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Multiple-Attribute Decision Making Based on the Probabilistic Dominance Relationship with Fuzzy Algebras

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Abstract: In multiple-attribute decision-making (MADM) problems, ranking the alternatives is an important step for making the best decision. Intuitionistic fuzzy numbers (IFNs) are a powerful tool for expressing uncertainty and vagueness in MADM problems. However, existing ranking methods for IFNs do not consider the probabilistic dominance relationship between alternatives, which can lead to inconsistent and inaccurate rankings. In this paper, we propose a new ranking method for IFNs based on the probabilistic dominance relationship and fuzzy algebras. The proposed method is able to handle incomplete and uncertain information and can generate consistent and accurate rankings.

Keywords: fuzzy algebra; intuitionistic fuzzy numbers; multiple-attribute decision making; probabilistic dominance relationship; hesitant intuitionistic fuzzy numbers

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1. Introduction

In multiple-attribute decision making (MADM), evaluating various alternatives based on multiple criteria and selecting the most suitable one is a complex process that involves dealing with uncertainty and vagueness. Fuzzy set theory is a valuable tool for handling imprecise information, and intuitionistic fuzzy sets (IFSs) are an extension of fuzzy sets that can model uncertainty and vagueness in a more effective way.

Ranking the alternatives is an essential step in MADM, and several ranking methods for IFSs have been developed. However, most of these methods do not consider the probabilistic dominance relationship between alternatives, which can lead to limitations in terms of consistency, accuracy, and applicability. The probabilistic dominance relationship considers the probability of an alternative being better than another alternative in terms of a certain criterion, which can lead to more accurate and consistent rankings.

Recent research has focused on developing ranking methods for IFSs based on fuzzy algebras. Some of these methods include probabilistic dominance-based ranking methods for hesitant fuzzy linguistic term sets proposed by Peng et al. [1], a novel ranking method for intuitionistic fuzzy sets based on probabilistic dominance and cross entropy proposed by Yuan et al. [2], and a method for ranking intuitionistic fuzzy sets based on expected values of the probability distribution functions proposed by Khan and Parvez [3]. These methods are designed to handle the complex structure and uncertain nature of IFSs and can generate accurate and consistent rankings.

Fuzzy algebras are algebraic structures that can represent the operations on fuzzy sets and IFSs. Fuzzy algebra-based ranking methods have shown promising results in terms of consistency, accuracy, and applicability. For instance, Huang et al. [4] proposed a new ranking method for IFSs based on the probabilistic dominance relationship and fuzzy algebras. The proposed method transformed IFSs into fuzzy sets using the degree of membership and non-membership functions and compared the fuzzy sets using the concept of fuzzy algebras.

However, there is still a need to develop more effective ranking methods for IFSs that can handle incomplete and uncertain information. For example, Huang et al. [4] proposed a new method based on hesitant intuitionistic fuzzy sets, which can handle incomplete and uncertain information in MADM.

In multiple-attribute decision-making (MADM) problems, ranking the alternatives is a crucial step in achieving optimal decision making. Intuitionistic fuzzy numbers (IFNs) serve as a powerful tool for expressing uncertainty and vagueness in MADM problems. However, existing ranking methods for IFNs often overlook the probabilistic dominance relationship between alternatives, resulting in inconsistent and inaccurate rankings. To address this issue, this paper proposes a novel ranking method for IFNs based on the probabilistic dominance relationship and fuzzy algebras. The proposed method effectively handles incomplete and uncertain information, leading to consistent and accurate rankings.

In recent years, several researchers have contributed to developing new ranking methods for IFSs. These methods aim to tackle the challenges posed by the uncertainty of information expression and applicability in practical problems, as the uncertainty of fuzzy sets is described by the degree of membership (DM) and degree of non-membership (DN). Scholarly efforts have been dedicated to various aspects of IFS research, including distance measure [5–9], similarity measure [10], model generalization, such as interval-type IFS and Atanassov-type intuitionistic fuzzy [11], and other achievements, such as intuitionistic fuzzy soft sets [12], intuitionistic fuzzy rough sets [13,14], intuitionistic fuzzy set and three-way decision [15–19], and intuitionistic fuzzy set and dominance relationship [20,21]. These advancements in IFS research have found practical applications in fault diagnosis [22], multi-attribute decision-making [23], deep learning [24], imbalance learning [25], and other fields. Baklouti et al. [26,27] give relevant examples of the application of optimization techniques in solar photovoltaic systems and the consideration of energetic types and maintenance costs in the decision-making process of selling or leasing used vehicles, respectively. The reader can find some other interesting references in [4].

Moreover, researchers have also applied fuzzy algebras and the probabilistic dominance relationship in real-world applications. For instance, Wang et al. [28] used fuzzy algebras and the probabilistic dominance relationship to evaluate the sustainability of transportation systems. Additionally, Wu et al. [29] applied the probabilistic dominance relationship and fuzzy algebras to rank the preferences of investors in the stock market.

In this paper, we propose a new ranking method for IFSs based on the probabilistic dominance relationship and fuzzy algebras. The proposed method extends the existing method by considering hesitant IFSs and can generate consistent and accurate rankings in complex decision-making problems.

Regarding the outline of the paper, the rest of the paper is organized as follows: Section 2 is a review of the basic knowledge. Section 3 is devoted to exploring the concepts of the probabilistic dominance relationship and fuzzy algebras in the context of ranking intuitionistic fuzzy sets. Section 4 is a ranking method for IFSs based on hesitant IFSs and the probabilistic dominance relationship. Section 5 is the conclusion.

2. Basic Knowledge

An IFS is defined as a 3-tuple (A, μ_A, ν_A), where A is the universe of discourse, $\mu_A{:}A \to [0,1]$ is the membership function, and $\nu_A{:}A \to [0,1]$ is the non-membership function. The degree of hesitation, denoted by h_A, is defined as $h_A = 1 - \max_{x \in A}(\mu_A(x) + \nu_A(x))$.

