be the positive cone defined by

$$K = \{u \in C([1, e]) : u(t) \geq 0, t \in [1, e]\}.$$

The main result of this paper is in the following.

Theorem 1. *Assume that*

(a) *$\aleph : [1, e] \times [0, +\infty) \to [0, +\infty)$ and $\Re : [1, e] \times [0, +\infty) \to [0, +\infty)$ are continuous with*

$$m(\{t \in [1, e] : \Re(t, 1) \neq 0\}) > 0,$$

where for some measurable set Ψ, $m(\Psi)$ denotes the Legesgue measure of Ψ;

(b) *$\aleph(t, u, v)$ is increasing on $u \in [0, +\infty)$ for fixed $t \in [1, e]$ and is decreasing on $v \in [0, +\infty)$ for fixed $t \in [1, e]$ and $u \in [0, +\infty)$, and $\Re(t, u)$ is increasing on $u \in [0, +\infty)$ for fixed $t \in [1, e]$;*

(c) *$\Re(t, \lambda u) \geq \lambda \Re(t, u)$ for all $\lambda \in (0, 1), t \in [1, e], u \in [0, +\infty)$;*

(d) *There exists a constant $\gamma \in (0, 1)$ such that*

$$\aleph(t, \lambda u, \lambda^{-1} v) \geq \lambda^\gamma \aleph(t, u, v), \lambda \in (0, 1), t \in [1, e], u, v \in [0, +\infty);$$

(e) *There exists a constant $\delta_0 > 0$ such that*

$$\aleph(t, u, v) \geq \delta_0 \Re(t, u), t \in [1, e], u, v \in\in [0, +\infty).$$

Then:

(1) *There exist $x_0, y_0 \in K_h$ and $r \in (0, 1)$ such that $rx_0 \leq y_0 \prec y_0$ and*

$$x_0(t) \leq \int_1^e \Xi(t, s) \aleph(s, x_0(s), y_0(s)) \frac{ds}{s} + \int_1^e \Xi(t, s) \Re(s, x_0(s)) \frac{ds}{s},$$
$$y_0(t) \geq \int_1^e \Xi(t, s) \aleph(s, y_0(s), x_0(s)) \frac{ds}{s} + \int_1^e \Xi(t, s) \Re(s, y_0(s)) \frac{ds}{s},$$

where $h(t) = (\ln t)^i, t \in [1, e]$;

(2) *Equations (1) and (2) has a unique positive solution $u^\star \in K_h$;*

(3) *For any* $u_0, v_0 \in P_h$*, constructing successively the sequences*

$$u_n(t) = \int_1^e \Xi(t,s)\aleph(s, u_{n-1}(s), v_{n-1}(s))\frac{ds}{s} + \int_1^e \Xi(t,s)\Re(s, u_{n-1}(s))\frac{ds}{s},$$
$$v_n(t) \geq \int_1^e \Xi(t,s)\aleph(s, v_{n-1}(s), u_{n-1}(s))\frac{ds}{s} + \int_1^e \Xi(t,s)\Re(s, v_{n-1}(s))\frac{ds}{s},$$

we have

$$\lim_{n\to\infty} \|u_n - u^\star\| = \lim_{n\to\infty} \|v_n - u^\star\| = 0.$$

Proof. From Lemma 2, Equation (3) has an integral formulation given by

$$u(t) = \int_1^e \Xi(t,s)\aleph(s, u(s), v(s))\frac{ds}{s} + \int_1^e \Xi(t,s)\Re(s, u(s))\frac{ds}{s}.$$

