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Surfaces with Constant Negative Curvature

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Abstract: In this paper, we have considered surfaces with constant negative Gaussian curvature in the simply isotropic 3-Space by defined Sauer and Strubeckerr. Firstly, we have studied the isotropic II-flat, isotropic minimal and isotropic II-minimal, the constant second Gaussian curvature, and the constant mean curvature of surfaces with constant negative curvature (SCNC) in the simply isotropic 3-space. Surfaces with symmetry are obtained when the mean curvatures are equal. Further, we have investigated the constant Casorati, the tangential and the amalgamatic curvatures of SCNC.

Keywords: negative Gaussian curvature; constant negative Gaussian curvature; simply isotropic 3-space

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1. Introduction

Constant curvature for surfaces was one of the top subjects regarding differential geometry in the 19th century (see [1,2]). Surfaces with curvature $\mathbf{K} = -1$ are denoted as $\mathbf{K}$-surfaces, and this topic is one of the main studies in differential geometry. The hyperbolic plane's intrinsic geometry is provided on $\mathbf{K}$-surfaces with a model [3,4] and the pseudosphere is the oldest known example of that geometry [5,6].

One of the most substantial problems in differential geometry is to construct a surface with constant negative Gaussian curvature in Euclidean space. From a known surface with $\mathbf{K} = -1$ [7], the Bäcklund's theorem provides a geometrical method to build a family of surfaces with Gaussian curvature. For the Gaussian $\mathbf{K}$-surfaces, Bäcklund transformation is given by Tian [8]. For pseudospherical surfaces, Bäcklund transformation can be limited to a transformation on space curves [9]. Many studies have been conducted in other spaces, such as Minkowski space [10,11]. $\mathbf{K}$-surfaces in an isotropic 3-space were studied extensively by K.Strubeckerr, as in [12–14]. Decu and Verstraelen defined isotropic Casorati curvature [15]. Suceava investigated the tangential and amalgamatic curvatures in Euclidean 3-space [16].

Casorati proposed the Casorati curvature over Gauss and mean curvatures since this correlates better with the general intuition of curvature [16,17]. Human/computer vision and geometry are investigated using Casorati curvature [18,19]. The idea behind the amalgamatic curvature is to expand the ratio $\frac{\tau}{\kappa}$ to the higher dimensions [20]. This idea can be traced back to papers [21,22], with the improvements provided in [23,24]. A recent application can be found in [16]. An important development for the invariant curvature is studied in [25] and named tangential curvature [16].

In this paper, we present the Gaussian, the second Gaussian, the mean, the second mean, the Casorati, the tangential and the amalgamatic curvatures of surfaces with a constant negative curvature defined by K.Strubeckerr and V.R.Sauer. The symmetry on the surfaces can be seen in the figures below.

2. Preliminaries

An absolute figure is an ordered triple (w, f_1, f_2) consisting of an absolute plane w with f_1 and f_2, which are its two complex–conjugate straight lines from the projective three-space $\mathcal{P}(\mathbb{R}^3)$. These are required to define the simply isotropic space $\mathbb{I}_3^1$, which is a Cayley–Klein space. $x_0 = 0$ give the absolute plane w and $x_0 = x_1 + ix_2 = 0$, $x_0 = x_1 - ix_2 = 0$ are the absolute lines f_1, f_2. They are called the homogeneous coordinates or projective coordinates in $\mathcal{P}(\mathbb{R}^3)$ [26]. Homogeneous coordinates played an important role in capturing the projection of a 3D view for use in monitors, T.V., etc. [27] Further information about Cayley–Klein spaces can be acquired from [28].

The absolute point $\mathbb{F}(0:0:0:1)$ is defined as the crossing node of these two lines. A motion in $\mathbb{I}_3^1$ is a given with corresponding coordinates, as $x = \frac{x_1}{x_0}, y = \frac{x_2}{x_0}, z = \frac{x_3}{x_0}$ can be found at [26] and given as

$$\begin{aligned} x' &= c_1 + x\cos\alpha - y\sin\alpha \\ y' &= c_2 + x\sin\alpha + y\cos\alpha \\ z' &= c_3 + c_4 x + c_5 y + z, \end{aligned} \tag{1}$$

where $c_1, c_2, c_3, c_4, c_5, \alpha \in \mathbb{R}$. These are named isotropic congruence transformations [26]. Isotropic congruence transformations looks like Euclidean motions (combination of a translation and a rotation) in the projection onto the xy-plane. This projection is named as a "top view" [29–31]. The combination of a Euclidean motion in the xy-plane and affine transformation with shearing along the z-direction is called an isotropic motion [32].

