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Extended Exton's Triple and Horn's Double Hypergeometric Functions and Associated Bounding Inequalities

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Abstract: This paper introduces extensions $H_{4,p}$ and $X_{8,p}$ of Horn's double hypergeometric function H_4 and Exton's triple hypergeometric function X_8, taking into account recent extensions of Euler's beta function, hypergeometric function, and confluent hypergeometric function. Among the numerous extended hypergeometric functions, the primary rationale for choosing H_4 and X_8 is their comparable extension type. Next, we present various integral representations of Euler and Laplace types, Mellin and inverse Mellin transforms, Laguerre polynomial representations, transformation formulae, and a recurrence relation for the extended functions. In particular, we provide a generating function for $X_{8,p}$ and several bounding inequalities for $H_{4,p}$ and $X_{8,p}$. We explore the utilization of the $H_{4,p}$ function within a probability distribution. Most special functions, such as the generalized hypergeometric function, the Beta function, and the p-extended Beta integral, exhibit natural symmetry.

Keywords: extended beta function; extended hypergeometric function; extended confluent hypergeometric function; extended Appell function; Mellin transform; inverse Mellin transform; H-functions; Laguerre polynomials; transformation formulae; recurrence relation; generating function; bounding inequalities

MSC: 33B20; 33C20; 33B15; 33C05

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1. Introduction and Preliminaries

The generalized hypergeometric function with the r numerator and s denominator parameters, as the series, reads

$$ {}_rF_s\left(\tau_1, \cdots, \tau_r; \upsilon_1, \cdots, \upsilon_s; z\right) = {}_rF_s\left(\tau_r; \upsilon_s; z\right) := \sum_{m=0}^{\infty} \frac{(\tau_1)_m \cdots (\tau_r)_m}{(\upsilon_1)_m \cdots (\upsilon_s)_m} \frac{z^m}{m!}, \qquad (1)$$

where $(\mu)_n = \mu(\mu+1)\cdots(\mu+n-1)$ $(n \in \mathbb{N})$, and $(\mu)_0 = 1$ signifies the Pochhammer symbol; moreover, $\tau_j \in \mathbb{C}$ and $\upsilon_j \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$, $j \in \overline{1,s} := \{1, 2, \ldots, s\}$ for a given $s \in \mathbb{N}$. The symbol $(\mu)_m$ is represented by $(\mu)_m = \Gamma(\mu+m)/\Gamma(\mu)$ $(\mu \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0},\ m \in \mathbb{N}_0)$, with Γ being the familiar Gamma function whose acquainted integral is

$$\Gamma(\mu) = \int_0^{\infty} e^{-t}\, t^{\mu-1}\, dt \quad (\Re(\mu) > 0). \qquad (2)$$

In this and other instances, the sets of positive integers, integers, real numbers, and complex numbers will be denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. Moreover, let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{Z}_{\leqslant 0} := \mathbb{Z} \setminus \mathbb{N}$. The series in (1) converges for all $z \in \mathbb{C}$ if $r \leqslant s$. It is divergent for all $z \in \mathbb{C} \setminus \{0\}$ when $r > s+1$, unless at least one numerator parameter is in $\mathbb{Z}_{\leqslant 0}$, in which case, (1) is a polynomial. For the case when $r = s + 1$, the series converges on the unit circle $|z| = 1$ under the constraints $\tau_j \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$ $(j \in \overline{1, r})$ and

$$\Re\Big(\sum_{j=1}^{s} v_j - \sum_{j=1}^{r} \tau_j\Big) > 0.$$

For the noted particular cases, ${}_2F_1$ is referred to as Gauss's hypergeometric function, and ${}_1F_1$, which is also denoted by Φ, is referred to as the confluent (Kummer's) hypergeometric function.

In 1997, Chaudhry et al. ([1], p. 20, Equation (1.7)) introduced and explored the p-extended Beta integral:

$$\mathrm{B}(\eta,\, \xi\,;\, p) := \int_0^1 t^{\eta-1}\,(1-t)^{\xi-1}\,\exp\Big(-\frac{p}{t(1-t)}\Big)\,dt \quad (\Re(p) > 0), \tag{3}$$

which follows a series of investigations of generalized incomplete gamma functions and their applications (see [2–4]; see also [5]). The p-extended Beta integral in (3) is connected to the Macdonald error and Whittaker functions. The case $p = 0$ of (3) becomes the classical Beta function given by (see, for example, ([6], p. 8, Equation (43)):

$$\mathrm{B}(\eta,\, \xi) = \begin{cases} \displaystyle\int_0^1 t^{\eta-1}(1-t)^{\xi-1}\,dt & (\Re(\eta) > 0,\ \Re(\xi) > 0) \\ \dfrac{\Gamma(\eta)\,\Gamma(\xi)}{\Gamma(\eta+\xi)} & (\eta,\, \xi \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}). \end{cases} \tag{4}$$

Making use of the subsequent transformation

$$\frac{(v)_m}{(\phi)_m} = \frac{\mathrm{B}(v+m, \phi - v)}{\mathrm{B}(v, \phi - v)} \quad (\Re(\phi) > \Re(v) > 0,\ m \in \mathbb{N}_0), \tag{5}$$

in which the numerator Beta function is replaced by the p-extended Beta function in (3), Chaudhry et al. [7] introduced the p-Gauss hypergeometric function and the p-Kummer confluent hypergeometric function, which are, respectively, given as follows:

$$F_p\,(\tau,\, v;\, \phi;\, z) := \sum_{m=0}^{\infty} (\tau)_m\, \frac{\mathrm{B}(v+m,\, \phi - v\,;\, p)}{\mathrm{B}(v,\, \phi - v)}\, \frac{z^m}{m!} \tag{6}$$

$$(p \geqslant 0,\ |z| < 1\,;\, \Re(\phi) > \Re(v) > 0)$$

and

$$\Phi_p\,(v;\, \phi;\, z) := \sum_{m=0}^{\infty} \frac{\mathrm{B}(v+m,\, \phi - v\,;\, p)}{\mathrm{B}(v,\, \phi - v)}\, \frac{z^m}{m!} \quad (p \geqslant 0\,;\, \Re(\phi) > \Re(v) > 0). \tag{7}$$

The functions were studied by Chaudhry et al. [7], who revealed numerous intriguing identities and formulae. These include integral representations, differentiation properties, Mellin transforms, transformations, recurrence relations, summation formulae, and asymptotic formulae. There are several further extensions of the Beta function and the hypergeometric function of types (3) and (6) (see, for example, [8–11]).

Özarslan and Özergin [12] introduced and investigated the p-extensions of the two-variable Appell hypergeometric functions F_1 and F_2 and the three variable Lauricella hypergeometric functions $F_D^{(3)}$ (see, for example, ([13], Chapter 1), among which, the p-extended F_2 function is recalled:

$$F_2(\tau, v, v'; \phi, \phi'; x, y; p) = \sum_{m,n=0}^{\infty} (\tau)_{m+n} \frac{\mathrm{B}(v+m, \phi-v; p)\mathrm{B}(v'+n, \phi'-v'; p)}{\mathrm{B}(v, \phi-v)\mathrm{B}(v', \phi'-v')} \frac{x^m}{m!} \frac{y^n}{n!} \tag{8}$$

$$\big(|x|+|y|<1;\ \Re(p) \geqslant 0\big).$$

They [12] also introduced a new extended Riemann–Liouville fractional derivative to present several intriguing generating relations for the p-Gauss hypergeometric function (6).

Similar to (3), the p-extensions (6) in (7) and (8), when $p = 0$, return to Gauss's hypergeometric function ${}_2F_1$, the confluent hypergeometric function ${}_1F_1$, and the Appell hypergeometric function F_2 of two variables, respectively.

Our investigation is primarily motivated by the vast range of potential applications of the extended Gauss hypergeometric, confluent hypergeometric, and Appell functions in various fields of mathematical, physical, engineering, and statistical sciences (as detailed in [1,7,12], and the references therein). In this study, we undertake a systematic exploration of the extended Horn's double hypergeometric function $H_{4,p}$ and the extended Exton's triple hypergeometric functions $X_{8,p}$. Specifically, we aim to present various integral representations of the Euler and Laplace types, as well as certain integral representations involving Bessel and modified Bessel functions, the Mellin transform, the Laguerre polynomial representation, transformation formulae, and recurrence relations. Additionally, we provide a generating function for $X_{8,p}$ and several bounding inequalities for $H_{4,p}$ and $X_{8,p}$. Further, we investigate the application of the $H_{4,p}$ function within a probability distribution.

It should be mentioned that symmetry can manifest itself in various fields and aspects of human life, either explicitly or implicitly. For instance, the Beta function, the p-extended Beta integral, and ${}_rF_s$ function are obvious examples of symmetrical phenomena. Explicitly,

$$B(\eta,\, \xi) = B(\xi, \eta), \quad \mathrm{B}(\eta,\, \xi\,;\, p) = \mathrm{B}(\xi\,,\, \eta\,;\, p),$$

for instance,

$${}_rF_s(\tau_1,\, \ldots,\, \tau_r;\, v_1,\, \ldots,\, v_s;\, z) = {}_rF_s(\tau_r,\, \ldots,\, \tau_1;\, v_s,\, \ldots,\, v_1;\, z),$$

where the function remains unchanged regardless of how the numerator parameters are reordered; similarly, the function remains the same regardless of how the denominator parameters are reordered.