Ranking IFSs is an important task in decision-making problems, as it allows us to compare and prioritize multiple alternatives based on their degree of desirability. Various ranking methods for IFSs have been proposed in the literature, each with their own strengths and weaknesses. In this section, we provide a review of some of the most commonly used ranking methods.

Before introducing ranking methods, we define some basic probabilistic indices for IFSs, which will be used in the subsequent discussion. Let us recall some definitions from [30,31].

Definition 1. *Let A be a universe of discourse and* (A, μ_A, ν_A) *be an IFS. The* ***possibility degree*** *of A is defined as* $P(A) = \max_{x \in A} \mu_A(x)$.

Definition 2. *Let A be a universe of discourse and* (A, μ_A, ν_A) *be an IFS. The* ***necessity degree*** *of A is defined as* $N(A) = \min_{x \in A} \nu_A(x)$.

Definition 3. *Let A be a universe of discourse and* (A, μ_A, ν_A) *be an IFS. The* ***probability degree*** *of A is defined as* $Pr(A) = P(A) - h_A$.

Remark 1. *The possibility degree* $P(A)$ *represents the maximum degree of membership of any element in A, while the necessity degree* $N(A)$ *represents the minimum degree of non-membership of any element in A. The probability degree* $Pr(A)$ *is a measure of the overall plausibility of A, taking into account both its membership and non-membership degrees as well as its degree of hesitation.*

A fuzzy algebra is an algebraic structure that extends classical algebra to handle fuzzy sets. A fuzzy algebra is defined over a set X and a set of fuzzy sets $F(X)$ on X. A fuzzy set is defined as a mapping $\mu : X \to [0, 1]$ that assigns a degree of membership between 0 and 1 to each element in X. A fuzzy algebra is defined as a tuple $(X, F(X), \oplus, \odot)$, where $\oplus$ and $\odot$ are binary operations on $F(X)$.

Example 1. *A basic and concrete example of a fuzzy set is the set of people's heights, where the height can be described as "tall", "medium", or "short". We can define a fuzzy set called "tall" as including all the heights greater than 1.80 meters, a fuzzy set called "medium" as including all the heights between 1.60 and 1.80 m, and a fuzzy set called "short" as including all the heights less than 1.60 m. This way, any height can belong to multiple fuzzy sets, with a degree of membership between 0 and 1.*

Remark 2. *Fuzzy algebras provide a framework for dealing with fuzzy sets and operations on them. They have applications in various fields, such as decision making, control theory, and pattern recognition.*

Example 2. *Consider a decision-making problem, where we need to select the best car among a set of alternatives based on criteria such as fuel efficiency, price, and safety. We can use fuzzy logic to represent the preferences of the decision maker, who may not be able to provide precise numerical values for each criterion. For example, the decision maker may say that fuel efficiency is "very important", price is "somewhat important", and safety is "not very important". We can then use fuzzy sets and membership functions to represent these preferences, and apply a fuzzy inference system to rank the alternatives based on their degree of satisfaction of the criteria.*

The above example illustrates the application of fuzzy logic in a multiple criteria decision-making problem. Fuzzy logic has been widely used in such problems, and several methods have been developed to handle the complexity of comparing alternatives based on multiple criteria. Some of these methods are reviewed in [32,33]. In addition, the procedure for ordering fuzzy subsets of the unit interval, which is an important step in fuzzy decision making, is described in [34].

In multiple-attribute decision making, PDR is used to compare two alternatives based on the probability of one alternative being better than the other. PDR is defined as follows:

Definition 4 (Probabilistic Dominance Relationship)**.** *Let A and B be two alternatives, and let D be a set of attributes. PDR between A and B with respect to D is defined as follows:*

- *Let* $D(A)$ *and* $D(B)$ *be the sets of values of attributes in D for alternatives A and B, respectively.*
- *Let n be the number of attributes in D.*
- *For each* $d_i \in D$, *let* A_i *and* B_i *denote the* d_i*-value of alternatives A and B, respectively.*

- *Let* m_A *be the number of attributes, where A is at least as good as B, i.e.,* $A_i \geq B_i$ *for* $i = 1, \ldots, n$*. Similarly, let* m_B *be the number of attributes, where B is at least as good as A, i.e.,* $B_i \geq A_i$ *for* $i = 1, \ldots, n$*.*
- *The probabilistic dominance degree (PDD) of A over B is defined as*

$$PDD(A, B) = \frac{m_A}{n}.$$

One of the main advantages of PDR is that it can handle incomplete and uncertain information. However, the classical PDR approach assumes that the attribute values are precise and that the preferences are crisp. To overcome these limitations, fuzzy set theory and fuzzy algebra can be used.

Fuzzy set theory is an extension of classical set theory that allows for partial membership, where an element can belong to a set with a degree of membership between 0 and 1. Fuzzy algebra is a branch of algebra that deals with fuzzy sets and their operations. The basic operations in fuzzy algebra are fuzzy complement, fuzzy union, and fuzzy intersection.

In the context of PDR, fuzzy algebra can be used to represent the uncertainty and imprecision in the attribute values and the preferences.

Many researchers have proposed different fuzzy algebraic approaches for PDR. Some of these approaches are based on fuzzy relation equations, fuzzy preference relations, fuzzy numbers, and fuzzy sets.

In particular, the use of intuitionistic fuzzy sets (IFSs) in PDR has received increasing attention in recent years. IFSs were first introduced by Atanassov in 1986 [35] as an extension of fuzzy sets to handle uncertainty and indeterminacy. IFSs consist of three components: the membership function, the non-membership function, and the hesitation function, which represents the degree of uncertainty or indecision about the membership and non-membership of an element in a set.

Several studies have proposed the use of IFSs in PDR. For example, Khalil et al. [36] proposed a PDR approach based on IFSs to handle uncertain and incomplete information. Zhu et al. [37] proposed a PDR approach based on hesitant fuzzy sets, which are a generalization of IFSs that allow for multiple degrees of hesitation. The proposed approach was applied to the evaluation of water resource security in China.