The equation

$$ds^2 = dx^2 + dy^2$$

is defined as the metric of $\mathbb{I}_3^1$. Let U = (u_1, u_2, u_3) and V = (v_1, v_2, v_3) be vectors in $\mathbb{I}_3^1$; then, the inner product of U and V is defined as,

$$< U, V >_i = \begin{cases} u_3 v_3 & \text{if } u_{1,2} = 0 \text{ and } v_{1,2} = 0 \\ u_1 v_1 + u_2 v_2 & \text{if otherwise} \end{cases}$$

This metric is induced by the absolute figure. If a line is not parallel to the z-direction, it is called non-isotropic; otherwise, it is isotropic. Isotropic planes are the planes that contain an isotropic line. Consider a C^r-surface **M**, $r \geq 1$, in $\mathbb{I}_3^1$ parameterized by

$$\mathbf{M} : \mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v)).$$

Let an arbitrary surface in $\mathbb{I}_3^1$ be called **M**. If a surface has no isotropic tangent planes, then it is called an admissible surface. The first and second forms **I** and **II**, called fundamental forms, of **M**, have the coefficients E, F, G and L, N, M, respectively, which can be easily stated with the induced matrix, given by [32] as,

$$\begin{aligned} I =& E du^2 + 2F du dv + G dv^2, \\ II =& L du^2 + 2M du dv + N dv^2, \end{aligned}$$

where (with $\delta = \sqrt{EG - F^2}$)

$$\begin{aligned} E &= < \mathbf{x}_u, \mathbf{x}_u >_i, F =< \mathbf{x}_u, \mathbf{x}_v >_i, G =< \mathbf{x}_v, \mathbf{x}_v >_i, \\ L &= \frac{det(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uu})}{\delta}, M = \frac{det(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uv})}{\delta}, N = \frac{det(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{vv})}{\delta}. \end{aligned}$$

Then, **K** and **H** , the isotropic Gaussian curvature and mean curvature, can be defined as

$$\mathbf{K} = k_1 k_2 = \frac{LN - M^2}{EG - F^2}, 2\mathbf{H} = k_1 + k_2 = \frac{EN - 2FM + GL}{EG - F^2}, \tag{2}$$

where k_1, k_2 are principal curvatures.Therefore, the extrema of the normal curvatures, k_1 and k_2, are determined with the non-isotropic section of the surface. Here, when $\mathbf{K} = 0$, and $\mathbf{H} = 0$, the surfaces **M** are called, respectively, isotropic flat and isotropic minimal [26]. Isotropic **II**-flat and isotropic **II**-minimal are named after the moment when a non-developable surface's second Gaussian curvature and second mean curvature are zero, respectively. The second Gaussian curvature $\mathbf{K_{II}}$ of $\mathbf{M}$ is given by

$$\mathbf{K_{II}}=\frac{1}{(LN-M^2)^2}\left[\begin{vmatrix} -\frac{L_{uu}}{2}+M_{uv}-\frac{N_{vv}}{2} & \frac{L_u}{2} & M_u-\frac{L_v}{2} \\ M_v-\frac{N_u}{2} & L & M \\ \frac{N_v}{2} & M & N \end{vmatrix} - \begin{vmatrix} 0 & \frac{L_v}{2} & \frac{N_u}{2} \\ \frac{L_v}{2} & L & M \\ \frac{N_u}{2} & M & N \end{vmatrix}\right]. \tag{3}$$

The second mean curvature for a surface in simply isotropic 3-space is given by [15]

$$\mathbf{H_{II}}=\mathbf{H}-\frac{1}{2\sqrt{LN-M^2}}\sum_{i,j=1}^{2}\frac{\partial}{\partial u_i}\left(\sqrt{LN-M^2}L^{ij}\frac{\partial}{\partial u_i}\left(\ln\sqrt{|\mathbf{K}|}\right)\right), \tag{4}$$

where $u_1=u$, $u_2=v$ and $\left(L^{ij}\right)$ is the inverse of the matrix L_{ij} of the second fundamental form [33]. The isotropic Casorati curvature is defined by

$$\mathbf{C}=\frac{k_1^2+k_2^2}{2}=2\mathbf{H}^2-\mathbf{K} \tag{5}$$

The tangential curvature and the amalgamatic curvature are given by

$$\tau=\frac{|k_1k_2|-1+\sqrt{(1+k_1^2)(1+k_2^2)}}{|k_1|+|k_2|}=\frac{|\mathbf{K}|-1+\sqrt{(\mathbf{K}-1)^2+4\mathbf{H}^2}}{\mathbf{H}}, \tag{6}$$

$$A=\frac{2k_1k_2}{k_1+k_2}=\frac{\mathbf{K}}{\mathbf{H}}, \tag{7}$$

respectively [16].

3. Curvatures of SCNC in $\mathbb{I}_3^1$

In this chapter, we provide the curvatures of SCNC in $\mathbb{I}_3^1$. The general form of SCNC can be derived as follows. Let $\mathbf{x}$ be a surface with

$$\mathbf{x}(u,v)=(x_1(u,v),x_2(u,v),x_3(u,v))$$

when we have

$$x_1(u,v)=f_1+g_1$$
$$x_2(u,v)=f_2+g_2$$

where $f_1'g_2'-f_2'g_1'\neq 0$ with f_1,f_2 and g_1,g_2 are twice continuous differentiable functions of u and v, respectively. The parametric curves are asymptotic if and only if the two conditions

$$\begin{vmatrix} f_1' & f_2' & x_{3u} \\ g_1' & g_2' & x_{3v} \\ f_1'' & f_2'' & x_{3uu} \end{vmatrix}=0, \qquad \begin{vmatrix} f_1' & f_2' & x_{3u} \\ g_1' & g_2' & x_{3v} \\ g_1'' & g_2'' & x_{3vv} \end{vmatrix}=0$$

are fulfilled. From the above equations with $f_1'g_2'-f_2'g_1'\neq 0$ condition, the existence of two functions $a(u,v)$ and $b(u,v)$ can be uniquely determined by $x_3(u,v)$, which satisfy the equations

$$x_{3u}=af_1'+bf_2', \qquad x_{3v}=ag_1'+bg_2',$$
$$x_{3uu}=af_1''+bf_2'', \qquad x_{3vv}=ag_1''+bg_2''.$$