2. Extended Horn's Double Hypergeometric Function

In terms of the extended Beta function $B(\eta,\, \xi\,;\, p)$ in (3), this section introduces the following extended Horn's double hypergeometric function $H_{4,p}$: For $\tau,\, v \in \mathbb{C}$ and $\phi, \phi' \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$,

$$H_{4,p}[\tau, v; \phi, \phi'; x, y] := \sum_{k,m=0}^{\infty} \frac{(\tau)_{2k+m}}{(\phi)_k} \frac{\mathrm{B}(v+m, \phi'-v; p)}{\mathrm{B}(v, \phi'-v)} \frac{x^k}{k!} \frac{y^m}{m!} \tag{9}$$
$$(p > 0;\ 2\sqrt{r_1} + r_2 < 1,\ |x| \leqslant r_1,\ |y| \leqslant r_2 \text{ when } p = 0),$$

where $\Re(\phi') > \Re(v) > 0$. In light of (5), when $p = 0$ in (9), it gives the classical Horn's double hypergeometric function H_4 (see, for example, ([13], pp. 24, 59, [14]): For $\tau,\, v \in \mathbb{C}$ and $\phi, \phi' \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$,

$$H_4[\tau, v; \phi, \phi'; x, y] = \sum_{k,m=0}^{\infty} \frac{(\tau)_{2k+m}(v)_m}{(\phi)_k(\phi')_m} \frac{x^k}{k!} \frac{y^m}{m!} \tag{10}$$
$$(2\sqrt{r_1} + r_2 < 1,\ |x| \leqslant r_1,\ |y| \leqslant r_2).$$

2.1. Integral Representations

Theorem 1. *The following integral representation for $H_{4,p}$ in (9) holds true:*

$$\begin{aligned} H_{4,p}[\tau, v; \phi, \phi'; x, y] = & \frac{1}{\mathrm{B}(v, \phi' - v)} \int_0^1 \frac{u^{v-1}(1-u)^{\phi'-v-1}}{(1-yu)^{\tau}} \\ & \times {}_2F_1\left[\frac{\tau}{2}, \frac{\tau}{2} + \frac{1}{2}; \phi; \frac{4x}{(1-yu)^2}\right] \exp\left(-\frac{p}{u(1-u)}\right) du \end{aligned} \tag{11}$$

$$\big(\Re(p) > 0;\ \Re(\phi') > \Re(v) > 0 \ \textit{when}\ \ p = 0\big).$$

Proof. By making use of the identity

$$(\tau)_{2k+m} = (\tau)_{2k}(\tau + 2k)_m$$

and the extended Gauss hypergeometric function (6), the extended Horn's double hypergeometric function (9) can be expressed as a single series:

$$H_{4,p}[\tau, v; \phi; \phi'; x, y] = \sum_{k=0}^{\infty} \frac{(\tau)_{2k}}{(\phi)_k} F_p\left[\begin{array}{r} \tau + 2k, v; \\ \phi'; \end{array} y\right] \frac{x^k}{k!}. \tag{12}$$

Applying the integral representation of the extended Gauss hypergeometric function ([7], p. 592, Equation (3.2))

$$F_p(\tau, v; \phi; z) = \frac{1}{\mathrm{B}(v, \phi - v)} \int_0^1 u^{v-1}(1-u)^{\phi-v-1}(1-zu)^{-\tau} \exp\left(-\frac{p}{u(1-u)}\right) du \tag{13}$$

$$\big(\Re(p) > 0;\ p = 0 \text{ and } |\arg(1-z)| < \pi;\ \Re(\phi) > \Re(v) > 0\big)$$

to (12), one finds

$$\begin{aligned} H_{4,p}[\tau, v; \phi; \phi'; x, y] = & \sum_{k=0}^{\infty} \int_0^1 \frac{u^{v-1}(1-u)^{\phi'-v-1}}{\mathrm{B}(v, \phi' - v)\,(1-yu)^{\tau}} \\ & \times \exp\left(-\frac{p}{u(1-u)}\right) \frac{(\tau)_{2k}}{(\phi)_k\, k!} \left\{\frac{x}{(1-yu)^2}\right\}^k du. \end{aligned} \tag{14}$$

Changing the order of summation and integration in (14), which is guaranteed under the restrictions, and using the identity

$$(\tau)_{2k} = 2^{2k} \left(\frac{\tau}{2}\right)_k \left(\frac{\tau+1}{2}\right)_k \quad (\tau \in \mathbb{C},\ k \in \mathbb{N}_0) \tag{15}$$

and the Gauss hypergeometric function ${}_2F_1$, we obtain the desired integral representation (11). □

The following corollary is obtained by setting $p = 0$ in Theorem 1.

Corollary 1. *The following integral representation for H_4 holds true:*

$$\begin{aligned} H_4[\tau, v; \phi, \phi'; x, y] = & \frac{1}{\mathrm{B}(v, \phi' - v)} \int_0^1 \frac{u^{v-1}(1-u)^{\phi'-v-1}}{(1-yu)^{\tau}} \\ & \times {}_2F_1\left[\frac{\tau}{2}, \frac{\tau}{2} + \frac{1}{2}; \phi; \frac{4x}{(1-yu)^2}\right] du \end{aligned} \tag{16}$$

$$\big(\Re(\phi') > \Re(v) > 0\big),$$

where the additional restrictions for the other parameters and variables would follow from those in (10).

Theorem 2. *The following Laplace-type integral representation for* $H_{4,p}$ *in* (9) *holds true:*

$$H_{4,p}[\tau, v; \phi, \phi'; x, y] = \frac{1}{\Gamma(\tau)} \int_0^\infty t^{\tau-1} e^{-t} \, {}_0F_1(-; \phi; xt^2) \, \Phi_p(v; \phi'; yt) \, dt \tag{17}$$

$$\big(\Re(p) > 0; \ \Re(\tau) > 0 \text{ when } p = 0\big).$$

Proof. Using the integral representation

$$(\tau)_n := \frac{1}{\Gamma(\tau)} \int_0^\infty t^{\tau+n-1} \, e^{-t} \, dt \quad \big(\Re(\tau) > 0, \, n \in \mathbb{N}_0\big) \tag{18}$$

for the Pochhammer symbol $(\tau)_{2k+m}$ in (9) and interchanging the order of summations and integrals, we have

$$\begin{aligned} &H_{4,p}[\tau, v; \phi, \phi'; x, y] \\ &\quad = \frac{1}{\Gamma(\tau)} \sum_{k,m=0}^\infty \int_0^\infty t^{\tau-1} \, e^{-t} \, \frac{1}{(\phi)_k} \, \frac{\mathrm{B}(v+m, \phi'-v; p)}{\mathrm{B}(v, \phi'-v)} \, \frac{(xt^2)^k}{k!} \, \frac{(yt)^m}{m!} \, dt \\ &\quad = \frac{1}{\Gamma(\tau)} \int_0^\infty t^{\tau-1} \, e^{-t} \left(\sum_{k=0}^\infty \frac{1}{(\phi)_k} \frac{(xt^2)^k}{k!} \right) \left(\sum_{m=0}^\infty \frac{\mathrm{B}(v+m, \phi'-v; p)}{\mathrm{B}(v, \phi'-v)} \frac{(yt)^m}{m!} \right) dt. \end{aligned}$$

Now, using (1) and (7) in each summation enclosed in parentheses yields the desired result (17). □

Remark 1. *The Bessel function* $J_\nu(z)$ *and the modified Bessel function* $I_\nu(z)$ *are expressible in terms of hypergeometric functions, as follows (see, for example, [15,16]; see also ([17], p. 265, Equation (3.2), [18,19]); in particular, [20]):*

$$J_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1)} \, {}_0F_1\left(\text{---}\,; \nu+1; -\frac{1}{4} z^2 \right) \qquad (\nu \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}) \tag{19}$$

and

$$I_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1)} \, {}_0F_1\left(\text{---}\,; \nu+1; \frac{1}{4} z^2 \right) \qquad (\nu \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}), \tag{20}$$

where $\mathbb{Z}_{\leqslant -1} := \mathbb{Z} \setminus \mathbb{N}_0$ *and* $z \in \mathbb{C} \setminus (-\infty, 0]$.

Now, applying the relationships (19) and (20) to (17), we can deduce certain interesting integral representations for the extended Horn's double hypergeometric function in (9) as asserted by Corollary 2 below. Here, we state the resulting integral representations.

Corollary 2. *Each of the following integral representations holds true:*

$$\begin{aligned} &H_{4,p}[\tau, v; \phi, \phi'; -x, y] \\ &\quad = \frac{\Gamma(\phi) \, x^{\frac{1-\phi}{2}}}{\Gamma(\tau)} \int_0^\infty t^{\tau-\phi} e^{-t} J_{\phi-1}(2\sqrt{x}t) \, \Phi_p(v; \phi'; yt) \, dt \end{aligned} \tag{21}$$

and

$$\begin{aligned} &H_{4,p}[\tau, v; \phi, \phi'; x, y] \\ &\quad = \frac{\Gamma(\phi) \, x^{\frac{1-\phi}{2}}}{\Gamma(\tau)} \int_0^\infty t^{\tau-\phi} e^{-t} I_{\phi-1}(2\sqrt{x}t) \, \Phi_p(v; \phi'; yt) \, dt. \end{aligned} \tag{22}$$

Here, all parameters and variables would be restricted, so that the representations can be meaningful and convergent. For example, $\Re(\tau-\phi) > -1$, $\phi \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$ *and* $x \in \mathbb{C} \setminus (-\infty, 0]$.

2.2. Transformation Formula

Theorem 3. *The following transformation formula for* $H_{4,p}$ *holds true:*

$$\begin{aligned} &H_{4,p}[\tau, v; \phi, \phi'; x, y] \\ &\quad = (1-y)^{-\tau} H_{4,p}\left[\tau, \phi' - v; \phi, \phi'; \frac{x}{(1-y)^2}, \frac{y}{y-1}\right]. \end{aligned} \tag{23}$$

Proof. If we first apply the extended Kummer transformation formula (see, for example, ([7], p. 596, Equation (6.3)):

$$\Phi_p(v; \phi; z) = e^z\, \Phi_p(\phi - v; \phi; -z) \tag{24}$$

to (17) and then set

$$t(1-y) = u \qquad \text{and} \qquad du = (1-y)dt$$

in the resulting integral, we obtain the transformation Formula (23). □

2.3. Recurrence Relation

The following lemma gives a recurrence relation for ${}_0F_1$, which is deducible from ([21], p. 19, Equation (2.2.2)) or ([21], p. 20, Equation (2.2.7)).

Lemma 1. *The following contiguous relation for the function* ${}_0F_1$ *holds true:*

$${}_0F_1(-; \phi - 1; x) - {}_0F_1(-; \phi; x) - \frac{x}{\phi(\phi-1)}\, {}_0F_1(-; \phi + 1; x) = 0. \tag{25}$$

Proof. Recall a contiguous relation for the function ${}_1F_1$ (see ([21], p. 19, Equation (2.2.2)):

$$v(v-1)\, {}_1F_1(\tau; v-1; x) - v(v-1+x)\, {}_1F_1(\tau; v; x) + (v-\tau)x\, {}_1F_1(\tau; v+1; x) = 0. \tag{26}$$

Substituting $\frac{x}{\tau}$ for x in (26) and taking the limit in the resulting identity as $|\tau| \to \infty$ with the aid of

$$\lim_{|\tau| \to \infty} \left\{ (\tau)_n \left(\frac{x}{\tau}\right)^n \right\} = x^n \quad (n \in \mathbb{N}_0,\ |x| < \infty), \tag{27}$$

and replacing v by ϕ in the final identity, we obtain the contiguous relation for the function ${}_0F_1$ in (25).

Additionally, it is worth mentioning that (25) can be proven through a straightforward computation. □

Theorem 4. *The following recurrence relation for* $H_{4,p}$ *holds true:*

$$\begin{aligned} H_{4,p}[\tau, v; \phi, \phi'; x, y] &= H_{4,p}[\tau, v; \phi - 1, \phi'; x, y] \\ &\quad + \frac{\tau(\tau+1)x}{\phi(1-\phi)} H_{4,p}[\tau, v; \phi + 1, \phi'; x, y]. \end{aligned} \tag{28}$$

Proof. By utilizing (25) on the integral form given in (17), we arrive at the relation (28). □

2.4. Mellin Transform and Inverse Mellin Transform

The Mellin transform of a function $f(t)$ with index s is defined by

$$\mathcal{M}\{f(\tau) : \tau \to s\} := \int_0^\infty \tau^{s-1}\, f(\tau)\, d\tau, \tag{29}$$

provided that the improper integral exists (see, for example, [17,22]).

