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Existence Results for Wardoski-Type Convex Contractions and the Theory of Iterated Function Systems

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Abstract: The purpose of this paper is to define the notion of extended convex $\mathcal{F}$ contraction by imposing less conditions on the function $\mathcal{F}$ satisfying certain contractive conditions. We prove the existence of fixed points for these types of mappings in the setting of b-metric spaces. In addition, some illustrative examples are provided to show the usability of the obtained results. Lastly, we use the obtained fixed-point results to find the fractals with respect to the iterated function systems in the framework of b-metric spaces. Furthermore, the variables involved in the b-metric space are symmetric, and symmetry plays an important role in solving the nonlinear problems defined in operator theory.

Keywords: b-metric space; fixed point; Wardowski (or $\mathcal{F}$) contraction; convex contraction; fractals; iterated function systems

MSC: primary 28A80; secondary 47H10; secondary 54E50

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1. Introduction and Preliminaries

The well-known Banach's fixed-point theorem (BFPT) [1] is the most important basic fixed-point result. Because this principle has numerous applications in various disciplines of mathematics, several writers have generalised, extended, and improved it in a variety of ways by considering various types of mappings or spaces. One such remarkable generalisation was given by Wardowski [2]. He introduced the notion of $\mathcal{F}$ contraction as follows:

Definition 1. *Let $(\mathcal{Z}, d)$ be a metric space (MS). A mapping $\mathrm{T} : \mathcal{Z} \to \mathcal{Z}$ is said to be an $\mathcal{F}$ contraction if there exists $\mathcal{F} \in \Delta(\mathcal{F})$ and $\lambda > 0$ such that for all $\nu, \mu \in \mathcal{Z}$, the following is true:*

$$\lambda + \mathcal{F}(d(\mathrm{T}\nu, \mathrm{T}\mu)) \leq \mathcal{F}(d(\nu, \mu)), \tag{1}$$

where $\Delta(\mathcal{F})$ is the set of all mappings $\mathcal{F} : (0, +\infty) \to \mathbb{R}$ that meets the following criteria:

($\mathcal{F}_1$) *$\mathcal{F}(\nu) < \mathcal{F}(\mu)$ for all $\nu < \mu$;*
($\mathcal{F}_2$) *For any sequence $\{\psi_p\} \subseteq (0, +\infty)$, $\lim_{p\to+\infty} \psi_p = 0$ if and only if $\lim_{p\to+\infty} \mathcal{F}(\psi_p) = -\infty$;*
($\mathcal{F}_3$) *There exists $0 < \wp < 1$ such that $\lim_{\psi\to0^+} \psi^{\wp}\mathcal{F}(\psi) = 0$.*

Theorem 1 ([2])**.** *Consider a complete MS $(\mathcal{Z}, d)$ and $\psi : \mathcal{Z} \to \mathcal{Z}$ to be an $\mathcal{F}$ contraction. Then, $\nu^* \in \mathcal{Z}$ is a unique fixed point of ψ, and for every $\nu_0 \in \mathcal{Z}$, a sequence $\{\psi^p \nu_0\}_{p\in\mathbb{N}}$ is convergent to ν^*.*

In [3], Secelean demonstrated that condition $(\mathcal{F}_2)$ can be modified with an equivalent and simpler one $((\mathcal{F}_2')$: $\inf \mathcal{F} = -\infty)$. Following that, Piri and Kumam [4] established Wardowski's theorem utilising $(\mathcal{F}_2')$ and the continuity rather than $(\mathcal{F}_2)$ and $(\mathcal{F}_3)$, respectively. Wardowski [5] later proved a fixed-point theorem for $\mathcal{F}$ contractions when λ is treated as a function:

Theorem 2 (Theorem 2.1 of [5])**.** *Let (z, d) be a complete MS and $\mathrm{T} : z \to z$. Let us say that there exist functions $\mathcal{F} :]0, \infty[\to \mathbb{R}$ and $\lambda :]0, \infty[\to]0, \infty[$ such that the following are true:*

(λ_1) *$\mathcal{F}$ satisfies $(\mathcal{F}_1)$ and $(\mathcal{F}_2')$.*

(λ_2) $\lim_{t\to\eta^+} \inf \lambda(t) > 0$ *for all* $\eta \geq 0$.

(λ_3) $\lambda(d(\nu, \mu)) + \mathcal{F}(d(\mathrm{T}\nu, \mathrm{T}\mu)) \leq \mathcal{F}(d(\nu, \mu))$ *for all $\nu, \mu \in z$ such that $\mathrm{T}\nu \neq \mathrm{T}\mu$.*

Then, T has only one fixed point in z.

From here onward, we denote with Λ the set of all functions $\lambda :]0, \infty[\to]0, \infty[$ satisfying condition (λ_2).

Recently, other authors demonstrated (in various methods) Wardowski's original results in the absence of both requirements $(\mathcal{F}_2)$ and $(\mathcal{F}_3)$ (see [6,7]). For more on this direction, consult [8–15]. Cosentino and Vetro [16] created a new concept, an $\mathcal{F}$ contraction of the Hardy–Rogers type, and derived the fixed-point theorem. Later, Vetro [14] expanded the notion of the Hardy–Rogers-type $\mathcal{F}$ contraction by switching λ with a function and proposed the notion of a Suzuki–Hardy–Rogers-type $\mathcal{F}$ contraction.

The concept of symmetry is characteristic of a Banach space, which is deeply related to the fixed-point problems [17] and has importance. Well-known researchers are observing it properly and working on it worldwide. This unwavering interest has been known to stem from the practical application of this area of research to several fields of research. Now, we should recall that symmetry is a mapping on some object X, which is supposed to be structured onto itself such that the structure is preserved. Saleem et al. [18] and Sain [19] provided several ways this mapping could occur. Neugebaner [17], using the concept of symmetry, obtained several applications of a layered compression–expansion fixed-point theorem in the existence of solutions of a second-order difference equation with Dirichlet boundary conditions.

On the other hand, Bakhtin [20] developed the concept of b-metric spaces as a generalisation of metric spaces in 1989 (also see the work of Czerwik [21]). Articles have been published that address results in b-metric spaces (see [22–31] and some related references therein). We will explain the definition of a b-metric space again:

Definition 2 ([21])**.** *Let z be a non-empty set, and let $\hbar \geq 1$ be a certain real number. A mapping $b : z \times z \to [0, \infty)$ is claimed to be b-metric if for any ν, μ, ω, the following requirements are met:*

(b_1) $b(\nu, \mu) = 0$ *if and only if* $\nu = \mu$;

(b_2) $b(\nu, \mu) = b(\mu, \nu)$;

(b_3) $b(\nu, \omega) \leq \hbar[b(\nu, \mu) + b(\mu, \omega)]$.

The pair (z, b) is called a b-metric space (b-MS) with a constant $\hbar \geq 1$.

The preceding definition makes it clear that a b-MS is standard metric space when $\hbar = 1$. Nonetheless, the converse is false (see [32,33]). It is important to remember that a b-metric space is not always continuous (see Example 3.3 in [34]). The lemmas listed below are quite helpful for handling this issue:

Lemma 1 ([22])**.** *Let (z, b) be a b-MS with a constant $\hbar \geq 1$ and $\{\nu_p\}$ be a sequence in z such that $\lim_{p\to\infty} \nu_p = \nu$. Then, for each $\mu \in z$, we have*

$$\frac{1}{\hbar} b(\nu, \mu) \leq \liminf_{p\to\infty} b(\nu_p, \mu) \leq \limsup_{p\to\infty} b(\nu_p, \mu) \leq \hbar b(\nu, \mu).$$

Lemma 2 ([35])**.** *Let $(\mathcal{Z}, b)$ be a b-MS with a constant $\hbar \geq 1$ and $\{\nu_p\}$ be a sequence in $\mathcal{Z}$ such that $\lim_{p\to\infty} b(\mathrm{T}\nu_p, \mathrm{T}\nu_{p+1}) = 0$. If $\{\nu_p\}$ is not Cauchy sequence in $(\mathcal{Z}, b)$, then there exist $\epsilon > 0$ and two sequences $\{q(r)\}$ and $\{p(r)\}$ of positive integers such that the following items hold:*

$$\epsilon^+ \leq \liminf_{r\to\infty} b(\nu_{q(r)}, \nu_{p(r)}) \leq \limsup_{r\to\infty} b(\nu_{q(r)}, \nu_{p(r)}) \leq \hbar\epsilon^+;$$

$$\frac{\epsilon}{\hbar}\epsilon^+ \leq \liminf_{r\to\infty} b(\nu_{q(r)}, \nu_{p(r)+1}) \leq \limsup_{r\to\infty} b(\nu_{q(r)}, \nu_{p(r)+1}) \leq \hbar^2\epsilon^+;$$

$$\frac{\epsilon}{\hbar}\epsilon^+ \leq \liminf_{r\to\infty} b(\nu_{q(r)+1}, \nu_{p(r)}) \leq \limsup_{r\to\infty} b(\nu_{q(r)+1}, \nu_{p(r)}) \leq \hbar^2\epsilon^+;$$

$$\frac{\epsilon}{\hbar^2}\epsilon^+ \leq \liminf_{r\to\infty} b(\nu_{q(r)+1}, \nu_{p(r)+1}) \leq \limsup_{r\to\infty} b(\nu_{q(r)+1}, \nu_{p(r)+1}) \leq \hbar^3\epsilon^+.$$

