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Some q-Symmetric Integral Inequalities Involving s-Convex Functions

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Abstract: The q-symmetric analogues of Hölder, Minkowski, and power mean inequalities are presented in this paper. The obtained inequalities along with a Montgomery identity involving q-symmetric integrals are used to extend some Ostrowski-type inequalities. The q-symmetric derivatives of the functions involved in these Ostrowski-type inequalities are convex or s-convex. Moreover, some Hermite–Hadamard inequalities for convex functions as well as for s-convex functions are also acquired with the help of q-symmetric calculus in the present work. Some examples are included to support the effectiveness of the proved results.

Keywords: quantum calculus; q-symmetric quantum calculus; Ostrowski-type inequalities; Hermite–Hadamard-type inequalities

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1. Introduction

Convex functions play an important role in many areas of mathematics. They are important for the study of optimization problems, where they are distinguished by a number of convenient properties. Classical convex function have several extensions. Some concepts in this regard include Pseudo-convex functions [1], E-convex functions [2], s-convex functions [3], and m-convex functions [4]. Among others, the present study is restricted to s-convex functions. W. Orlicz [5] introduced s-convexity in the first sense in 1961. In 1978, Breckner [6] provided a slight modification to it, which is known as s-convexity in the second sense. H. Hudzik and L. Maligranda discussed both of these two kinds of s-convexity in [3]. They showed that s-convexity in the second sense ($0 < s \leq 1$) is stronger than s-convexity in the original sense (first sense) whenever $0 < s < 1$.

Quantum calculus, which is known as q-calculus (where q stands for quantum), was introduced by Euler and Jacobi before F.H. Jackson in the early twentieth century. Numerous mathematical fields, including number theory, combinatorics, orthogonal polynomials, fundamental hyper-geometric functions, and other sciences, including physics and the theory of relativity, have used it successfully [7–12].

Da Cruz et al. [13] introduced the concept of q-symmetric variational calculus. The q-symmetric derivative has essential qualities for the q-exponential function. Many researchers have applied the idea of q-symmetric calculus from different perspectives and have established certain subclasses of analytic functions, geometric functions, and, most particulary, quantum mechanics [14–16].

In many practical problems, it is necessary to restrict one quantity from another. For this, classical inequalities, including Hermite-Hadamard-, Jensen-, and Ostrowski-type inequalities are quite helpful. Many researchers have demonstrated various inequalities with error estimates of the functions of bounded variation, Lipschitzian, monotone, absolutely continuous, and convex functions, as well as n-times differentiable mappings [17–20].

Additionally, the research on q-integral inequalities is quite significant, and several researchers have explored integral inequalities in depth, whether in classical analysis or quantum mechanics. More precisely, some inequalities involving convexity or s-convexity [20–22] and q-integrals are studied in [7,18,23–26].

The aim of the present work is the study of some Ostrowski and Hermite–Hadamard inequalities in the framework of q-symmetric calculus. The paper is organized as follows. In Section 2, convex functions, s-convex functions (in the second sense), and q-symmetric derivatives and integrals are recalled along with their properties. In Section 3, Hölder, Minkowski, and power mean inequalities are studied with the help of q-symmetric integrals. In Section 4, some Ostrowski-type inequalities are proved with the help of a q-symmetric analogue of the Montgomery identity together with the inequalities proved in Section 3. Section 5 deals with q-symmetric Hermite–Hadamard inequalities for convex as well as for s-convex functions (in the second sense). A summary of the findings is discussed in Section 6.

2. Preliminaries

Some basic concepts of convexity, s-convexity, and q-symmetric derivatives and integrals are recalled in this section, which have been used in rest of the paper.

Mid-Point Convex Function:

A function $H_1 : J_1 \to \mathbb{R}$ is said to be mid-point convex function if

$$\frac{H_1(\theta_1 + \delta_1)}{2} \leq \frac{1}{2}(H_1(\theta_1) + H_1(\delta_1)), \tag{1}$$

holds for all $\theta_1, \delta_1 \in J_1$.

Convex Functions:

Assume that $H_1 : J_1 \subset \mathbb{R}^+ \to \mathbb{R}$ is said to be a convex function if

$$H_1(t_1\theta_1 + (1 - t_1)\delta_1) \leq t_1 H_1(\theta_1) + (1 - t_1)H_1(\delta_1), \tag{2}$$

holds for all $\theta_1, \delta_1 \in J_1$ and $t_1 \in [0, 1]$.
By choosing $t_1 = \frac{1}{2}$ in (2), one obtains (1).

The definition of an s-convex function in the second sense is defined by Breckner [6] as:

s-Convex Functions:

A function $H_1 : \mathbb{R}^+ \to \mathbb{R}$ is said to be an s-convex function in the second sense if

$$H_1(t_1\theta_1 + (1 - t_1)\delta_1) \leq t_1^s H_1(\theta_1) + (1 - t_1)^s H_1(\delta_1), \tag{3}$$

for each $\theta_1, \delta_1 \in \mathbb{R}^+$ where $t_1 \in [0, 1]$ and $s \in (0, 1]$. Moreover, by choosing $s \in (0, 1)$, (3) defines s-convexity in the first sense [5]. By taking $s = 1$ in (3), one obtains (2). Therefore, all convex functions are s-convex functions.

The following definitions and related properties are recalled from [9].

The q-derivative measures the rate of change with respect to a dilatation of its argument by a factor q. It is clear that if H_1 is differentiable at $x_1 \neq 0$, then

$$D_q H_1(x_1) = \lim_{q \to 1} \frac{H_1(qx_1) - H_1(x_1)}{(q - 1)x_1}.$$

q-Derivative:

For a continuous mapping $H_1 : [\theta_1, \delta_1] \to \mathbb{R}$, the q-derivative at $x_1 \in [\theta_1, \delta_1]$ is defined by

$$_{\theta_1}D_qH_1(x_1) = \frac{H_1(x_1) - H_1(qx_1 + (1-q)\theta_1)}{(1-q)(\delta_1 - \theta_1)}, x_1 \neq \theta_1. \tag{4}$$

q-Symmetric Derivative:

Let $H_1 : [\theta_1, \delta_1] \to \mathbb{R}$ be a continuous function. Then, the q-symmetric derivative at $x_1 \in [\theta_1, \delta_1]$ is

$$_{\theta_1}\tilde{D}_qH_1(x_1) = \frac{H_1(q^{-1}x_1 + (1-q^{-1})\theta_1) - H_1(qx_1 + (1-q)\theta_1)}{(q^{-1} - q)(\delta_1 - \theta_1)}, x_1 \neq \theta_1. \tag{5}$$

The q-symmetric analogue of power $(a_1 - b_1)^{k_1}$, defined in [15], is

$$(a_1 - b_1)^{k_1} = a_1^{k_1} \frac{\prod_{i=0}^{\infty}(a_1 - b_1q^{2i+1})}{\prod_{i=0}^{\infty}(a_1 - b_1q^{2(i+k_1)+1})}, a_1 \neq 0, k_1 \in \mathbb{R}, \tag{6}$$

and for the real parameter $q \in \mathbb{R}^+ \setminus \{1\}$, the q-real number $[n]$ is defined by

$$[n] = \frac{1 - q^{2n}}{1 - q^2}, n \in \mathbb{R}. \tag{7}$$

When n is a positive integer, we have

$$(\theta_1 - x_1)^n_{\tilde{q}} = \prod_{i=0}^{n-1}(\theta_1 - q^{2i+1}x_1). \tag{8}$$

For $n \geq 1$, we have the following evaluation:

$$\begin{aligned}
&(\theta_1 - x_1)^n_{\tilde{q}} = (\theta_1 - qx_1)(\theta_1 - q^3x_1)(\theta_1 - q^5x_1)\cdots(\theta_1 - q^{2n-1}x_1); \\
&_{\theta_1}\tilde{D}_q(\theta_1 - x_1)^n_{\tilde{q}} = -[n]q(\delta_1 - qx_1)^{n-1}_{\tilde{q}}; \\
&(\theta_1 - qx_1)^n_{\tilde{q}} = -\frac{1}{q[n+1]}\,{}_{\theta_1}\tilde{D}_q(\theta_1 - x_1)^{n+1}_{\tilde{q}}; \\
&\int(\theta_1 - x_1)^n_{\tilde{q}}\,{}_{\theta_1}\tilde{d}_qx_1 = -\frac{(\theta_1 - q^{-1}x_1)^{n+1}_{\tilde{q}}}{[n+1]}, (\theta_1 \neq -1).
\end{aligned} \tag{9}$$

Properties of q-Symmetric Derivative:

Let H_1 and G_1 be q-symmetric differentiable functions on J_1. Let $\alpha, \beta \in \mathbb{R}$, and $t_1 \in J_1$; then,

(i) $_{\theta_1}\tilde{D}_q[H_1] \equiv 0$ if H_1 is constant on J_1;
(ii) $_{\theta_1}\tilde{D}_q[\alpha H_1 + \beta G_1](t_1) = \alpha\,{}_{\theta_1}\tilde{D}_q[H_1](t_1) + \beta\,{}_{\theta_1}\tilde{D}_q[G_1](t_1)$;
(iii) $_{\theta_1}\tilde{D}_q[H_1G_1](t_1) = H_1(qt_1)\,{}_{\theta_1}\tilde{D}_q[G_1](t_1) + G_1(q^{-1}t_1)\,{}_{\theta_1}\tilde{D}_q[H_1](t_1)$;
(iv) $_{\theta_1}\tilde{D}_q\left[\frac{H_1}{G_1}\right](t_1) = \frac{G_1(q^{-1}t_1)\,{}_{\theta_1}\tilde{D}_q[H_1](t_1) - H_1(q^{-1}t_1)\,{}_{\theta_1}\tilde{D}_q[G_1](t_1)}{G(qt_1)G(q^{-1}t_1)}$ if $G(qt_1)G(q^{-1}t_1) \neq 0$.

q-Symmetric Antiderivative:

Suppose that $H_1 : [\theta_1, \delta_1] \to \mathbb{R}$ is a continuous function. Then, the q-symmetric definite integral on $[\theta_1, \delta_1]$ is defined as

$$\int_{\theta_1}^{x_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1 = (q^{-1}-q)(x_1-\theta_1)\sum_{n=0}^{\infty} q^{2n+1} H_1(q^{2n+1}x_1+(1-q^{2n+1})\theta_1)$$
$$= (1-q^2)(x_1-\theta_1)\sum_{n=0}^{\infty} q^{2n} H_1(q^{2n+1}x_1+(1-q^{2n+1})\theta_1), \quad (10)$$

for $x_1 \in [\theta_1, \delta_1]$.

Formula for Integration by Parts:

Let $H_1, G_1 : [\theta_1, \delta_1] \to \mathbb{R}$ be continuous mappings $\theta_1 \in \mathbb{R}$ with $x_1, x_2 \in [\theta_1, \delta_1]$; then,

$$\int_{x_1}^{x_2} H_1(qt_1+(1-q)\theta_1)G_1(t_1)_{\theta_1}\tilde{d}_q t_1 = H_1(x_2)G_1(x_2) - H_1(x_1)G_1(x_1)$$
$$- \int_{x_1}^{x_2} G_1(q^{-1}t_1+(1-q^{-1})\theta_1)_{\theta_1}\tilde{D}_q H_1(t_1)\tilde{d}_q t_1. \quad (11)$$

3. Some q-Symmetric Preliminary Inequalities

In this section, Hölder, Minkowski, and power mean inequalities are established using q-symmetric integrals. These act as helpful tools to prove Ostrowski-type inequalities in the next section.