Theorem 5. *The following Mellin transform representation of $H_{4,p}$ in (9) holds true:*

$$\begin{aligned} F(s) :&= \mathcal{M}\{H_{4,p}[\tau,v;\phi,\phi';x,y] : p \to s\} \\ &= \frac{\Gamma(s)\mathrm{B}(v+s,\phi'-v+s)}{\mathrm{B}(v,\phi'-v)}\, H_4[\tau,v+s;\phi,\phi'+2s;x,y] \end{aligned} \tag{30}$$

$$(\Re(s) > 0,\ \Re(\phi') > \Re(v) > 0).$$

Moreover, the restrictions of the other parameters and variables would follow from those in (9).

Proof. Using the Mellin transform (29) in (9), and interchanging the order of integral and summations, which is guaranteed under the restrictions, we have

$$\begin{aligned} &\mathcal{M}\{H_{4,p}[\tau,v;\phi,\phi';x,y] : p \to s\} \\ &= \frac{1}{\mathrm{B}(v,\phi'-v)} \sum_{k,m=0}^{\infty} \frac{(\tau)_{2k+m}}{(\phi)_k} \frac{x^k}{k!} \frac{y^m}{m!} \int_0^\infty p^{s-1}\mathrm{B}(v+m,\phi'-v;p)\,dp. \end{aligned} \tag{31}$$

Applying the known result (see ([1], p. 21, Equation (2.1)):

$$\begin{aligned} &\int_0^\infty p^{s-1}\mathrm{B}(x,\,y;\,p)\,dp = \Gamma(s)\,\mathrm{B}(x+s,\,y+s) \\ &(\Re(s) > 0,\ \Re(x+s) > 0,\ \Re(y+s) > 0) \end{aligned} \tag{32}$$

to the improper integral in (31), we obtain

$$\begin{aligned} &\mathcal{M}\{H_{4,p}[\tau,v;\phi,\phi';x,y] : p \to s\} \\ &= \Gamma(s) \sum_{k,m=0}^{\infty} \frac{(\tau)_{2k+m}}{(\phi)_k} \frac{\mathrm{B}(v+m+s,\phi'-v+s)}{\mathrm{B}(v,\phi'-v)} \frac{x^k}{k!} \frac{y^m}{m!}, \end{aligned}$$

which, upon using (10), yields the desired representation (30). □

Theorem 6. *The following Mellin–Barnes-type integral holds true: For a fixed $\mu > 0$,*

$$\begin{aligned} H_{4,p}[\tau,v;\phi,\phi';x,y] &= \frac{1}{\mathrm{B}(v,\phi'-v)} \sum_{k,m=0}^{\infty} \frac{(\tau)_{2k+m}}{(\phi)_k} \frac{x^k}{k!} \frac{y^m}{m!} \\ &\quad \times \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} \frac{\Gamma(s)\,\Gamma(\phi'-v+s)\,\Gamma(v+m+s)}{\Gamma(\phi'+m+2s)} p^{-s} ds \\ &= \frac{1}{\mathrm{B}(v,\phi'-v)} \sum_{k,m=0}^{\infty} \frac{(\tau)_{2k+m}}{(\phi)_k} H_{1,3}^{3,0}\left[p \,\middle|\, \begin{matrix} (\phi'+m,2) \\ (0,1),(\phi'-v,1),(v+m,1) \end{matrix}\right] \frac{x^k}{k!} \frac{y^m}{m!} \end{aligned} \tag{33}$$

$$\left(p > 0,\ \Re(\phi') > \Re(v) > 0,\ 2\sqrt{r_1} + r_2 < 1,\ |x| \leqslant r_1,\ |y| \leqslant r_2\right),$$

where $i = \sqrt{-1}$ and $H_{1,3}^{3,0}$ denotes the H-function (see, for example, [23], Section 1.2).

Proof. It follows from (9) that

$$\begin{aligned} &f(p) := H_{4,p}[\tau,v;\phi,\phi';x,y] \\ &= \frac{1}{\mathrm{B}(v,\phi'-v)} \sum_{k,m=0}^{\infty} \frac{(\tau)_{2k+m}}{(\phi)_k} \frac{x^k}{k!} \frac{y^m}{m!} \int_0^1 t^{v+m-1}(1-t)^{\phi'-v-1} e^{-\frac{p}{t(1-t)}}\,dt. \end{aligned}$$

The following asymptotic conditions are satisfied:

$$f(p) = H_4[\tau, v; \phi, \phi'; x, y] = O(1) = O\left(p^0\right) \quad \text{as } p \to 0,$$

and $f(p) = o(1) = O(p^{-\mu})$ for every $\mu > 0$, as $p \to \infty$. Therefore, one finds (see, for example, ([22], p. 559) that the Mellin transform $F(s) = \mathcal{M}\{f(p) : p \to s\}$ is analytic in the strip $0 < \Re(p) < \infty$ and the inverse Mellin transform is given as follows:

$$f(p) = \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} F(s)\, p^{-s} ds \quad (0 < \mu < \infty). \tag{34}$$

Using (30) for $F(s)$ in (34), we obtain the first equality in (33). The second equality in (33) is found by employing the H-function. □

Remark 2. *The case of* (30) *when* $s = 1$ *yields a relation between the extended Horn's double hypergeometric* $H_{4,p}$ *and the classical Horn's double hypergeometric* H_4*, is as follows:*

$$\int_0^\infty H_{4,p}[\tau, v; \phi, \phi'; x, y]\, dp = \frac{v(\phi' - v)}{\phi'(\phi' + 1)} H_4[\tau, v + 1; \phi, \phi' + 2; x, y]. \tag{35}$$

Moreover, setting $s = 1$ *in* (32) *gives*

$$\int_0^\infty \mathrm{B}(x,\, y\,;\, p)\, dp = \mathrm{B}(x + 1,\, y + 1) \quad (\Re(x + 1) > 0,\ \Re(y + 1) > 0). \tag{36}$$

The Mellin–Barnes-type integral in (33) *converges (see ([23], Section 1.2). Using the duplication formula for the Gamma function (see, for example, ([6], p. 6, Equation (29)) in the Mellin–Barnes-type integral in* (33)*, we obtain*

$$\begin{aligned} &H_{1,3}^{3,0}\left[p \,\middle|\, \begin{array}{c} (\phi' + m, 2) \\ (0,1), (\phi' - v, 1), (v + m, 1) \end{array}\right] \\ &= \frac{\sqrt{\pi}}{2^{\phi' + m - 1}} H_{2,3}^{3,0}\left[4p \,\middle|\, \begin{array}{c} \left(\frac{\phi' + m}{2}, 1\right), \left(\frac{\phi' + m + 1}{2}, 1\right) \\ (0,1), (\phi' - v, 1), (v + m, 1) \end{array}\right]. \end{aligned} \tag{37}$$

Comparing (9) *with the first equality in* (33) *reveals that* $\mathrm{B}(v + m, \phi' - v; p)$*, as a function in* p*, is the inverse Mellin transform of the function:*

$$\frac{\Gamma(s)\,\Gamma(\phi' - v + s)\,\Gamma(v + m + s)}{\Gamma(\phi' + m + 2s)}.$$

2.5. Laguerre Polynomial Representation

Theorem 7. *The following Laguerre polynomial representation for* $H_{4,p}$ *holds true: For* $\Re(p) > 0$ *and* $\Re(\phi' - v) > 0$,

$$\begin{aligned} H_{4,p}[\tau, v; \phi, \phi'; x, y] = \frac{e^{-2p}}{\mathrm{B}(v, \phi' - v)} \sum_{m,n=0}^{\infty} \mathrm{B}(v + m + 1, \phi' - v + n + 1) \\ \times H_4[\tau, v + m + 1; \phi, \phi' + m + n + 2; x, y]\, L_m(p)\, L_n(p), \end{aligned} \tag{38}$$

where $L_n(p)$ *are Laguerre polynomials (see, for example, ([16], Chapter 12).*

Proof. Using the identity due to Miller ([24], p. 30, Equation (3.5)):

$$\exp\left(-\frac{p}{t(1 - t)}\right) = e^{-2p} \sum_{m,\, n=0}^{\infty} L_m(p)\, L_n(p)\, t^{m+1}\, (1 - t)^{n+1} \tag{39}$$

in (11), we have

$$H_{4,p}[\tau, v; \phi, \phi'; x, y] = \frac{e^{-2p}}{\mathrm{B}(v, \phi' - v)} \int_0^1 u^{v-1}(1-u)^{\phi'-v-1}(1-uy)^{-\tau} \\ \times \, {}_2F_1\left[\frac{\tau}{2}, \frac{\tau}{2} + \frac{1}{2}; \phi; \frac{4x}{(1-yu)^2}\right] \left\{ \sum_{m,n=0}^{\infty} L_m(p)\, L_n(p)\, u^{m+1}\, (1-u)^{n+1} \right\} du. \tag{40}$$

Now, changing summations and integrals in (40) and using (16), we obtain the desired identity (38). □

3. Extended Exton's Triple Hypergeometric Function

In terms of the extended beta function $\mathrm{B}(x, y; p)$ in (3), this section introduces the following extended Exton's triple hypergeometric function $X_{8,p}$: For $\tau, v, v' \in \mathbb{C}$ and $\phi_1, \phi_2, \phi_3 \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$,

$$X_{8,p}[\tau, v, v'; \phi_1, \phi_2, \phi_3; x, y, z] \\ := \sum_{k,m,n=0}^{\infty} \frac{(\tau)_{2k+m+n}}{(\phi_1)_k} \frac{\mathrm{B}(v+m, \phi_2 - v; p)}{\mathrm{B}(v, \phi_2 - v)} \frac{\mathrm{B}(v'+n, \phi_3 - v'; p)}{\mathrm{B}(v', \phi_3 - v')} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \tag{41}$$

$$\left(p \geqslant 0;\ 2\sqrt{r_1} + r_2 + r_3 < 1,\ |x| \leqslant r_1,\ |y| \leqslant r_2,\ |z| \leqslant r_3, \ \text{when} \ p = 0\right).$$

Setting $p = 0$ in (41) yields the Exton's triple hypergeometric function X_8 (see, for example, ([13], p. 84, 41a and p. 101): For $\tau, v, v' \in \mathbb{C}$ and $\phi_1, \phi_2, \phi_3 \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$,

$$X_8[\tau, v, v'; \phi_1, \phi_2, \phi_3; x, y, z] = \sum_{k,m,n=0}^{\infty} \frac{(\tau)_{2k+m+n} (v)_m (v')_n}{(\phi_1)_k (\phi_2)_m (\phi_3)_n} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \tag{42}$$

$$\left(2\sqrt{r_1} + r_2 + r_3 < 1,\ |x| \leqslant r_1,\ |y| \leqslant r_2,\ |z| \leqslant r_3\right).$$

Remark 3. *It is observed that if $v' = \phi_3$ and $z = 0$ is put in (42), the Exton's triple hypergeometric function X_8 reduces to the Horn's double hypergeometric function (10). In this sense, two extensions share a common type and their respective p-extensions, $H_{4,p}$ (9) and $X_{8,p}$ (41), are interrelated, as depicted in (43).*

3.1. Integral Representations

This section explores certain integral representations for the extended Exton's triple hypergeometric function in (41) of Euler and Laplace types. Integral representations incorporating Bessel and modified Bessel functions are provided as corollaries.