Proposition 1 (Proposition 3.11 of [23])**.** *Let $(\mathcal{Z}, b)$ be a b-MS with $\hbar \geq 1$. If b is continuous in one variable, then it is also continuous in the other.*

Lukács and Kajántá [6] refined Wardowski's theorem in the context of b-MS and dropped condition $(\mathcal{F}_2)$. Following that, several authors demonstrated (through various methods) Wardowski's original results in the absence of both conditions $(\mathcal{F}_2)$ and $(\mathcal{F}_3)$ (see [7,36]). Derouiche and Ramoul [35] recently introduced the notions of the extended $\mathcal{F}$ contraction of the Hardy–Rogers type, extended $\mathcal{F}$ contraction of the Suzuki–Hardy–Rogers type, and generalised $\mathcal{F}$-weak contraction of the Hardy–Rogers type by employing a relaxed version of condition $(\mathcal{F}_2)$ and eliminating condition $(\mathcal{F}_3)$, and they established some new fixed-point results for such kinds of mappings in the setting of complete b-metric spaces by using the following lemma:

Lemma 3 (Proposition 3.6 of [35])**.** *Let $(\mathcal{Z}, b)$ be a b-MS with $\hbar \geq 1$ and $\aleph$ be a certain real number such that $1 \leq \aleph \leq \hbar$. Let $\mathrm{T} : \mathcal{Z} \to \mathcal{Z}$ be a mapping and $\{\nu_p\}$ be the Picard sequence of T based on an arbitrary point $\nu_0 \in \mathcal{Z}$. Consider that there exists an increasing function $\mathcal{F}$ and $\lambda \in \Lambda_1$ such that for each $z \in \mathcal{Z}$ with $\mathrm{T}z \neq \mathrm{T}^2z$, the following holds:*

$$\lambda(b(z, \mathrm{T}z)) + \mathcal{F}(\aleph b(\mathrm{T}z, \mathrm{T}^2z)) \leq \mathcal{F}\Big((\rho_1 + \rho_2)b(z, \mathrm{T}z) + \rho_3 b(\mathrm{T}z, \mathrm{T}^2z) + \rho_4 b(z, \mathrm{T}^2z)\Big), \quad (2)$$

where ρ_i, in which $i = 1, 2, 3, 4$, represents nonnegative real numbers satisfying $\rho_1 + \rho_2 + \rho_3 + 2\rho_4\hbar = \frac{\aleph}{\hbar}$ and $\rho_3 \neq \frac{\aleph}{\hbar}$. Then, $\lim_{p\to\infty} b(\mathrm{T}\nu_p, \mathrm{T}\nu_{p+1}) = 0$.

Not long ago, in 2021, Huang et al. [37] introduced the notion of a convex $\mathcal{F}$ contraction and established some fixed-point results for such contractions in the context of b-MS.

Motivated by the works in [35,37], in this paper, we refine the notion of the convex $\mathcal{F}$ contraction in the setting of b-MS by introducing the extended convex $\mathcal{F}$ contraction. Our results unify and generalise many existing results in the literature, including those in [5,14,35,37].

2. Fundamental Results

We start this section by providing the following helpful lemma:

Lemma 4 ([35])**.** *Let $\vartheta \geq 1$ be a specific real number. Let $\{g_n\}$ be a sequence, and let $\beta, \alpha :]0, \infty[\to]-\infty, \infty[$ be functions that meet the following requirements:*

(i) $\alpha(\vartheta g_p) \leq \beta(g_{p-1})$ *for all* $p \in \mathbb{N}$*;*
(ii) α *is increasing;*
(iii) $\beta(g) < \alpha(g)$ *for all* $t > 0$*;*
(iv) $\limsup_{g\to\rho^+} \beta(g) < \alpha(\rho^+)$ *for all* $\rho > 0$*.*

Then, $\lim_{n\to\infty} g_n = 0$.

Consistent with [35], we have

$$\nabla(\mathcal{F}_c) = \{\mathcal{F} :]0,\infty[\to]-\infty,+\infty[\mid \mathcal{F} \text{ is a continuous increasing function}\}.$$

Let $\varpi \geq 1$ be a particular real number. We denote with Λ_ϖ the family of all functions $\lambda :]0,\infty[\to]0,\infty[$ which meet the criteria listed below:

$$\lim_{g\to h} \inf \lambda(g) > 0, \quad \text{where } h \in [\xi^+, \xi^+\varpi], \text{ for all } \xi > 0. \tag{3}$$

Obviously, if $\varpi = 1$, then Equation (3) becomes the following:

$$\lim_{g\to \xi^+} \inf \lambda(g) > 0, \quad \text{for all } \xi > 0. \tag{4}$$

From here onward, we denote with Λ_1 the set Λ_ϖ when $\varpi = 1$. Definitively, we have $\Lambda_\varpi \subseteq \Lambda_1$. Additionally, observe that in the sense of standard metric space, it is sufficient to employ the condition that $\lambda \in \Lambda_1$ rather than the condition $\lambda \in \Lambda_\varpi$.

Example 1. *Consider the function $\mathcal{F} : (0,+\infty) \to \mathbb{R}$ defined by $\mathcal{F}(g) = g$. Then, $\mathcal{F}$ is increasing and continuous, and thus $\mathcal{F} \in \nabla(\mathcal{F}_c)$.*

Example 2. *Consider the function $\mathcal{F} : (0,+\infty) \to \mathbb{R}$ defined by $\mathcal{F}(g) = \ln(g+1)$. Definitively, $\mathcal{F} \in \nabla(\mathcal{F}_c)$, but $\mathcal{F}$ does not satisfy condition $(\mathcal{F}_2)$. Indeed, for any sequence $\pi_p \in (0,+\infty)$ such that $\lim_{p\to\infty} \pi_p = 0$, we have*

$$\begin{aligned}\lim_{p\to\infty} \mathcal{F}(\pi_p) &= \lim_{p\to\infty} \ln(1+\pi_p) \\ &= \ln(1 + \lim_{p\to\infty} \pi_p) \\ &= 0 \neq -\infty.\end{aligned}$$

More precisely, $\Delta(\mathcal{F}) \subseteq \nabla(\mathcal{F}_c)$.

Example 3 ([35])**.** *Let $\lambda_i :]0,\infty[\to]0,\infty[$ be functions defined by the following conditions:*

(a) $\lambda_1(g) = \lambda$ *for each* $g \in]0,\infty[$, *where* $\lambda > 0$ *is a constant real number;*
(b) $\lambda_2(g) = \ln(1+g)$ *for each* $g \in]0,\infty[$;
(c) $\lambda_3(g) = \varrho g$ *for each* $g \in]0,\infty[$, *where* $\varrho > 0$.

Then, $\lambda_i \in \Lambda_\varpi$ for all $i = 1,2,3$, but $\lambda_2 \notin \Lambda$.

We now prove the following lemmas, which significantly contribute to the proofs of our results:

Lemma 5. *Let (z, b) be a b-MS with a constant $\hbar \geq 1$ and $\aleph$ be a given real number such that $1 \leq \aleph \leq \hbar$. Let $\mathrm{T} : z \to z$ be a mapping and $\{\nu_p\}$ be the Picard sequence of T based on an arbitrary point $\nu_0 \in z$. Assume that there exists an increasing function $\mathcal{F}$ and $\lambda \in \Lambda_1$ such that for all $p \in \mathbb{N}_0$ $(\mathbb{N}_0 = \mathbb{N} \cup \{0\})$ with $\mathrm{T}(\nu_p, \nu_{p+1}) > 0$, the following is true:*

$$\lambda(b(\nu_{p-1}, \nu_p)) + \mathcal{F}(\aleph b(\nu_p, \nu_{p+1})) \leq \mathcal{F}\left(\kappa b(\nu_p, \nu_{p+1}) + \left(\frac{\aleph}{\hbar} - \kappa\right) b(\nu_{p-1}, \nu_p)\right), \tag{5}$$

where $\kappa \in \left[0, \frac{\aleph}{\hbar}\right)$. Then, $\lim_{p\to\infty} b(\mathrm{T}\nu_p, \mathrm{T}\nu_{p+1}) = 0$.