The operators $\Theta : K \times K \to E$ and $B : K \to E$ defined by

$$\Theta(u,v)(t) = \int_1^e \Xi(t,s)\aleph(s, u(s), v(s))\frac{ds}{s}, t \in [1,e],$$

$$Bu(t) = \int_1^e \Xi(t,s)\Re(s, u(s))\frac{ds}{s}, t \in [1,e].$$

Clearly, u is a solution to Equations (1) and (2) if and only if $\Theta(u,u) + Bu = u$. Furthermore, it follows from (b) of Theorem 1 that Θ is mixed monotone and B is increasing. On the other hand, for any $\lambda \in (0,1)$, $u, v \in K$, from (d) of Theorem 1, we have

$$\begin{aligned}\Theta(\lambda u, \lambda^{-1}v)(t) &= \int_1^e \Xi(t,s)\aleph(s, \lambda u(s), \lambda^{-1}v(s))\frac{ds}{s} \\ &\geq \lambda^\gamma \int_1^e \Xi(t,s)\aleph(s, u(s), v(s))\frac{ds}{s} = \lambda^\gamma\Theta(u,v)(t).\end{aligned}$$

Thus, we have

$$\Theta(\lambda u, \lambda^{-1}v) \succeq \lambda^\gamma\Theta(u,v), \lambda \in (0,1), u, v \in K.$$

From (c) of Theorem 1, for all $\lambda \in (0,1), u \in K$, we have

$$B(\lambda u)(t) = \int_1^e \Xi(t,s)\Re(s, \lambda u(s))\frac{ds}{s} \geq \lambda \int_1^e \Xi(t,s)\Re(s, u(s))\frac{ds}{s} = \lambda Bu(t).$$

Thus, we have

$$B(\lambda u) \succeq \lambda Bu, \lambda \in (0,1), u \in K,$$

which implies that B is a sub-homogeneous operator. Let $h \in K$ be defined by

$$h(t) = (\ln t)^i, t \in [1,e].$$

Using Lemma 3 and (b) of Theorem 1, we obtain

$$\begin{aligned}\Theta(h,h)(t) &= \int_1^e \Xi(t,s)\aleph(s, h(s), h(s))\frac{ds}{s} \\ &\leq \frac{(\ln t)^i}{\Delta} \int_1^e \vartheta(s)(\ln\frac{e}{s})^{\alpha-i-1}\aleph(s, e, 1)\frac{ds}{s}.\end{aligned}$$

Using Lemma 3 and (b) of Theorem 1, we obtain

$$\begin{aligned} re(h,h)(t) &= \int_1^e \Xi(t,s)\aleph(s,h(s),h(s))\frac{ds}{s} \\ &\geq \frac{(\ln t)^i}{\Gamma(\sigma)}\int_1^e (\ln\frac{e}{s})^{\sigma-i-1}[1-(\ln\frac{e}{s})^i]\aleph(s,e,1)\frac{ds}{s}, t\in[1,e]. \end{aligned}$$

Denote

$$\mu_1 = \frac{1}{\Gamma(\sigma)}\int_1^e (\ln\frac{e}{s})^{-i-1}[1-(\ln\frac{e}{s})^i]\aleph(s,1,e)\frac{ds}{s},$$

and

$$\mu_2 = \frac{1}{\Delta}\int_1^e \vartheta(s)(\ln\frac{e}{s})^{\sigma-i-1}\aleph(s,e,1)\frac{ds}{s}.$$

Then, we have

$$\mu_1 h \preceq \Theta(h,h) \leq \mu_2 h.$$

On the other hand, from (b) and (c) of Theorem 1, we have

$$\aleph(s,e,1) \geq \aleph(s,1,e) \geq \delta_0\Re(s,1) \geq 0.$$

Since $m(\{t\in[1,e]:\Re(t,e)\neq 0\})>0$, we obtain

$$\begin{aligned} \mu_2 =& \frac{1}{\Delta}\int_1^e \vartheta(s)(\ln\frac{e}{s})^{\sigma-i-1}\aleph(s,e,1)\frac{ds}{s} \\ \geq& \frac{\delta_0}{\Delta}\int_1^e \frac{\Delta}{\Gamma(\sigma)}(\ln\frac{e}{s})^{\sigma-i-1}\Re(s,1)\frac{ds}{s} \\ =& \frac{\delta_0}{\Gamma(\sigma)}\int_1^e (\ln\frac{e}{s})^{\sigma-i-1}\Re(s,1)\frac{ds}{s} > 0, \end{aligned}$$

and

$$\begin{aligned} \mu_1 =& \frac{1}{\Gamma(\sigma)}\int_1^e (\ln\frac{e}{s})^{\sigma-i-1}[1-(\ln\frac{e}{s})^i]\aleph(s,1,e)\frac{ds}{s} \\ \geq& \frac{\delta_0}{\Gamma(\sigma)}\int_1^e (\ln\frac{e}{s})^{\sigma-i-1}[1-(\ln\frac{e}{s})^i]\Re(s,1)\frac{ds}{s} > 0. \end{aligned}$$