Theorem 1 (q-Symmetric Hölder's Inequality)**.** *Suppose that H_1 and G_1 are q-symmetric integrable functions on $[\theta_1, \delta_1]$, $0 < q < 1$ and $\frac{1}{n} + \frac{1}{m} = 1$ with $m > 1$; then,*

$$\int_{\theta_1}^{\delta_1} |H_1(t_1)G_1(t_1)|_{\theta_1}\tilde{d}_q t_1 \leq \left\{\int_{\theta_1}^{\delta_1} |H_1(t_1)|^n{}_{\theta_1}\tilde{d}_q t_1\right\}^{\frac{1}{n}} \left\{\int_{\theta_1}^{\delta_1} |G_1(t_1)|^m{}_{\theta_1}\tilde{d}_q t_1\right\}^{\frac{1}{m}}. \quad (12)$$

Proof. Consider

$$\int_{\theta_1}^{\delta_1} |H_1(t_1)G_1(t_1)|_{\theta_1}\tilde{d}_q t_1 = \int_{\theta_1}^{\delta_1} |H_1(t_1)||G_1(t_1)|_{\theta_1}\tilde{d}_q t_1.$$

Apply the definition of a q-symmetric integral to obtain

$$\int_{\theta_1}^{\delta_1} |H_1(t_1)||G_1(t_1)|_{\theta_1}\tilde{d}_q t_1$$
$$= (1-q^2)(\delta_1-\theta_1)\sum_{n=0}^{\infty} q^{2n}|H_1(q^{2n+1}\delta_1+(1-q^{2n+1})\theta_1)||G_1(q^{2n+1}\delta_1+(1-q^{2n+1})\theta_1)|. \quad (13)$$

Using the discrete Hölder's inequality, one obtains

$$\sum_{n=0}^{\infty}(1-q^2)(\delta_1-\theta_1)q^{2n}|H_1(q^{2n+1}\delta_1+(1-q^{2n+1})\theta_1)||G_1(q^{2n+1}\delta_1+(1-q^{2n+1})\theta_1)|$$
$$\leq (1-q^2)(\delta_1-\theta_1)\sum_{n=0}^{\infty} q^{2n}\left(|H_1(q^{2n+1}\delta_1+(1-q^{2n+1})\theta_1)|^n\right)^{\frac{1}{n}}$$
$$\times (1-q^2)(\delta_1-\theta_1)\sum_{n=0}^{\infty} q^{2n}\left(|G_1(q^{2n+1}\delta_1+(1-q^{2n+1})\theta_1)|^m\right)^{\frac{1}{m}}. \quad (14)$$

Using (14) in (13), one obtains

$$\int_{\theta_1}^{\delta_1} |H_1(t_1)||G_1(t_1)|_{\theta_1}\tilde{d}_q t_1 = \left(\int_{\theta_1}^{\delta_1} |H_1(t_1)|^n{}_{\theta_1}\tilde{d}_q t_1\right)^{\frac{1}{n}} \left(\int_{\theta_1}^{\delta_1} |G_1(t_1)|^m{}_{\theta_1}\tilde{d}_q t_1\right)^{\frac{1}{m}}. \quad (15)$$

Hence, we obtain the result. □

Theorem 2 (q-Symmetric Minkowski Inequality)**.** *Assume that $\theta_1, \delta_1 \in \mathbb{R}$ and $H_1, G_1 : [\theta_1, \delta_1] \to \mathbb{R}$ are continuous functions, where n and m are positive real numbers and $n > 1$ such that $\frac{1}{n} + \frac{1}{m} = 1$. Therefore,*

$$\left\{ \int_{\theta_1}^{\delta_1} |H_1(t_1) + G_1(t_1)|^n{}_{\theta_1}\tilde{d}_q t_1 \right\}^{\frac{1}{n}} \leq \left\{ \int_{\theta_1}^{\delta_1} |H_1(t_1)|^n{}_{\theta_1}\tilde{d}_q t_1 \right\}^{\frac{1}{n}} + \left\{ \int_{\theta_1}^{\delta_1} |G_1(t_1)|^n{}_{\theta_1}\tilde{d}_q t_1 \right\}^{\frac{1}{n}}. \tag{16}$$

Proof. Consider the following expression and apply the properties of absolute value, as follows:

$$\begin{aligned}
&\int_{\theta_1}^{\delta_1} |H_1(t_1) + G_1(t_1)|^n{}_{\theta_1}\tilde{d}_q t_1 = \int_{\theta_1}^{\delta_1} |(H_1 + G_1)(t_1)|^{n-1}|(H_1 + G_1)(t_1)|_{\theta_1}\tilde{d}_q t_1 \\
&\leq \int_{\theta_1}^{\delta_1} |(H_1 + G_1)(t_1)|^{n-1}|H_1(t_1)|_{\theta_1}\tilde{d}_q t_1 + \int_{\theta_1}^{\delta_1} |(H_1 + G_1)(t_1)|^{n-1}||G_1(t_1)|_{\theta_1}\tilde{d}_q t_1.
\end{aligned} \tag{17}$$

Apply the discrete Hölder's inequality on the right hand side of (17) to obtain

$$\begin{aligned}
&\int_{\theta_1}^{\delta_1} |(H_1 + G_1)(t_1)|^{n-1}|H_1(t_1)|_{\theta_1}\tilde{d}_q t_1 + \int_{\theta_1}^{\delta_1} |(H_1 + G_1)(t_1)|^{n-1}||G_1(t_1)|_{\theta_1}\tilde{d}_q t_1 \\
&\quad \leq \left\{ \int_{\theta_1}^{\delta_1} |(H_1 + G_1)(t_1)|^{m(n-1)}|_{\theta_1}\tilde{d}_q t_1 \right\}^{\frac{1}{m}} \left\{ \int_{\theta_1}^{\delta_1} |H_1(t_1)|^n{}_{\theta_1}\tilde{d}_q t_1 \right\}^{\frac{1}{n}} \\
&\quad + \left\{ \int_{\theta_1}^{\delta_1} |(H_1 + G_1)(t_1)|^{m(n-1)}|_{\theta_1}\tilde{d}_q t_1 \right\}^{\frac{1}{m}} \left\{ \int_{\theta_1}^{\delta_1} |G_1(t_1)|^n{}_{\theta_1}\tilde{d}_q t_1 \right\}^{\frac{1}{n}} \\
&\quad = \left[\left\{ \int_{\theta_1}^{\delta_1} |H_1(t_1)|^n{}_{\theta_1}\tilde{d}_q t_1 \right\}^{\frac{1}{n}} + \left\{ \int_{\theta_1}^{\delta_1} |G_1(t_1)|^n{}_{\theta_1}\tilde{d}_q t_1 \right\}^{\frac{1}{n}} \right] \\
&\qquad \times \left\{ \int_{\theta_1}^{\delta_1} |(H_1 + G_1)(t_1)|^{m(n-1)}|_{\theta_1}\tilde{d}_q t_1 \right\}^{\frac{1}{m}}.
\end{aligned} \tag{18}$$

Therefore, (17) together with (18) gives

$$\left\{ \int_{\theta_1}^{\delta_1} |H_1(t_1) + G_1(t_1)|^n{}_{\theta_1}\tilde{d}_q t_1 \right\}^{\frac{1}{n}} \leq \left\{ \int_{\theta_1}^{\delta_1} |H_1(t_1)|^n{}_{\theta_1}\tilde{d}_q t_1 \right\}^{\frac{1}{n}} + \left\{ \int_{\theta_1}^{\delta_1} |G_1(t_1)|^n{}_{\theta_1}\tilde{d}_q t_1 \right\}^{\frac{1}{n}}.$$

□

Theorem 3 (q-Symmetric Power Mean Inequality)**.** *Suppose that $\frac{1}{n} + \frac{1}{m} = 1$, where $n, m > 1$ are real numbers. If $\theta_1, \delta_1 \in \mathbb{R}$ and $H_1, G_1 : [\theta_1, \delta_1] \to \mathbb{R}$ are continuous functions, then*

$$\int_{\theta_1}^{\delta_1} |H_1(t_1)G_1(t_1)|_{\theta_1}\tilde{d}_q t_1 \leq \left\{ \int_{\theta_1}^{\delta_1} |H_1(t_1)|_{\theta_1}\tilde{d}_q t_1 \right\}^{1-\frac{1}{m}} \left\{ \int_{\theta_1}^{\delta_1} |H_1(t_1)||G_1(t_1)|^m{}_{\theta_1}\tilde{d}_q t_1 \right\}^{\frac{1}{m}}. \tag{19}$$

Proof. Consider the following integral and apply the property of absolute value along with the definition of q-symmetric integrals to find

$$\begin{aligned}
&\int_{\theta_1}^{\delta_1} |H_1(t_1)G_1(t_1)|_{\theta_1}\tilde{d}_q t_1 = \int_{\theta_1}^{\delta_1} |H_1(t_1)||G_1(t_1)|_{\theta_1}\tilde{d}_q t_1 \\
&= (1 - q^2)(\delta_1 - \theta_1) \sum_{n=0}^{\infty} q^{2n}|H_1(q^{2n+1}\delta_1 + (1 - q^{2n+1})\theta_1)||G_1(q^{2n+1}\delta_1 + (1 - q^{2n+1})\theta_1)|.
\end{aligned}$$

Using the discrete power mean inequality, we have

$$
\begin{aligned}
&(1-q^2)(\delta_1-\theta_1)\sum_{n=0}^{\infty}q^{2n}|H_1(q^{2n+1}\delta_1+(1-q^{2n+1})\theta_1)||G_1(q^{2n+1}\delta_1+(1-q^{2n+1})\theta_1)| \\
&\leq (1-q^2)(\delta_1-\theta_1)\sum_{n=0}^{\infty}q^{2n}\left(|H_1(q^{2n+1}\delta_1+(1-q^{2n+1})\theta_1)|\right)^{1-\frac{1}{m}} \\
&\times (1-q^2)(\delta_1-\theta_1)\sum_{n=0}^{\infty}q^{2n}\left(|H_1(t_1)||G_1(q^{2n+1}\delta_1+(1-q^{2n+1})\theta_1)|^m\right)^{\frac{1}{m}} \\
&= \left(\int_{\theta_1}^{\delta_1}|H_1(t_1)|_{\theta_1}\tilde{d}_q t_1\right)^{1-\frac{1}{m}}\left(\int_{\theta_1}^{\delta_1}|H_1(t_1)||G_1(t_1)|^m{}_{\theta_1}\tilde{d}_q t_1\right)^{\frac{1}{m}}.
\end{aligned}
$$

□

4. q-Symmetric Ostrowski-Type Inequalities

In this section, some Ostrowski-type inequalities are extended for those functions whose derivatives are either convex or s-convex in the second sense. For this purpose, first, we have to establish the following Montgomery identity for q-symmetric integrals.