Proof. Start with $b_p = b(\nu_p, \nu_{p+1})$. If $\nu_p = \nu_{p+1}$ for some $p \in \mathbb{N}_0$, then the proof is conclusive. Therefore, assume that $\nu_p \neq \nu_{p+1}$ for all $p \in \mathbb{N}_0$. By applying the inequality in Equation (5), we have for all $p \in \mathbb{N}$

$$\lambda(b_{p-1}) + \mathcal{F}(\aleph b_p) \leq \mathcal{F}\left(\kappa b_p + \left(\frac{\aleph}{\hbar} - \kappa\right) b_{p-1}\right). \tag{6}$$

By virtue of the fact that $\lambda(g) > 0$ for all $g > 0$, we have

$$\mathcal{F}(\aleph b_p) < \mathcal{F}\left(\kappa b_p + \left(\frac{\aleph}{\hbar} - \kappa\right) b_{p-1}\right).$$

Since $\mathcal{F}$ is increasing, then

$$\aleph b_p < \kappa b_p + \left(\frac{\aleph}{\hbar} - \kappa\right) b_{p-1},$$

which further implies that

$$(\aleph - \kappa) b_p < \left(\frac{\aleph}{\hbar} - \kappa\right) b_{p-1}.$$

Since

$$\frac{\aleph}{\hbar} - \kappa \leq \aleph - \kappa,$$

then we have

$$\left(\frac{\aleph}{\hbar} - \kappa\right) b_p \leq (\aleph - \kappa) b_p < \left(\frac{\aleph}{\hbar} - \kappa\right) b_{p-1}.$$

Consequently, we have

$$0 < b_p < b_{p-1}. \tag{7}$$

Hence, $\{b_p\}$ is a convergent sequence. Now, from Equations (6) and (7), we have

$$\mathcal{F}(\aleph b_p) \leq \mathcal{F}(b_{p-1}) - \lambda(b_{p-1}). \tag{8}$$

By taking $\alpha(g) = \mathcal{F}(g)$ and $\beta(g) = \mathcal{F}(g) - \lambda(g)$ for all $g \in]0, \infty[$, the inequality in Equation (8) can be written as

$$\alpha(\aleph b_p) \leq \beta(b_{p-1}), \quad \text{for all } \; p \in \mathbb{N}. \tag{9}$$

As $\mathcal{F}$ is increasing, then in light of the inequality in Equation (9), and using the fact that $\lambda \in \Lambda_1$, it is clear that all of Lemma 4's criteria with $\vartheta = \aleph \geq 1$ are satisfied. Thus, $\lim_{p\to\infty} b_p = 0$. □

Remark 1. *Lemma 5 greatly extends and improves Lemma 3. Indeed, let all hypotheses of Lemma 3 hold true and $\{\nu_p\}$ be a Picard sequence of* T *based on an arbitrary $\nu_0 \in z$. Assume that $\nu_p \neq \nu_{p+1}$ for all $p \in \mathbb{N}_0$ and $b_p > 0$ for all $p \in \mathbb{N}_0$. Then, from Equation (2), for all $p \in \mathbb{N}_0$, we have*

$$\begin{aligned} \lambda(b_{p-1}) + \mathcal{F}(\aleph b \nu_p) \leq & \mathcal{F}\left((d_1 + d_2) b_{p-1} + d_3 b_p + d_4 b(\nu_{p-1}, \nu_{p+1})\right) \\ \leq & \mathcal{F}\left((d_1 + d_2 + d_4\hbar) b_{p-1} + (d_3 + d_4\hbar) b_p\right). \end{aligned} \tag{10}$$

By letting $\kappa = d_3 + d_4\hbar$, the inequality in Equation (10) turns into Equation (5). Hence, by using Lemma 5, we have $\lim_{p\to\infty} b_p = 0$.

Lemma 6. *Let $(\mathcal{z}, b)$ be a b-MS with a constant $\hbar \geq 1$ and $\mathrm{T} : \mathcal{z} \to \mathcal{z}$ be a mapping that satisfies Equation (5) for an increasing function $\mathcal{F}$ and $\lambda \in \Lambda_1$. If $\kappa\hbar^4 - \kappa\hbar^2 + \aleph\hbar \leq 1$, then for every $\nu \in \mathcal{z}$, the sequence $\{\mathrm{T}^p\nu\}_{p\in\mathbb{N}_0}$ is a Cauchy sequence.*

Proof. Start with $b_p = b(\nu_p, \nu_{p+1})$. Choose an arbitrary point $\nu \in \mathcal{z}$, and construct a Picard sequence $\nu_p = \mathrm{T}^p\nu$ for all $p \in \mathbb{N}_0$. If $\nu_p = \nu_{p+1}$ for some $p_0 \in \mathbb{N}_0$, then

$$\{\mathrm{T}^p\nu\} = \{\nu, \mathrm{T}\nu, \mathrm{T}^2\nu, \cdots, \mathrm{T}^{p_0-1}\nu, \nu_{p_0}, \nu_{p_0}, \cdots\}.$$

Hence, $\{\mathrm{T}^p\nu\}$ is a Cauchy sequence. Assume that $\nu_p \neq \nu_{p+1}$ for all $p \in \mathbb{N}_0$ and $b_p > 0$ for all $p \in \mathbb{N}_0$. Then, we can apply the contractive condition in Equation (5). Hence, we obtain the following for all $p \in \mathbb{N}$:

$$\lambda(b(\nu_{p-1}, \nu_p)) + \mathcal{F}(\aleph b(\nu_p, \nu_{p+1})) \leq \mathcal{F}\left(\kappa b(\nu_p, \nu_{p+1}) + \left(\frac{\aleph}{\hbar} - \kappa\right) b(\nu_{p-1}, \nu_p)\right), \tag{11}$$

Hence, from Lemma 5, we have

$$\lim_{p\to\infty} b_p = 0. \tag{12}$$

Now suppose, on the contrary, that $\{\mathrm{T}_p = \mathrm{T}^p\nu\}$ is not a Cauchy sequence. Then, from Equation (12) and the first item of Lemma 2, there exist $\epsilon > 0$ and two sequences $\{q(r)\}$ and $\{p(r)\}$ of positive integers such that the following item holds:

$$\epsilon^+ \leq \liminf_{r\to\infty} b(\nu_{q(r)}, \nu_{p(r)}) \leq \limsup_{r\to\infty} b(\nu_{q(r)}, \nu_{p(r)}) \leq \hbar\epsilon^+.$$

Thus, we infer that there exists $r_0 \in \mathbb{N}$ such that $\{b(\nu_{q(r)}, \nu_{p(r)})\}$ is bounded for all $r \geq r_0$ and thereby has a convergent subsequence. It follows that there exist a real number η and a subsequence $\{r(\wp)\}_{\wp \geq r_0}$ of $\{r\}_{r\geq r_0}$ such that

$$\lim_{\wp\to\infty} b(\nu_{q(r(\wp))}, \nu_{p(r(\wp))}) = \eta \tag{13}$$

with

$$0 < \epsilon^+ \leq \liminf_{r\to\infty} b(\nu_{q(r)}, \nu_{p(r)}) \leq \limsup_{r\to\infty} b(\nu_{q(r)}, \nu_{p(r)}) \leq \hbar\epsilon^+. \tag{14}$$

On the other hand, using condition (b_3), we obtain the following for all $\wp \geq r_0$:

$$\begin{aligned} b(\nu_{q(r(\wp))}, \nu_{p(r(\wp))}) &\leq \hbar b(\nu_{q(r(\wp))}, \nu_{q(r(\wp))+1}) + \hbar b(\nu_{q(r(\wp))+1}, \nu_{p(r(\wp))}) \\ &\leq \hbar b(\nu_{q(r(\wp))}, \nu_{q(r(\wp))+1}) + \hbar^2 b(\nu_{q(r(\wp))+1}, \nu_{p(r(\wp))+1}) + \\ &\hbar^2 b(\nu_{p(r(\wp))}, \nu_{p(r(\wp))+1}) \\ &= \hbar b_{q(r(\wp))} + \hbar^2 b(\nu_{q(r(\wp))+1}, \nu_{p(r(\wp))+1}) + \hbar^2 b_{p(r(\wp))}. \end{aligned} \tag{15}$$

This leads to

$$b(\nu_{q(r(\wp))+1}, \nu_{p(r(\wp))+1}) \geq \frac{1}{\hbar^2}\left(b(\nu_{q(r(\wp))}, \nu_{p(r(\wp))}) - \hbar b_{q(r(\wp))} - \hbar^2 b_{p(r(\wp))}\right), \tag{16}$$

for all $\wp \geq r_0$. By letting the lower limit be $\wp \to \infty$ in Equation (16) and using Equation (12), we obtain

$$\liminf_{\wp\to\infty} b(\nu_{q(r(\wp))+1}, \nu_{p(r(\wp))+1}) \geq \frac{\eta}{s^2}. \tag{17}$$

As a result, there exist $N \geq r_0$ such that

$$b(\mathrm{T}\nu_{q(r(\wp))}, \mathrm{T}\nu_{p(r(\wp))}) = b(\nu_{q(r(\wp))+1}, \nu_{p(r(\wp))+1}) > 0, \quad \text{for all} \quad \wp \geq N. \tag{18}$$