Theorem 8. *The following Euler-type integral representation for $X_{8,p}$ in (41) holds true:*

$$X_{8,p}[\tau, v, v'; \phi_1, \phi_2, \phi_3; x, y, z] = \frac{1}{\mathrm{B}(v', \phi_3 - v')} \int_0^1 \frac{v^{v'-1}(1-v)^{\phi_3 - v' - 1}}{(1-zv)^{\tau}} \\ \times \, H_{4,p}\left[\tau, v; \phi_1, \phi_2; \frac{x}{(1-zv)^2}, \frac{y}{1-zv}\right] \exp\left(-\frac{p}{v(1-v)}\right) dv \tag{43}$$

$$\left(\Re(p) > 0;\ \Re(\phi') > \Re(v') > 0 \ \textit{when} \ p = 0\right).$$

Proof. The extended Exton's triple hypergeometric function in (41) can be expressed as a double series involving the extended Gauss hypergeometric function in (6) by making use of the Pochhammer symbol identity $(\tau)_{2k+m+n} = (\tau)_{2k+m}(\tau+2k+m)_n$:

$$X_{8,p}[\tau, v, v'; \phi_1, \phi_2, \phi_3; x, y, z] = \sum_{k,m=0}^{\infty} \frac{(\tau)_{2k+m}}{(\phi_1)_k} \frac{\mathrm{B}(v+m, \phi_2 - v; p)}{\mathrm{B}(v, \phi_2 - v)} F_p \left[\begin{matrix} \tau + 2k + m, v'; \\ \phi_3; \end{matrix} z \right] \frac{x^k}{k!} \frac{y^m}{m!}, \tag{44}$$

Employing the integral representation for the extended Gauss hypergeometric function in (13) (see also ([7], p. 592, Equation (3.2)) in (44), we have

$$\begin{aligned} X_{8,p}&[\tau, v, v'; \phi_1, \phi_2, \phi_3; x, y, z] \\ &= \frac{1}{\mathrm{B}(v', \phi_3 - v')} \sum_{k,m=0}^{\infty} \int_0^1 v^{v'-1}(1-v)^{\phi_3 - v' - 1}(1-vz)^{-\tau} \exp\left(-\frac{p}{v(1-v)}\right) \\ &\times \frac{(\tau)_{2k+m}}{(\phi_1)_k} \frac{\mathrm{B}(v+m, \phi_2 - v; p)}{\mathrm{B}(v, \phi_2 - v) k!\, m!} \left\{ \frac{x}{(1-vz)^2} \right\}^k \left(\frac{y}{1-vz} \right)^m dv. \end{aligned} \tag{45}$$

Changing the order of summations and integration in (45) and using the extended Horn's function in (9), we obtain the desired integral representation (43). □

Theorem 9. *The following Euler-type integral representation for $X_{8,p}$ in (41) holds true:*

$$\begin{aligned} X_{8,p}[\tau, v, v'; \phi_1, \phi_2, \phi_3; x, y, z] &= \int_0^1 \int_0^1 \frac{u^{v-1} v^{v'-1} (1-u)^{\phi_2 - v - 1} (1-v)^{\phi_3 - v' - 1}}{\mathrm{B}(v, \phi_2 - v)\, \mathrm{B}(v', \phi_3 - v')(1 - yu - zv)^{\tau}} \\ \times\ {}_2F_1 &\left[\frac{\tau}{2}, \frac{\tau}{2} + \frac{1}{2}; \phi_1; \frac{4x}{(1-yu-zv)^2} \right] \exp\left(-\frac{p}{u(1-u)} - \frac{p}{v(1-v)} \right) du\, dv \end{aligned} \tag{46}$$

$$\big(\Re(p) > 0;\ \Re(\phi_2) > \Re(v) > 0,\ \Re(\phi_3) > \Re(v') > 0 \text{ when } p = 0\big).$$

Proof. The extended Exton's triple hypergeometric function in (41) is expressed as a single series involving the extended second Appell hypergeometric function in (8) by making use of the Pochhammer symbol identity $(\tau)_{2k+m+n} = (\tau)_{2k}(\tau+2k)_{m+n}$:

$$X_{8,p}[\tau, v, v'; \phi_1, \phi_2, \phi_3; x, y, z] = \sum_{k=0}^{\infty} \frac{(\tau)_{2k}}{(\phi_1)_k} F_2[\tau + 2k, v, v'; \phi_2, \phi_3; y, z; p] \frac{x^k}{k!}. \tag{47}$$

Employing the integral representation of the extended second Appell hypergeometric function (see [12], Theorem 2.2):

$$\begin{aligned} F_2&(a, b, b'; c, c'; x, y; p) \\ &= \frac{1}{\mathrm{B}(b, c-b)\mathrm{B}(b', c'-b')} \int_0^1 \int_0^1 \frac{u^{b-1}(1-u)^{c-b-1}\, v^{b'-1}(1-v)^{c'-b'-1}}{(1-xu-yv)^a} \\ &\qquad \times \exp\left(-\frac{p}{u(1-u)} \right) \exp\left(-\frac{p}{v(1-v)} \right) du\, dv \end{aligned} \tag{48}$$

$$\big(\Re(p) > 0;\ p = 0 \text{ and } |x| + |y| < 1;\ \Re(c) > \Re(b) > 0,\ \Re(c') > \Re(b') > 0,\ \Re(a) > 0\big)$$

in (47), we obtain

$$
\begin{aligned}
&X_{8,p}[\tau, v, v'; \phi_1, \phi_2, \phi_3; x, y, z] \\
&= \sum_{k=0}^{\infty} \int_0^1 \int_0^1 \frac{u^{v-1} v^{v'-1} (1-u)^{\phi_2 - v - 1} (1-v)^{\phi_3 - v' - 1}}{\mathrm{B}(v, \phi_2 - v)\, \mathrm{B}(v', \phi_3 - v') (1 - yu - zv)^{\tau}} \\
&\times \exp\left(-\frac{p}{u(1-u)}\right) \exp\left(-\frac{p}{v(1-v)}\right) \frac{(\tau)_{2k}}{(\phi_1)_k\, k!} \left\{\frac{x}{(1-yu-zv)^2}\right\}^k du\, dv.
\end{aligned} \tag{49}
$$

Interchanging the order of summation and integrations and using the identity (15) with the choice of ${}_2F_1$ from (1) in (49), we obtain the desired integral representation (46). □

Theorem 10. *The following Laplace-type integral representation for $X_{8,p}$ in (41) holds true:*

$$
\begin{aligned}
&X_{8,p}[\tau, v, v'; \phi_1, \phi_2, \phi_3; x, y, z] \\
&\quad = \frac{1}{\Gamma(\tau)} \int_0^{\infty} t^{\tau-1} e^{-t}\, {}_0F_1(-; \phi_1; xt^2)\, \Phi_p(v; \phi_2; yt)\, \Phi_p(v'; \phi_3; zt)\, dt
\end{aligned} \tag{50}
$$

$$\big(\Re(p) > 0;\ \Re(\tau) > 0 \ \textit{when} \ \ p = 0\big).$$

Proof. Applying the integral representations for the Pochhammer symbol $(\tau)_{2k+m+n}$ in (18) to (41) and interchanging the order of summations and integrals, we have

$$
\begin{aligned}
&X_{8,p}[\tau, v, v'; \phi_1, \phi_2, \phi_3; x, y, z] \\
&= \frac{1}{\Gamma(\tau)} \int_0^{\infty} t^{\tau-1}\, e^{-t} \left(\sum_{k=0}^{\infty} \frac{1}{(\phi_1)_k} \frac{(xt^2)^k}{k!}\right) \left(\sum_{m=0}^{\infty} \frac{\mathrm{B}(v+m, \phi_2 - v; p)}{\mathrm{B}(v, \phi_2 - v)} \frac{(yt)^m}{m!}\right) \\
&\qquad \times \left(\sum_{n=0}^{\infty} \frac{\mathrm{B}(v'+n, \phi_3 - v'; p)}{\mathrm{B}(v', \phi_3 - v')} \frac{(zt)^n}{n!}\right) dt.
\end{aligned} \tag{51}
$$

Then, using the generalized hypergeometric function (1) (with $r = 0$ and $s = 1$) and extended confluent hypergeometric function (7) in (51), we obtain the desired result (50). □

Likewise, as in Corollary 2, we can deduce integral expressions for Exton's extended triple hypergeometric function in (41) by utilizing (19) and (20) in (50). This is stated in Corollary 3, and we present the resulting integral representations here, without demonstrating their derivations.

Corollary 3. *Each of the following integral representations holds true:*

$$
\begin{aligned}
X_{8,p}[\tau, v, v'; \phi_1, \phi_2, \phi_3; -x, y, z] \ &= \frac{\Gamma(\phi_1)\, x^{\frac{1-\phi_1}{2}}}{\Gamma(\tau)} \\
&\times \int_0^{\infty} t^{\tau-\phi_1} e^{-t} J_{\phi_1 - 1}(2\sqrt{x}t)\, \Phi_p(v; \phi_2; yt)\, \Phi_p(v'; \phi_3; zt) dt
\end{aligned} \tag{52}
$$

and

$$
\begin{aligned}
X_{8,p}[\tau, v, v'; \phi_1, \phi_2, \phi_3; x, y, z] \ &= \frac{\Gamma(\phi_1)\, x^{\frac{1-\phi_1}{2}}}{\Gamma(\tau)} \\
&\times \int_0^{\infty} t^{\tau-\phi_1} e^{-t} I_{\phi_1 - 1}(2\sqrt{x}t)\, \Phi_p(v; \phi_2; yt)\, \Phi_p(v'; \phi_3; zt)\, dt,
\end{aligned} \tag{53}
$$

Here, all parameters and variables would be restricted so that the representations can be meaningful and convergent: For example, $\Re(\tau - \phi_1) > -1$, $\phi_1 \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$ and $x \in \mathbb{C} \setminus (-\infty, 0]$.

3.2. Transformation Formulae

This subsection derives transformation formulae for the extended Exton's triple hypergeometric functions $X_{8,p}$. One can consult [25] for transformations of certain hypergeometric functions of three variables.