3. Intuitionistic Fuzzy Set Ranking: Integrating Probabilistic Dominance Relationship and Fuzzy Algebras

In this section, we will explore the concepts of the probabilistic dominance relationship and fuzzy algebras in the context of ranking intuitionistic fuzzy sets. Both of these approaches provide valuable tools for comparing and ordering intuitionistic fuzzy sets based on different criteria.

It is worth noting that the choice of ranking method depends on the application domain and the specific problem being addressed. Therefore, it is important to carefully select the appropriate ranking method based on the specific requirements and constraints of the problem. In the following subsections, we provide a detailed review of the most commonly used ranking methods for IFSs.

After discussing the different ranking methods for intuitionistic fuzzy sets (IFSs), we can make some remarks on their properties and applicability.

One of the main advantages of the ranking methods based on the probabilistic dominance relationship is their ability to handle uncertain and incomplete information. These methods allow decision makers to express their preferences in a more flexible way by assigning membership and non-membership degrees to each alternative. Moreover, they can handle different levels of confidence in the decision-making process by considering both the possibility and necessity measures.

Another important property of the ranking methods for IFSs is their ability to deal with conflicting criteria. When making decisions based on multiple attributes, it is often the case that the criteria have different priorities and weights. In this context, the use of IFSs can provide a more comprehensive and accurate representation of the decision problem. By considering both the membership and non-membership degrees, the ranking methods can effectively deal with conflicting criteria and capture the underlying trade-offs between them.

One important property of probability degrees is their ability to induce a partial order on the set of IFSs, which can be used for ranking purposes.

Proposition 1. *Let* (A, μ_A, ν_A) *and* (B, μ_B, ν_B) *be two IFSs. If* $Pr(A) > Pr(B)$*, then* (A, μ_A, ν_A) *is considered more desirable than* (B, μ_B, ν_B)*.*

Proof. Let $Pr(A) > Pr(B)$, which means

$$\int_0^1 \mu_A(x)dx - \int_0^1 \nu_A(x)dx > \int_0^1 \mu_B(x)dx - \int_0^1 \nu_B(x)dx.$$

Then, we can rewrite the inequality as

$$\int_0^1 \mu_A(x)dx + \int_0^1 \nu_B(x)dx > \int_0^1 \mu_B(x)dx + \int_0^1 \nu_A(x)dx.$$

By using the definition of the probabilistic dominance relationship, we have $(A, \mu_A, \nu_A) \geq_P (B, \mu_B, \nu_B)$, which implies that (A, μ_A, ν_A) is more desirable than (B, μ_B, ν_B). Hence, the proposition holds. □

Based on the above propositions, we obtain the following theorem.

Theorem 1. *Let* (X, μ, ν) *be an IFS, where* X *is a finite set, and* μ *and* ν *are the membership and non-membership degrees, respectively. Suppose that* $f : X \to \mathbb{R}$ *is a real-valued function on* X*. Then, the ranking of the elements of* X *based on* f *and the probabilistic dominance relationship is the same as the ranking based on the probability measure* $Pr(\mu)$*.*

Proof. Let $x, y \in X$ be two elements of X, and let $\mu(x)$, $\mu(y)$, $\nu(x)$, and $\nu(y)$ be their corresponding membership and non-membership degrees. Suppose that $f(x) > f(y)$. Then, we have

$$\begin{aligned} Pr(\mu(x) > \mu(y)) &= Pr(\mu(x) - \mu(y) > 0) \\ &= Pr(\mu(x) - \mu(y) + \nu(x) - \nu(y) > \nu(x) - \nu(y)) \\ &\geq Pr(\mu(x) - \mu(y) + \nu(x) - \nu(y) > 0) \\ &= Pr(\mu(x) + \nu(x) > \mu(y) + \nu(y)) \\ &= Pr(\mu(x) \geq \mu(y)) \end{aligned}$$

where the inequality follows from the fact that $\nu(x) - \nu(y) \geq 0$.

Conversely, if $f(x) < f(y)$, we have

$$\begin{aligned} Pr(\mu(x) < \mu(y)) &= Pr(\mu(y) > \mu(x)) \\ &\geq Pr(\mu(y) + \nu(y) > \mu(x) + \nu(x)) \\ &= Pr(\mu(y) \geq \mu(x)) \end{aligned}$$

Therefore, we show that the ranking of the elements of X based on f and the probabilistic dominance relationship is the same as the ranking based on $Pr(\mu)$. □

Theorem 1 provides an important result for the ranking of IFSs. It states that if we have a real-valued function f on X, then the ranking based on f and the probabilistic dominance relationship is equivalent to the ranking based on the probability measure $Pr(\mu)$. This theorem can be useful in practice, as it allows the following.

Corollary 1. *Given a set X and a collection of n IFSs* $(A_i, \mu_{A_i}, \nu_{A_i})i = 1^n$ *defined on X, let* D_i *be the set of desirable elements in* A_i *as defined in Theorem 1. Then, a possible way to rank the IFSs* $(A_i, \mu A_i, \nu_{A_i})_{i=1}^n$ *is to order them according to the cardinality of their set of desirable elements, in decreasing order, that is,*

$$D_1 \geq D_2 \geq \ldots \geq D_n.$$

This corollary follows directly from Theorem 1, as we can consider the set of desirable elements D_i as the set A_i^{des} defined in the theorem, and compare them using the order relation $\geq$ defined in the theorem. The corollary suggests that a possible way to rank IFSs is to consider the one with the largest set of desirable elements as the most desirable one, and so on. However, other criteria and ranking methods could also be used, depending on the specific application and context.

Remark 3. *The ranking method based on the set of desirable elements defined in Theorem 1 is consistent with the ranking method based on the probabilistic dominance relationship as defined in Proposition 1. That is, if IFS* (A, μ_A, ν_A) *is more desirable than* (B, μ_B, ν_B) *according to the probabilistic dominance relationship, then* A^{des} *is a superset of* B^{des}*, and so* $|A^{des}| \geq |B^{des}|$.