Therefore, by applying the contractive inequality in Equation (5), for all $\wp \geq N$, we obtain

$$\begin{aligned} &\lambda(b(\nu_{q(r(\wp))}, \nu_{p(r(\wp))})) + \mathcal{F}(\aleph b(\nu_{q(r(\wp))+1}, \nu_{p(r(\wp))+1})) \\ \leq &\mathcal{F}\left(\kappa b(\nu_{q(r(\wp))+1}, \nu_{p(r(\wp))+1}) + \left(\frac{\aleph}{\hbar} - \kappa\right) b(\nu_{q(r(\wp))}, \nu_{p(r(\wp))})\right). \end{aligned} \tag{19}$$

In addition, by using condition (b_3), for all $\wp \geq N$, we have

$$\begin{aligned} &\kappa b(\nu_{q(r(\wp))+1}, \nu_{p(r(\wp))+1}) + \left(\frac{\aleph}{\hbar} - \kappa\right) b(\nu_{q(r(\wp))}, \nu_{p(r(\wp))}) \\ \leq &\kappa\hbar b_{q(r(\wp))} + \kappa\hbar^2 b(\nu_{q(r(\wp))}, \nu_{p(r(\wp))}) + \kappa\hbar^2 b_{p(r(\wp))} + \left(\frac{\aleph}{\hbar} - \kappa\right) b(\nu_{q(r(\wp))}, \nu_{p(r(\wp))}) \\ = &\kappa\hbar b_{q(r(\wp))} + \kappa\hbar^2 b_{p(r(\wp))} + \left(\kappa\hbar^2 + \frac{\aleph}{\hbar} - \kappa\right) b(\nu_{q(r(\wp))}, \nu_{p(r(\wp))}). \end{aligned} \tag{20}$$

Using Equation (20), the monotonicity of $\mathcal{F}$, and $\kappa\hbar^4 - \kappa\hbar^2 + \aleph\hbar \leq 1$, for all $\wp \geq N$, we obtain

$$\begin{aligned} &\lambda(b(\nu_{q(r(\wp))}, \nu_{p(r(\wp))})) + \mathcal{F}(b(\nu_{q(r(\wp))+1}, \nu_{p(r(\wp))+1})) \\ \leq &\lambda(b(\nu_{q(r(\wp))}, \nu_{p(r(\wp))})) + \mathcal{F}(\aleph b(\nu_{q(r(\wp))+1}, \nu_{p(r(\wp))+1})) \\ \leq &\mathcal{F}\left(\kappa b(\nu_{q(r(\wp))+1}, \nu_{p(r(\wp))+1}) + \left(\frac{\aleph}{\hbar} - \kappa\right) b(\nu_{q(r(\wp))}, \nu_{p(r(\wp))})\right) \\ \leq &\mathcal{F}\left(\kappa\hbar b_{q(r(\wp))} + \kappa\hbar^2 b_{p(r(\wp))} + \left(\kappa\hbar^2 + \frac{\aleph}{\hbar} - \kappa\right) b(\nu_{q(r(\wp))}, \nu_{p(r(\wp))})\right) \\ \leq &\mathcal{F}\left(\kappa\hbar b_{q(r(\wp))} + \kappa\hbar^2 b_{p(r(\wp))} + \frac{1}{s^2} b(\nu_{q(r(\wp))}, \nu_{p(r(\wp))})\right). \end{aligned} \tag{21}$$

Now, by combining Equation (21) with Equations (13) and (17), and by virtue of the fact that $\mathcal{F} \in \nabla(\mathcal{F}_c)$, we obtain

$$\begin{aligned} &\liminf_{t \to \eta} \lambda(t) + \mathcal{F}\left(\frac{\eta}{\hbar^2}\right) \\ \leq &\liminf_{\wp \to \infty} \lambda(b(\nu_{q(r(\wp))}, \nu_{p(r(\wp))})) + \mathcal{F}\left(\frac{\eta}{\hbar^2}\right) \\ \leq &\liminf_{\wp \to \infty} \lambda(b(\nu_{q(r(\wp))}, \nu_{p(r(\wp))})) + \mathcal{F}\left(\liminf_{\wp \to \infty} b(\nu_{q(r(\wp))+1}, \nu_{p(r(\wp))+1})\right) \\ = &\liminf_{\wp \to \infty} \left[\lambda(b(\nu_{q(r(\wp))}, \nu_{p(r(\wp))})) + \mathcal{F}\left(b(\nu_{q(r(\wp))+1}, \nu_{p(r(\wp))+1})\right)\right] \\ \leq &\lim_{\wp \to \infty} \mathcal{F}(\kappa\hbar b_{q(r(\wp))} + \kappa\hbar^2 b_{p(r(\wp))} + \frac{1}{s^2} b(\nu_{q(r(\wp))}, \nu_{p(r(\wp))})) \\ = &\mathcal{F}\left(\frac{\eta}{s^2}\right). \end{aligned}$$

The preceding inequality implies that

$$\liminf_{t \to m} \lambda(t) \leq 0, \quad \text{where } t \in [\varepsilon^+, \varepsilon^+\omega], \text{ for all } \varepsilon > 0,$$

which is a contradiction with Equation (3). This contradiction shows that $\{\mathrm{T}^p\nu\}$ is a Cauchy sequence. □

3. Fixed-Point Theorems

Definition 3. *Let $(\mathcal{z}, b)$ be a b-MS with a constant $\hbar \geq 1$. A mapping $\mathrm{T} : \mathcal{z} \to \mathcal{z}$ is said to be an extended convex Wardowski contraction (or extended convex $\mathcal{F}$ contraction) if there exist $\mathcal{F} :]0, \infty[\to \mathbb{R}$, $\lambda \in \Lambda_{\varpi}$ and $\kappa \in \left[0, \frac{1}{\hbar}\right)$ such that for all $\nu, \mu \in \mathcal{z}$, the following is true:*

$$b(\mathrm{T}\nu, \mathrm{T}\mu) > 0 \ \textit{implies} \ \lambda(b(\nu, \mu)) + \mathcal{F}(b(\mathrm{T}\nu, \mathrm{T}\mu)) \leq \mathcal{F}\left(\kappa b(\mathrm{T}\nu, \mathrm{T}\mu) + \left(\frac{1}{\hbar} - \kappa\right) b(\nu, \mu)\right). \quad (22)$$

Remark 2. *If $\mathcal{F}$ is an increasing function, then Definition 3 implies that every extended convex $\mathcal{F}$ contraction T satisfies the condition*

$$b(\mathrm{T}\nu, \mathrm{T}\mu) < b(\nu, \mu). \quad (23)$$

for all $\nu, \mu \in \mathcal{z}$ with $\mathrm{T}\nu \neq \mathrm{T}\mu$.

Theorem 3. *Let $(\mathcal{z}, b)$ be a complete b-MS with a constant $\hbar \geq 1$ and $\mathrm{T} : \mathcal{z} \to \mathcal{z}$ be an extended convex $\mathcal{F}$ contraction for $\mathcal{F} \in \nabla(\mathcal{F}_c)$. Assume that $\kappa\hbar^4 - \kappa\hbar^2 + \hbar \leq 1$. Then, T has a unique fixed point in $\mathcal{z}$.*

Proof. Let $\{\nu_p\}$ be a Picard sequence based on an arbitrary $\nu_0 \in \mathcal{z}$. If $\nu_p = \nu_{p+1}$ for some $p \in \mathbb{N}_0$, then ν_p is a fixed point of T, and the proof is conclusive. Therefore, assume that $\nu_p \neq \nu_{p+1}$ for all $p \in \mathbb{N}_0$. Then, we have

$$b_p := b(\nu_p, \nu_{p+1}) = b(\mathrm{T}\nu_p, \mathrm{T}\nu_{p+1}) > 0, \quad \text{for all} \quad p \in \mathbb{N}. \quad (24)$$

By using the inequality in Equation (22) with $\nu = \nu_{p-1}$ and $\mu = \nu_p$, for all $p \in \mathbb{N}$, we obtain

$$\lambda(b(\nu_{p-1}, \nu_p)) + \mathcal{F}(b(\nu_p, \nu_{p+1})) \leq \mathcal{F}\left(\kappa b(\nu_p, \nu_{p+1}) + \left(\frac{1}{\hbar} - \kappa\right) b(\nu_{p-1}, \nu_p)\right), \quad (25)$$

which is the inequality in Equation (5). Therefore, by virtue of $\Lambda_{\varpi} \subseteq \Lambda_1$ and Lemma 5 with $\aleph = 1$, we have

$$\lim_{p \to \infty} b_p = 0. \quad (26)$$

Since $\kappa\hbar^4 - \kappa\hbar^2 + 1 \leq \frac{1}{s^2}$, from Lemma (6) with $\aleph = 1$, we conclude that $\{\nu_p\} = \{\mathrm{T}^p \nu_0\}$ is a Cauchy sequence. With the completeness of $(\mathcal{z}, b)$, $\{\nu_p\}$ converges to some point $\nu* \in \mathcal{z}$; that is, we have

$$\lim_{p \to \infty} b(\nu_p, \nu*) = 0. \quad (27)$$

Next, we show that $\nu*$ is a fixed point of T. Suppose, on the contrary, that $b(\nu*, \mathrm{T}\nu*) > 0$. Then, from Equation (27), there exists $p_0 \in \mathbb{N}$ such that

$$b(\nu_p, \nu*) \leq \frac{b(\nu*, \mathrm{T}\nu*)}{2s}, \quad \text{for all} \quad p \geq p_0. \quad (28)$$