Lemma 1 (q-Symmetric Montgomery identity)**.** *Let $H_1 : J_1 \subset \mathbb{R} \to \mathbb{R}$ be a q-symmetric differentiable function on J_1^o and $\theta_1, \delta_1 \in J_1$ for $\theta_1 < \delta_1$. If ${}_{\theta_1}\tilde{D}_q H_1 \in L_1[\theta_1, \delta_1]$. Then, the following q-symmetric integral equality is valid:*

$$
\begin{aligned}
\frac{1}{q}\left(H_1(x_1)-\frac{q}{\delta_1-\theta_1}\int_{\theta_1}^{\delta_1}H_1(t_1)_{\theta_1}\tilde{d}_q t_1\right) &= \frac{(x_1-\theta_1)^2}{\delta_1-\theta_1}\int_0^1 t_{1\theta_1}\tilde{D}_q(t_1x_1+(1-t_1)\theta_1)_0\tilde{d}_q t_1 \\
&+ \frac{(\delta_1-x_1)^2}{\delta_1-\theta_1}\int_0^1 t_{1\theta_1}\tilde{D}_q(t_1x_1+(1-t_1)\delta_1)_0\tilde{d}_q t_1. \quad (20)
\end{aligned}
$$

Proof. Denote

$$
I_1+I_2=\frac{(x_1-\theta_1)^2}{\delta_1-\theta_1}\int_0^1 t_{1\theta_1}\tilde{D}_q(t_1x_1+(1-t_1)\theta_1)_0\tilde{d}_q t_1+\frac{(\delta_1-x_1)^2}{\delta_1-\theta_1}\int_0^1 t_{1\theta_1}\tilde{D}_q(t_1x_1+(1-t_1)\delta_1)_0\tilde{d}_q t_1. \quad (21)
$$

Using the definition of a q-symmetric derivative, one can write

$$
{}_{\theta_1}\tilde{D}_q H_1(t_1x_1+(1-t_1)\theta_1)=\frac{H_1(qt_1x_1+(1-qt_1)\theta_1)-H_1(q^{-1}t_1x_1+(1-q^{-1}t_1)\theta_1)}{(q-q^{-1})(x_1-\theta_1)t_1}.
$$

First, we simplify the integrals I_1 and I_2 as follows:

$$
\begin{aligned}
I_1 &= \frac{(x_1-\theta_1)^2}{\delta_1-\theta_1}\int_0^1 t_{1\theta_1}\tilde{D}_q(t_1x_1+(1-t_1)\theta_1)_0\tilde{d}_q t_1 \\
&= \frac{(x_1-\theta_1)^2}{\delta_1-\theta_1}\int_0^1 t_1\left(\frac{H_1(qt_1x_1+(1-qt_1)\theta_1)-H_1(q^{-1}t_1x_1+(1-q^{-1}t_1)\theta_1)}{(q-q^{-1})(x_1-\theta_1)t_1}\right)_0\tilde{d}_q t_1 \\
&= \frac{(x_1-\theta_1)}{\delta_1-\theta_1}\int_0^1\left(\frac{H_1(qt_1x_1+(1-qt_1)\theta_1)-H_1(q^{-1}t_1x_1+(1-q^{-1}t_1)\theta_1)}{(q-q^{-1})}\right)_0\tilde{d}_q t_1 \\
&= \frac{(x_1-\theta_1)}{\delta_1-\theta_1}\left[\int_0^1\frac{H_1(qt_1x_1+(1-qt_1)\theta_1)}{q-q^{-1}}{}_0\tilde{d}_q t_1-\int_0^1\frac{H_1(q^{-1}t_1x_1+(1-q^{-1}t_1)\theta_1)}{q-q^{-1}}{}_0\tilde{d}_q t_1\right] \\
&= \frac{(x_1-\theta_1)}{(\delta_1-\theta_1)}\left[\frac{(q^{-1}-q)}{(q-q^{-1})}\sum_{n=0}^{\infty}q^{2n+1}H_1(q^{2n+2}x_1+(1-q^{2n+2})\theta_1)-\frac{(q^{-1}-q)}{(q-q^{-1})}\right. \\
&\left.\sum_{n=0}^{\infty}q^{2n+1}H_1(q^{2n}x_1+(1-q^{2n})\theta_1)\right]
\end{aligned}
$$

$$= \frac{(x_1-\theta_1)}{(\delta_1-\theta_1)}\left[-\frac{1}{q}\sum_{n=0}^{\infty} q^{2n+2}H_1(q^{2n+2}x_1+(1-q^{2n+2})\theta_1)+q\sum_{n=0}^{\infty} q^{2n}H_1(q^{2n}x_1+(1-q^{2n})\theta_1)\right]$$
$$= \frac{(x_1-\theta_1)}{(\delta_1-\theta_1)}\left[-\frac{1}{q}\sum_{n=1}^{\infty} q^{2n}H_1(q^{2n}x_1+(1-q^{2n})\theta_1)+q\sum_{n=0}^{\infty} q^{2n}H_1(q^{2n}x_1+(1-q^{2n})\theta_1)\right]$$
$$= \frac{(x_1-\theta_1)}{(\delta_1-\theta_1)}\left[-\frac{1}{q}\sum_{n=0}^{\infty} q^{2n+1}H_1(q^{2n+1}x_1+(1-q^{2n+1})\theta_1)+q\sum_{n=0}^{\infty} q^{2n+1}H_1(q^{2n+1}x_1+(1-q^{2n+1})\theta_1)+\frac{1}{q}H_1(x_1)\right].$$

Therefore,

$$\begin{aligned} I_1 &= \frac{(x_1-\theta_1)}{(\delta_1-\theta_1)}\left[\left(q-\frac{1}{q}\right)\sum_{n=0}^{\infty} q^{2n+1}H_1(q^{2n+1}x_1+(1-q^{2n+1})\theta_1)+\frac{1}{q}H_1(x_1)\right] \\ &= \frac{(x_1-\theta_1)}{(\delta_1-\theta_1)}\left[\left(\frac{q^2-1}{q}\right)\frac{1}{(q^{-1}-q)(x_1-\theta_1)}\int_{\theta_1}^{x_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1+\frac{1}{q}H_1(x_1)\right] \\ &= \frac{(x_1-\theta_1)}{(\delta_1-\theta_1)q}H_1(x_1)-\frac{1}{(\delta_1-\theta_1)}\int_{\theta_1}^{x_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1. \end{aligned} \tag{22}$$

In a similar fashion, we have

$$I_2 = \frac{(\delta_1-x_1)}{(\delta_1-\theta_1)q}H_1(x_1)-\frac{1}{(\delta_1-\theta_1)}\int_{x_1}^{\delta_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1. \tag{23}$$

Substituting (22) and (23) in (21), we have

$$\begin{aligned} cI_1+I_2 &= \frac{(x_1-\theta_1)}{(\delta_1-\theta_1)q}H_1(x_1)-\frac{1}{(\delta_1-\theta_1)}\int_{\theta_1}^{x_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1+\frac{(\delta_1-x_1)}{(\delta_1-\theta_1)q}H_1(x_1)-\frac{1}{(\delta_1-\theta_1)}\int_{x_1}^{\delta_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1 \\ &= \frac{1}{q}H_1(x_1)-\frac{1}{(\delta_1-\theta_1)}\left[\int_{\theta_1}^{x_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1+\int_{x_1}^{\delta_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1\right] \\ &= \frac{1}{q}H_1(x_1)-\frac{1}{(\delta_1-\theta_1)}\left[\left(\int_0^{x_1} H_1(t_1)_0\tilde{d}_q t_1-\int_0^{\theta_1} H_1(t_1)_0\tilde{d}_q t_1\right)+\left(\int_0^{\delta_1} H_1(t_1)_0\tilde{d}_q t_1\right.\right. \\ &\qquad \left.\left.-\int_0^{x_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1\right)\right] \\ &= \frac{1}{q}H_1(x_1)-\frac{1}{(\delta_1-\theta_1)}\left(\int_0^{\delta_1} H_1(t_1)_0\tilde{d}_q t_1-\int_0^{\theta_1} H_1(t_1)_0\tilde{d}_q t_1\right) \\ &= \frac{1}{q}H_1(x_1)-\frac{1}{(\delta_1-\theta_1)}\int_{\theta_1}^{\delta_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1 = \frac{1}{q}\left[H_1(x_1)-\frac{q}{(\delta_1-\theta_1)}\int_{\theta_1}^{\delta_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1\right]. \end{aligned} \tag{24}$$

Hence, (24) completes the proof. □

Remark 1. *Lemma 1 extends Lemma 3.1 of [25]. If we choose $q=1$ in Lemma 1, it becomes Lemma 1 of [21].*

Now, we are ready to construct the following Ostrowski-type inequalities with the help of Lemma 1.

Theorem 4. *Suppose that $H_1 : J_1 \subset \mathbb{R} \to \mathbb{R}$ is q-symmetric differentiable for (θ_1,δ_1) and ${}_{\theta_1}\tilde{D}_q H_1 \in L_1[\theta_1,\delta_1]$, in which $\theta_1,\delta_1 \in J_1$ for $\theta_1 < \delta_1$. If $|{}_{\theta_1}\tilde{D}_q H_1(x_1)|$ is a convex function on $[\theta_1,\delta_1]$ for some $q \in (0,1)$ and $|{}_{\theta_1}\tilde{D}_q H_1(x_1)| \le M$, then the following q-symmetric integral inequality is obtained:*

$$\left|\frac{1}{q}\left(H_1(x_1)-\frac{q}{\delta_1-\theta_1}\int_{\theta_1}^{\delta_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1\right)\right| \le \frac{M_1}{\delta_1-\theta_1}\left[\frac{(x_1-\theta_1)^2+(\delta_1-x_1)^2}{1+q^2}\right], \tag{25}$$

for each $x_1 \in [\theta_1,\delta_1]$.