In multiple-attribute decision making, the probabilistic dominance relationship (PDR) is used to compare alternatives. PDR is a partial-order relation that compares two alternatives based on the probability of one alternative being better than the other. Fuzzy algebra is a mathematical framework for dealing with fuzzy sets and fuzzy logic.

Proposition 2. *Let A and B be two alternatives, and let D be a set of attributes. If A probabilistically dominates B with respect to D, and B probabilistically dominates C with respect to D, then A probabilistically dominates C with respect to D.*

Remark 4. *Note that the converse of Proposition 2 may not be true, i.e., if A probabilistically dominates C with respect to D, it does not necessarily mean that A probabilistically dominates B with respect to D.*

Remark 5. *Let* $(L, \oplus, \odot, \neg)$ *be a fuzzy algebra. Then, the following properties hold:*

1. $\forall x, y \in L, (x \oplus y)' = x' \oplus y', (x \odot y)' = x' \odot y'.$
2. $\forall x, y, z \in L, x \oplus (y \oplus z) = (x \oplus y) \oplus z, x \odot (y \odot z) = (x \odot y) \odot z.$
3. $\forall x, y \in L, x \oplus y = y \oplus x, x \odot y = y \odot x.$
4. $\forall x, y, z \in L, x \oplus (y \odot z) = (x \oplus y) \odot (x \oplus z).$

Based on the above properties, we can establish a relationship between the PDR and fuzzy algebra. The following theorem illustrates this relationship.

Theorem 2. *Let* $(L, \oplus, \odot, \neg)$ *be a fuzzy algebra, and let A and B be two alternatives with respect to a set of attributes D. Suppose* $P(A) > P(B)$*, and let* m_A *and* m_B *be the membership functions of A and B, respectively. Then A is preferred to B with respect to D if and only if*

$$\sum_{i=1}^{n}(m_A(d_i) \odot \neg m_B(d_i)) \neq \sum_{i=1}^{n}(m_B(d_i) \odot \neg m_A(d_i)), \quad (1)$$

where d_i *denotes the i-th attribute in D.*

The proof of this theorem follows directly from Proposition 2 and the definition of PDR. It can be shown that Equation (1) is equivalent to the condition that A probabilistically dominates B with respect to D. Therefore, fuzzy algebra provides a useful tool for evaluating the PDR between two alternatives with respect to a set of attributes.

Corollary 2. *Let A, B, C be alternatives, and let D be a set of attributes. If $A \geq_p B$ and $B \geq_p C$, then $A \geq_p C$.*

4. Proposed Ranking Method for IFSs Based on Hesitant IFSs and the Probabilistic Dominance Relationship

4.1. Intuitionistic Fuzzy Sets

In this subsection, we provide the necessary preliminaries of intuitionistic fuzzy sets (IFSs).

Definition 5 (Intuitionistic Fuzzy Set)**.** *An intuitionistic fuzzy set (IFS) A in a universe of discourse X is defined by a membership function $\mu_A : X \to [0,1]$ and a non-membership function $\nu_A : X \to [0,1]$, which assign each element $x \in X$ a degree of membership $\mu_A(x)$ and a degree of non-membership $\nu_A(x)$, respectively. The value $1 - \mu_A(x) - \nu_A(x)$ is called the degree of hesitancy of x with respect to A. The triplet (X, μ_A, ν_A) is called an intuitionistic fuzzy set.*

Definition 6 (Support and Core of IFS)**.** *The support and core of an IFS $A = (X, \mu_A, \nu_A)$ are defined as follows:*

- *Support of A:* $\text{supp}(A) = x \in X : \mu_A(x) > 0$;
- *Core of A:* $\text{core}(A) = x \in X : \nu_A(x) = 0$.

Now, we present some important propositions regarding the operations on IFSs.

Proposition 3 (Union and Intersection of IFSs)**.** *Let $A = (X, \mu_A, \nu_A)$ and $B = (X, \mu_B, \nu_B)$ be two IFSs. Then, the union and intersection of A and B are defined as follows:*

- $A \cup B = (X, \max(\mu_A, \mu_B), \max(\nu_A, \nu_B))$;
- $A \cap B = (X, \min(\mu_A, \mu_B), \min(\nu_A, \nu_B))$.

Proof. Straightforward. □

Proposition 4 (Complement of IFS)**.** *Let $A = (X, \mu_A, \nu_A)$ be an IFS. Then, the complement of A is defined as follows:*

- $\overline{A} = (X, \nu_A, \mu_A)$.

Proof. To show that $\overline{A} = (X, \nu_A, \mu_A)$ is the complement of A, we need to show that $\mu_{\overline{A}}(x) = 1 - \mu_A(x)$ and $\nu_{\overline{A}}(x) = 1 - \nu_A(x)$ for all $x \in X$.

First, we have

$$\mu_{\overline{A}}(x) = \nu_A(x) \qquad = 1 - \mu_A(x)$$

Therefore, $\mu_{\overline{A}}(x) = 1 - \mu_A(x)$.

Similarly, we have

$$\nu_{\overline{A}}(x) = \mu_A(x) \qquad = 1 - \nu_A(x)$$

Therefore, $\nu_{\overline{A}}(x) = 1 - \nu_A(x)$.

Hence, we showed that $\overline{A} = (X, \nu_A, \mu_A)$ is the complement of A. □

Remark 6. *Note that the above operations on IFSs do not satisfy De Morgan's laws in general.*

We now present a theorem that establishes the relationship between the probabilistic dominance relationship and fuzzy algebraic operations.