By solving these equations, we can obtained

$$a_{uv} = b_{uv} = 0, a = \Phi_1(u) + \Psi_2(v), b = \Phi_2(u) + \Psi_2(v).$$

For the arbitrary functions $\Phi_1, \Phi_2, \Psi_1, \Psi_2$, from the solution of the previous equation, we obtain,

$$\Phi_1' f_1' + \Phi_2' f_2' = 0, \Psi_1' g_1' + \Psi_1' g_2' = 0,$$

results in

$$\Phi_1' = \Lambda(u) f_2', \Phi_2' = -\Lambda(u) f_1', \Psi_1' = M(v) g_2', \Psi_2' = -M(v) g_1'.$$

Then,

$$x_{3uv} = M(v)(g_2' f_1' - g_1' f_2'),$$
$$x_{3vu} = \Lambda(u)(g_1' f_2' - g_2' f_1')$$

by choosing

$$\Lambda(u) = -M(v) = const. = k_1$$

we obtain

$$a = k_1(f_2 - g_2) + k_2, b = -k_1(f_1 - g_1) + k_3$$

by using these in the equations,

$$x_{3u} = k_1(f_1' f_2 - f_1 f_2') + k_1(f_2' g_1 - f_1' g_2) + k_2 f_1' + k_3 f_2',$$

$$x_{3v} = k_1(g_2' g_1 - g_1' g_2) + k_1(g_1' f_2 - g_2' f_1) + k_2 g_1' + k_3 g_2'.$$

The integrability condition $x_{3uv} = x_{3vu}$ is fulfilled and, from integration, results in $x_3 = k_1\{(f_2' g_1 - f_1' g_2) + \int(f_1' f_2 - f_1 f_2')du + \int(g_2' g_1 - g_1' g_2)dv\} + k_2(f_1 + g_1) + k_3(f_2 + g_2) + k_4$, from this, $k_1 = 1, k_2 = k_3 = k_4 = 0$, we can obtained $\mathbf{x}(u,v) = (x_1, x_2, x_3)$ as

$$\mathbf{x}(u,v) = \begin{pmatrix} f_1 + g_1, \\ f_2 + g_2, \\ (f_2 g_1 - f_1 g_2) + \int(f_1' f_2 - f_2' f_1)du \\ + \int(g_1 g_2' - g_2 g_1')dv \end{pmatrix} \tag{8}$$

where f_1, f_2, f are functions of u and g_1, g_2, g are functions of v [12,34]. The isotropic curvature $\mathbf{K}$ and the mean curvature $\mathbf{H}$ of the surface (8) is given by

$$\mathbf{K} = -1, \ \mathbf{H} = \frac{f_1' g_1' + f_2' g_2'}{f_2' g_1' - f_1' g_2'}. \tag{9}$$

Assume the mean curvature of (8) is constant. Then,

$$\mathbf{H}_0 = \frac{f_1' g_1' + f_2' g_2'}{f_2' g_1' - f_1' g_2'}, \tag{10}$$

where $\mathbf{H}_0 \in \mathbb{R}$. If we use the separation of variables method, the mean curvature and the mean curvature $\mathbf{H}_0 \neq 0 \in \mathbb{R}$ if and only if

$$\frac{f_2'}{f_1'} = \frac{g_1' + \mathbf{H}_0 g_2'}{\mathbf{H}_0 g_1' - g_2'},$$

where u, v are independent variables and both sides of the Equation (10) are constant. If we show that this constant is equal to p, we can obtain

$$\frac{f_2'}{f_1'} = p = \frac{g_1' + \mathbf{H}_0 g_2'}{\mathbf{H}_0 g_1' - g_2'}. \tag{11}$$

Hence, we can write

$$\begin{cases} f_1 = c_1 + \frac{f_2}{p} \\ f_2 = c_2 + p f_1 \\ g_1 = c_3 + \frac{(\mathbf{H}_0 + p) g_2}{\mathbf{H}_0 p - 1} \\ g_2 = c_4 + \frac{(\mathbf{H}_0 p - 1) g_1}{\mathbf{H}_0 + p} \end{cases}, \tag{12}$$

where $c_i \in \mathbb{R}$.

If the surface (8) is isotropic minimal, then, from (10), we have

$$f_1' g_1' + f_2' g_2' = 0$$

and as a result of the solution of this equation,

$$\begin{cases} f_1 = c_1 + p f_2 \\ f_2 = c_2 + \frac{f_1}{p} \\ g_1 = c_3 - \frac{g_2}{p} \\ g_2 = c_4 - p g_2 \end{cases}.$$

Figures 1 and 2 are drawn for Equation (12); their functions are shown in their respective figures.