On the other side, from (b_3), we have

$$b(\nu*, \mathrm{T}\nu*) \leq \hbar b(\nu*, \mathrm{T}\nu_p) + \hbar b(\mathrm{T}\nu_p, \mathrm{T}\nu*). \quad (29)$$

The inequalities in Equations (28) and (29) yield

$$
\begin{aligned}
b(\mathrm{T}\nu_p, \mathrm{T}\nu*) &\geq \frac{1}{\hbar}(b(\nu*, \mathrm{T}\nu*) - \hbar b(\nu*, \mathrm{T}\nu_p)) \\
&= \frac{1}{\hbar} b(\nu*, \mathrm{T}\nu*) - b(\nu*, \nu_{p+1}) \\
&\geq \frac{b(\nu*, \mathrm{T}\nu*)}{2s} > 0,
\end{aligned} \tag{30}
$$

for all $p \geq p_0$. Now, owing to Equation (23) with $\nu = \nu_p$ and $\mu = \nu*$, for all $p \geq p_0$, Equation (29) gives

$$
\begin{aligned}
0 < b(\nu*, \mathrm{T}\nu*) &\leq \hbar b(\nu*, \nu_{p+1}) + \hbar b(\mathrm{T}\nu_p, \mathrm{T}\nu*) \\
&< \hbar b(\nu*, \nu_{p+1}) + \hbar b(\nu_p, \nu*) \\
&= 0,
\end{aligned} \tag{31}
$$

which is a contradiction. Hence, $\mathrm{T}\nu* = \nu*$.

Lastly, we prove that T has a maximum of one fixed point. Assume that $\nu*$ and $\mu*$ are two distinct fixed points of T. Then, we have

$$
b(\mathrm{T}\nu*, \mathrm{T}\mu*) = b(\nu*, \mu*) > 0.
$$

From Equation (22), we obtain

$$
\begin{aligned}
\lambda(b(\nu*, \mu*)) + \mathcal{F}(b(\nu*, \mu*)) &\leq \mathcal{F}\left(\kappa b(\nu*, \mu*) + \left(\frac{1}{\hbar} - \kappa\right) b(\nu*, \mu*)\right) \\
&= \mathcal{F}\left(\frac{1}{\hbar} b(\nu*, \mu*)\right) \\
&\leq \mathcal{F}(b(\nu*, \mu*)).
\end{aligned} \tag{32}
$$

The inequality in Equation (32) implies that $\lambda(b(\nu*, \mu*)) \leq 0$, which is a contradiction, and the proof is conclusive. □

Remark 3. *Observe that in Theorem 3, conditions $(\mathcal{F}_2^{\kappa})$ and $(\mathcal{F}_3^{\kappa})$ are omitted. In addition, the strictness of the monotonicity of $\mathcal{F}$ is not considered.*

Moreover, Theorem 3 gives the answer to Problem 1 in [37], as conditions $(\mathcal{F}_3^{\kappa})$ and $(\mathcal{F}_3)$ are not used to prove Theorem 3.

Since a standard metric space is a b-MS for $\hbar = 1$, then by virtue of Theorem 3, we obtain the following:

Corollary 1. *Let $(\mathcal{Z}, d)$ be a complete MS and $\mathrm{T} : \mathcal{Z} \to \mathcal{Z}$. If there exist $\mathcal{F} \in \nabla(\mathcal{F}_c)$, $\lambda \in \Lambda_1$, and $\kappa \in [0, 1)$ such that for all $\nu, \mu \in \mathcal{Z}$ with $\mathrm{T}\nu \neq \mathrm{T}\mu$, the following is true:*

$$
\lambda(d(\nu, \mu)) + \mathcal{F}(d(\mathrm{T}\nu, \mathrm{T}\mu)) \leq \mathcal{F}(\kappa d(\mathrm{T}\nu, \mathrm{T}\mu) + (1 - \kappa) d(\nu, \mu)). \tag{33}
$$

then T *has only one fixed point in $\mathcal{Z}$.*

Remark 4. *Note that in Corollary 1, conditions $(\mathcal{F}_2)$, $(\mathcal{F}_2')$, and $(\mathcal{F}_3)$ are omitted. Furthermore, the strictness of the monotonicity of $\mathcal{F}$ is not considered, and $\lambda \in \Lambda_{\varpi}$ is weakened to the condition $\lambda \in \Lambda_1$. Additionally, by using $\kappa = 0$ in Equation (33), we recover Equation (1), and thus Corollary 1 significantly enhances and broadens Theorem 2 in [5].*

Example 4. *Let $z = \{0, 3, 8\}$ be endowed with the Euclidean metric b. Then, (z, b) is a complete b-MS with $\hbar = 1$. Define the mapping $\mathrm{T} : z \to z$ as follows:*

$$\mathrm{T}(0) = \mathrm{T}(3) = 0 \quad \text{and} \quad \mathrm{T}(8) = 3.$$

Define $\mathcal{F}(g) = g$ and $\lambda(g) = \frac{g}{11}$ for all $g \in]0, \infty[$. Then, $\mathcal{F} \in \nabla(\mathcal{F}_c)$ and $\lambda \in \Lambda_1$. Consider $b(\mathrm{T}\nu, \mathrm{T}\mu) = 3 > 0$. Then, t the following cases arise:
Case-I:
If $\nu = 0$ and $\mu = 8$, then

$$\begin{aligned}
\lambda(b(\nu, \mu)) + \mathcal{F}(b(\mathrm{T}\nu, \mathrm{T}\mu)) &= \frac{b(\nu, \mu)}{11} + b(\mathrm{T}\nu, \mathrm{T}\mu) \\
&= 3.72 \\
&< 5.5 \\
&= \frac{1}{2} \times 3 + \frac{1}{2} \times 8 \\
&= \mathcal{F}(\kappa b(\mathrm{T}\nu, \mathrm{T}\mu) + (1 - \kappa) b(\nu, \mu)).
\end{aligned}$$

Case-II:
If $\nu = 8$ and $\mu = 0$, then

$$\lambda(b(\nu, \mu)) + \mathcal{F}(b(\mathrm{T}\nu, \mathrm{T}\mu)) = 3.72 < 5.5 = \mathcal{F}(\kappa b(\mathrm{T}\nu, \mathrm{T}\mu) + (1 - \kappa) b(\nu, \mu)).$$

Case-III:
If $\nu = 3$ and $\mu = 8$, then

$$\begin{aligned}
\lambda(b(\nu, \mu)) + \mathcal{F}(b(\mathrm{T}\nu, \mathrm{T}\mu)) &= \frac{b(\nu, \mu)}{11} + b(\mathrm{T}\nu, \mathrm{T}\mu) \\
&= 3.5 \\
&< 4 \\
&= \frac{1}{2} \times 3 + \frac{1}{2} \times 5 \\
&= \mathcal{F}(\kappa b(\mathrm{T}\nu, \mathrm{T}\mu) + (1 - \kappa) b(\nu, \mu)).
\end{aligned}$$

Case-IV:
If $\nu = 8$ and $\mu = 3$, then

$$\lambda(b(\nu, \mu)) + \mathcal{F}(b(\mathrm{T}\nu, \mathrm{T}\mu)) = 3.5 < 4 = \mathcal{F}(\kappa b(\mathrm{T}\nu, \mathrm{T}\mu) + (1 - \kappa) b(\nu, \mu)).$$

Hence, in all cases, T is an extended convex $\mathcal{F}$ contraction for $\kappa = \frac{1}{2}$. In addition, note that $\kappa\hbar^4 - \kappa\hbar^2 + \hbar = 1$ for $\kappa = \frac{1}{2}$ and $\hbar = 1$. Thus, all of the requirements for Theorem 3 are met, and zero is the only fixed point of T.