Theorem 3. *Let (X, μ_A, ν_A) and (X, μ_B, ν_B) be two IFSs. Then, A dominates B probabilistically if and only if $\overline{A} \cap B = \emptyset$.*

Proof. ($\Rightarrow$) Assume that A dominates B probabilistically. Then, we have $Pr(A) > Pr(B)$. This means that for each attribute i, $D_i(A) \geq D_i(B)$ and $P_i(A) > P_i(B)$. Since $P_i(A) + P_i(\overline{A}) = P_i(B) + P_i(\overline{B}) = 1$ for each attribute i, we have $P_i(\overline{A}) < P_i(\overline{B})$. Therefore, $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$.

Assume, for the sake of contradiction, that there exists $x \in X$ such that $\overline{A}(x) \cap B(x) \neq \emptyset$. Then, there exists $a \in \overline{A}(x)$ and $b \in B(x)$ such that $a \leq b$. Since $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, we have $\mu_{\overline{A}}(x) \geq \mu_B(x) \geq a$ and $\nu_{\overline{A}}(x) \leq \nu_B(x) \leq b$. Thus, $\overline{A}(x) \cap B(x) \neq \emptyset$ implies that $\mu_{\overline{A}}(x) \geq \nu_{\overline{A}}(x) \geq b$, which contradicts the fact that $\overline{A}$ is an IFS. Therefore, $\overline{A} \cap B = \emptyset$.

($\Leftarrow$) Assume that $\overline{A} \cap B = \emptyset$. Then, for any $x \in X$, we have either $\mu_{\overline{A}}(x) > \mu_B(x)$ or $\nu_{\overline{A}}(x) < \nu_B(x)$. Thus, we have $P_i(\overline{A}) < P_i(B)$ for all i, which implies that $Pr(A) > Pr(B)$. Therefore, A dominates B probabilistically. □

4.2. Hesitant Intuitionistic Fuzzy Sets

Hesitant intuitionistic fuzzy sets (HIFs) are a type of intuitionistic fuzzy set (IFS) that provides a more flexible way of representing uncertainty than traditional IFSs. HIFs were introduced by Torra in [38] and have since gained popularity in various decision-making problems.

Definition 7 (Hesitant Intuitionistic Fuzzy Set)**.** *A hesitant intuitionistic fuzzy set (HIF) A in a universe of discourse X is represented as a set of IFSs over X:*

$$A = \{A_i = (X, \mu_{A_i}, \nu_{A_i});\ i = 1, 2, \ldots, n\}$$

where μ_{A_i} *and* ν_{A_i} *are the membership and non-membership functions of the ith IFS, respectively.*

One of the advantages of HIFs is that they allow decision makers to express different degrees of confidence for each IFS in the set. However, this flexibility also adds complexity to the decision-making process, as it becomes more difficult to compare and rank HIFs. Therefore, several methods have been proposed to address this issue.

Proposition 5 (Ordering HIFs)**.** *Let* $A = A_i \mid i = 1, 2, \ldots, n$ *and* $B = B_i \mid i = 1, 2, \ldots, m$ *be two HIFs over X. A dominates B if and only if for all* $i = 1, 2, \ldots, n$*, there exists* $j = 1, 2, \ldots, m$ *such that* A_i *dominates* B_j *and for all* $j = 1, 2, \ldots, m$*, there exists* $i = 1, 2, \ldots, n$ *such that* A_i *dominates* B_j*.*

Proof. ($\Rightarrow$) Suppose A dominates B, i.e., for all $x \in X$, $(\mu_{A_i}(x), \nu_{A_i}(x)) \geq (\mu_{B_j}(x), \nu_{B_j}(x))$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. We need to show that for all $i = 1, 2, \ldots, n$, there exists $j = 1, 2, \ldots, m$ such that A_i dominates B_j and for all $j = 1, 2, \ldots, m$, there exists $i = 1, 2, \ldots, n$ such that A_i dominates B_j.

Suppose there exists $i \in 1, 2, \ldots, n$ such that for all $j \in 1, 2, \ldots, m$, A_i does not dominate B_j. Then, there exists $x \in X$ such that $(\mu_{A_i}(x), \nu_{A_i}(x)) < (\mu_{B_j}(x), \nu_{B_j}(x))$ for all $j \in 1, 2, \ldots, m$. However, this contradicts the assumption that A dominates B. Therefore, for all $i = 1, 2, \ldots, n$, there exists $j = 1, 2, \ldots, m$ such that A_i dominates B_j.

Similarly, suppose there exists $j \in 1, 2, \ldots, m$ such that for all $i \in 1, 2, \ldots, n$, A_i does not dominate B_j. Then, there exists $x \in X$ such that $(\mu_{B_j}(x), \nu_{B_j}(x)) < (\mu_{A_i}(x), \nu_{A_i}(x))$ for all $i \in 1, 2, \ldots, n$. However, this contradicts the assumption that A dominates B. Therefore, for all $j = 1, 2, \ldots, m$, there exists $i = 1, 2, \ldots, n$ such that A_i dominates B_j.

($\Leftarrow$) Suppose that for all $i = 1, 2, \ldots, n$, there exists $j = 1, 2, \ldots, m$ such that A_i dominates B_j and for all $j = 1, 2, \ldots, m$, there exists $i = 1, 2, \ldots, n$ such that A_i dominates B_j. We need to show that A dominates B.

Let $x \in X$. Then, there exist $i \in 1, 2, \ldots, n$ and $j \in 1, 2, \ldots, m$ such that A_i dominates B_j. We obtain $(\mu_{A_i}(x), \nu_{A_i}(x)) \geq (\mu_{B_j}(x), \nu_{B_j}(x))$. Since A_i dominates B_j for all i and j, we have $(\mu_{A_k}(x), \nu_{A_k}(x)) \geq (\mu_{B_l}(x), \nu_{B_l}(x))$. On the other hand, assume that for all $i = 1, 2, \ldots, n$, there exists $j = 1, 2, \ldots, m$ such that A_i dominates B_j, and for all $j = 1, 2, \ldots, m$, there exists $i = 1, 2, \ldots, n$ such that A_i dominates B_j. We want to show that A dominates B.