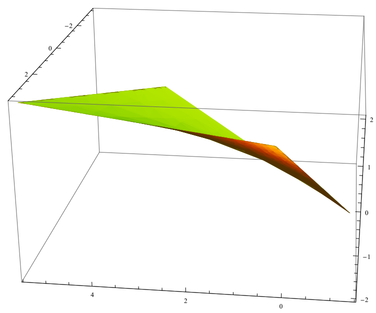

Figure 1. $f_1 = \sin u$ and $g_1 = \cos v$ with $c_2 = c_4 = 1$, $H_0 = 5/2$, $p = 2$.

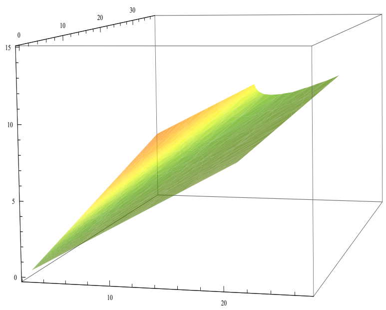

Figure 2. $f_1 = e^u$ and $g_1 = e^v$ with $c_2 = c_4 = 1$, $H_0 = 5/2$, $p = 2$.

If we choose

$$\begin{cases} f_1 = u \\ f_2 = f' \\ g_1 = v \\ g_2 = g' \end{cases}, \tag{13}$$

the surface (8) turns into the following form

$$\mathbf{x}(u,v) = \big(u + v, f' + g', 2(f - g) + (v - u)(f' + g')\big). \tag{14}$$

The Figure 3 with symmetry is drawn for (14) as an example where the constants c_2, c_4, H_0 and p are from Equation (12) and f_1, g_1 are from (13) [12–14].

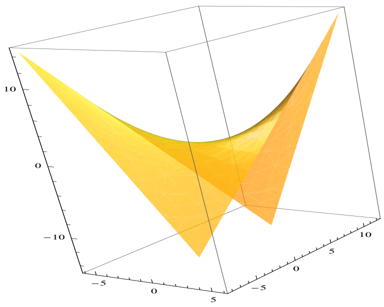

Figure 3. $f_1 = u$ and $g_1 = v$ with $c_2 = c_4 = 1$, $H_0 = 5/2$, $p = 2$.

Let us consider that the surface (14). Then, using the surface (14), the coefficients of the first and the second fundamental forms are given by

$$E = 1 + f''^2,\ G = 1 + g''^2,\ F = 1 + f''g'', \tag{15}$$

and

$$L = 0, N = 0,\ M = -(f'' - g''), \tag{16}$$

respectively. The isotropic Gaussian curvature $\mathbf{K}, \mathbf{K_{II}}$ and the mean curvatures $\mathbf{H}, \mathbf{H_{II}}$, the isotropic Casorati curvature, the tangential curvature and the amalgamatic curvature of the surface (14) are given by

$$\mathbf{K} = -1,\ \mathbf{K_{II}} = \frac{f'''g'''}{(f'' - g'')^3}, \tag{17}$$

$$\mathbf{H} = \mathbf{H_{II}} = \frac{1 + f''g''}{f'' - g''}, \tag{18}$$

$$\mathbf{C} = 1 + \frac{2(1 + f''g'')^2}{(f'' - g'')^2}, \tag{19}$$

$$\tau = \frac{\sqrt{(1 + f'')^2(1 + g'')^2}}{1 + f''g''}, \tag{20}$$

$$\mathbf{A} = \frac{\sqrt{(1 + f'')^2(1 + g'')^2}}{1 + f''g''}, \tag{21}$$

respectively. Let us assume that the surface (14) has a constant mean curvature. Then,

$$\frac{1 + f''g''}{f'' - g''} = \mathbf{H}_0, \tag{22}$$

where $\mathbf{H}_0 \in \mathbb{R}$. If we use the separation of variables method, the mean curvature $\mathbf{H}_0$ is a constant but 0, if and only if

$$\frac{1 - \mathbf{H}_0 f''}{-(\mathbf{H}_0 + f'')} = p_1 = g'', \tag{23}$$

where $p_1 \neq 0 \in \mathbb{R}$. We can easily obtain

$$\begin{cases} f = c_1 + uc_2 + \frac{\frac{u^2}{2}(1+\mathbf{H}_0 p_1)}{\mathbf{H}_0 - p_1} \\ g = c_3 + vc_4 + \frac{p_1 v^2}{2} \end{cases}, \tag{24}$$

where $c_i \in \mathbb{R}$.

We can obtain the next Figure 4 by choosing $c_1 = c_2 = c_3 = c_4 = 1$, $H_0 = 5/2$ and $p_1 = 2$ at (24) and by using (14).

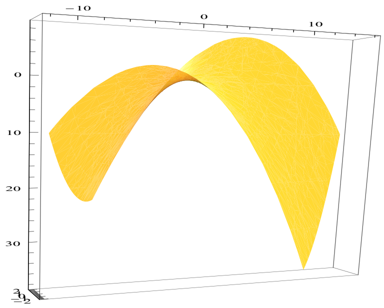

Figure 4. $f = 1 + u + 6u^2$ and $g = 1 + v + v^2$.

Suppose that the mean curvature $\mathbf{H}$ of the constant negative curvature surface vanishes identically; then, from (22), we have

$$1 + f''g'' = 0. \tag{25}$$

Here, u and v are independent variables, so each side of (25) is equal to a constant, called p_2. Hence, the two equations are

$$-\frac{1}{f''} = p_2 = g''. \tag{26}$$

With appropriate solutions to these differential equations, we have

$$\begin{cases} f = c_1 + uc_2 - \frac{u^2}{2p_2} \\ g = c_3 + vc_4 + \frac{p_2 v^2}{2} \end{cases}, \tag{27}$$

where $c_i \in \mathbb{R}$.