Proof. Using (20), we have

$$c\frac{1}{q}\left(H_1(x_1)-\frac{q}{\delta_1-\theta_1}\int_{\theta_1}^{\delta_1}H_1(t_1)_{\theta_1}\tilde{d}_qt_1\right)$$
$$=\frac{(x_1-\theta_1)^2}{\delta_1-\theta_1}\int_0^1 t_{1\theta_1}\tilde{D}_q(t_1x_1+(1-t_1)\theta_1)_0\tilde{d}_qt_1+\frac{(\delta_1-x_1)^2}{\delta_1-\theta_1}\int_0^1 t_{1\theta_1}\tilde{D}_q(t_1x_1+(1-t_1)\delta_1)_0\tilde{d}_qt_1. \quad (26)$$

Taking the modulus on both sides of (26), we have

$$\left|\frac{1}{q}\left(H_1(x_1)-\frac{q}{\delta_1-\theta_1}\int_{\theta_1}^{\delta_1}H_1(t_1)_{\theta_1}\tilde{d}_qt_1\right)\right|$$
$$\leq\frac{(x_1-\theta_1)^2}{\delta_1-\theta_1}\int_0^1|t_{1\theta_1}\tilde{D}_q(t_1x_1+(1-t_1)\theta_1)_0\tilde{d}_qt_1|+\frac{(\delta_1-x_1)^2}{\delta_1-\theta_1}\int_0^1|t_{1\theta_1}\tilde{D}_q(t_1x_1+(1-t_1)\delta_1)_0\tilde{d}_qt_1|$$
$$\leq\frac{(x_1-\theta_1)^2}{\delta_1-\theta_1}\int_0^1 t_1|_{\theta_1}\tilde{D}_q(t_1(x_1)+(1-t_1)\theta_1)|_0\tilde{d}_qt_1+\frac{(\delta_1-x_1)^2}{\delta_1-\theta_1}\int_0^1 t_{1\theta_1}|\tilde{D}_q(t_1x_1+(1-t_1)\delta_1)|_0\tilde{d}_qt_1.$$

Using the convexity of q-symmetric derivatives, we obtain

$$|_{\theta_1}\tilde{D}_q(t_1x_1+(1-t_1)\theta_1)|\leq t_1|_{\theta_1}\tilde{D}_qH_1(x_1)|+(1-t_1)|\tilde{D}_qH_1(\theta_1)|$$
$$\leq\frac{(x_1-\theta_1)^2}{\delta_1-\theta_1}\int_0^1\left[t_1^2|_{\theta_1}\tilde{D}_qH_1(x_1)|+t_1(1-t_1)|_{\theta_1}\tilde{D}_qH_1(\theta_1)|\right]_0\tilde{d}_qt_1$$
$$+\frac{(\delta_1-x_1)^2}{\delta_1-\theta_1}\int_0^1\left[t_1^2|_{\theta_1}\tilde{D}_qH_1(x_1)|+t_1(1-t_1)|\tilde{D}_qH_1(\delta_1)|\right]_0\tilde{d}_qt_1$$
$$\leq\frac{M_1(x_1-\theta_1)^2}{\delta_1-\theta_1}\left[\int_0^1 t_{1\,0}^2\tilde{d}_qt_1+\int_0^1 t_1(1-t_1)_0\tilde{d}_qt_1\right]+\frac{M_1(\delta_1-x_1)^2}{\delta_1-\theta_1}\left[t_{1\,0}^2\tilde{d}_qt_1+\int_0^1 t_1(1-t_1)_0\tilde{d}_qt_1\right]$$
$$=M_1\left(\frac{(x_1-\theta_1)^2+(\delta_1-x_1)^2}{\delta_1-\theta_1}\right)\left[\int_0^1 t_{1\,0}\tilde{d}_qt_1\right]=\frac{M_1}{\delta_1-\theta_1}\left[\frac{(x_1-\theta_1)^2+(\delta_1-x_1)^2}{1+q^2}\right].$$

□

Remark 2. *Theorem 4 extends Theorem 3.1 of [25].*

Example 1. *Set $H(x_1)=1-x_1$, $\delta_1=1$, $\theta_1=2$, $q=1/2$, and $M=2$ in Theorem 4 to obtain the following estimate:*

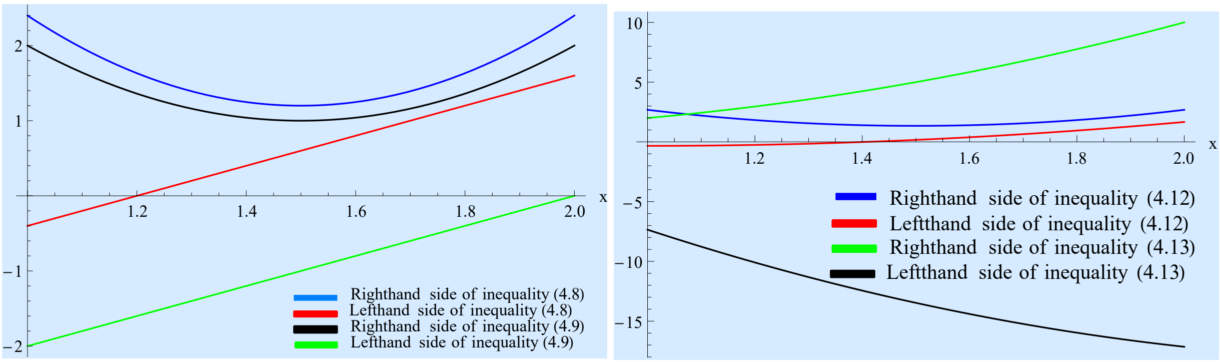
$$\left|2\left((1-x_1)-\frac{1}{2}\int_1^2(1-t_1)_0\tilde{d}_qt_1\right)\right|\leq 2\left(\frac{(x-1)^2+(2-x_1)^2}{1+(1/2)^2}\right)$$
$$\left|2\left((1-x_1)+\frac{1}{2}\left|\frac{(1-q^{-1}t_1)^2_{\tilde{q}}}{[2]}\right|_1^2\right)\right|\leq\left(\frac{8(2x_1{}^2-6x_1+5)}{5}\right)$$

$$\left|2\left((1-x_1)-\frac{1}{5}\right)\right|\leq 8\left(\frac{2x_1{}^2-6x_1+5}{5}\right)$$
$$\left|\frac{2}{5}(4-5x_1)\right|\leq\frac{8}{5}(2x_1{}^2-6x_1+5). \quad (27)$$

However, using the same substitution, Theorem 3.1 of [25] yields

$$\left|2\left((1-x_1)-\int_1^2(1-t_1)_0d_qt_1\right)\right|\leq 2\left(\frac{(x-1)^2+(2-x_1)^2}{1+(1/2)}\right)$$
$$\left|2\left((1-x)+\left|\frac{q(1-q^{-1}t_1)^2_{\tilde{q}}}{[2]}\right|_1^2\right)\right|\leq\left(\frac{6(2x_1{}^2-6x_1+5)}{3}\right) \quad (28)$$
$$\left|(4-2x_1)\right|\leq 6\left(\frac{2x_1{}^2-6x_1+5}{3}\right).$$

Clearly, Figure 1a shows that Inequality (27) gives a better approximation than Inequality (28).



(**a**) (**b**)

Figure 1. (**a**) presents comparsion of (27) with (28); (**b**) presents comparison of (31) with (32).

Theorem 5. *Assume that $H_1 : J_1 \subset \mathbb{R} \to \mathbb{R}$ is a q-symmetric differentiable mapping on (θ_1, δ_1), and ${}_{\theta_1}\tilde{D}_q H_1 \in L_1[\theta_1, \delta_1]$, where $\theta_1, \delta_1 \in J_1$, and $\theta_1 < \delta_1$. If $|_{\theta_1}\tilde{D}_q H_1|^m$ is a convex function on $[\theta_1, \delta_1]$ for some static $q \in (0,1)$, $m > 1$, $n = m/m-1$, and $|\tilde{D}_q H_1(x_1)| \leq M_1$, then the following inequality is valid:*

$$\left|\frac{1}{q}\left(H_1(x_1) - \frac{q}{\delta_1 - \theta_1}\int_{\theta_1}^{\delta_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1\right)\right| \leq \frac{M_1}{([n+1])^{\frac{1}{n}}}\left[\frac{(x_1-\theta_1)^2 + (\delta_1 - x_1)^2}{\delta_1 - \theta_1}\right], \quad (29)$$

for each $x_1 \in [\theta_1, \delta_1]$.

Proof. From (20), we have

$$\left|\frac{1}{q}\left(H_1(x_1) - \frac{q}{\delta_1 - \theta_1}\int_{\theta_1}^{\delta_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1\right)\right| \leq \frac{(x_1-\theta_1)^2}{\delta_1-\theta_1}\int_0^1 t_1 |_{\theta_1}\tilde{D}_q(t_1(x_1) + (1-t_1)\theta_1)|_0\tilde{d}_q t_1 + \frac{(\delta_1 - x_1)^2}{\delta_1 - \theta_1}\int_0^1 t_1|_{\theta_1}\tilde{D}_q(t_1 x_1 + (1-t_1)\delta_1)|_0\tilde{d}_q t_1.$$

Using the q-symmetric Hölder inequality (12) on the right-hand side, we have

$$\leq \frac{(x_1-\theta_1)^2}{\delta_1-\theta_1}\left[\left(\int_0^1 t_1^n{}_0\tilde{d}_q t_1\right)^{\frac{1}{n}}\left(\int_0^1 |_{\theta_1}\tilde{D}_q(t_1(x_1) + (1-t_1)\theta_1)|^m{}_0\tilde{d}_q t_1\right)^{\frac{1}{m}}\right] + \frac{(\delta_1-x_1)^2}{\delta_1-\theta_1}\left[\left(\int_0^1 t_1^n{}_0\tilde{d}_q t_1\right)^{\frac{1}{n}}\left(\int_0^1 |_{\theta_1}\tilde{D}_q(t_1(x_1) + (1-t_1)\delta_1)|^m{}_0\tilde{d}_q t_1\right)^{\frac{1}{m}}\right]. \quad (30)$$

From (16), the right-hand side of (30) satisfies the following:

$$
\begin{aligned}
&\frac{(x_1-\theta_1)^2}{\delta_1-\theta_1}\left[\left(\frac{1}{n+1}\right)^{\frac{1}{n}}\left(\int_0^1 |_{\theta_1}\tilde{D}_q(t_1(x_1)+(1-t_1)\theta_1)|^m{}_0\tilde{d}_q t_1^{\frac{1}{m}}\right)\right] \\
&+\frac{(\delta_1-x_1)^2}{\delta_1-\theta_1}\left[\left(\frac{1}{n+1}\right)^{\frac{1}{n}}\left(\int_0^1 |_{\theta_1}\tilde{D}_q(t_1(x_1)+(1-t_1)\delta_1)|^m{}_0\tilde{d}_q t_1\right)^{\frac{1}{m}}\right] \\
&\le \frac{(x_1-\theta_1)^2}{\delta_1-\theta_1}\left[\left(\frac{1}{n+1}\right)^{\frac{1}{n}}\left(\int_0^1 |_{\theta_1}t_1|\tilde{D}_q(x_1)|^m+|(1-t_1)\tilde{D}_q(\theta_1)|^m{}_0\tilde{d}_q t_1\right)^{\frac{1}{m}}\right] \\
&+\frac{(\delta_1-x_1)^2}{\delta_1-\theta_1}\left[\left(\frac{1}{n+1}\right)^{\frac{1}{n}}\left(\int_0^1 |t_{1\theta_1}\tilde{D}_q(x_1)|^m+|(1-t_1)\tilde{D}_q(\delta_1)|^m{}_0\tilde{d}_q t_1\right)^{\frac{1}{m}}\right] \\
&\le \frac{(x_1-\theta_1)^2}{\delta_1-\theta_1}\left[\frac{1}{[n+1]^{\frac{1}{n}}}\left(M_1^m\int_0^1 1_0\tilde{d}_q t_1\right)^{\frac{1}{m}}\right]+\frac{(\delta_1-x_1)^2}{\delta_1-\theta_1}\left[\frac{1}{[n+1]^{\frac{1}{n}}}\left(M_1^m\int_0^1 1_0\tilde{d}_q t_1\right)^{\frac{1}{m}}\right] \\
&= \frac{(x_1-\theta_1)^2}{\delta_1-\theta_1}\left[\frac{1}{[n+1]^{\frac{1}{n}}}(M_1^m)^{\frac{1}{m}}\right]+\frac{(\delta_1-x_1)^2}{\delta_1-\theta_1}\left[\frac{1}{[n+1]^{\frac{1}{n}}}(M^m)^{\frac{1}{m}}\right] \\
&= \frac{M_1}{[n+1]^{\frac{1}{n}}}\left[\frac{(x_1-\theta_1)^2+(\delta_1-x_1)^2}{\delta_1-\theta_1}\right].
\end{aligned}
$$