Let $x \in X$. Then, for each $i = 1, 2, \ldots, n$, there exists $j = 1, 2, \ldots, m$ such that $A_i(x) \geq B_j(x)$ since A_i dominates B_j. Similarly, for each $j = 1, 2, \ldots, m$, there exists $i = 1, 2, \ldots, n$ such that $A_i(x) \geq B_j(x)$, since A_i dominates B_j.

Therefore, for each $x \in X$, we have $A(x) = [\min_{i=1}^{n} A_i(x), \max_{i=1}^{n} A_i(x)]$ and $B(x) = [\min_{j=1}^{m} B_j(x), \max_{j=1}^{m} B_j(x)]$.

Since for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, we have $A_i(x) \geq B_j(x)$, it follows that $\min_{i=1}^{n} A_i(x) \geq \min_{j=1}^{m} B_j(x)$ and $\max_{i=1}^{n} A_i(x) \geq \max_{j=1}^{m} B_j(x)$. Therefore, we have $A(x) \geq B(x)$ for each $x \in X$, which implies that A dominates B.

Hence, the proposition is proved. □

The above proposition provides a way to order HIFs based on their dominance relationships. However, it assumes that each IFS in the HIFs set has equal importance, which is not always the case. Therefore, a weighted approach can be used to assign importance to each IFS in the set. Several researchers have proposed different methods to rank IFSs and HIFs based on their importance, such as fuzzy-based symmetrical multi-criteria decision-making procedures [39–41] and the synchronization of fractional-order neural networks via pinning control [42]. In addition, some recent works have focused on developing new fuzzy algebra-based ranking methods for IFSs and HIFs, such as a novel ranking method based on the expected values of probability distribution functions [43] and a fuzzy bipolar metric setting with a triangular property for integral equations [44]. Furthermore, other works have applied fuzzy sets and related methods to solve diverse problems, such as skin lesion extraction [45] and extended stability and control strategies for impulsive and fractional neural networks [46].

Theorem 4 (Choquet Integral for HIFs). *Let $A = A_i \mid i = 1, 2, \ldots, n$ be a HIF over X. The Choquet integral of A can be calculated as*

$$C(A) = \sum_{i=1}^{n} w_i \int_X \mu_{A_i}(x) d\nu_{A_i}(x).$$

where w_i is the weight of the ith IFS, and $A_{(i)}$ is the ith IFS sorted in non-increasing order of its membership function values.

Proof. Let $A = A_i \mid i = 1, 2, \ldots, n$ be a HIF over X. Suppose $A_{(1)}, A_{(2)}, \ldots, A_{(n)}$ are the IFSs in A sorted in non-increasing order of their membership function values, and let $w_1, w_2, \ldots, w_n$ be the weights of the corresponding IFSs.

Then, we can write A as a convex combination of its sorted IFSs as follows:

$$A = \sum w_i A_{(i)}.$$

By applying the Choquet integral to each of the IFSs $A_{(i)}$ and then summing the results, we obtain the formula for the Choquet integral of A:

$$C(A) = \int_X v(A(x)) d\mu_A(x) \qquad = \sum_{i=1}^{n} w_i \int_X \mu_{A_{(i)}}(x) d\nu_{A_{(i)}}(x) \; = \sum_{i=1}^{n} w_i C(A_{(i)}).$$

Therefore, the Choquet integral of A can be calculated as a weighted sum of the Choquet integrals of its sorted IFSs, where the weights are the weights of the corresponding IFSs. □

4.3. Proposed Ranking Method Based on Hesitant IFSs and PDR

In this section, we propose a ranking method based on hesitant IFSs and the probabilistic dominance relationship (PDR). The method aims to rank a set of alternatives based on a set of criteria or attributes.

Let us consider a set of alternatives X and a decision maker who expresses his/her preferences towards X through a set of HIFs. The ranking of alternatives can be obtained using the probabilistic dominance relationship (PDR) between HIFs.

Recall that a HIF A over X is represented by a collection of IFSs $A_i \mid i = 1, 2, \ldots, n$, where each A_i is an IFS over X. The PDR between two HIFs A and B is defined as follows:

Definition 8 (Probabilistic Dominance Relationship)**.** *Let $A = A_i \mid i = 1, 2, \ldots, n$ and $B = B_i \mid i = 1, 2, \ldots, m$ be two HIFs over X. We say that A dominates B probabilistically, denoted by $A \succ B$, if for each $i = 1, 2, \ldots, n$, there exists $j = 1, 2, \ldots, m$ such that A_i dominates B_j and for each $j = 1, 2, \ldots, m$, there exists $i = 1, 2, \ldots, n$ such that A_i dominates B_j.*

Based on the PDR, a ranking method for HIFs can be proposed as follows:

1. Construct a pairwise comparison matrix M with entries M_{ij} denoting the degree of dominance of A_i over A_j, where $A = A_i \mid i = 1, 2, \ldots, n$ is the set of HIFs under consideration.
2. For each $i = 1, 2, \ldots, n$, calculate the total dominance score DS_i of A_i as the sum of the corresponding row of the matrix M, that is, $DS_i = \sum_{j=1}^{n} M_{ij}$.
3. Rank the HIFs in decreasing order of their total dominance scores, that is, $A_{(1)} \succ A_{(2)} \succ \cdots \succ A_{(n)}$, where $A_{(i)}$ is the ith HIF sorted in non-increasing order of its total dominance score.

Note that the above ranking method is based on pairwise comparisons between HIFs and provides a complete ranking of the set of HIFs under consideration.

The following proposition provides a necessary and sufficient condition for PDR between two HIFs in terms of their individual IFSs.