Thus, we have the following results:

Theorem 1. *Let* $\mathbf{M}$ *be the isotropic surface (14) with the constant isotropic mean curvature* $\mathbf{H} \neq 0$ *in* $\mathbb{I}_3^1$*. Then, the functions* f *and* g *are given by (24).*

Theorem 2. *Let* $\mathbf{M}$ *be the isotropic surface (14) with zero isotropic mean curvature (isotropic minimal* $\mathbf{H} \neq 0$*) in* $\mathbb{I}_3^1$*. Then, the functions* f *and* g *are given by (27).*

Theorem 3. *If the isotropic surface parametrized by (14) in* $\mathbb{I}_3^1$ *has* $\mathbf{K_{II}} = 0$*, then*

$$\begin{cases} f = c_1 + c_2 u + c_3 u^2 \\ g = c_4 + c_5 v + c_6 v^2 \end{cases}, \tag{28}$$

and if $\mathbf{K_{II}}$ *is a non-zero constant,*

$$\begin{cases} f = \frac{4\sqrt{2}(u+c_7)\sqrt{-c_2(u+c_7)} - 3puc_8(u+2c_7)}{6c_2} + c_9 + uc_{10} \\ g = c_{11} + vc_{12} - \frac{pv^2}{2} \end{cases},$$

where $c_i, p \in \mathbb{R}$ *where* $c_2 \neq 0$ *and* $c_2(u + c_7) \in \mathbb{R}^+$*.*

Proof. From (14), we have

$$\frac{f'''g'''}{(f''-g'')^3}=c, \tag{29}$$

where $c \in \mathbb{R}$.

If $\mathbf{K_{II}} = 0$, from the solution of the Equation (29), we have

$$f'''g''' = 0. \tag{30}$$

By solving (30), the functions f and g are obtained as follows

$$\begin{cases} f = c_1 + c_2 u + c_3 u^2 \\ g = c_4 + c_5 v + c_6 v^2 \end{cases},$$

where $c_i \in \mathbb{R}$.

When we choose the constants at (28) as $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 1$, we can obtain the figure with two different intervals as Figure 5.

Now, suppose that the second Gaussian curvature $\mathbf{K_{II}} = c$ is a non-zero constant. The partial derivative of (29) with respect to u gives

$$g'''\left(-\frac{3f'''^2}{f''-g''}+f^{(IV)}\right)=0. \tag{31}$$

Therefore, either, i.e., $g''' = 0$ or

$$\left(-\frac{3f'''^2}{f''-g''}+f^{(IV)}\right)=0. \tag{32}$$

If $g''' = 0$, then we have

$$g = c_1 + c_2 v + c_3 v^2,$$

where $c_i \in \mathbb{R}$. If $\left(-\frac{3f'''^2}{f''-g''}+f^{(IV)}\right)=0$, then we have

$$-g'' = \frac{3f'''^2 - f''f^{(IV)}}{f^{(IV)}}. \tag{33}$$

By solving (33), we find

$$\begin{cases} f = \frac{4\sqrt{2}(u+c_1)\sqrt{-c_2(u+c_1)}-3p_3uc_2(u+2c_1)}{6c_2}+c_3+uc_4 \\ g = c_5 + vc_6 - \frac{p_3 v^2}{2} \end{cases}, \tag{34}$$

where $c_i, p_3 \in \mathbb{R}$, where $c_2 \neq 0$ and $c_2(u+c_7) \in \mathbb{R}^-$.

As we choose $c_1 = -c_2 = c_3 = c_4 = c_5 = c_6 = p_3 = 1$ and $H_0 = 5/2$, we can obtain the Figure 6, shown from two different angles. □

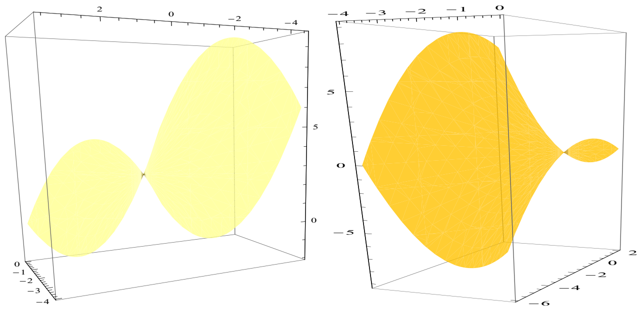

Figure 5. $f = 1 + u + u^2$ and $g = 1 + v + v^2$.