Theorem 11. *The following transformation formulae for* $X_{8,p}$ *hold true:*

$$\begin{aligned}&\mathrm{X}_{8,p}[\tau,\upsilon,\upsilon';\phi_1,\phi_2,\phi_3;x,y,z]\\&\quad=(1-y)^{-\tau}\mathrm{X}_{8,p}\left[\tau,\phi_2-\upsilon,\upsilon';\phi_1,\phi_2,\phi_3;\frac{x}{(1-y)^2},\frac{y}{y-1},\frac{z}{1-y}\right];\end{aligned}\tag{54}$$

$$\begin{aligned}&\mathrm{X}_{8,p}[\tau,\upsilon,\upsilon';\phi_1,\phi_2,\phi_3;x,y,z]\\&\quad=(1-z)^{-\tau}\mathrm{X}_{8,p}\left[\tau,\upsilon,\phi_3-\upsilon';\phi_1,\phi_2,\phi_3;\frac{x}{(1-z)^2},\frac{y}{1-z},\frac{z}{z-1}\right];\end{aligned}\tag{55}$$

$$\begin{aligned}&\mathrm{X}_{8,p}[\tau,\upsilon,\upsilon';\phi_1,\phi_2,\phi_3;x,y,z]=(1-y-z)^{-\tau}\\&\quad\times\mathrm{X}_{8,p}\left[\tau,\phi_2-\upsilon,\phi_3-\upsilon';\phi_1,\phi_2,\phi_3;\frac{x}{(1-y-z)^2},\frac{y}{y+z-1},\frac{z}{y+z-1}\right].\end{aligned}\tag{56}$$

Proof. Applying the extended Kummer transformation Formula (24) to $\Phi_p(\upsilon;\phi_2;yt)$ in (50) and then setting

$$t(1-y)=u \qquad \text{and} \qquad du=(1-y)dt$$

in the resulting integral, we obtain the first transformation Formula (54). A similar argument $\Phi_p(\upsilon';\phi_3;zt)$ will establish the second transformation Formula (55). Finally, using the extended Kummer transformation Formula (24), simultaneously, for both $\Phi_p(\upsilon;\phi_2;yt)$ and $\Phi_p(\upsilon';\phi_3;zt)$, we obtain the third transformation Formula (56). □

3.3. Recurrence Relation and Generating Function

This subsection investigates a recurrence relation and a generating function for the extended Exton's triple hypergeometric functions $X_{8,p}$. One can see [26] the contiguous relations between certain hypergeometric functions of three variables.

Theorem 12. *The following recurrence relation for* $X_{8,p}$ *holds true:*

$$\begin{aligned}\mathrm{X}_{8,p}[\tau,\upsilon,\upsilon';\phi_1,\phi_2,\phi_3;x,y,z]&=\mathrm{X}_{8,p}[\tau,\upsilon,\upsilon';\phi_1-1,\phi_2,\phi_3;x,y,z]\\&\quad+\frac{\tau(\tau+1)x}{\phi_1(1-\phi_1)}\mathrm{X}_{8,p}[\tau,\upsilon,\upsilon';\phi_1+1,\phi_2,\phi_3;x,y,z].\end{aligned}\tag{57}$$

Proof. Applying the contiguous relation for the function ${}_0F_1$ in (25) to the integral representation (50), we obtain the desired result. □

Theorem 13. *The following generating function for* $X_{8,p}(x,y;z)$ *in* (41) *holds true:*

$$\begin{aligned}&\sum_{r=0}^{\infty}\frac{(\lambda)_r}{r!}\,\mathrm{X}_{8,p}[\lambda+r,\upsilon,\upsilon';\phi_1,\phi_2,\phi_3;x,y,z]\,t^r\\&\quad=(1-t)^{-\lambda}\,\mathrm{X}_{8,p}\left(\lambda,\upsilon,\upsilon';\phi_1,\phi_2,\phi_3;\frac{x}{(1-t)^2},\frac{y}{1-t},\frac{z}{1-t}\right)\end{aligned}\tag{58}$$

$$(\Re(p)\geqq 0,\ \lambda\in\mathbb{C}\ \textit{and}\ |t|<1).$$

Proof. Let $\mathcal{L}$ be the left-handed member of (58). Using the binomial theorem:

$$(1-t)^{-\lambda} = \sum_{r=0}^{\infty} (\lambda)_r \frac{t^r}{r!} \quad (|t| < 1) \tag{59}$$

and (41), we have

$$\begin{aligned}
\mathcal{L} &= \sum_{r=0}^{\infty} \frac{(\lambda)_r t^r}{r!} \left(\sum_{k,m,n=0}^{\infty} \frac{(\lambda + r)_{2k+m+n}}{(\phi_1)_k} \frac{\mathrm{B}(v+m, \phi_2 - v; p)}{\mathrm{B}(v, \phi_2 - v)} \frac{\mathrm{B}(v'+n, \phi_3 - v'; p)}{\mathrm{B}(v', \phi_3 - v')} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \right) \\
&= \sum_{k,m,n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(\lambda)_r (\lambda + r)_{2k+m+n}}{(\phi_1)_k} \frac{\mathrm{B}(v+m, \phi_2 - v; p)}{\mathrm{B}(v, \phi_2 - v)} \frac{\mathrm{B}(v'+n, \phi_3 - v'; p)}{\mathrm{B}(v', \phi_3 - v')} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{t^r}{r!}.
\end{aligned}$$

Employing the relation $(\lambda)_r (\lambda + r)_{2k+m+n} = (\lambda + 2k + m + n)_r (\lambda)_{2k+m+n}$, we obtain

$$\begin{aligned}
\mathcal{L} &= \sum_{k,m,n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(\lambda + 2k + m + n)_r (\lambda)_{2k+m+n}}{(\phi_1)_k} \frac{\mathrm{B}(v+m, \phi_2 - v; p)}{\mathrm{B}(v, \phi_2 - v)} \frac{\mathrm{B}(v'+n, \phi_3 - v'; p)}{\mathrm{B}(v', \phi_3 - v')} \\
&\qquad \times \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{t^r}{r!} \\
&= \sum_{k,m,n=0}^{\infty} \frac{(\lambda)_{2k+m+n}}{(\phi_1)_k} \frac{\mathrm{B}(v+m, \phi_2 - v; p)}{\mathrm{B}(v, \phi_2 - v)} \frac{\mathrm{B}(v'+n, \phi_3 - v'; p)}{\mathrm{B}(v', \phi_3 - v')} \\
&\qquad \times \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \left(\sum_{r=0}^{\infty} (\lambda + 2k + m + n)_r \frac{t^r}{r!} \right).
\end{aligned}$$

Using the binomial theorem

$$\sum_{r=0}^{\infty} (\lambda + 2k + m + n)_r \frac{t^r}{r!} = (1-t)^{-\lambda - 2k - m - n} \quad (|t| < 1),$$

we find

$$\begin{aligned}
\mathcal{L} = (1-t)^{-\lambda} &\sum_{k,m,n=0}^{\infty} \frac{(\lambda)_{2k+m+n}}{(\phi_1)_k} \frac{\mathrm{B}(v+m, \phi_2 - v; p)}{\mathrm{B}(v, \phi_2 - v)} \frac{\mathrm{B}(v'+n, \phi_3 - v'; p)}{\mathrm{B}(v', \phi_3 - v')} \\
&\times \frac{1}{k!} \left(\frac{x}{(1-t)^2} \right)^k \frac{1}{m!} \left(\frac{y}{1-t} \right)^m \frac{1}{n!} \left(\frac{z}{1-t} \right)^n,
\end{aligned}$$

which, in terms of (41), corresponds to the right-handed member of (58). □

4. Bounding Inequalities for $H_{4,p}$ and $X_{8,p}$

This section explores bounding inequalities for the extended Horn's double hypergeometric function $H_{4,p}$ and the extended Exton's triple hypergeometric function $X_{8,p}$. The first auxiliary lemma is a simple but sharp estimate ([27], p. 224, Equation (5.78)):

$$\max_{0<t<1} e^{-\frac{p}{t(1-t)}} = e^{-4p} \quad (p \geqslant 0), \tag{60}$$

which can be proven by noticing that the function $g(t) = -\frac{1}{t(1-t)}$ has the maximum value -4 at $t = \frac{1}{2}$ on the interval $0 < t < 1$.

The following lemma provides an inequality that is readily verifiable by using (3) and the observation (60).

Lemma 2. *Let $p \geqslant 0$ and $\eta, \xi \in \mathbb{R}_{>0}$. Then*

$$\mathrm{B}(\eta, \xi; p) \leq e^{-4p} \mathrm{B}(\eta, \xi). \tag{61}$$

Let $\mathbb{R}_{>0}$ stand for the set of positive real numbers, both here and in other contexts.

4.1. Bounds for the Extended Functions

The following theorem provides bounding inequalities for the extended Gaussian hypergeometric F_p, the extended Kummer confluent hypergeometric Φ_p, the extended second Appell F_2, the extended Horn's double hypergeometric $H_{4,p}$, and the extended Exton's triple hypergeometric function $X_{8,p}$, by using their series representations.

Theorem 14. *Let $p \geqslant 0$. Moreover, let the numerator parameters be nonnegative real numbers and the denominator parameters be positive real numbers. Furthermore, let the variables be nonnegative real numbers. Then*

$$\begin{aligned} &F_p(\tau, v; \phi; z) \leqslant e^{-4p}\, {}_2F_1(\tau, v; \phi; z) \\ &(z < 1,\ \phi > v;\ z = 1,\ \phi > \tau + v); \end{aligned} \tag{62}$$

$$\Phi_p(v; \phi; z) \leqslant e^{-4p}\, \Phi(v; \phi; z); \tag{63}$$

$$\begin{gathered} F_2(\tau, v, v'; \phi, \phi'; x, y; p) \leqslant e^{-8p}\, F_2(\tau, v, v'; \phi, \phi'; x, y) \\ (x + y < 1,\ \phi > v,\ \phi' > v'); \end{gathered} \tag{64}$$

$$\begin{gathered} H_{4,p}[\tau, v; \phi, \phi'; x, y] \leqslant e^{-4p}\, H_4[\tau, v; \phi, \phi'; x, y] \\ (2\sqrt{x} + y < 1,\ \phi' > v); \end{gathered} \tag{65}$$

$$\begin{gathered} X_{8,p}[\tau, v, v'; \phi_1, \phi_2, \phi_3; x, y, z] \leqslant e^{-8p}\, X_8[\tau, v, v'; \phi_1, \phi_2, \phi_3; x, y, z] \\ (2\sqrt{x} + y + z < 1,\ \phi_2 > v,\ \phi_3 > v'). \end{gathered} \tag{66}$$

Each equality of the inequalities holds when $p = 0$.