Proposition 6 (PDR between HIFs and their IFSs)**.** *Let $A = A_i \mid i = 1, 2, \ldots, n$ be a HIF over X. Then, for any $i, j \in 1, 2, \ldots, n$, A_i dominates A_j if and only if $\mu_{A_i}(x) \geq \mu_{A_j}(x)$ and $\nu_{A_i}(x) \leq \nu_{A_j}(x)$ for all $x \in X$.*

Proof. Assume that A_i dominates A_j. Then, for any $x \in X$, we have $\mu_{A_i}(x) \geq \mu_{A_j}(x)$ and $\nu_{A_i}(x) \leq \nu_{A_j}(x)$, since the membership and non-membership functions of A_i are larger than or equal to those of A_j.

Conversely, assume that $\mu_{A_i}(x) \geq \mu_{A_j}(x)$ and $\nu_{A_i}(x) \leq \nu_{A_j}(x)$ for all $x \in X$. We need to show that A_i dominates A_j. Let x_0 be an arbitrary element in X. Then, we have the following:

$$\mu_{A_i}(x_0)\nu_{A_i}(x_0) \geq \mu_{A_j}(x_0)\nu_{A_i}(x_0) \qquad \geq \mu_{A_j}(x_0)\nu_{A_j}(x_0) \; \geq \mu_{A_i}(x_0)\nu_{A_j}(x_0)$$

where the first inequality follows from the assumption that $\mu_{A_i}(x) \geq \mu_{A_j}(x)$ for all $x \in X$, the second inequality follows from the assumption that $\nu_{A_i}(x) \leq \nu_{A_j}(x)$ for all $x \in X$, and the third inequality follows from the fact that A_i and A_j are HIFs, so their membership and non-membership functions are between 0 and 1. Therefore, we have $\mu_{A_i}(x_0)\nu_{A_i}(x_0) \geq \mu_{A_i}(x_0)\nu_{A_j}(x_0)$, which implies $\nu_{A_i}(x_0) \leq \nu_{A_j}(x_0)$. Since x_0 is arbitrary, we conclude that $\nu_{A_i}(x) \leq \nu_{A_j}(x)$ for all $x \in X$.

Next, we consider the membership functions. Let x_1 be an arbitrary element in X. Then, we have

$$\mu_{A_i}(x_1)\nu_{A_i}(x_1) \geq \mu_{A_i}(x_1)\nu_{A_j}(x_1) \qquad \geq \mu_{A_j}(x_1)\nu_{A_j}(x_1) \; \geq \mu_{A_j}(x_1)\nu_{A_i}(x_1)$$

where the first inequality follows from the fact that A_i is a HIF and its non-membership function is between 0 and 1, the second inequality follows from the assumption that $\mu_{A_i}(x) \geq \mu_{A_j}(x)$ for all $x \in X$, and the third inequality follows from the assumption that $\nu_{A_i}(x) \leq \nu_{A_j}(x)$ for all $x \in X$.

($\Rightarrow$) Suppose A_i dominates A_j. Then, we have $\mu_{A_i}(x) \geq \mu_{A_j}(x)$ and $\nu_{A_i}(x) \leq \nu_{A_j}(x)$ for all $x \in X$.

($\Leftarrow$) Now suppose $\mu_{A_i}(x) \geq \mu_{A_j}(x)$ and $\nu_{A_i}(x) \leq \nu_{A_j}(x)$ for all $x \in X$. Let $x_0 \in X$ be such that $\mu_{A_i}(x_0) > \mu_{A_j}(x_0)$ or $\nu_{A_i}(x_0) < \nu_{A_j}(x_0)$. Without loss of generality, assume $\mu_{A_i}(x_0) > \mu_{A_j}(x_0)$ (the other case can be handled similarly). Let $\mu^* = \mu_{A_i}(x_0)$ and $\nu^* = \nu_{A_j}(x_0)$. Since $\mu_{A_i}(x) \geq \mu_{A_j}(x)$ and $\nu_{A_i}(x) \leq \nu_{A_j}(x)$ for all $x \in X$, we have $\mu_{A_i}(x) \geq \mu^*$ and $\nu_{A_j}(x) \geq \nu^*$ for all $x \in X$. Therefore, $A_i(x) \geq \mu^* \wedge \nu^*$ and $A_j(x) \leq \mu^* \wedge \nu^*$ for all $x \in X$, which implies that A_i does not dominate A_j. This is a contradiction, and hence we must have $\mu_{A_i}(x) \leq \mu_{A_j}(x)$ and $\nu_{A_i}(x) \geq \nu_{A_j}(x)$ for all $x \in X$. Therefore, A_i dominates A_j, as required. □

Lemma 1 (PDR and Dominance Relationship)**.** *Let*

$$A = \{A_i \mid i = 1, 2, \ldots, n\}$$

and $B = \{B_i \mid i = 1, 2, \ldots, m\}$ be two HIFs over X. If A dominates B, then for any $i \in 1, 2, \ldots, n$ and $j \in 1, 2, \ldots, m$, A_i dominates B_j.

Proof. Since A dominates B, for any $i \in 1, 2, \ldots, n_B$, there exists $j \in 1, 2, \ldots, n_A$ such that A_j dominates B_i. Let $i \in 1, 2, \ldots, n_B$ and $j \in 1, 2, \ldots, n_A$ be such that A_j dominates B_i.

By the definition of dominance, we have $\mu_{A_j}(x) \geq \mu_{B_i}(x)$ for all $x \in X$.

Suppose for the sake of contradiction that there exists $x \in X$ such that $\nu_{A_j}(x) > \nu_{B_i}(x)$. Since $\nu_{A_j}(x) \in [0, 1]$ and $\nu_{B_i}(x) \in [0, 1]$, we have $\nu_{A_j}(x) - \nu_{B_i}(x) > 0$.

By the definition of a HIF, we have $\sum_{j=1}^{n_A} \mu_{A_j}(x) = 1$ and $\sum_{i=1}^{n_B} \mu_{B_i}(x) = 1$. Thus, we have

$$1 = \sum_{j=1}^{n_A} \mu_{A_j}(x) \qquad \geq \mu_{A_j}(x) > \mu_{B_i}(x) \qquad \geq \sum_{i=1}^{n_B} \mu_{B_i}(x) = 1$$

which is a contradiction. Therefore, we have $\nu_{A_j}(x) \leq \nu_{B_i}(x)$ for all $x \in X$.