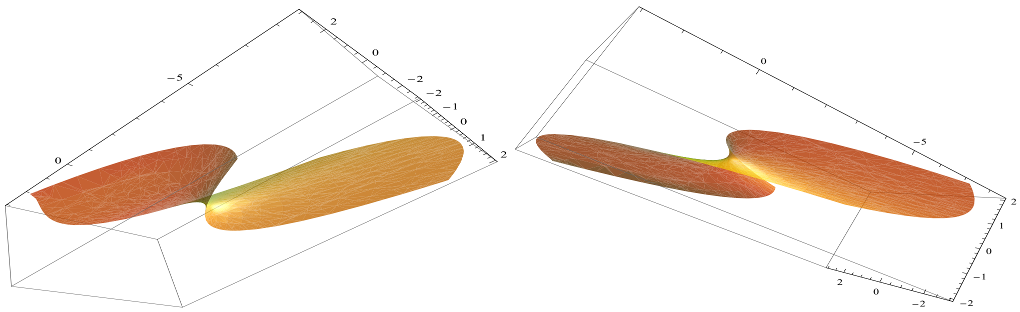

Figure 6. $f = 1 + u - \frac{1}{6}(4\sqrt{2}(1+u)^{\frac{3}{2}} + 3u(2+u)) \qquad g = 1 + v - \frac{v^2}{2}$.

Now, we consider the isotropic surface (14) in $\mathbb{I}_3^1$ as satisfying $\mathbf{C} = 2\mathbf{H}^2 - \mathbf{K}$ as constant. From (21), we have

$$1 + \frac{2(1 + f''g'')^2}{(f'' - g'')^2} = \mathbf{C}_0, \tag{35}$$

where $\mathbf{C}_0 \in \mathbb{R}$. Taking the partial derivative of (35) with respect to u gives

$$\Big((1 + \mathbf{C}_0)g'' + f''(1 - \mathbf{C}_0 + 2g''^2)\Big)f''' = 0. \tag{36}$$

If $f''' = 0$, then we have

$$f = c_1 + c_2 u + c_3 u^2, \tag{37}$$

where $c_i \in \mathbb{R}$. If $(1 + \mathbf{C}_0)g'' + f''(1 - \mathbf{C}_0 + 2g''^2) = 0$, then we have

$$\begin{cases} f = c_4 + c_5 u - p_4 \frac{u^2}{2} \\ g = c_6 + c_7 v + \frac{\left(1 + \mathbf{C}_0 - \sqrt{(1+\mathbf{C}_0)^2 + 8p_4^2(\mathbf{C}_0 - 1)}\right)v^2}{8p_4} \end{cases}, \tag{38}$$

where $c_i, p_4 \neq 0, \mathbf{C}_0 \in \mathbb{R}$, where $(1 + \mathbf{C}_0)^2 + 8p_4^2(\mathbf{C}_0 - 1)$ is positive.

$c_4 = c_5 = c_6 = c_7 = p_4 = 1$ and $\mathbf{C}_0 = 5/2$ are chosen for a Figure 7 at Equation (38).

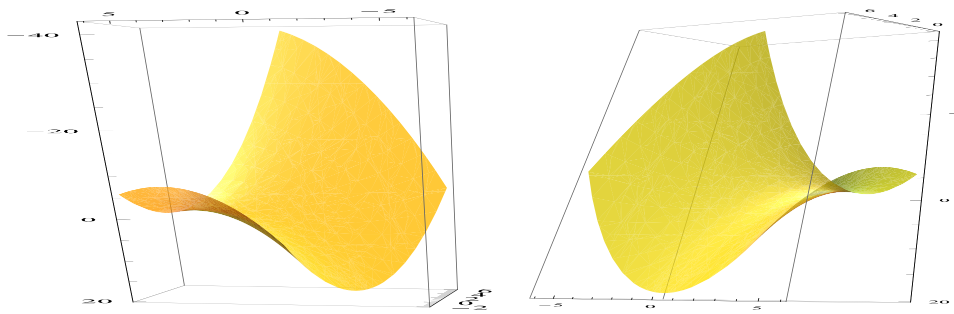

Figure 7. $f = 1 + u - u^2/2$ and $g = 1 + v + 1/8(7/2 - \sqrt{97}/2)v^2$.

If the surface (14) has zero Casorati curvature ($\mathbf{C} = 0$), then, by using a similar technique for the solution, we have

$$\begin{cases} f = c_4 + c_5 u + p_5 \frac{u^2}{2} \\ g = c_6 + c_7 v + \frac{\left(-1+\sqrt{1-8p_5^2}\right)v^2}{8p_5} \end{cases}, \tag{39}$$

where $c_i, p_5 \in \mathbb{R}$.

$c_4 = c_5 = c_6 = c_7 = 1$ and $p_5 = 0.35$ values are provided for the Figure 8 at Equation (39).

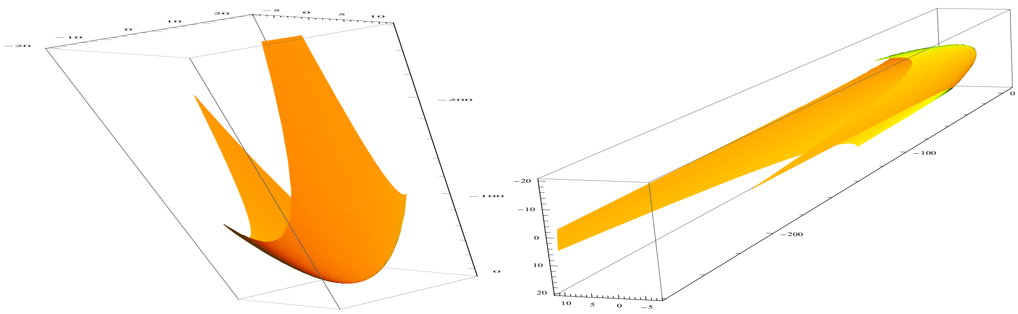

Figure 8. $f = 1 + u - 0.175u^2$ and $g = 1 + v + 0.306635v^2$.