Proof. We prove only (62). Applying (61) to the extended Gaussian hypergeometric function (6), we have

$$\begin{aligned} F_p(\tau, v; \phi; z) &\leqslant e^{-4p} \sum_{n \geqslant 0} (\tau)_n \frac{\mathrm{B}(v + n, \phi - v)}{\mathrm{B}(v, \phi - v)} \frac{z^n}{n!} \\ &= e^{-4p}\, {}_2F_1(\tau, v; \phi; z). \end{aligned}$$

This proves (62). The other inequalities can be verified using an argument similar to the one presented in the proof of (62). However, the specifics are omitted. □

4.2. Bounds Obtained via Integral Representations

In this subsection, we investigate the bounds of the extended Horn's double hypergeometric function $H_{4,p}$ and the extended Exton's triple hypergeometric function $X_{8,p}$, which were introduced in Sections 2 and 3, respectively. To accomplish this, we review and recall certain inequalities pertaining to the generalized hypergeometric function, Bessel function, and modified Bessel function, as follows:

- For $b_j \geqslant \tau_j > 0$ ($j \in \overline{1, r}$) and $x \in \mathbb{R}_{>0}$, the following two-sided inequalities for ${}_rF_r$ hold true according to Luke's theorem (see ([28], Theorem 16, Equation (5.6)):

$$e^{\theta x} < {}_rF_r(\tau_r;\, v_r;\, x) < 1 - \theta(1 - e^x), \tag{67}$$

where

$$\theta = \frac{\max\limits_{1 \leqslant j \leqslant r} \tau_j}{\min\limits_{1 \leqslant j \leqslant r} v_j}. \tag{68}$$

For $\phi \geqslant v > 0$, the two-sided inequalities for Kummer's confluent hypergeometric function $\Phi(v;\phi;x) = {}_1F_1(v;\phi;x)$ easily follow:

$$e^{\frac{v}{\phi}x} < \Phi(v;\,\phi;\,x) < 1 - \frac{v}{\phi}\big(1 - e^x\big). \tag{69}$$

- Bounding inequalities for J_ν and I_ν:
  (i) Lommel's bounds (see, for example, ([18], pp. 31 and 406, [29], [30], pp. 548–549);

$$|J_\nu(t)| \leqslant 1, \qquad |J_{\nu+1}(t)| \leqslant \frac{1}{\sqrt{2}} \quad (\nu \in \mathbb{R}_{>0},\, t \in \mathbb{R}) \tag{70}$$

  (ii) The Minakshisundaram and Szász bound (see [31], Equation (1.8); see also [32], pp. 36–37; cf. [18], p. 16);

$$|J_\nu(t)| \leqslant \frac{1}{\Gamma(\nu+1)}\left(\frac{|t|}{2}\right)^\nu \quad (\nu \geqslant 0,\; t \in \mathbb{R}) \tag{71}$$

  (iii) For $\nu \geqslant 0$ and $t \in \mathbb{R}$, Landau bounds [33]

$$|J_\nu(t)| \leqslant b_L\, \nu^{-1/3}, \qquad b_L := \sqrt[3]{2}\sup_{t\geqslant 0} \mathrm{Ai}(t), \tag{72}$$

$$|J_\nu(t)| \leqslant c_L\, |t|^{-1/3}, \qquad c_L := \sup_{t\geqslant 0} t^{1/3} J_0(t)\,, \tag{73}$$

  where $\mathrm{Ai}(\cdot)$ stands for the Airy function

$$\mathrm{Ai}(t) := \frac{\pi}{2}\sqrt{\frac{t}{3}}\Big(J_{-1/3}\Big\{2(t/3)^{3/2}\Big\} + J_{-1/3}\Big\{2(t/3)^{3/2}\Big\}\Big). \tag{74}$$

  (iv) The Olenko bound ([34], Theorem 2.1)

$$\sup_{t\geqslant 0} \sqrt{t}|J_\nu(t)| \leqslant b_L\sqrt{\nu^{1/3} + \frac{\tau_1}{\nu^{1/3}} + \frac{3\tau_1^2}{10\nu}} =: d_O, \qquad \nu > 0\,, \tag{75}$$

  where τ_1 is the smallest positive zero of the Airy function Ai in (74) and b_L is Landau's constant in (72). This bound is asymptotically precise and the constant b_L is the best possible.

  (v) Luke ([28], Equation (6.25)) gave the following inequality for the modified Bessel function I_μ:

$$I_\mu(t) < \frac{\left(\frac{t}{2}\right)^\mu}{\Gamma(\mu+1)}\cosh t \quad \left(t > 0,\; \mu > -\frac{1}{2}\right). \tag{76}$$

The following theorem states and proves our second set of findings for the bounded inequalities of $H_{4,p}$.

Theorem 15. *The following inequalities hold true:*

$$\begin{aligned}\Big|H_{4,p}[\tau, v;\phi,\phi';-x,y]\Big| \leqslant & \frac{\Gamma(\phi)\,\Gamma(\tau-\phi+1)\,|x|^{\frac{1-\phi}{2}}}{\Gamma(\tau)\,e^{4p}} \\ & \times\left[1 - \frac{v}{\phi'}\left\{1-(1-y)^{-\tau+\phi-1}\right\}\right]\end{aligned} \tag{77}$$

$$\left\{\begin{matrix} p > 0,\, \tau+1 > \phi > 0,\, \phi' \geqslant v > 0,\, x \geqslant 0,\, y \leqslant 0; \\ p = 0,\, \tau+1 > \phi > 0,\, \phi' \geqslant v > 0,\, y \leqslant 0,\, 2\sqrt{|x|} + |y| < 1 \end{matrix}\right\};$$

$$\left|H_{4,p}[\tau,v;\phi,\phi';-x,y]\right| \leq \frac{\Gamma(\phi)\,\Gamma(\tau-\phi+1)\,b'_L\,|x|^{\frac{1-\phi}{2}}}{\sqrt[3]{\phi-1}\,\Gamma(\tau)\,e^{4p}} \times \left[1-\frac{v}{\phi'}\left(1-(1-y)^{-\tau+\phi-1}\right)\right] \tag{78}$$

$$\left\{\begin{matrix} p>0,\ \tau+1>\phi>0,\ \phi'\geqslant v>0,\ x\geqslant 0,\ y\leqslant 0; \\ p=0,\ \tau+1>\phi>0,\ \phi'\geqslant v>0,\ y\leqslant 0,\ 2\sqrt{|x|}+|y|<1 \end{matrix}\right\},$$

where

$$b'_L := \sqrt[3]{2}\sup_{t\geqslant 0}\mathrm{Ai}(2\sqrt{x}t);$$

$$\left|H_{4,p}[\tau,v;\phi,\phi';x,y]\right| \leq e^{-4p}\left\{\frac{1-\frac{v}{\phi'}}{(1-2\sqrt{x})^{\tau}}+\frac{\frac{v}{\phi'}}{(1-2\sqrt{x}-y)^{\tau}}\right\} \tag{79}$$

$$\left\{p\geqslant 0,\ \tau>0,\ \phi>\frac{1}{2},\ \phi'\geqslant v>0,\ 0<x<\frac{1}{4},\ 2\sqrt{x}+y<1\right\}.$$

The equality in (77) holds when $p=0$, $\phi=1$, $x\to 0+$ and $v\to 0+$.

Proof. Applying the estimate (63) in Theorem 14 to the integral representations (21) and (22), respectively, we obtain

$$R_1 := \left\{\begin{matrix} \left|H_{4,p}[\tau,v;\phi,\phi';-x,y]\right| \\ \left|H_{4,p}[\tau,v;\phi,\phi';x,y]\right| \end{matrix}\right\} \leqslant \frac{\Gamma(\phi)\,e^{-4p}}{\Gamma(\tau)\,|x|^{\frac{\phi-1}{2}}}\int_0^\infty e^{-t}\,t^{\tau-\phi}\left\{\begin{matrix} |J_{\phi-1}(2\sqrt{x}t)| \\ |I_{\phi-1}(2\sqrt{x}t)| \end{matrix}\right\}\Phi(v;\phi';yt)\,dt. \tag{80}$$

Employing Luke's upper bound (69) in (80) gives the following estimate:

$$R_1 \leqslant \frac{\Gamma(\phi)\,e^{-4p}}{\Gamma(\tau)\,|x|^{\frac{\phi-1}{2}}}\int_0^\infty e^{-t}\,t^{\tau-\phi}\left\{\begin{matrix} |J_{\phi-1}(2\sqrt{x}t)| \\ |I_{\phi-1}(2\sqrt{x}t)| \end{matrix}\right\}\left[1-\frac{v}{\phi'}\{1-e^{yt}\}\right]dt =: R_2. \tag{81}$$

Using the first one in (70) in the first one of R_2 in (81), we find

$$R_2 \leqslant \frac{\Gamma(\phi)\,e^{-4p}}{\Gamma(\tau)\,|x|^{\frac{\phi-1}{2}}}\int_0^\infty e^{-t}\,t^{\tau-\phi}\left[1-\frac{v}{\phi'}\{1-e^{yt}\}\right]dt,$$

which, upon employing (2) and the following integral formula:

$$\int_0^\infty e^{-\mu t}\,t^{\tau-1}\,dt = \frac{\Gamma(\tau)}{\mu^{\tau}} \quad (\Re(\tau)>0,\ \mu>0) \tag{82}$$

to evaluate the right-sided integral, and combine the resulting inequality into (81), we obtain the desired inequality (77).

By utilizing the first Landau result ((72), we can derive inequality (78) in a manner similar to obtaining inequality (77).

Applying the inequality ($\cosh t \leqslant e^t$ $(t\in\mathbb{R})$) to (76) offers the following inequality:

$$I_\mu(t) < \frac{\left(\frac{t}{2}\right)^{\mu}}{\Gamma(\mu+1)}\,e^t \quad \left(t>0,\ \mu>-\frac{1}{2}\right),$$

which gives

$$|I_{\phi-1}(2\sqrt{x}t)| < \frac{x^{\frac{\phi-1}{2}}\, t^{\phi-1}}{\Gamma(\phi)}\, e^{2\sqrt{x}t} \quad \left(x>0,\ t>0,\ \phi>\frac{1}{2}\right). \tag{83}$$

Employing (83) at the second inequality of (80), using a similar process as in the proof of (77), we obtain inequality (79). The involved details are omitted.

We prove the sharpness of the inequality in (77): Note that $|J_\nu(t)| = 1$ when $\nu = 0$ and $t \to +0$ in (19). Then, setting $p = 0$, $\phi = 1$, and taking $x \to +0$ in (21) gives

$$\left|H_{4,0}[\tau, v; 1, \phi'; 0, y]\right| = \frac{1}{\Gamma(\tau)} \int_0^\infty t^{\tau-1} e^{-t}\, \Phi(v;\phi';yt)\, dt,$$

which, upon taking $v \to +0$ and using (2), proves the equality of (77). □

The following theorem states and proves our third set of findings for bounded inequalities of $H_{4,p}$.