Hence, we obtain the result. □

Remark 3. *Theorem 5 extends Theorem 3.2 of [25].*

Example 2. *Set $H(x_1)=(1-x_1)^2$, $\delta_1=1$, $\theta_1=2$, $q=1/2$, $n=1$, and $M=3$ in Theorem 5 to obtain the following:*

$$
\begin{aligned}
&\left|2\left((1-x_1)^2-\frac{1}{2}\int_1^2(1-t_1)^2{}_0\tilde{d}_q t_1\right)\right| \le 3\left(\frac{(x_1-1)^2+(2-x_1)^2}{[2]^{1/2}}\right) \\
&\left|2\left((1-x_1^2)+\frac{1}{2}\left|\frac{(1-q^{-1}t_1)^3_{\tilde{q}}}{[3]}\right|_1^2\right)\right| \le 3\left(\frac{2x_1{}^2-6x_1+5}{(5/4)^{1/2}}\right) \\
&\left|\left(\frac{6x_1{}^2-12x_1+5}{3}\right)\right| \le 3\left(\frac{2x_1{}^2-6x_1+5}{(5/4)^{1/2}}\right).
\end{aligned}
\tag{31}
$$

Using the same substitution in Theorem 3.2 of [25] yields

$$
\begin{aligned}
&\left|2\left((1-x_1)^2-\frac{1}{2}\int_1^2(1-t_1)^2{}_0d_q t_1\right)\right| \le 3\left(\frac{(x_1-1)^2+(2-x_1)^2}{[2]^{1/2}}\right) \\
&\left|2\left((1-x_1^2)+\frac{1}{2}\left|\frac{(1-q^{-1}t_1)^3_q}{[3]}\right|_1^2\right)\right| \le 3\left(\frac{2x_1{}^2-6x_1+5}{(3/2)^{1/2}}\right) \\
&\left|2(x_1{}^2-2x_1+1)\right| \le 3\left(\frac{2x_1{}^2-5x_1+5}{(3/2)^{1/2}}\right).
\end{aligned}
\tag{32}
$$

The validity and a comparison of (31) and (32) can be seen in Figure 1b.

Theorem 6. *Assume that $H_1: J_1\subset\mathbb{R}^+\to\mathbb{R}$ is a q-symmetric differentiable function on ${J_1}^0$ and ${}_{\theta_1}\tilde{D}_qH_1\in L_1[\theta_1,\delta_1]$ for $\theta_1,\delta_1\in J_1$ with $\theta_1<\delta_1$. If the absolute value of $\tilde{D}_qH_1(x_1)$ is s-convex in the second sense on $[\theta_1,\delta_1]$, and if a static $s\in(0,1]$ and ${}_{\theta_1}\tilde{D}_qH_1(x_1)$ are bounded by M_1, then for $x_1\in[\theta_1,\delta_1]$, the following inequality is valid:*

$$
\left|\frac{1}{q}\left(H_1(x_1)-\frac{q}{\delta_1-\theta_1}\int_{\theta_1}^{\delta_1}H_1(t_1)_{\theta_1}\tilde{d}_q t_1\right)\right| \le M_1\left(\frac{(x_1-\theta_1)^2+(\delta_1-x_1)^2}{\delta_1-\theta_1}\right)
$$

$$\times \left[-\frac{1}{[s+1]} \left((1-q^{-1})_{\tilde{q}}^{s+1} + \frac{(1-q^{-1})_{\tilde{q}}^{s+2}}{[s+2]} - \frac{1}{[s+2]} \right) + \frac{1}{[s+2]} \right]. \quad (33)$$

Proof. Since $|_{\theta_1}\tilde{D}_q H_1|$ is an s-convex function in the second sense on $[\theta_1, \delta_1]$, therefore, using (20), we have the following:

$$\begin{aligned}
\left| \frac{1}{q} \left(H_1(x_1) - \frac{q}{\delta_1 - \theta_1} \int_{\theta_1}^{\delta_1} H_1(t_1)_{\theta_1} \tilde{d}_q t_1 \right) \right| &\leq \frac{(x_1-\theta_1)^2}{\delta_1-\theta_1} \int_0^1 t_1 |_{\theta_1}\tilde{D}_q H_1(t_1(x_1) + (1-t_1)\theta_1)|_0 \tilde{d}_q t_1 \\
&+ \frac{(\delta_1-x_1)^2}{\delta_1-\theta_1} \int_0^1 t_1 |_{\theta_1}\tilde{D}_q H_1(t_1(x_1) + (1-t_1)\delta_1)|_0 \tilde{d}_q t_1 \\
\leq \frac{(x_1-\theta_1)^2}{\delta_1-\theta_1} &\left[\int_0^1 (t_1)_{\tilde{q}}^{s+1} |_{\theta_1}\tilde{D}_q H_1(x_1)|_0 \tilde{d}_q t_1 + \int_0^1 t_1 (1-t_1)_{\tilde{q}}^{s} |_{\theta_1}\tilde{D}_q H_1(\theta_1)|_0 \tilde{d}_q t_1 \right] \\
+ \frac{(\delta_1-x_1)^2}{\delta_1-\theta_1} &\left[\int_0^1 (t_1)_{q}^{s+1} |_{\theta_1}\tilde{D}_q H_1(x_1)|_0 \tilde{d}_q t_1 + \int_0^1 t_1 (1-t_1)_{\tilde{q}}^{s} |_{\theta_1}\tilde{D}_q H_1(\delta_1)|_0 \tilde{d}_q t_1 \right] \\
&= \frac{M_1(x_1-\theta_1)^2}{\delta_1-\theta_1} \left[\int_0^1 (t_1)_{\tilde{q}}^{s+1}{}_0 \tilde{d}_q t_1 + \int_{\theta_1}^{\delta_1} t_1 (1-t_1)_{\tilde{q}0}^{s} \tilde{d}_q t_1 \right] \\
&\quad + \frac{M_1(\delta_1-x_1)^2}{\delta_1-\theta_1} \left[\int_0^1 (t_1)_{\tilde{q}}^{s+1}{}_0 \tilde{d}_q t_1 + \int_0^1 t_1 (1-t_1)_{\tilde{q}0}^{s} \tilde{d}_q t_1 \right]
\end{aligned}$$

$$= M_1 \frac{(x_1-\theta_1)^2 + (\delta_1-x_1)^2}{\delta_1-\theta_1} \left[\int_0^1 (t_1)_{\tilde{q}}^{s+1}{}_0 \tilde{d}_q t_1 + \int_0^1 t_1 (1-t_1)_{\tilde{q}0}^{s} \tilde{d}_q t_1 \right]. \quad (34)$$

Since

$$\int_0^1 (t_1)_{\tilde{q}}^{s+1}{}_0 \tilde{d}_q t_1 = \frac{1}{[s+2]}, \quad (35)$$

and

$$\begin{aligned}
\int_0^1 t_1 (1-t_1)_{\tilde{q}0}^{s} \tilde{d}_q t_1 &= -\frac{1}{[s+1]} \int_0^1 t_{1\,\theta_1}\tilde{D}_q (1-q^{-1}t_1)_{\tilde{q}}^{s+1}{}_0 \tilde{d}_q t_1 \\
&= -\frac{1}{[s+1]} \left[\left| t_1 (1-q^{-1}t_1)_{\tilde{q}}^{s+1} \right|_0^1 - \int_0^1 t_1 (1-q^{-1}t_1)_{\tilde{q}}^{s+1} (1)_0 \tilde{d}_q t_1 \right] \\
&= -\frac{1}{[s+1]} \left[(1-q^{-1})_{\tilde{q}}^{s+1} + \frac{(1-q^{-1})_{\tilde{q}}^{s+2}}{[s+2]} - \frac{1}{[s+2]} \right], \quad (36)
\end{aligned}$$

therefore, by substituting (35) and (36) in (34), we have

$$\begin{aligned}
M_1 \frac{(x_1-\theta_1)^2 + (\delta_1-x_1)^2}{\delta_1-\theta_1} &\left[\int_0^1 (t_1)_{\tilde{q}}^{s+1}{}_0 \tilde{d}_q t_1 + \int_0^1 t_1 (1-t_1)_{\tilde{q}0}^{s} \tilde{d}_q t_1 \right] \\
&= M_1 \left(\frac{(x_1-\theta_1)^2 + (\delta_1-x_1)^2}{\delta_1-\theta_1} \right) \\
&\left[-\frac{1}{[s+1]} \left((1-q^{-1})_{\tilde{q}}^{s+1} + \frac{(1-q^{-1})_{\tilde{q}}^{s+2}}{[s+2]} - \frac{1}{[s+2]} \right) + \frac{1}{[s+2]} \right].
\end{aligned}$$

□

Remark 4. *Theorem 6 is an extension of Theorem 15 of [18].*
If we set $q = 1$ in Inequality (33), then Theorem 6 leads to Theorem 2 of [21].
If we set $s = 1$ in Inequality (33), then it becomes Inequality (25) of Theorem 4.

Example 3. *If we set $H(x_1) = 1 - x_1^2$, $\delta_1 = 1$, $\theta_1 = 2$, $q = 1/2$, $s = 1$, and $M = 1$ in Theorem 6, we have the following estimate:*

$$\begin{aligned}&\left|2\left(1-(x_1)^2\right)-\frac{1}{2}\int_1^2(1-t_1^2)_0\tilde{d}_qt_1\right)\right|\leq\left(\frac{((x_1)-1)^2+(2-(x_1))^2}{2-1}\right)\\&\times\left[-\frac{1}{[2]}\left((1-q^{-1})^2_{\tilde{q}}+\frac{(1-q^{-1})^3_{\tilde{q}}}{[3]}-\frac{1}{[3]}\right)+\frac{1}{[3]}\right]\\&\left|2\left(1-(x_1)^2\right)-\frac{1}{2}+\frac{1}{2}\left|\frac{t_1^3}{[3]}\right|_1^2\right)\right|\leq\left(2{x_1}^2-6x_1+5\right)\left(-\frac{4}{5}\left(-\frac{16}{21}\right)+\frac{16}{21}\right)\\&\left|\frac{14-6x_1^2}{3}\right|\leq 189\left(\frac{2{x_1}^2-6x_1+5}{80}\right).\end{aligned}\tag{37}$$

However, using the same substitution, Theorem 15 of [18] yields

$$\begin{aligned}&\left|2\left(1-(x_1)^2\right)-\frac{1}{2}\int_1^2(1-t_1^2)d_qt_1\right)\right|\leq\left(\frac{((x_1)-1)^2+(2-(x_1))^2}{2-1}\right)\\&\times\left[-\frac{1}{[2]}\left((1-q^{-1})^2_{q}+\frac{(1-q^{-1})^3_{q}}{[3]}-\frac{1}{[3]}\right)+\frac{1}{[3]}\right]\\&\left|2\left(1-(x_1)^2\right)-\frac{1}{2}+\frac{1}{2}\left|\frac{t_1^3}{[3]}\right|_1^2\right)\right|\leq\left(2{x_1}^2-6x_1+5\right)\left(-\frac{2}{3}\left(-\frac{4}{7}\right)+\frac{4}{7}\right)\\&\left|2(4-2x_1^2)\right|\leq 20\left(\frac{2{x_1}^2-6x_1+5}{21}\right).\end{aligned}\tag{38}$$

Figure 2a shows that new estimates are better than existing ones.