Remark 5. *Note that in Example 4, if for any sequence $\{\pi_p\} \subseteq (0, +\infty)$, we have $\lim_{p \to +\infty} \pi_p = 0$, then $\lim_{p \to +\infty} \mathcal{F}(\pi_p) = 0 \neq -\infty$. Thus, $\mathcal{F}$ does not satisfy conditions $(\mathcal{F}_2)$ or $(\mathcal{F}_2')$, and $\lambda \notin \Lambda$.*

Remark 6. *In Example 4, for all cases, T is an extended $\mathcal{F}$ contraction (see [35]) for $\alpha = \frac{7}{8}$, $\theta = \frac{1}{8}$, and $\beta = \zeta = \delta = 0$. However, $\alpha + \beta + \zeta + 2\delta = \frac{7}{8} \neq 1$, and $\alpha + \beta + \zeta + 2\theta = \frac{9}{8} \neq 1$. Therefore, Theorem 3.13 in [35] is not applicable to Example 4.*

Theorem 4. *Let (z, b) be a complete b-MS with a constant $\hbar \geq 1$ and $\mathrm{T} : z \to z$ be a mapping. If there exist $\mathcal{F} \in \nabla(\mathcal{F}_c)$ and $\lambda \in \Lambda_\omega$ such that for all $\nu, \mu \in z$ with $\mathrm{T}\nu \neq \mathrm{T}\mu$, we have*

$$\lambda(b(\nu, \mu)) + \mathcal{F}(b(\mathrm{T}\nu, \mathrm{T}\mu)) \leq \mathcal{F}\left(\frac{1}{2\hbar}[b(\nu, \mathrm{T}\mu) + b(\mu, \mathrm{T}\nu)]\right). \tag{34}$$

and if $\frac{\hbar^2+1}{2} \leq \frac{1}{\hbar}$*, then* T *has a unique fixed point in* z.

Proof. Let $\{\nu_p\}$ be a Picard sequence based on an arbitrary $\nu_0 \in z$. If $\nu_p = \nu_{p+1}$ for some $p \in \mathbb{N}_0$, then ν_p is a fixed point of T, and the proof is conclusive. Therefore, assume that $\nu_p \neq \nu_{p+1}$ for all $p \in \mathbb{N}_0$. Then, we have

$$d_p := b(\nu_p, \nu_{p+1}) = b(\mathrm{T}\nu_p, \mathrm{T}\nu_{p+1}) > 0, \quad \text{for all} \quad p \in \mathbb{N}.$$

Thus, by using the inequality in Equation (34) with $\nu = \nu_{p-1}$ and $\mu = \nu_p$, for all $p \in \mathbb{N}$, we obtain

$$\lambda(b(\nu_{p-1}, \nu_p)) + \mathcal{F}(b(\nu_p, \nu_{p+1})) \leq \mathcal{F}\left(\frac{1}{2\hbar} b(\nu_{p-1}, \nu_{p+1})\right). \tag{35}$$

By using condition (b_3), Equation (35) implies

$$\lambda(b_{p-1}) + \mathcal{F}(b_p) \leq \mathcal{F}\left(\frac{1}{2}(b_{p-1} + b_p)\right). \tag{36}$$

By using $\kappa = \frac{1}{2}$ and $\aleph = 1$, the inequality in Equation (36) turns into Equation (5). Therefore, by virtue of $\Lambda_\varpi \subseteq \Lambda_1$ and Lemma 5 with $\aleph = 1$, we have

$$\lim_{p \to \infty} b_p = 0. \tag{37}$$

If $\frac{\hbar^2+1}{2} \leq \frac{1}{\hbar}$, then for $\kappa = \frac{1}{2}$, we have $\kappa\hbar^4 - \kappa\hbar^2 + \hbar \leq 1$. Thus, by using Lemma 6, $\{\nu_p\}_{p \in \mathbb{N}_0}$ is a Cauchy sequence, and consequently, $\{\nu_p\}$ converges to some point $\nu* \in z$; that is, we have

$$\lim_{p \to \infty} d(\nu_p, \nu*) = 0. \tag{38}$$

Now, if $\nu* \neq \mathrm{T}\nu*$, then by using Equation (34), we have

$$\begin{aligned} b(\nu*, \mathrm{T}\nu*) &\leq \hbar(b(\nu*, \nu_{p+1}) + b(\nu_{p+1}, \mathrm{T}\nu*)) \\ &\leq \hbar\left(b(\nu*, \nu_{p+1}) + \frac{1}{2\hbar}(b(\nu_p, \nu_{p+1}) + b(\nu*, \mathrm{T}\nu*))\right). \end{aligned} \tag{39}$$

By letting $p \to \infty$ in the inequality in Equation (39), we obtain

$$b(\nu*, \mathrm{T}\nu*) \leq \frac{1}{2} b(\nu*, \mathrm{T}\nu*) < b(\nu*, \mathrm{T}\nu*),$$

which is a contradiction, and consequently, $\nu* = \mathrm{T}\nu*$.

Next, if T has two fixed points $\nu*$ and $\mu*$ such that $\nu* \neq \mu*$, then by using Equation (34), we obtain

$$b(\nu*, \mu*) = b(\mathrm{T}\nu*, \mathrm{T}\mu*) \leq \frac{1}{2s}(b(\nu*, \mathrm{T}\nu*) + b(\mu*, \mathrm{T}\mu*)) = 0,$$

which is a contradiction, and this completes the proof. □

Corollary 2. *Let* (z, d) *be a complete MS and* $\mathrm{T} : z \to z$ *be a mapping. If there exist* $\mathcal{F} \in \nabla(\mathcal{F}_c)$ *and* $\lambda \in \Lambda_1$ *such that for all* $\nu, \mu \in z$ *with* $\mathrm{T}\nu \neq \mathrm{T}\mu$*, it is true that*

$$\lambda(d(\nu, \mu)) + \mathcal{F}(d(\mathrm{T}\nu, \mathrm{T}\mu)) \leq \mathcal{F}\left(\frac{1}{2}[d(\nu, \mathrm{T}\mu) + d(\mu, \mathrm{T}\nu)]\right). \tag{40}$$

then T *has a unique fixed point in* z.

Theorem 5. *Let (z, d) be a complete MS and $\mathrm{T} : z \to z$ be a mapping. If there exist $\mathcal{F} \in \nabla(\mathcal{F}_c)$ and $\lambda \in \Lambda_1$ such that for all $\nu, \mu \in z$ with $\mathrm{T}\nu \neq \mathrm{T}\mu$, it is true that*

$$\begin{aligned} &\lambda(d(\nu, \mu)) + \mathcal{F}(d(\mathrm{T}\nu, \mathrm{T}\mu)) \\ &\qquad \leq \mathcal{F}(\alpha b(\nu, \mu) + \beta b(\nu, \mathrm{T}\nu) + \zeta b(\mu, \mathrm{T}\mu) + \delta b(\nu, \mathrm{T}\mu), \theta b(\mu, \mathrm{T}\nu)), \end{aligned} \tag{41}$$

where $\alpha, \beta, \zeta, \delta, \theta \in [0, \infty)$, $\zeta \neq 1$, and $\alpha + \delta + \theta \leq 1$, then assume either $\alpha + \beta + \zeta + 2\theta = 1$ or $\alpha + \beta + \zeta + 2\delta = 1$ holds, and T has a unique fixed point in z.

Proof. First, we prove that there is at most one fixed point of T in z. Assume that $\nu*, z \in z$ are fixed points of T with $\nu* \neq z$. Now, if $\alpha + \delta + \theta > 0$, by using $\nu = z$ and $\mu = \nu*$ in Equation (41), we have

$$\lambda(d(z, \nu*)) + \mathcal{F}(d(\mathrm{T}z, \mathrm{T}\nu*)) \leq \mathcal{F}((\alpha + \delta + \theta)d(z, \nu*)),$$

which is a contradiction since $\alpha + \delta + \theta \leq 1$, and hence $z = \nu*$. On the other hand, if $\alpha + \delta + \theta = 0$, by using Equations (3) and (41), we obtain

$$d(z, \nu*) = d(\mathrm{T}z, \mathrm{T}\nu*) < \beta d(z, \mathrm{T}z) + \zeta d(\nu*, \mathrm{T}\nu*) = 0,$$

which is a contradiction, and thus $z = \nu*$.

Let $\{\nu_p\}$ be a Picard sequence based on an arbitrary $\nu_0 \in z$. If $\nu_p = \nu_{p+1}$ for some $p \in \mathbb{N}_0$, then ν_p is a fixed point of T, and the proof is conclusive. Therefore, when assuming that $\nu_p \neq \nu_{p+1}$ for all $p \in \mathbb{N}_0$, we then have

$$d_p := d(\nu_p, \nu_{p+1}) = d(\mathrm{T}\nu_p, \mathrm{T}\nu_{p+1}) > 0, \quad \text{for all} \quad p \in \mathbb{N}.$$

If $\alpha + \beta + \zeta + 2\theta = 1$, then by using the inequality in Equation (41) with $\nu = \nu_p$ and $\mu = \nu_{p-1}$, for all $p \in \mathbb{N}$, we obtain

$$\begin{aligned} &\lambda(d(\nu_p, \nu_{p-1})) + \mathcal{F}(d(\nu_{p+1}, \nu_p)) \\ &\qquad \leq \mathcal{F}\big((\alpha + \zeta)d(\nu_{p-1}, \nu_p) + \beta d(\nu_p, \nu_{p+1}) + \theta d(\nu_{p-1}, \nu_{p+1})\big). \end{aligned} \tag{42}$$

By using a triangular inequality, Equation (42) implies that

$$\lambda(d_{p-1}) + \mathcal{F}(d\nu_p) \leq \mathcal{F}\big((\alpha + \zeta + \theta)d_{p-1} + (\beta + \theta)d_p\big). \tag{43}$$

By using $\kappa = \beta + \theta$, $\aleph = 1$, and $\hbar = 1$, the inequality in Equation (42) turns into Equation (5).