Theorem 16. *The following inequalities hold true:*

For $\tau>0$, $\phi>0$, $v \geqslant \phi' > 0$ and $x>0$, $y \leqslant 0$, we have

$$\left|H_{4,p}[\tau, v; \phi, \phi'; -x, y]\right| \leqslant \mathrm{e}^{-4p}\left\{1 - \frac{v}{\phi'}\left(1-(1-y)^{-\tau}\right)\right\}. \tag{84}$$

For $\tau>0$, $\phi>0$, $v \geqslant \phi' > 0$ and $x>0$, $y \leqslant 0$, we have

$$\begin{aligned}\left|H_{4,p}[\tau, v; \phi, \phi'; -x, y]\right| &\leqslant \frac{\Gamma(\phi)\,\mathrm{e}^{-4p}}{\Gamma(\tau)} \\ &\times \begin{cases} \dfrac{c_L\, x^{-\frac{\phi}{2}+\frac{1}{3}}}{\sqrt[3]{2}}\,\Gamma\!\left(\tau-\phi+\frac{2}{3}\right)\left\{1-\frac{v}{\phi'}\left(1-(1-y)^{-\tau+\phi-\frac{2}{3}}\right)\right\}, \\ \dfrac{d_O\, x^{-\frac{\phi}{2}+\frac{1}{4}}}{\sqrt{2}}\,\Gamma\!\left(\tau-\phi+\frac{1}{2}\right)\left\{1-\frac{v}{\phi'}\left(1-(1-y)^{-\tau+\phi-\frac{1}{2}}\right)\right\},\end{cases}\end{aligned} \tag{85}$$

where the first bound needs the additional restriction, $3\tau - 3\phi + 2 > 0$, while the second one needs an additional restriction, $2\tau - 2\phi + 1 > 0$. Moreover, in view of (9), when $p = 0$, we assume that $2\sqrt{|x|} + |y| < 1$.

The equality in (84) holds when $p = 0$, $\phi = 1$, $x \to 0+$ and $v \to 0+$.

Proof. Here, we first see that the estimates of the Bessel function in (71), (73), and (75), are of the magnitude $|J_{\phi-1}(t)| \le \mathfrak{C}\, t^\kappa$, where $\mathfrak{C} \in \{\frac{1}{2^{\phi-1}\Gamma(\phi)}, c_L, d_O\}$, and $\kappa \in \{\phi-1, -\frac{1}{3}, -\frac{1}{2}\}$, respectively. Now, applying the estimate (71) in Theorem 14 to the integral representation (21), denoted by R_1', gives

$$\begin{aligned} R_1' &:= \left|H_{4,p}[\tau, v; \phi, \phi'; -x, y]\right| \\ &\leqslant \frac{\Gamma(\phi)\, e^{-4p}}{\Gamma(\tau)\, |x|^{\frac{\phi-1}{2}}} \int_0^\infty e^{-t}\, t^{\tau-\phi} \left|J_{\phi-1}(2\sqrt{x}t)\right| \Phi(v;\phi';yt)\, dt =: R_2'. \end{aligned} \tag{86}$$

$$
\begin{aligned}
R_2' &\leq \frac{\mathfrak{C} e^{-4p}\,\Gamma(\phi)\,|x|^{\frac{1-\phi+\kappa}{2}}}{\Gamma(\tau)} \int_0^\infty \mathrm{e}^{-t}\, t^{\tau+\kappa-\phi} \left\{ 1 - \frac{v}{\phi'}\left(1-\mathrm{e}^{yt}\right) \right\} \mathrm{d}t \\
&= \frac{\mathfrak{C} e^{-4p}\,\Gamma(\phi)\,|x|^{\frac{1-\phi+\kappa}{2}}}{\Gamma(\tau)} \left\{ \left(1-\frac{v}{\phi'}\right) \int_0^\infty \mathrm{e}^{-t} t^{\tau+\kappa-\phi}\, \mathrm{d}t + \frac{v}{\phi'} \int_0^\infty \mathrm{e}^{-(1-y)t} t^{\tau+\kappa-\phi}\, \mathrm{d}t \right\} \\
&= \frac{\mathfrak{C} e^{-4p}\,\Gamma(\phi)\,|x|^{\frac{1-\phi+\kappa}{2}}}{\Gamma(\tau)} \Gamma(\tau+\kappa-\phi+1) \left\{ 1 - \frac{v}{\phi'} + \frac{v'}{\phi'}\, \frac{1}{(1-y)^{\tau+\kappa-\phi}} \right\} \\
&= \frac{\mathfrak{C} e^{-4p}\,\Gamma(\phi)\,|x|^{\frac{1-\phi+\kappa}{2}}\,\Gamma(\tau+\kappa-\phi+1)}{\Gamma(\tau)} \left\{ 1 - \frac{v}{\phi'}\left(1-(1-y)^{-\tau-\kappa+\phi}\right) \right\}.
\end{aligned}
$$

Then, choosing $\mathfrak{C} = \frac{1}{2^{\phi-1}\Gamma(\phi)}$, c_L, d_O and $\kappa = \phi - 1$, $\kappa = -\frac{1}{3}, -\frac{1}{2}$, we realize the bounds affiliated with Minakshisundaram and Szász, as well as the second Landau and Olenko estimates, respectively, given in Theorem 16.

The process of demonstrating the sharpness of the inequality in (84) can be carried out using the same method as that in (77), but without including the specifics. □

Bounding inequalities for $X_{8,p}$ can be obtained by using an argument similar to the one used in Theorem 15. The following theorem presents the first two results in parallel with those in Theorem 15.

Theorem 17. *The following inequalities hold true:*

$$
\begin{aligned}
\left| X_{8,p}[\tau, v, v'; \phi_1, \phi_2, \phi_3; -x, y, z] \right| &\leqslant \frac{\Gamma(\phi_1)\,\Gamma(\tau-\phi_1+1)\,|x|^{\frac{1-\phi_1}{2}}}{\Gamma(\tau)\, e^{4p}} \\
&\times \left[1 - \frac{v}{\phi_2}\left(1-(1-y)^{-\tau+\phi_2-1}\right) - \frac{v}{\phi_3}\left(1-(1-z)^{-\tau+\phi_2-1}\right) \right. \\
&\left. + \frac{vv'}{\phi_2\phi_3}\left(1-(1-y)^{-\tau+\phi_2-1} - (1-z)^{-\tau+\phi_2-1} - (1-y-z)^{-\tau+\phi_2-1}\right) \right],
\end{aligned}
\tag{87}
$$

provided that

(i) $p > 0$, $\tau > 0$, $\tau + 1 > \phi_k > 0$ $(k = 1, 2)$, $\phi_j \geqslant v > 0$ $(j = 2, 3)$,
$0 < vv' \leqslant \phi_2\phi_3$, $x \geqslant 0$, $y \leqslant 0$, $z \leqslant 0$.

(ii) $p = 0$, $\tau + 1 > \phi_k > 0$ $(k = 1, 2)$, $\phi_j \geqslant v > 0$ $(j = 2, 3)$,
$0 < vv' \leqslant \phi_2\phi_3$, $x \geqslant 0$, $y \leqslant 0$, $z \leqslant 0$, $2\sqrt{|x|} + |y| + |z| < 1$.

In both cases, additional adjustments are made to the involved variables and parameters so that the right member of (87) *is nonnegative.*

$$
\begin{aligned}
\left| X_{8,p}[\tau, v, v'; \phi_1, \phi_2, \phi_3; -x, y, z] \right| &\leqslant \frac{\Gamma(\phi_1)\,\Gamma(\tau-\phi_1+1)\, b_L'\,|x|^{\frac{1-\phi_1}{2}}}{\sqrt[3]{\phi_1-1}\,\Gamma(\tau)\, e^{4p}} \\
&\times \left[1 - \frac{v}{\phi_2}\left(1-(1-y)^{-\tau+\phi_2-1}\right) - \frac{v}{\phi_3}\left(1-(1-z)^{-\tau+\phi_2-1}\right) \right. \\
&\left. + \frac{vv'}{\phi_2\phi_3}\left(1-(1-y)^{-\tau+\phi_2-1} - (1-z)^{-\tau+\phi_2-1} - (1-y-z)^{-\tau+\phi_2-1}\right) \right],
\end{aligned}
\tag{88}
$$

provided that

(i) $p > 0$, $\tau > 0$, $\tau + 1 > \phi_k > 0$ $(k = 1, 2)$, $\phi_j \geqslant v > 0$ $(j = 2, 3)$,
$\phi_1 > 1$, $0 < vv' \leqslant \phi_2\phi_3$, $x \geqslant 0$, $y \leqslant 0$, $z \leqslant 0$.

(ii) $p = 0,\ \tau > 0,\ \tau + 1 > \phi_k > 0\ (k = 1, 2),\ \phi_j \geqslant v > 0\ (j = 2, 3),$

$\phi_1 > 1,\ 0 < vv' \leqslant \phi_2\phi_3,\ x \geqslant 0,\ y \leqslant 0,\ z \leqslant 0,\ 2\sqrt{|x|} + |y| + |z| < 1.$

The variables and parameters are further adjusted in both cases to ensure that the right-hand side of (88) *is nonnegative.*

Here,

$$b'_L := \sqrt[3]{2} \sup_{t\geqslant 0} \mathrm{Ai}(2\sqrt{xt}).$$

The equality in (87) *holds when* $p = 0$, $\phi_1 = 1$, $x \to 0+$, $v \to 0+$ *and* $v' \to 0+$.

Proof. Applying the estimate (63) in Theorem 14 to the integral representation (52), we obtain

$$\begin{aligned} R''_1 &:= \left| X_{8,p}[\tau, v, v'; \phi_1, \phi_2, \phi_3; -x, y, z] \right| \\ &\leqslant \frac{\Gamma(\phi_1)\, e^{-4p}}{\Gamma(\tau)\, |x|^{\frac{\phi_1 - 1}{2}}} \int_0^\infty e^{-t}\, t^{\tau - \phi_1} \left| J_{\phi_1 - 1}(2\sqrt{xt}) \right| \Phi_p(v; \phi_2; yt)\ \Phi_p(v'; \phi_3; zt)\, dt. \end{aligned} \tag{89}$$

Employing Luke's upper bound (69) in (89) gives the following estimate:

$$\begin{aligned} R''_1 \leqslant & \frac{\Gamma(\phi_1)\, e^{-4p}}{\Gamma(\tau)\, |x|^{\frac{\phi_1 - 1}{2}}} \int_0^\infty e^{-t}\, t^{\tau - \phi_1} \\ & \times \left| J_{\phi_1 - 1}(2\sqrt{xt}) \right| \left[1 - \frac{v}{\phi_2}\{1 - e^{yt}\} \right] \left[1 - \frac{v'}{\phi_3}\{1 - e^{zt}\} \right] dt =: R''_2. \end{aligned} \tag{90}$$

Using the first one in (70) in the first one of R''_2 in (90), we find

$$R''_2 \leqslant \frac{\Gamma(\phi_1)\, e^{-4p}}{\Gamma(\tau)\, |x|^{\frac{\phi_1 - 1}{2}}} \int_0^\infty e^{-t}\, t^{\tau - \phi_1} \left[1 - \frac{v}{\phi_2}\{1 - e^{yt}\} \right] \left[1 - \frac{v'}{\phi_3}\{1 - e^{zt}\} \right] dt,$$

which, upon employing (2) and the following integral formula (82) to evaluate the right-sided integral, and combining the resulting inequality into (90), we obtain the desired inequality (87).