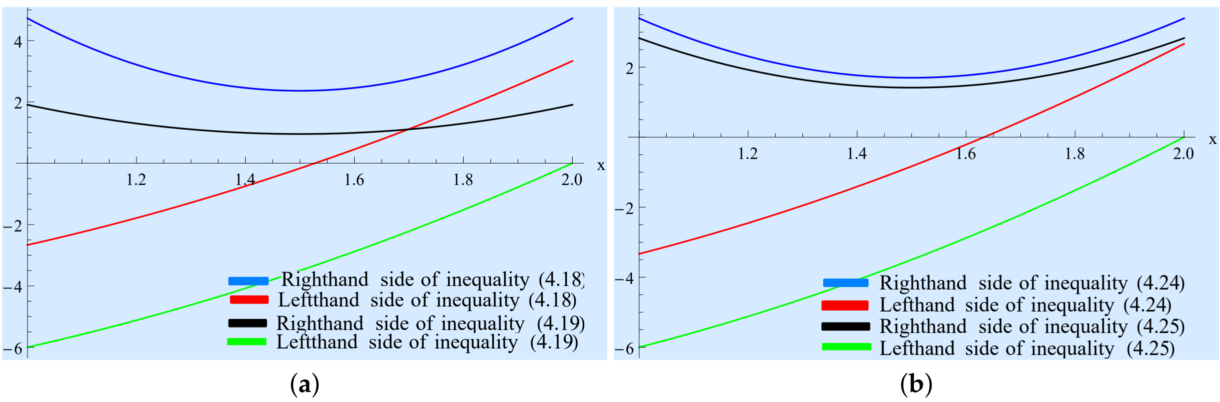


Figure 2. (**a**) presents comparsion of (37) with (38); (**b**) presents comparison of (43) with (44).

Theorem 7. *Let $H_1 : J_1 \subset \mathbb{R}^+ \to \mathbb{R}$ be a q-symmetric differentiable function on J_1^o and ${}_{\theta_1}\tilde{D}_qH_1 \in L_1[\theta_1, \delta_1]$, in which $\theta_1, \delta_1 \in J_1$ for $\theta_1 < \delta_1$. If $|{}_{\theta_1}\tilde{D}_qH_1(x_1)|^m$ is an s-convex function in the second sense on $[\theta_1, \delta_1]$ for a unique $s \in (0, 1]$, and if $m > 1$, $n = m/m-1$, and ${}_{\theta_1}\tilde{D}_qH_1(x_1)$ is bounded by M_1, then the inequality*

$$\left|\frac{1}{q}\left(H_1(x_1)-\frac{q}{\delta_1-\theta_1}\int_{\theta_1}^{\delta_1}H_1(t_1)_{\theta_1}\tilde{d}_qt_1\right)\right|\leq\frac{M_1}{[n+1]^{\frac{1}{n}}}\left[\frac{(x_1-\theta_1)^2+(\delta_1-x_1)^2}{\delta_1-\theta_1}\right]\times\left[\frac{2-(1-q^{-1})^{s+1}_{\tilde{q}})}{[s+1]}\right]^{1/m},\tag{39}$$

is valid for each $x_1 \in [\theta_1, \delta_1]$.

Proof. From (20) and using the q-symmetric analogue of the Hölder inequality, we obtain

$$\left|\frac{1}{q}\left(H_1(x_1) - \frac{q}{\delta_1-\theta_1}\int_{\theta_1}^{\delta_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1\right)\right| \leq \frac{(x_1-\theta_1)^2}{\delta_1-\theta_1}\int_0^1 t_1|_{\theta_1}\tilde{D}_q H_1(t_1(x_1)+(1-t_1)\theta_1)|_0\tilde{d}_q t_1$$
$$+ \frac{(\delta_1-x_1)^2}{\delta_1-\theta_1}\int_0^1 t_1|_{\theta_1}\tilde{D}_q H_1(t_1(x_1)+(1-t_1)\delta_1)|_0\tilde{d}_q t_1$$

From (12), the right-hand side of the above inequality becomes

$$\leq \frac{(x_1-\theta_1)^2}{\delta_1-\theta_1}\left(\int_0^1 (t_1)^n_{q\,0}\tilde{d}_q t_1\right)^{\frac{1}{n}}\left(\int_0^1 |_{\theta_1}\tilde{D}_q H_1(t_1(x_1)+(1-t_1)\theta_1|^m{}_0\tilde{d}_q t_1\right)^{\frac{1}{m}}$$
$$+ \frac{(\delta_1-x_1)^2}{\delta_1-\theta_1}\int_0^1 (t_1)^n_q\tilde{d}_q t_1\Big)^{\frac{1}{n}}\left(\int_0^1 |_{\theta_1}\tilde{D}_q H_1(t_1(x_1)+(1-t_1)\delta_1)|^m{}_0\tilde{d}_q t_1\right)^{1/m}. \quad (40)$$

Additionally, using the definition of an s-convexity,

$$\begin{aligned}\int_0^1 |_{\theta_1}\tilde{D}_q H_1(t_1(x_1)+(1-t_1)\theta_1)|^m{}_0\tilde{d}_q t_1 &\leq \int_0^1 (t_1)^s_{\tilde{q}}|_{\theta_1}\tilde{D}_q H_1(t_1(x_1))|^m{}_0\tilde{d}_q t_1 \\ &+ \int_0^1 (1-t_1)^s_{\tilde{q}}|_{\theta_1}\tilde{D}_q H_1(t_1(\theta_1))|^m{}_0\tilde{d}_q t_1 \\ &\leq {M_1}^m\left[\left|\frac{(t_1)^{s+1}_{\tilde{q}}}{[s+1]}\right|_0^1 - \left|\frac{(1-q^{-1}t_1)^{s+1}_{\tilde{q}}}{[s+1]}\right|_0^1\right] \\ &= {M_1}^m\left[\frac{1}{[s+1]} - \frac{(1-q^{-1})^{s+1}_{\tilde{q}}}{[s+1]} + \frac{1}{[s+1]}\right] \\ &= {M_1}^m\left[\frac{2-(1-q^{-1})^{s+1}_{\tilde{q}})}{[s+1]}\right].\end{aligned} \quad (41)$$

In a similar fashion, we have

$$\int_0^1 |_{\theta_1}\tilde{D}_q H_1(t_1(x_1)+(1-t_1)\delta_1)|^m{}_0\tilde{d}_q t_1 \leq {M_1}^m\left[\frac{2-(1-q^{-1})^{s+1}_{\tilde{q}})}{[s+1]}\right]. \quad (42)$$

Substituting (41) and (42) in (40), we obtain

$$\left|\frac{1}{q}\left(H_1(x_1) - \frac{q}{\delta_1-\theta_1}\int_{\theta_1}^{\delta_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1\right)\right| \leq M_1\left(\frac{1}{[n+1]}\right)^{\frac{1}{n}}\left[\frac{2-(1-q^{-1})^{s+1}_{\tilde{q}})}{[s+1]}\right]^{1/m}$$
$$\times\left[\frac{(x_1-\theta_1)^2+(\delta_1-x_1)^2}{\delta_1-\theta_1}\right].$$

□

Remark 5. *Theorem 7 extends Theorem 16 of [18].*
For $q = 1$, Theorem 7 reduces to Theorem 3 of [21].
For $s = 1$, then Inequality (39) reduces to Inequality (29) of Theorem 5.

Figure 2b shows a comparison of (43) and (44) in Example 4.

Example 4. *Consider $H(x_1) = x_1^2$, $\delta_1 = 1$, $\theta_1 = 2$, $q = 1/2$, s = 1, m = 2, n = 1, and $M = 3$ in Theorem 7 to obtain the following:*

$$\begin{aligned}
&\left|2\left((x_1)^2\right)-\frac{1}{2}\int_1^2 t_{10}^2\tilde{d}_q t_1\right)\right| \leq 3\left(\frac{(x_1-1)^2+(2-x_1)^2}{([2])^{1/2}}\right)\left[\frac{2-(1-q^{-1})^2_{\tilde{q}})}{[2]}\right]^{1/2}\\
&\left|2\left(x_1^2-\frac{1}{2}\left|\frac{t_1^3}{[3]}\right|_1^2\right)\right| \leq 3\left(\frac{2x_1^2-6x_1+5}{(5/4)^{1/2}}\right)(2)^{1/2}\\
&\left|2\left(x_1^2-\frac{8}{3}\right)\right| \leq 12\left(\frac{2x_1^2-5x_1+5}{(5/2)^{1/2}}\right)\\
&\left|\frac{2}{3}(3x_1^2-8)\right| \leq 12\left(\frac{2x_1^2-6x_1+5}{(5/2)^{1/2}}\right).
\end{aligned} \tag{43}$$

Using the above substitution in Theorem 16 of [18], we have

$$\begin{aligned}
&\left|2\left((x_1)^2\right)-\frac{1}{2}\int_1^2 t_1^2 d_q t_1\right)\right| \leq 3\left(\frac{(x_1-1)^2+(2-x_1)^2}{([2])^{1/2}}\right)\left[\frac{2-(1-q^{-1})^2_{\tilde{q}})}{[2]}\right]^{1/2}\\
&\left|2\left(x_1^2-\frac{1}{2}\left|\frac{t_1^3}{[3]}\right|_1^2\right)\right| \leq 3\left(\frac{2x_1^2-6x_1+5}{(3/2)^{1/2}}\right)(2)^{1/2}\\
&\left|2\left(x_1^2-4\right)\right| \leq 6\left(\frac{2x_1^2-6x_1+5}{(3/2)^{1/2}}\right)\\
&\left|2(x_1^2-4)\right| \leq 6\left(\frac{2x_1^2-6x_1+5}{(3/2)^{1/2}}\right).
\end{aligned} \tag{44}$$

Theorem 8. *Let $H_1 : J_1 \subset \mathbb{R}^+ \to \mathbb{R}$ be a q-symmetric differentiable mapping on ${J_1}^0$ with ${}_{\theta_1}\tilde{D}_q H_1 \in L_1[\theta_1,\delta_1]$, in which $\theta_1,\delta_1 \in J_1$ for $\theta_1 < \delta_1$. If the absolute value of $({}_{\theta_1}\tilde{D}_q H_1(x_1))^m$ is an s-convex mapping in the second sense on $[\theta_1,\delta_1]$, some unique $s \in (0,1]$, $m \geq 1$, and $|{}_{\theta_1}\tilde{D}_q H_1(x_1)| \leq M_1$, then*

$$\begin{aligned}
&\left|\frac{1}{q}\left(H_1(x_1)-\frac{q}{\delta_1-\theta_1}\int_{\theta_1}^{\delta_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1\right)\right| \leq\ M_1\left(\frac{(x_1-\theta_1)^2+(\delta_1-x_1)^2}{\delta_1-\theta_1}\right)\times\\
&\quad\left(\frac{1}{[2]}\right)^{1-\frac{1}{m}}\left[-\frac{1}{[s+1]}\left((1-q^{-1})^{s+1}_{\tilde{q}}+\frac{(1-q^{-1})^{s+2}_{\tilde{q}}}{[s+2]}-\frac{1}{[s+2]}\right)+\frac{1}{[s+2]}\right]^{\frac{1}{m}}
\end{aligned} \tag{45}$$

holds for each $x_1 \in [\theta_1,\delta_1]$.