Let us assume that the isotropic surface (14) has a constant tangential curvature. Then from (22), we have

$$\tau_0 = \frac{\sqrt{(1+f'')^2(1+g'')^2}}{1+f''g''}, \tag{40}$$

where $\tau_0 \in \mathbb{R}$. By solving (22) we can find

$$\begin{cases} f = c_1 + c_2 u + c_3 u^2 \\ f = c_4 + c_5 u + p_6 \frac{u^2}{2} \\ g = c_7 \pm v\left(c_8 + \frac{v p_6}{\tau_0^2 + \sqrt{-4p_6^2 + 4\tau_0^2 p_6^2 + \tau_0^4}}\right) \end{cases}, \tag{41}$$

where $c_i, p_6 \in \mathbb{R}$ where $-4p_6^2 + 4\tau_0^2 p_6^2 + \tau_0^4 \geq 0$

Figure 9 for (41) is constructed with the constants $c_4 = c_5 = c_6 = c_7 = c_8 = 1$, $p_6 = 0.8$ and $\tau_0 = 1$.

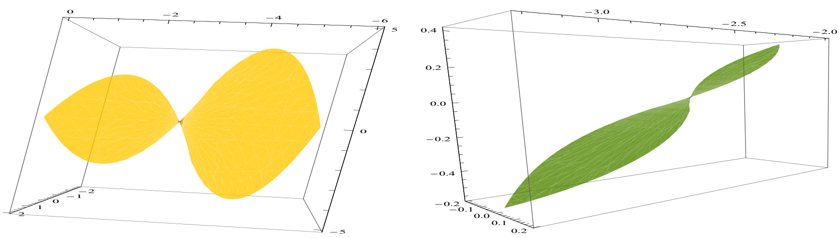

Figure 9. $f = 1 + u + 0.4u^2$ and $g = 1 + v + 0.4v^2$.

If the (14) has the constant amalgamatic curvature. Then, from (40), we obtain

$$\mathbf{A}_0 = \frac{\sqrt{(1+f'')^2(1+g'')^2}}{1+f''g''}, \tag{42}$$

where $A_0 \in \mathbb{R}$. By solving (42), we find

$$\begin{cases} f = c_1 + c_2 u + p_7 \frac{u^2}{2} \\ g = c_3 + c_4 v - \frac{v^2(2p_7 + \mathbf{A}_0)}{2(p_7 \mathbf{A}_0 - 2)} \end{cases}, \tag{43}$$

where $c_i, p_7 \in \mathbb{R}$ with $p_7 \mathbf{A}_0 - 2 \neq 0$.

The Figure 10 is for Equation (43), where the constants are $c_1 = c_2 = c_3 = c_4 = \mathbf{A}_0 = 1$ and $p_7 = 0.8$.

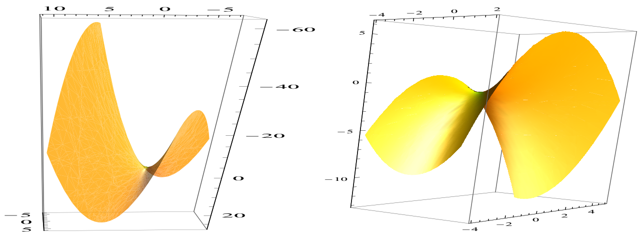

Figure 10. $f = 1 + u + 0.4u^2$ and $g = 1 + v + 1.08333v^2$.

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References

1. Bianchi, L. *Lezioni di Geometria Differenziale*; Ecole Bibliotheque: Pisa, Italy, 1902.
2. Darboux, G. *Lecons sur la Theorie Generate des Surfaces et les Applications Geometriques du Calcul Infinitesimal*; Gauthier-Villars: Paris, France, 1887–1896; pp. 1–4.
3. Pinkall, U. Designing Cylinders with Constant Negative Curvature. In *Discrete Differential Geometry. Oberwolfach Seminars*; Birkhäuser: Basel, Switzerland, 2008; Volume 38.
4. Bobenko, A.; Pinkall, U.J. Discrete surfaces with constant negative Gaussian curvature and the Hirota equation. *Differ. Geom.* **1996**, *43*, 527–611. [CrossRef]
5. Melko, M.; Sterling, I. Application of soliton theory to the construction of pseudospherical surfaces in R3. *Ann. Glob. Anal. Geom.* **1993**, *11*, 65–107. [CrossRef]
6. Cieslinski, J.L. Pseudospherical surfaces on time scales: A geometric definition and the spectral approach. *J. Phys. A Math. Theor.* **2007**, *40*, 12525–12538. [CrossRef]
7. Aldossary, M.T. Bäcklund's Theorem for Spacelike Surfaces in Minkowski 3-Space. *OSR J. Math.* **2013**, *5*, 24–30.
8. Tian, C. Bäcklund transformations for surfaces with $K = -1$ in $\mathbb{R}^{2,1}$. *J. Geom. Phys.* **1997**, *22*, 212–218. [CrossRef]
9. Erdoğdu, M.; Özdemir, M. On Bäcklund Transformation Between Timelike Curves in Minkowski Space-Time. *SDU J. Nat. Appl. Sci.* **2018**, *22*, 1143–1150.
10. Kim, Y.H.; Yoon, D.W. Classification of ruled surfaces in Minkowski 3-spaces. *J. Geom. Phys.* **2004**, *49*, 89–100. [CrossRef]
11. López, R. Differential Geometry of Curves and Surfaces in Lorentz-Minkowski Space. *Int. Electron. J. Geom.* **2014**, *7*, 44–107. [CrossRef]
12. Strubeckerr, K. Theorie der flachentreuen Abbildungen der Ebene, Contributions to Geometry. In Proceedings of the Geometry-Symposium, Siegen, Germany, 28 June–1 July 1978.
13. Strubeckerr, K. Die Flächen konstanter Relativkrümmung $K = rt - s^2$. Differentialgeometrie des isotropen Raumes II. *Math. Z.* **1942**, *47*, 743–777. [CrossRef]
14. Strubecker, K. Die Flächen, deren Asymptotenlinien ein Quasi-Rückungsnetz bilden. *Sitzungsberichte* **1952**, *12*, 103–110.
15. Decu, S.; Verstraelen, L. A note on the isotropical Geometry of production surfaces. *Kragujev. J. Math.* **2013**, *37*, 217–220.
16. Brubaker, N.D.; Camero, J.; Rocha, O.R.; Suceava, B.D. A ladder of curvatures in the geometry of surfaces. *Int. Electron. J. Geom.* **2018**, *11*, 28–33.