By utilizing the first Landau result (72), we can derive inequality (88) in a manner akin to obtaining inequality (87).

The same method used to prove Theorem (15) can be used to demonstrate the equality of inequality (87). The details are omitted. □

5. An Application

Special functions are important in studying probability distribution and statistical inference (see, e.g., ([35], Chapter 17, [36–38]). Recently, researchers have been studying McKay Bessel-type distributions, which are related to special functions, such as Exton's and Horn's confluent functions (see [39–42]). The extended Horn's double hypergeometric function (17) is expected to have many applications, similar to the generalized Beta and Gamma functions. One potential application is in statistics, and it can also be applied in inequality theory to derive novel bilateral bounds for the generalized Horn's function $H_{4,p}$ using probabilistic methods.

Consider the random variable ξ defined on a standard probability space $(\Omega, \mathfrak{F}, \mathbf{P})$, where Ω is a sample space, $\mathfrak{F}$ is the event space in Ω, and $\mathbf{P}$ is a probability function, characterized by the following probability density function (PDF):

$$f_\xi(u) := \begin{cases} \dfrac{\lambda^\tau\, u^{\tau-1} e^{-\lambda u}\, {}_0F_1(-;\phi;xu^2)\, \Phi_p(v;\phi';yu)}{\Gamma(\tau)\, H_{4,p}\big[\tau, v;\phi,\phi';x/\lambda^2, y/\lambda\big]}, & u > 0 \\ 0 & \text{elsewhere,} \end{cases} \tag{91}$$

where it is assumed that $\Re(\lambda) > 0$, $\Re(\tau) > 0$, the positive arguments (x, y), and the parameters p, λ, μ and v are suitably constrained so that $f_\xi(u)$ remains nonnegative. Denoting

$$C_p(\lambda,\tau) := \frac{\lambda^\tau}{\Gamma(\tau)\, H_{4,p}\big[\tau, v;\phi,\phi';x/\lambda^2, y/\lambda\big]},$$

we find that

$$\int_0^\infty f_\xi(u)\, \mathrm{d}u = C_p(\lambda,\tau) \int_0^\infty e^{-\lambda u} u^{\tau-1}\, {}_0F_1(-;\phi;xu^2)\, \Phi_p(v;\phi';yu)\, \mathrm{d}u$$

is a Laplace–Mellin transform of the function $u \mapsto {}_0F_1(-;\phi;xu^2)\ \Phi_p(v;\phi';yu)$. After calculating, we confirm that

$$\int_0^\infty f_\xi(u)\, \mathrm{d}u = 1,$$

proving $f_\xi(u)$ is a PDF.

We define the generalized Horn's gamma distribution ξ as $\mathbf{GHG}(\theta)$, where $\theta = (p;\tau,v,\phi,\phi',\lambda;x,y)$ is the parameter vector. Alternatively, we denote this as $\xi \sim f_\xi(u)$. Hereafter, we will derive some statistical functions for the random variable $\xi \sim \mathbf{GHG}(\theta)$.

Raw Moments

The sth fractional-order moments m_s, $s > 0$ equal

$$m_s = \mathsf{E}\xi^s = \int_0^\infty u^s f_\xi(u)\, \mathrm{d}u = \frac{(\tau)_s}{\lambda^\tau} \frac{H_{4,p}[\tau+s, v;\phi,\phi';x/\lambda^2, y/\lambda]}{H_{4,p}[\tau, v;\phi,\phi';x/\lambda^2, y/\lambda]}. \tag{92}$$

As the first application of (92), we derive a Turán-type bounding inequality for the extended Horn's double hypergeometric function $H_{4,p}$ by virtue of the moment inequality, which holds for the nonnegative random variable $\xi \sim f_\xi(u)$. Lukacs reported on the moment inequality ([42], p. 28, Equation (1.4.6)):

$$m_s^2 \leqslant m_s\, m_{s+2r} \quad (\min\{s,r\} > 0). \tag{93}$$

By inserting the expression (92) in (93), we obtain the bounding inequality for the extended Horn's double hypergeometric function $H_{4,p}$:

$$H_{4,p}^2[s+r] \leqslant \frac{\Gamma(\tau+s)\,\Gamma(\tau+s+2r)}{\Gamma^2(\tau+s+r)}\, H_{4,p}[s] \cdot H_{4,p}[s+2r] \quad (2s > -\tau,\ s+2r > -\tau). \tag{94}$$

Here, and in the following,

$$H_{4,p}[a] := H_{4,p}[\tau+a, v;\phi,\phi';x/\lambda^2, y/\lambda].$$

Lukacs ([42], p. 393, a) states that for $0 < r \leqslant s$, the moment inequality $m_{s+r}^2 \leqslant m_{2s} \cdot m_{2r}$ holds, which can be inferred using the Cauchy–Bunyakovsky–Schwarz inequality. This inequality implies a variant of the Turán–type inequality:

$$H_{4,p}^2[s+r] \leqslant \frac{\Gamma(\tau+2s)\,\Gamma(\tau+2r)}{\Gamma^2(\tau+s+r)}\, H_{4,p}[2s] \cdot H_{4,p}[2r] \quad (2\min\{s,r\} > -\tau). \tag{95}$$

Characteristic Function

The Fourier transform of the PDF $f_\xi(u)$ is the characteristic function (CHF) $\varphi_\xi(t)$, say, of the random variable ξ. Hence,

$$\begin{aligned}\varphi_\xi(t) &= \mathsf{E}e^{\mathrm{i}t\xi} = \int_0^\infty e^{\mathrm{i}tu} f_\xi(u)\,\mathrm{d}u \\ &= C_p(\lambda,\tau)\int_0^\infty e^{-(\lambda-\mathrm{i}t)u}u^{\tau-1}{}_0F_1(-;\phi;xu^2)\,\Phi_p(v;\phi';yu)\,\mathrm{d}u\,.\end{aligned} \tag{96}$$

The recognition of the definition of the generalized Horn's double hypergeometric function's integral form (17) leads to the derivation of the characteristic function

$$\varphi_\xi(t) = \frac{\lambda^\tau\, H_{4,p}\big[\tau, v; \phi, \phi'; x/(\lambda - it)^2, y/(\lambda-\mathrm{i}t)\big]}{(\lambda-\mathrm{i}t)^\tau\, H_{4,p}\big[\tau, v; \phi, \phi'; x/\lambda^2, y/\lambda\big]}\,. \tag{97}$$

The surprising summation result in the following theorem establishes a connection between the probability density function (PDF) and the characteristic function (CHF) through the corresponding integer-order moments.

Theorem 18. *For the positive parameter vector, $\theta = (p;\tau, v, \phi, \phi', \lambda; x, y)$, we have*

$$\begin{aligned}&\sum_{n\geqslant 0}(\tau)_n\, H_{4,p}\left[\tau+n, v; \phi, \phi'; \frac{x}{\lambda^2}, \frac{y}{\lambda}\right]\frac{(\mathrm{i}t)^n}{n!} \\ &\qquad = \frac{\lambda^{2\tau}}{(\lambda-\mathrm{i}t)^\tau}\, H_{4,p}\left[\tau, v; \phi, \phi'; \frac{x}{(\lambda-\mathrm{i}t)^2}, \frac{y}{\lambda-\mathrm{i}t}\right].\end{aligned} \tag{98}$$

Proof. It is well-known that the Maclaurin series of the CHF reads ([42], p. 41)

$$\varphi_\xi(t) = \sum_{n\geqslant 0} m_n\,\frac{(\mathrm{i}t)^n}{n!}\,.$$

By inserting (92) and (97) into this expansion, the routine steps lead to the assertion. □

6. Concluding Remarks

In 1997, Chaudhry et al. ([1], p. 20, Equation (1.7)) introduced and explored the p-extended Beta integral (3) of the classical Beta function (4). The p-extended Beta integral (3) is proved to be connected to the Macdonald error and Whittaker functions. Since then, a number of such p-extensions of the hypergeometric function and its various generalizations of one and several variables have been presented and investigated (see, for example, [7,12,27]).

This paper explored extensions $H_{4,p}$ (9) and $X_{8,p}$ (41) of Horn's double hypergeometric function H_4 (10) and Exton's triple hypergeometric function X_8 (42), taking into account recent extensions of Euler's beta function, the hypergeometric function, the confluent hypergeometric function, two-variable Appell hypergeometric functions F_1 and F_2, and three-variable Lauricella hypergeometric functions $F_D^{(3)}$. Out of the many extended hypergeometric functions (see, for example, ([13], Chapters 2 and 3), the primary rationale for selecting H_4 and X_8 is their comparable extension type (see Remark 3). We presented various integral representations of Euler and Laplace types, Mellin transforms, the Laguerre polynomial representation, transformation formulae, and a recurrence relation for the extended functions. In particular, we provided a generating function for the $X_{8,p}$ and several bounding inequalities for the $H_{4,p}$ and $X_{8,p}$.

The hypergeometric series in one and several variables naturally appears in a wide range of problems across various fields of applied mathematics, including statistical distributions, operation research, theoretical physics, communication engineering, the theory of Lie algebras and Lie groups, perturbation theory, queuing theory, and engineering sciences. Multiple hypergeometric functions have been applied in these diverse fields, and Exton's monograph [35] and Srivastava and Karlsson's work [13] provide detailed accounts of these applications.

It seems possible that several applications of extended $H_{4,p}$ in (9) and $X_{8,p}$ in (41) of Horn's double hypergeometric function H_4 in (10) and Exton's triple hypergeometric function X_8 in (42) can be connected to certain fields mentioned in ([35], Chapters 7 and 8) and ([13], p. 47, Section (1.7)). However, there may be potential applications of the content in this work (as discussed in Section 5) that have yet to be explored and require further investigation.

We conclude this paper by providing a differential equation and posing a question, as follows:

- The following differential equation is derivable from (3):
$$\frac{d^n}{dp^n}\mathrm{B}(x,y;p) = (-1)^n\,\mathrm{B}(x-n,y-n;p) \quad (p>0,\ n\in\mathbb{N}_0). \tag{99}$$
- Can other bounding inequalities for $H_{4,p}$ and $X_{8,p}$ be given?

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