Proof. Using (20) and using the q-symmetric-analogue of the power mean inequality, we obtain

$$\begin{aligned}
\left|\frac{1}{q}\left(H_1(x_1)-\frac{q}{\delta_1-\theta_1}\int_{\theta_1}^{\delta_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1\right)\right| &\leq \frac{(x_1-\theta_1)^2}{\delta_1-\theta_1}\int_0^1 t_1|{}_{\theta_1}\tilde{D}_q H_1(t_1(x_1)+(1-t_1)\theta_1)|_0\tilde{d}_q t_1\\
&\quad+\frac{(\delta_1-x_1)^2}{\delta_1-\theta_1}\int_0^1 t_1|{}_{\theta_1}\tilde{D}_q H_1(t_1(x_1)+(1-t_1)\delta_1)|_0\tilde{d}_q t_1,
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{(x_1-\theta_1)^2}{\delta_1-\theta_1}\left(\int_0^1 t_{10}\tilde{d}_q t_1\right)^{1-(1/m)}\left(\int_0^1 t_1|{}_{\theta_1}\tilde{D}_q H_1(t_1(x_1)+(1-t_1)\theta_1|^m{}_0\tilde{d}_q t_1\right)^{\frac{1}{m}}\\
&+\frac{(\delta_1-x_1)^2}{\delta_1-\theta_1}\int_0^1 t_{10}\tilde{d}_q t_1\right)^{1-(1/m)}\left(\int_0^1 t_1|{}_{\theta_1}\tilde{D}_q H_1(t_1(x_1)+(1-t_1)\delta_1)|^m{}_0\tilde{d}_q t_1\right)^{1/m}.
\end{aligned} \tag{46}$$

We use the definition of s-convexity in the second sense and (16) to obtain

$$\begin{aligned}\int_0^1 t_1 |_{\theta_1}\tilde{D}_q H_1(t_1(x_1)+(1-t_1)\theta_1)|^m {}_0\tilde{d}_q t_1 &\leq \int_0^1 (t_1)_{\tilde{q}}^{s+1} |_{\theta_1}\tilde{D}_q H_1(x_1)|^m {}_0\tilde{d}_q t_1 \\ &+ \int_0^1 t_1(1-t_1)_{\tilde{q}}^{s} |_{\theta_1}\tilde{D}_q H_1(\theta_1)|^m {}_0\tilde{d}_q t_1 \\ &\leq M_1{}^m \left[\int_0^1 (t_1)_{\tilde{q}}^{s+1} {}_0\tilde{d}_q t_1 + \int_0^1 t_1(1-t_1)_{\tilde{q}0}^{s}\tilde{d}_q t_1\right],\end{aligned} \tag{47}$$

and

$$\int_0^1 t_1 |_{\theta_1}\tilde{D}_q H_1(t_1(x_1)+(1-t_1)\delta_1|^m {}_0\tilde{d}_q t_1 \leq M_1{}^m \left[\int_0^1 (t_1)_{\tilde{q}}^{s+1} {}_0\tilde{d}_q t_1 + \int_0^1 t_1(1-t_1)_{\tilde{q}0}^{s}\tilde{d}_q t_1\right]. \tag{48}$$

We use (47) and (48) in (46) to obtain

$$\begin{aligned}\left|\frac{1}{q}\left(H_1(x_1) - \frac{q}{\delta_1-\theta_1}\int_{\theta_1}^{\delta_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1\right)\right| &\leq M_1 \left(\int_0^1 t_{10}\tilde{d}_q t_1\right)^{1-\frac{1}{m}} \\ &\left[\int_0^1 (t_1)_{\tilde{q}}^{s+1} {}_0\tilde{d}_q t_1 + \int_0^1 t_1(1-t_1)_{\tilde{q}0}^{s}\tilde{d}_q t_1\right]^{\frac{1}{m}} \times \left(\frac{(x_1-\theta_1)^2+(\delta_1-x_1)^2}{\delta_1-\theta_1}\right).\end{aligned} \tag{49}$$

Since

$$\int_0^1 (t_1)_{\tilde{q}}^{s+1} {}_0\tilde{d}_q t_1 = \frac{1}{[s+2]},$$

and

$$\begin{aligned}-\frac{1}{[s+1]}\int_0^1 t_{1\theta_1}\tilde{D}_q(1-q^{-1}t_1)_{\tilde{q}}^{s+1} {}_0\tilde{d}_q t_1 &= -\frac{1}{[s+1]}\left[\left|t_1(1-q^{-1}t_1)_{\tilde{q}}^{s+1}\right|_0^1 - \int_0^1 (1-t_1)_{\tilde{q}}^{s+1} {}_0\tilde{d}_q t_1\right] \\ &= -\frac{1}{[s+1]}\left[(1-q^{-1})_{\tilde{q}}^{s+1} + \frac{(1-q^{-1})_{\tilde{q}}^{s+2}}{[s+2]} - \frac{1}{[s+2]}\right],\end{aligned}$$

therefore, (49) becomes

$$\begin{aligned}\left|\frac{1}{q}\left(H_1(x_1) - \frac{q}{\delta_1-\theta_1}\int_{\theta_1}^{\delta_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1\right)\right| &\leq M_1 \left(\frac{(x_1-\theta_1)^2+(\delta_1-x_1)^2}{\delta_1-\theta_1}\right)\left(\frac{1}{[2]}\right)^{1-\frac{1}{m}} \\ &\times \left[-\frac{1}{[s+1]}\left((1-q^{-1})_{\tilde{q}}^{s+1} + \frac{(1-q^{-1})_{\tilde{q}}^{s+2}}{[s+2]} - \frac{1}{[s+2]}\right) + \frac{1}{[s+2]}\right]^{\frac{1}{m}}.\end{aligned}$$

□

Remark 6. *Theorem 8 extends Theorem 17 of [18].*
If we set $q=1$, then Theorem 8 becomes Theorem 4 of [21].

5. q-Symmetric Hermite–Hadamard Inequalities

In this section, we present q-symmetric analogues of Hermite–Hadamard inequalities for convex as well as for s-convex functions.

Theorem 9. *Suppose that $H_1 : J_1 \to \mathbb{R}$ is a q-symmetric differentiable function, ${}_{\theta_1}\tilde{D}_q H_1$ is continuous on $[\theta_1, \delta_1]$, and $0 < q < 1$. Then, we have*

$$H_1\left(\frac{\theta_1+\delta_1}{2}\right) \leq \frac{1}{\delta_1-\theta_1}\int_{\theta_1}^{\delta_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1 \leq \frac{qH_1(\delta_1)+(1-q+q^2)H_1(\theta_1)}{1+q^2}. \tag{50}$$

Proof. Using the definition of convexity, we have

$$H_1(t_1\delta_1 + (1-t_1)\theta_1) \le t_1 H_1(\delta_1) + (1-t_1)H_1(\theta_1), \tag{51}$$

Taking the q-symmetric integral of (51) with respect to t_1, where $t_1 \in (0,1)$, we have

$$\int_0^1 H_1(t_1\delta_1 + (1-t_1)\theta_1)_0\tilde{d}_q t_1 \le H_1(\delta_1)\int_0^1 t_{1 0}\tilde{d}_q t_1 + H_1(\theta_1)\int_0^1 (1-t_1)_0\tilde{d}_q t_1. \tag{52}$$

Using the q-symmetric Jackson's integral [15], we obtain

$$\int_0^x t_{1 0}\tilde{d}_q t_1 = (1-q^2)x\sum_{n=0}^{\infty} q^{2n}(q^{2n+1}x) = \frac{qx}{1+q^2}. \tag{53}$$

Therefore,

$$\int_0^1 t_{1 0}\tilde{d}_q t_1 = \frac{q}{1+q^2},$$

and

$$\int_0^1 (1-t_1)_0\tilde{d}_q t_1 = \frac{1-q+q^2}{1+q^2}.$$

Hence, (52) implies

$$\begin{aligned}\int_0^1 H_1(t_1\delta_1 + (1-t_1)\theta_1)_0\tilde{d}_q t_1 &\le H_1(\delta_1)\left(\frac{q}{1+q^2}\right) + H_1(\theta_1)\left(\frac{1-q+q^2}{1+q^2}\right)\\ &= \frac{qH_1(\delta_1) + (1-q+q^2)H_1(\theta_1)}{1+q^2}.\end{aligned} \tag{54}$$

Now consider

$$\begin{aligned}\int_0^1 H_1(t_1\delta_1 + (1-t_1)\theta_1)_0\tilde{d}_q t_1 &= (1-q^2)\sum_{n=0}^{\infty} q^{2n}H_1(q^{2n+1}\theta_1 + (1-q^{2n+1})\delta_1)\\ &= \frac{1}{\delta_1-\theta_1}\int_{\theta_1}^{\delta_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1.\end{aligned} \tag{55}$$

Equations (54) and (55) imply

$$\frac{1}{\delta_1-\theta_1}\int_{\theta_1}^{\delta_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1 \le \frac{qH_1(\delta_1) + (1-q+q^2)H_1(\theta_1)}{1+q^2}. \tag{56}$$

Using the definition of mid-convexity, we have

$$H_1\left(\frac{t_1\delta_1 + (1-t_1)\theta_1 + t_1\theta_1 + (1-t_1)\delta_1}{2}\right) \le \frac{1}{2}\Big[H_1(t_1\delta_1 + (1-t_1)\theta_1) + H_1(t_1\theta_1 + (1-t_1)\delta_1)\Big]$$
$$H_1\left(\frac{\theta_1+\delta_1}{2}\right) \le \frac{1}{2}\Big[H_1(t_1\delta_1 + (1-t_1)\theta_1) + H_1(t_1\theta_1 + (1-t)\delta_1)\Big].$$

Integrating from 0 to 1 with respect to t_1, we obtain

$$\begin{aligned}\int_0^1 H_1\left(\frac{\theta_1+\delta_1}{2}\right)_0\tilde{d}_q t_1 &\le \frac{1}{2}\left[\int_0^1 H_1(t_1\delta_1 + (1-t_1)\theta_1)_0\tilde{d}_q t_1 + \int_0^1 H_1(t_1\theta_1 + (1-t_1)\delta_1)_0\tilde{d}_q t_1\right].\\ &= \int_0^1 H_1(t_1\delta_1 + (1-t_1)\theta_1)_0\tilde{d}_q t_1 = \frac{1}{\delta_1-\theta_1}\int_{\theta_1}^{\delta_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1.\end{aligned} \tag{57}$$

From (56) and (57), we obtain the desired inequalities. □

Remark 7. *Theorem 9 is a suitable extension of Theorem 3.2 of [26].*
If we put $q = 1$, then inequalities (50) are reduced to classical Hermite–Hadamard inequalities.