17. Casorati, F. Mesure de la courbure des surfaces suivant l'idée commune. *Acta Math.* **1890**, *14*, 95–110. [CrossRef]
18. Koenderink, J.J. *Shadows of Shapes*; De Clootcrans Press: Utrecht, The Netherlands, 2012.
19. Verstraelen, L. Geometry of submanifolds I. The first Casorati curvature indicatrices. *Kragujev. J. Math.* **2013**, *37*, 5–23.
20. Conley, C.; Etnyre, R.; Gardener, B.; Odom, L.H.; Suceava, B.D. New Curvature Inequalities for Hypersurfaces in the Euclidean Ambient Space. *Taiwan. J. Math.* **2013**, *17*, 885–895. Available online: http://www.jstor.org/stable/taiwjmath.17.3.885 (accessed on 12 April 2023). [CrossRef]
21. Weingarten, J. Uber eine Klasse aufeinander abwickelbarer Flachen. *J. F'Ur Die Reine Angew. Math.* **1861**, *59*, 382–393.
22. Weingarten, J. Uber die Flachen, derer Normalen eine gegebene Flache beruhren. *J. Fur Die Reine Angew. Math.* **1863**, *62*, 61–63.
23. Hartman, P.; Wintner, A. Umbilical points and W-surfaces. *Am. J. Math.* **1954**, *76*, 502–508. [CrossRef]
24. Chern, S.S. On special W-surfaces. *Proc. Am. Math. Soc.* **1955**, *6*, 783–786. [CrossRef]
25. Chen, B.Y. *Pseudo-Riemannian Geometry, $\delta-$Invariants and Applications*; World Scientific: Singapore, 2011.
26. Šipuš, Ž.M. Translation surfaces of constant curvatures in a simply isotropic space. *Period Math. Hung.* **2014**, *68*, 160–175. [CrossRef]
27. Sangwine, S.J. Perspectives on Color Image Processing by Linear Vector Methods Using Projective Geometric Transformations. In *Advances in Imaging and Electron Physics*; Elsevier: Amsterdam, The Netherlands, 2013; Volume 175, pp. 283–307. [CrossRef]
28. Onishchik, A.L.; Sulanke, R. *Projective and Cayley-Klein Geometries*; Springer: Berlin, Germany, 2006. [CrossRef]
29. Pottmann, H.; Grohs, P.; Mitra, N.J. Laguerre minimal surfaces, isotropic geometry and linear elasticity. *Adv. Comput. Math.* **2009**, *31*, 391–419. [CrossRef]
30. Pottmann, H.; Liu, Y. Discrete Surfaces in Isotropic Geometry. *Mathematics of Surfaces XII*; Springer: Berlin, Germany, 2007; pp. 341–363.
31. Yoon, D.W.; Lee, J.W. Linear Weingarten Helicoidal Surfaces in Isotropic Space. *Symmetry* **2016**, *8*, 126. [CrossRef]
32. Lone, M.S.; Karacan, M.K. Dual Translation Surfaces in the Three Dimensional Simply Isotropic Space. *Tamkang J. Math.* **2018**, *49*, 67–77. [CrossRef]
33. Güler, E.; Vanli, A.T. On The Mean, Gauss, The Second Gaussian and The Second Mean Curvature of The Helicoidal Surfaces with Light-Like Axis in $\mathbb{R}^3_1$. *Tsukuba J. Math.* **2008**, *32*, 49–65. [CrossRef]
34. Sauer, V.R. Infinitesimale Verbiegung der Flächen, deren Asymptotenlinien ein Quasi-Rückungsnetz bilden. *Sitz. Ber. Math. Naturw. Kl. Bayr. Akad. Wiss.* **1950**, *1*, 1–12.