Thus, we prove that $\Theta(h,h)\in K_h$. On the other hand, using the same method as above, we have

$$\frac{h(t)}{\Gamma(\sigma)}\int_1^e (\ln\frac{e}{s})^{\sigma-i-1}[1-(\ln\frac{e}{s})^i]\Re(s,1)\frac{ds}{s} \leq Bh(t) \leq \frac{h(t)}{\Delta}\int_1^e \vartheta(s)(\ln\frac{e}{s})^{\sigma-i-1}\Re(s,e)\frac{ds}{s}.$$

Denote

$$\lambda_1 = \frac{1}{\Gamma(\sigma)}\int_1^e (\ln\frac{e}{s})^{\sigma-i-1}[1-(\ln\frac{e}{s})^i]\Re(s,1)\frac{ds}{s},$$

and

$$\lambda_2 = \frac{1}{\Delta}\int_1^e \frac{\vartheta(s)}{\Delta}(\ln\frac{e}{s})^{\sigma-i-1}\Re(s,e)\frac{ds}{s}.$$

Then, we obtain

$$\lambda_1 h \preceq Bh \preceq \lambda_2 h.$$

From (b) and the condition $m(\{t\in[1,e]:\Re(t,e)\neq 0\})>0$, we have $\lambda_1>0$ and $\lambda_2>0$.

Thus, we have $Bh\in K_h$. Let $u,v\in K$. From (e), we obtain

$$\Theta(u,v)(t) = \int_1^e \Xi(t,s)\aleph(s,u(s),v(s))\frac{ds}{s} \geq \delta_0\int_1^e \Xi(t,s)\Re(s,u(s))\frac{ds}{s} = \delta_0 Bu(t), t\in[1,e].$$

Hence, we obtain $\Theta(u,v) \succeq \delta_0 Bu, u \in K$. Then, we obtain the desired results by applying Lemma 4.

Now, we define an operator $T = \Theta + B$ by

$$T(u(t), v(t)) = u(t) = \int_1^e \Xi(t,s)\aleph(s,u(s),v(s))\frac{ds}{s} + \int_1^e \Xi(t,s)\Re(s,u(s))\frac{ds}{s}.$$

Then, $T : K \times K \to K$ is a mixed monotone operator and $T(h,h) \in K_h$. As the similar method in [43], we obtain that there exists $\varphi(t) \in (t,e]$ with respect to $t \in (1,e)$ such that

$$T(\ln tu, (\ln t)^{-1}v) \geq \varphi(t)T(u,v), \forall u,v \in K.$$

The proof is as follows:

$$f(t) = \frac{(\ln t)^\beta - \ln t}{(\ln t)^\alpha - (\ln t)^\beta}, \forall t \in (1,e) \text{ where } \beta \in (\alpha, 1).$$

It is easy to show that f is increasing in $(1,e)$ and

$$\lim_{t\to 1^+} f(t) = 0, \lim_{t \to e^-} f(t) = \frac{1-\beta}{\beta-\alpha}.$$

Moreover, fixing $t \in (1,e)$, we obtain

$$\lim_{\beta \to 1^-} f(t) = \lim_{\beta \to 1^-} \frac{(\ln t)^\beta - \ln t}{(\ln t)^\alpha - (\ln t)^\beta} = 0.$$

Hence, there exists $\beta_0(t) \in (\alpha, 1)$ on t, such that

$$\frac{(\ln t)^{\beta_0(t)} - (\ln t)}{(\ln t)^\alpha - (\ln t)^{\beta_0(t)}} \leq \delta_0, t \in (1,e),$$

then we obtain

$$\Theta(u,v) \geq \delta_0 Bx \geq \frac{(\ln t)^{\beta_0(t)} - (\ln t)}{(\ln t)^\alpha - (\ln t)^{\beta_0(t)}} Bu, \forall t \in (1,e), u,v \in K.$$