If $\alpha + \beta + \zeta + 2\delta = 1$, then by using the inequality in Equation (41) with $\nu = \nu_{p-1}$ and $\mu = \nu_p$, for all $p \in \mathbb{N}$, we obtain

$$\lambda(d_{p-1}) + \mathcal{F}(d\nu_p) \leq \mathcal{F}\big((\alpha + \zeta + \delta)d_{p-1} + (\zeta + \delta)d_p\big). \tag{44}$$

By using $\kappa = \zeta + \delta$, $\aleph = 1$, and $\hbar = 1$, the inequality in Equation (44) turns into Equation (5).

Therefore, in either case, by virtue of Lemma 5 with $\aleph = 1$ and $\hbar = 1$, we have

$$\lim_{p \to \infty} d_p = 0. \tag{45}$$

In addition, by using Lemma 6, $\{\nu_p\}_{p \in \mathbb{N}_0}$ is a Cauchy sequence, and consequently, $\{\nu_p\}$ converges to some point $\nu* \in z$; that is, we have

$$\lim_{p \to \infty} d(\nu_p, \nu*) = 0. \tag{46}$$

In the following, we show that $\nu*$ is a fixed point of T. Suppose, on contrary, that $d(\nu*, \mathrm{T}\nu*) > 0$. If $\mathrm{T}\nu_p = \mathrm{T}\nu*$ for infinite values of $p \in \mathbb{N}_0$, then the sequence $\{\nu_p\}$ has a subsequence that converges to $\mathrm{T}\nu*$, and the uniqueness of the limit implies $\mathrm{T}\nu* = \nu*$. Then, we can assume that $\mathrm{T}\nu_p \neq \mathrm{T}\nu*$ for all $p \in \mathbb{N}_0$. Now, by using Equations (3) and (41), we obtain

$$\begin{aligned} d(\nu*, \mathrm{T}\nu*) \leq & d(\nu*, \nu_{p+1}) + d(\mathrm{T}\nu_p, \mathrm{T}\nu*) \\ < & d(\nu*, \nu_{p+1}) + \alpha d(\nu_p, \nu*) + \beta d(\nu_p, \mathrm{T}\nu_p) + \zeta d(\nu*, \mathrm{T}\nu*) + \delta d(\nu_p, \mathrm{T}\nu*) \\ & + \theta d(\nu*, \mathrm{T}\nu_p). \end{aligned} \tag{47}$$

By letting $p \to \infty$ in the inequality in Equation (47), we obtain

$$d(\nu*, \mathrm{T}\nu*) \leq (\zeta + \delta) d(\nu*, \mathrm{T}\nu*) < d(\nu*, \mathrm{T}\nu*),$$

which is a contradiction, and hence $\mathrm{T}\nu* = \nu*$. □

Remark 7. *Theorem 5 is Theorem 3.13 in [35] for the case where $\hbar = 1$, but here, we re-proof this theorem by using Lemmas 5 and 6 and the note from Remark 3 that Lemma 5 greatly extends and improves Lemma 3.*

Moreover, Theorem 5 improves Theorem 1 in [14] as condition $(\mathcal{F}_2)$ is omitted and $\lambda \in \Lambda$ is weakened to the condition that $\lambda \in \Lambda_1$.

4. Application to the Theory of Iterated Function Systems

Let $(\mathcal{Z}, b)$ be a b-MS with a constant $\hbar \geq 1$. We denote with $P(\mathcal{Z})$ and $P_{cp}(\mathcal{Z})$ the family of all nonempty subsets of $\mathcal{Z}$ and the family of nonempty and compact subsets of $\mathcal{Z}$, respectively. For $\mathcal{G}, \mathcal{Q} \in P(\mathcal{Z})$, define D_b, ρ_b, and $H_b : P(\mathcal{Z}) \times P(\mathcal{Z}) \to [0, \infty) \cup \{+\infty\}$ as follows:

$$D_b(\mathcal{G}, \mathcal{Q}) = \begin{cases} \inf\{b(\hat{g}, \hat{j}) | \hat{g} \in \mathcal{G}, \hat{j} \in \mathcal{Q}\}, & \mathcal{G} \neq \emptyset \neq \mathcal{Q} \\ 0 & \mathcal{G} = \emptyset = \mathcal{Q} \\ +\infty, & \text{otherwise,} \end{cases}$$

$$\rho_b(\mathcal{G}, \mathcal{Q}) = \begin{cases} \sup\{D_b(\hat{g}, \mathcal{Q}) | \hat{g} \in \mathcal{G}\}, & \mathcal{G} \neq \emptyset \neq \mathcal{Q} \\ 0 & \mathcal{G} = \emptyset = \mathcal{Q} \\ +\infty, & \text{otherwise,} \end{cases}$$

$$H_b(\mathcal{G}, \mathcal{Q}) = \begin{cases} \max\{\rho_b(\mathcal{G}, \mathcal{Q}), \rho_b(\mathcal{Q}, \mathcal{G})\}, & \mathcal{G} \neq \emptyset \neq \mathcal{Q} \\ 0 & \mathcal{G} = \emptyset = \mathcal{Q} \\ +\infty, & \text{otherwise,} \end{cases}$$

Then, $(P(\mathcal{Z}), H_b)$ is a complete b-MS, provided that $(\mathcal{Z}, b)$ is complete [38].

Lemma 7 ([38])**.** *Let $(\mathcal{Z}, b)$ be a b-MS with a constant $\hbar \geq 1$ and $\mathcal{G}, \mathcal{Q} \in P_{cp}(\mathcal{Z})$. Then, for each $\hat{g} \in \mathcal{G}$, there exists $\hat{j} \in \mathcal{Q}$ such that*

$$b(\hat{g}, \hat{j}) \leq \hbar H_b(\mathcal{G}, \mathcal{Q}).$$

If $(\mathcal{Z}, b)$ is a b-MS, and $b : \mathcal{Z} \times \mathcal{Z} \to [0, \infty)$ is a continuous b metric, then for each $\hat{g} \in \mathcal{G}$, there exists $\hat{j} \in \mathcal{Q}$ such that

$$b(\hat{g}, \hat{j}) \leq H_b(\mathcal{G}, \mathcal{Q}).$$

Consider a finite family of continuous operators $\mathrm{T}_1, \mathrm{T}_2, \cdots, \mathrm{T}_m : \mathcal{Z} \to \mathcal{Z}$. The system $\mathrm{T} = (\mathrm{T}_1, \mathrm{T}_2, \cdots, \mathrm{T}_m)$ is called an iterated functions system (IFS) [39]. Define the fractal operator $T_{\mathrm{T}} : P_{cp}(\mathcal{Z}) \to P_{cp}(\mathcal{Z})$ generated by the IFS T with the following relation:

$$T_{\mathrm{T}}(\mathcal{P}) = \bigcup_{i=1}^{m} \mathrm{T}_1(\mathcal{P}), \quad \text{for all} \quad \mathcal{P} \in P_{cp}(\mathcal{Z}).$$

Then, a nonempty compact subset $\mathcal{G}*$ of $\mathcal{Z}$ is said to be a self-similar set or a fractal with respect to the IFS T if and only if it is a fixed point for the associated fractal operator (i.e., $T_{\mathrm{T}}(\mathcal{G}*) = \mathcal{G}*$). Note that $(P_{cp}(\mathcal{Z}), H_b)$ is a complete b-MS if $(\mathcal{Z}, b)$ is complete and is known as a fractal space. Now, we will prove the following lemma:

Lemma 8. *Let $(\mathcal{Z}, b)$ be a b-MS with a constant $\hbar \geq 1$ such that b is a continuous functional on $\mathcal{Z} \times \mathcal{Z}$. If $\mathrm{T} : \mathcal{Z} \to \mathcal{Z}$ is an extended convex $\mathcal{F}$ contraction for $\mathcal{F} \in \nabla(\mathcal{F}_c)$, and $\lambda(t) = \frac{1}{2\hbar}$ for all $t \in]0, \infty[$, then $T_{\mathrm{T}} : P_{cp}(\mathcal{Z}) \to P_{cp}(\mathcal{Z})$ is also an extended convex $\mathcal{F}$ contraction for the same $\mathcal{F} \in \nabla(\mathcal{F}_c)$, and $\lambda(t) = \frac{1}{2\hbar}$ for all $t \in]0, \infty[$; that is, there exist $\mathcal{F} \in \nabla(\mathcal{F}_c)$ and $\kappa \in \left[0, \frac{1}{\hbar}\right)$ such that for all $\mathcal{G}, \mathcal{Q} \in P_{cp}(\mathcal{Z})$, the following holds.*

$$\begin{aligned} &H_b(T_{\mathrm{T}}\mathcal{G}, T_b\mathcal{Q}) > 0 \ \textit{implies} \\ &\mathcal{F}(H_b(T_{\mathrm{T}}\mathcal{G}, T_{\mathrm{T}}\mathcal{Q})) \leq \mathcal{F}\left(\kappa H_b(T_{\mathrm{T}}\mathcal{G}, T_{\mathrm{T}}\mathcal{Q}) + \left(\frac{1}{2\hbar} - \kappa\right) H_b(\mathcal{G}, \mathcal{Q})\right), \end{aligned} \tag{48}$$

where for all $C \in P_{cp}(\mathcal{Z})$, $T_{\mathrm{T}}(C) := \mathrm{T}(C)$.