Example 5. *Choose $H_1(t_1) = 1 - t_1$, $q \in (0,1)$, $\theta_1 = 0$, and $\delta_1 = 1$ in Theorem 9 to obtain the following:*

$$\begin{aligned}
&H_1\left(\frac{0+1}{2}\right) \le \int_0^1 H_1(t_1)_0\tilde{d}_q t_1 \le \frac{qH_1(1) + (1-q+q^2)H(0)}{1+q^2}\\
&H_1\left(\frac{1}{2}\right) \le (1-q^2)\sum_{n=0}^{\infty} q^{2n} H_1(q^{2n+1}) \le \frac{(1-q+q^2)}{1+q^2}\\
&\frac{1}{2} \le \frac{1-q+q^2}{1+q^2} \le \frac{(1-q+q^2)}{1+q^2}.
\end{aligned}$$

Remark 8. *Note that it is shown in Example 5 of [7] that the left-hand inequality of Theorem 3.2 in [26] does not hold for $q = 1/2$. From Example 5, it is clear that q-symmetric analogues (both left and right) of Hermite–Hadamard inequalities are valid for the functions $H(t_1) = 1 - t_1$ and $[\theta_1, \delta_1] = [0,1]$, which are chosen in Example 5 of [7].*

In Example 5, the right-hand inequality becomes an equality for the function $H_1(t_1) = (1 - t_1)$ and $t_1 \in [0,1]$. Now, by choosing $t_1 \in [0,2]$, we have the following inequalities:

Example 6. *Let us set $H_1(t_1) = 1 - t_1$, $q \in (0,1)$, $\theta_1 = 0$, and $\delta_1 = 2$ in Theorem 5.1 to obtain the following:*

$$\begin{aligned}
&H_1\left(\frac{0+2}{2}\right) \le \int_0^2 H_1(t_1)_0\tilde{d}_q t_1 \le \frac{qH_1(2) + (1-q+q^2)H_1(0)}{1+q^2}\\
&H_1(1) \le (1-q^2)\sum_{n=0}^{\infty} q^{2n} H_1(2q^{2n+1}) \le \frac{-q-(1-q+q^2)}{1+q^2}\\
&0 \le \frac{1-q+q^2}{2(1+q^2)} \le \frac{(1-2q+q^2)}{1+q^2}.
\end{aligned}$$

Theorem 10. *Let $H_1 : \mathbb{R}^+ \to \mathbb{R}$ be an s-convex mapping in the second sense, for which $s, q \in (0,1)$, and let $\theta_1, \delta_1 \in \mathbb{R}^+$, and $\theta_1 < \delta_1$. If ${}_{\theta_1}\tilde{D}_q H_1 \in L_1([\theta_1, \delta_1])$, then the following inequality is valid:*

$$2^{s-1} H_1\left(\frac{\theta_1+\delta_1}{2}\right) \le \frac{1}{\delta_1-\theta_1}\int_{\theta_1}^{\delta_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1 \le \frac{H_1(\delta_1)(1-(1-q^{-1})^{s+1}) + H_1(\theta_1)}{[s+1]}. \tag{58}$$

Proof. From the definition of s-convex functions,

$$\begin{aligned}
&H_1(t_1\theta_1 + (1-t_1)\delta_1) \le t_1^s H_1(\theta_1) + (1-t_1)^s H_1(\delta_1),\\
&\qquad\int_0^1 H_1(t_1\theta_1 + (1-t_1)\delta_1)_0\tilde{d}_q t_1 \le H_1(t_1)\int_0^1 t_{1\,0}^s\tilde{d}_q t_1 + H_1(t_1)\int_0^1 (1-t_1)^s{}_0\tilde{d}_q t_1.
\end{aligned}\tag{59}$$

$$\begin{aligned}
\int_0^1 H_1(t_1\theta_1 + (1-t_1)\delta_1)_0\tilde{d}_q t_1 &= \frac{(1-q^2)(\delta_1-\theta_1)}{\delta_1-\theta_1}\sum_{n=0}^{\infty} q^{2n} H_1(q^{2n+1}\delta_1 + (1-q^{2n+1})\theta_1)\\
&= \frac{1}{\delta_1-\theta_1}\int_0^1 H_1(t_1)_0\tilde{d}_q t_1,
\end{aligned}\tag{60}$$

Additionally,

$$\int_0^1 t_{1\,0}^s\tilde{d}_q t_1 = \frac{1}{[s+1]}, \int_0^1 (1-t_1)^s{}_0\tilde{d}_q t_1 = \frac{(1-(1-q^{-1}))^{s+1}}{[s+1]}. \tag{61}$$

Substituting (60) and (61) in (59), we have

$$\frac{1}{\delta_1-\theta_1}\int_{\theta_1}^{\delta_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1 \le \frac{H_1(\theta_1)+(1-(1-q^{-1}))^{s+1}H(\delta_1)}{[s+1]}. \tag{62}$$

Let us consider $x_1 = t_1\theta_1 + (1-t_1)\delta_1$, $x_2 = t_1\delta_1 + (1-t_1)\theta_1$ and substitute in

$$H_1\left(\frac{x_1+x_2}{2}\right) \le \frac{H_1(x_1)+H_1(x_2)}{2^s},$$

to obtain

$$H_1\left(\frac{t_1\theta_1+(1-t_1)\delta_1+t_1\delta_1+(1-t_1)\theta_1}{2}\right) \le \frac{H_1(t_1\theta_1+(1-t_1)\delta_1)+H_1(t_1\delta_1+(1-t_1)\theta_1)}{2^s}.$$

This gives

$$\begin{aligned} H_1\left(\frac{\theta_1+\delta_1}{2}\right) &\le \frac{1}{2^s}\left(\int_0^1 H_1(t_1\theta_1+(1-t_1)\delta_1)_0\tilde{d}_q t_1 + \int_0^1 H_1(t_1\delta_1+(1-t_1)\theta_1)_0\tilde{d}_q t_1\right) \\ &= \frac{1}{2^s}\left(\frac{1}{\theta_1-\delta_1}\int_{\delta_1}^{\theta_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1 + \frac{1}{\delta_1-\theta_1}\int_{\theta_1}^{\delta_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1\right) = \frac{1}{2^{s-1}}\left(\frac{1}{\delta_1-\theta_1}\int_{\theta_1}^{\delta_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1\right). \end{aligned}$$

Hence,

$$2^{s-1}H_1\left(\frac{\theta_1+\delta_1}{2}\right) \le \frac{1}{\delta_1-\theta_1}\int_{\theta_1}^{\delta_1} H_1(t_1)_{\theta_1}\tilde{d}_q t_1. \tag{63}$$

Inequality (62) together with Inequality (63) complete the proof. □

Remark 9. *Theorem 10 is an extension of Theorem 15 of [18].*
If we set $q = 1$ in (58), we obtain Hermite–Hadamard inequalities for the s-convex function given in [22].

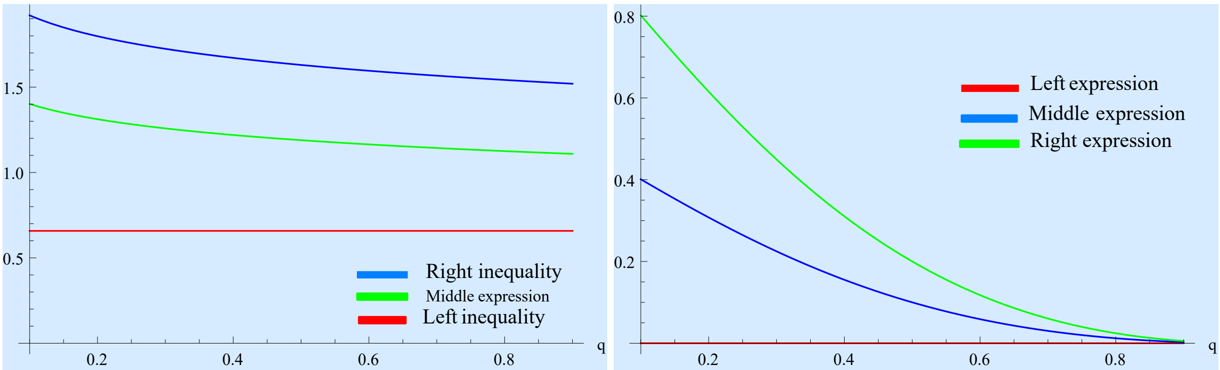
Example 7. *We can set $H_1(t_1) = t^s$, $s = 1/2, q \in (0,1)$, $\theta_1 = 0$, and $\delta_1 = 1$ in Theorem 10 to obtain*

$$2^{1/2-1}H_1\left(\frac{0+1}{2}\right) \le \int_0^1 H_1(t_1)_0\tilde{d}_q t_1 \le \frac{H_1(1)(1-(1-q^{-1})^{3/2})+H_1(0)}{[3/2]}$$

$$2^{-1/2}H_1\left(\frac{1}{2}\right)^{1/2} \le (1-q^2)\sum_{n=0}^{\infty} q^{2n}H_1(q^{2n+1}) \le \frac{(1-(1-q^{-1})^{3/2})}{[3/2]}$$

$$\frac{1}{2} \le \frac{(1+q)q^{1/2}}{1+q+q^2} \le \frac{1+q^2}{(1+q+q^2)}.$$

Figure 3a,b is a graphical representation of Example 6 and Example 7, respectively.



(**a**) (**b**)

Figure 3. (**a**) presents graphical representation of Example 6; (**b**) presents graphical representation of Example 7.

6. Conclusions

In this paper, q-symmetric Hölder, Minkowski, and power mean inequalities and the q-symmetric Montgomery identity are proved, which are keys to finding q-symmetric Ostrowski-type inequalities. Some Hermite–Hadamar-type inequalities are also established in this paper. The present results extend the Montgomery identities of [21,25]. Ostrowski-type inequalities for convex functions are proved in [18,25], and Ostrowski-type inequalities for s-convex functions are given in [18,21]. Hermite–Hadamar-type inequalities for convex functions are provided in [18,26], and Hermite–Hadamar-type inequalities for s-convex functions are provided in [18,22]. Several examples are included to show that the present results give better approximations in comparison with existing results in the literature. It can be seen from graphs that the differences in the left- and right-hand sides of the present inequalities are smaller than the differences in the left- and right-hand sides of the existing inequalities.

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