Then, we have $(\ln t)^\alpha \Theta(u,v) + (\ln t)Bu \geq (\ln t)^{\beta_0(t)}[\Theta(u,v) + Bu], \forall t \in [1,e], u,v \in K$. Hence, for any $t \in (1,e)$ and $u,v \in K$,

$$\begin{aligned} T((\ln t)u, (\ln t)^{-1}v) &= \Theta((\ln t)u, (\ln t)^{-1}v) + B(\ln tu) \geq (\ln t)^\alpha \Theta(u,v) + (\ln t)Bu \\ &\geq (\ln t)^{\beta_0(t)}[\Theta(u,v) + Bu] = (\ln t)^{\beta_0(t)} T(u,v). \end{aligned}$$

Let $\varphi(t) = (\ln t)^{\beta_0(t)}, t \in [1,e]$. Hence, for any $t \in (1,e)$ and $u,v \in K$. Hence, by Lemma 1 and Theorem 1 in [44], we obtain thte following: (1) There exist $x_0, y_0 \in K_h$ and $r \in (0,1)$ such that $rx_0 \leq y_0 \prec y_0$ and

$$x_0(t) \leq \int_1^e \Xi(t,s)\aleph(s,x_0(s),y_0(s))\frac{ds}{s} + \int_1^e \Xi(t,s)\Re(s,x_0(s))\frac{ds}{s},$$

$$y_0(t) \geq \int_1^e \Xi(t,s)\aleph(s,y_0(s),x_0(s))\frac{ds}{s} + \int_1^e \Xi(t,s)\Re(s,y_0(s))\frac{ds}{s},$$

where $h(t) = (\ln t)^i, t \in [1,e]$; (2) Equations (1) and (2) has a unique positive solution $u^\star \in K_h$; (3) For any $u_0, v_0 \in P_h$, constructing successively the sequences,

$$u_n(t) = \int_1^e \Xi(t,s)\aleph(s,u_{n-1}(s),v_{n-1}(s))\frac{ds}{s} + \int_1^e \Xi(t,s)\Re(s,u_{n-1}(s))\frac{ds}{s},$$

$$v_n(t) \geq \int_1^e \Xi(t,s)\aleph(s, v_{n-1}(s), u_{n-1}(s))\frac{ds}{s} + \int_1^e \Xi(t,s)\Re(s, v_{n-1}(s))\frac{ds}{s},$$

we have

$$\lim_{n\to\infty} \|u_n - u^\star\| = \lim_{n\to\infty} \|v_n - u^\star\| = 0.$$

The proof of Theorem 1 is completed. □

4. Discussion

Compared with Caputo and Riemann–Liouville fractional derivatives, the Caputo–Hadamard fractional order derivative contains a logarithmic function of arbitrary order, which is invariant to dilation on the half-axis. The integral of the Caputo and Riemann–Liouville fractional equations with respect to the time variable is $(0,1)$, and the interval of the Caputo–Hadamard fractional equation with respect to the time variable is $(1,e)$, and we obtain the iterative solution by an iterative method under the Caputo–Hadamard fractional differentiation on $(1,e)$, the following equation is taken as an example. Consider the following infinite-point Hadamard fractional differential equations

$$^{cH}D_{1^+}^{\frac{7}{2}} u(t) + 3(t^3 + (u(t))^{\frac{1}{4}}) + \frac{1}{\sqrt[4]{u(t)+1}} = 0,\ 1 < t < e, \tag{6}$$

with infinite-point boundary condition

$$u(1) = u'(1) = u'''(1) = 0, u''(e) = \sum_{j=1}^{\infty} \frac{1}{2j^2} u\left(\frac{1}{j^4}\right), \tag{7}$$

where $\alpha = \frac{7}{2}, \mu = \frac{1}{2}, p_1 = \frac{5}{2}, p_2 = \frac{3}{2}, \eta_j = j^4, \xi_j = e^{\frac{1}{j^4}}$. Consider the functions $\aleph : [1,e] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ and $\Re : [1,e] \times [0,+\infty) \to [0,+\infty)$ defined by

$$\aleph(t,u,v) = 2(t^2 + \sqrt[4]{u(t)}) + \frac{1}{\sqrt[4]{y+1}}, \Re(t,u) = \sqrt[4]{u} + t^2, t \in [1,e], u, v \geq 0.$$