Proof. Let$\mathcal{G}, \mathcal{Q} \in P_{cp}(\mathcal{Z})$ such that $H_b(T_{\mathrm{T}}\mathcal{G}, T_b\mathcal{Q}) > 0$ and b be a continuous functional on $\mathcal{Z} \times \mathcal{Z}$. Choose an arbitrary element $\hat{g}_0 \in \mathcal{G}$. Then, by the compactness of $\mathcal{G}$, there is $\hat{j}_{\hat{g}_0} \in \mathcal{Q}$ such that

$$\min_{\hat{j} \in \mathcal{Q}} b(\hat{g}_0, \hat{j}) = b(\hat{g}_0, \hat{j}_{\hat{g}_0}). \tag{49}$$

This implies that

$$\inf_{\hat{j} \in \mathcal{Q}} \{b(\hat{g}_0, \hat{j})\} \leq b(\hat{g}_0, \hat{j}_{\hat{g}_0}) = \min_{\hat{j} \in \mathcal{Q}} b(\hat{g}_0, \hat{j}). \tag{50}$$

By using Lemma 7 and the inequality in Equation (49), we obtain

$$\min_{\hat{j} \in \mathcal{Q}} b(\hat{g}_0, \hat{j}) \leq \left(\max_{\hat{g} \in \mathcal{G}} \min_{\hat{j} \in \mathcal{Q}} b(\hat{g}, \hat{j})\right) \leq H_b(\mathcal{G}, \mathcal{Q}). \tag{51}$$

Since $\hat{g}_0$ was arbitrary, we have

$$\sup_{\hat{g} \in \mathcal{G}} \left\{\min_{\hat{j} \in \mathcal{Q}} b(\hat{g}, \hat{j})\right\} \leq H_b(\mathcal{G}, \mathcal{Q}) \tag{52}$$

and hence

$$\sup_{\hat{g} \in \mathcal{G}} \inf_{\hat{j} \in \mathcal{Q}} \{b(\hat{g}, \hat{j})\} \leq \sup_{\hat{g} \in \mathcal{G}} \{\min_{\hat{j} \in \mathcal{Q}} b(\hat{g}, \hat{j})\} \leq H_b(\mathcal{G}, \mathcal{Q}). \tag{53}$$

Similarly, we have

$$\sup_{\hat{j} \in \mathcal{Q}} \inf_{\hat{g} \in \mathcal{G}} \{b(\hat{g}, \hat{j})\} \leq H_b(\mathcal{G}, \mathcal{Q}). \tag{54}$$

Since $\mathrm{T} : \mathcal{Z} \to \mathcal{Z}$ is an extended convex $\mathcal{F}$ contraction for $\mathcal{F} \in \nabla(\mathcal{F}_c)$, and $\lambda(t) = \frac{1}{2\hbar}$ for all $t \in]0, \infty[$, we have

$$b(\mathrm{T}\hat{g}, \mathrm{T}\hat{j}) \leq \frac{\left(\frac{1}{2\hbar} - \kappa\right)}{(1-\kappa)} b(\hat{g}, \hat{j}). \tag{55}$$

Therefore, by using the inequalities in Equations (53) and (55), we obtain

$$\begin{aligned} D_b(T_{\mathrm{T}}\mathcal{G}, T_{\mathrm{T}}\mathcal{Q}) &= \max_{\mathrm{T}(\hat{g}) \in T(\mathcal{G})} \min_{\mathrm{T}(\hat{j}) \in T(\mathcal{Q})} b(\mathrm{T}\hat{g}, \mathrm{T}\hat{j}) \\ &= \max_{\hat{g} \in \mathcal{G}} \min_{\hat{j} \in \mathcal{Q}} b(\mathrm{T}\hat{g}, \mathrm{T}\hat{j}) \\ &\leq \sup_{\hat{g} \in \mathcal{G}} \min_{\hat{j} \in \mathcal{Q}} \left\{ \frac{\left(\frac{1}{2\hbar} - \kappa\right)}{(1-\kappa)} b(\hat{g}, \hat{j}) \right\} \\ &= \frac{\left(\frac{1}{2\hbar} - \kappa\right)}{(1-\kappa)} \sup_{\hat{g} \in \mathcal{G}} \min_{\hat{j} \in \mathcal{Q}} b(\hat{g}, \hat{j}) \\ &\leq \frac{\left(\frac{1}{2\hbar} - \kappa\right)}{(1-\kappa)} H_b(\mathcal{G}, \mathcal{Q}). \end{aligned} \tag{56}$$

Similarly, we have

$$D_b(T_{\mathrm{T}}\mathcal{Q}, T_{\mathrm{T}}\mathcal{G}) \leq \frac{\left(\frac{1}{2\hbar} - \kappa\right)}{(1-\kappa)} H_b(\mathcal{G}, \mathcal{Q}). \tag{57}$$

Hence, we obtain

$$H_b(T_{\mathrm{T}}\mathcal{G}, T_{\mathrm{T}}\mathcal{Q}) = \max\{D_b(T_{\mathrm{T}}\mathcal{G}, T_b\mathcal{Q}), D_b(T_{\mathrm{T}}\mathcal{Q}, T_{\mathrm{T}}\mathcal{G})\} \leq \frac{\left(\frac{1}{2\hbar} - \kappa\right)}{(1-\kappa)} H_b(\mathcal{G}, \mathcal{Q}), \tag{58}$$

which further implies

$$\mathcal{F}(H_b(T_{\mathrm{T}}\mathcal{G}, T_{\mathrm{T}}\mathcal{Q})) \leq \mathcal{F}\left(\kappa H_b(T_{\mathrm{T}}\mathcal{G}, T_{\mathrm{T}}\mathcal{Q}) + \left(\frac{1}{2\hbar} - \kappa\right) H_b(\mathcal{G}, \mathcal{Q})\right) \tag{59}$$

for $\mathcal{F} \in \nabla(\mathcal{F}_c)$. □

Theorem 6. *Let $(\mathcal{Z}, b)$ be a complete b-MS with a constant $\hbar \geq 1$ such that b is a continuous functional on $\mathcal{Z} \times \mathcal{Z}$, $\mathrm{T} : \mathcal{Z} \to \mathcal{Z}$ is an extended convex $\mathcal{F}$ contraction for $\mathcal{F} \in \nabla(\mathcal{F}_c)$, and $\lambda(t) = \frac{1}{2\hbar}$ for all $t \in]0, \infty[$. Assume that $\kappa\hbar^4 - \kappa\hbar^2 + \hbar \leq 1$. Then, the fractal operator T_{T} has a unique fixed point $\mathcal{G}* \in P_{cp}(\mathcal{Z})$.*

Proof. Let $(\mathcal{Z}, b)$ be a complete b-MS. Then, $(P_{cp}(\mathcal{Z}), H_b)$ is a complete b-MS. Since $\mathrm{T} : \mathcal{Z} \to \mathcal{Z}$ is an extended convex $\mathcal{F}$ contraction for $\mathcal{F} \in \nabla(\mathcal{F}_c)$, and $\lambda(t) = \frac{1}{2\hbar}$ for all $t \in]0, \infty[$, then under Lemma 8, the fractal operator T_{T} is also an extended convex $\mathcal{F}$ contraction for $\mathcal{F} \in \nabla(\mathcal{F}_c)$, and $\lambda(t) = \frac{1}{2\hbar}$ for all $t \in]0, \infty[$. Hence, all conditions for Theorem 3 hold true, and T_{T} has a unique fixed point $\mathcal{G}* \in P_{cp}(\mathcal{Z})$. □

Finally, we pose the following problems:
Open Problem 1:
Does Theorem 6 hold if b is a non-continuous functional on $\mathcal{Z} \times \mathcal{Z}$?
Open Problem 2:

Does Theorem 6 hold if $\mathrm{T} : \mathcal{Z} \to \mathcal{Z}$ is an extended convex $\mathcal{F}$ contraction for $\mathcal{F} \in \nabla(\mathcal{F}_c)$ and for any $\lambda \in \Lambda_{\varpi}$?

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