Then, Equations (6) and (7) is equivalent to

$$\begin{aligned} &^{cH}D_{1^+}^{\frac{7}{2}} u(t) + \aleph(t, u(t), u(t)) + \Re(t, u(t)) = 0,\ 1 < t < e, \\ &u(1) = u'(1) = u'''(1) = 0, u''(e) = \sum_{j=1}^{\infty} \frac{1}{2j^2} u\left(\frac{1}{j^4}\right). \end{aligned} \tag{8}$$

Let us check that the conditions of Theorem 1 are satisfied. Clearly,

$$\Delta = i! - \sum_{j=1}^{\infty} \eta_j (\ln \xi_j)^i = \quad 2! - \sum_{j=1}^{\infty} \frac{1}{j^4} = 0.9177,$$

the functions $\aleph : [1,e] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ and $\Re : [1,e] \times [0,+\infty) \to [0,+\infty)$ are continuous with

$$m(\{t \in [1,e] : \Re(t,e) \neq 0\}) = 1.$$

We can observe easily that $\aleph(t,u,v)$ is increasing on $u \in [0,+\infty)$ for fixed $t \in [1,e]$ and $v \in [0,+\infty)$, decreasing on $v \in [0,+\infty)$ for fixed $t \in [1,e]$ and $u \in [0,+\infty)$, and $\Re(t,u)$ is increasing on $u \in [0,+\infty)$ for fixed $t \in [1,e]$. For all $\lambda \in (1,e)$, $t \in [1,e]$ and $u \geq 0$, we obtain

$$\Re(t,\lambda u) = \sqrt[4]{\lambda u} + t^2 = \lambda^{\frac{1}{4}} u^{\frac{1}{4}} + t^2 \geq \lambda(\sqrt[4]{u} + t^2) = \lambda \Re(t,u).$$

For all $\lambda \in (1, e), t \in [1, e], u, v \geq 0$, we obtain

$$\begin{aligned} \aleph(t, \lambda u, \lambda^{-1} v) = &\ 2t^2 + 2\sqrt[4]{\lambda u} + \frac{1}{\sqrt[4]{\lambda^{-1} v + 1}} \\ \geq &2\sqrt{\lambda}(2t^2 + 2\sqrt[4]{u} + \frac{1}{\sqrt[4]{v+1}}) \\ = &\lambda^{\frac{1}{2}} \aleph(t, u, v). \end{aligned}$$

For all $t \in [1, e], u, v \geq 0$, we obtain

$$\aleph(t, u, v) = 2(t^2 + \sqrt[4]{u(t)}) + \frac{1}{\sqrt[4]{v+1}} \geq t^2 + \sqrt[4]{u} = 1 \cdot \Re(t, u).$$

Thus, we prove that all the hypotheses of Theorem 1 are satisfied. Thus, we deduce that Equation (6–7) has one and only one positive solution $u^\star \in K_h$, where $h(t) = (\ln t)^{\frac{5}{2}}, t \in [1, e]$.

5. Conclusions

It is precisely because of the obvious advantages and precise results in the simulation of systems that the fractional differential model has attracted more and more scholars' research interest in recent years, and the determination of the structure of the solution of the boundary value problem of the fractional differential model, as one of the important research directions, has been developing rapidly. In this paper, we first deduce Green's function under Caputo–Hadamard fractional differentiation, and the properties of Green's function are deduced, then, based on the properties of Green's function, the existence of iterative positive solutions for a nonlinear Caputo–Hadamard infinite-point fractional differential equation are obtained by iterative methods for sum operators, an example is proved to illustrate the main result.

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