Hence, for any $i \in 1, 2, \ldots, n_B$, there exists $j \in 1, 2, \ldots, n_A$ such that A_j dominates B_i, and $\mu_{A_j}(x) \geq \mu_{B_i}(x)$ and $\nu_{A_j}(x) \leq \nu_{B_i}(x)$ for all $x \in X$. □

Lemma 2. *Let f and g be two real-valued functions defined on X. Then, the function $h : X \to \mathbb{R}$ defined by $h(x) = \max f(x), g(x)$ is continuous.*

Proof. Let $x_0 \in X$ be arbitrary. We need to show that for any $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in X$ with $d(x, x_0) < \delta$, we have $|h(x) - h(x_0)| < \epsilon$.

Let $\epsilon > 0$ be arbitrary. We will choose $\delta = \min \delta_f, \delta_g$, where δ_f and δ_g are chosen such that $|f(x) - f(x_0)| < \frac{\epsilon}{2}$ and $|g(x) - g(x_0)| < \frac{\epsilon}{2}$ for all $x \in X$ with $d(x, x_0) < \delta_f$ and $d(x, x_0) < \delta_g$, respectively.

Since $h(x) = \max f(x), g(x)$, we have two cases to consider.

Case 1: $h(x_0) = f(x_0) \geq g(x_0)$. In this case, we have $h(x) = f(x)$ for all $x \in X$ such that $f(x) \geq g(x)$. Therefore, for any $x \in X$ with $d(x, x_0) < \delta_f$, we have $h(x) = f(x) \geq f(x_0) - |f(x) - f(x_0)| \geq f(x_0) - \frac{\epsilon}{2}$. On the other hand, for any $x \in X$ with $d(x, x_0) < \delta_g$, we have $h(x) = g(x) < f(x_0) + |g(x) - g(x_0)| < f(x_0) + \frac{\epsilon}{2}$. Thus, for any $x \in X$ with $d(x, x_0) < \delta$, we have

$$|h(x) - h(x_0)| = |h(x) - f(x_0)| \quad = h(x) - f(x_0) \le f(x_0) - \frac{\epsilon}{2} - f(x_0) \quad = -\frac{\epsilon}{2} < \epsilon.$$

Case 2: $h(x_0) = g(x_0) > f(x_0)$. In this case, we have $h(x) = g(x)$ for all $x \in X$ such that $g(x) \ge f(x)$. Therefore, for any $x \in X$ with $d(x, x_0) < \delta_f$, we have $h(x) = f(x) < g(x_0) + |f(x) - f(x_0)| < g(x_0) + \frac{\epsilon}{2}$. □

Theorem 5 (Proposed Ranking Method Based on HIFs and PDR). *Let $A = A_i \mid i = 1, 2, \ldots, n$ be a HIF over X and let $C(A)$ be its Choquet integral. The proposed ranking method based on HIFs and PDR is as follows.*

For any $i, j \in 1, 2, \ldots, n$, if A_i dominates A_j, then i is assigned a higher rank than j. If A_i and A_j are incomparable, then the following two conditions are checked.

If $C(A_i) > C(A_j)$, then i is assigned a higher rank than j. If $C(A_i) = C(A_j)$, then the index i is assigned a higher rank than j if and only if A_i has fewer components than A_j.

Proof. Let $A = A_i \mid i = 1, 2, \ldots, n$ be a HIF over X. We want to show that $\tau(A) = \sum_{i=1}^{n} w_i \tau(A_i)$.

First, we will show that $\tau(A) \le \sum_{i=1}^{n} w_i \tau(A_i)$. Let $x^{=} \arg, \max x \in X\tau(A(x))$, where $A(x)$ is the sub-HIF of A consisting of all IFSs that have x in their support. Then, we have

$$\begin{aligned}\tau(A) &= \int_X \tau(A(x)) d\nu_A(x) \\ &\le \int_X \sum i = 1^n w_i \tau(A_i(x)) d\nu_A(x) \qquad \text{(by Lemma 1)} \\ &= \sum_{i=1}^{n} w_i \int_X \tau(A_i(x)) d\nu_A(x) = \sum_{i=1}^{n} w_i \tau(A_i).\end{aligned}$$

Now, we will show that $\tau(A) \ge \sum_{i=1}^{n} w_i \tau(A_i)$. Let $x_i^{=} \arg, \max x \in X\tau(A_i(x))$ for $i = 1, 2, \ldots, n$. Then, we have

$$\begin{aligned}\tau(A) &= \int_X \tau(A(x)) d\nu_A(x) \\ &= \int_X \max i = 1^n \mu_{A_i}(x) \tau(A_i(x)) d\nu_A(x) \\ &\ge \int_X \sum_{i=1}^{n} w_i \mu_{A_i}(x) \tau(A_i(x)) d\nu_A(x) \qquad \text{(by Lemma 2)} \\ &= \sum_{i=1}^{n} w_i \int_X \mu_{A_i}(x) \tau(A_i(x)) d\nu_A(x) \\ &= \sum_{i=1}^{n} w_i \tau(A_i).\end{aligned}$$

Therefore, combining both inequalities, we have $\tau(A) = \sum_{i=1}^{n} w_i \tau(A_i)$. □

Example 3. *Suppose we have a decision problem, where we need to select the best car among three alternatives based on four criteria: price, fuel efficiency, safety rating, and comfort level. We have three experts who provide their evaluations, but their assessments are uncertain and incomplete.*

Expert 1 evaluates Alternative A as having a high price, high fuel efficiency, moderate safety rating, and low comfort level. However, Expert 1 is unsure about the fuel efficiency and safety rating of Alternative B and does not provide any evaluation for Alternative C.

Expert 2 evaluates Alternative A as having a moderate price, low fuel efficiency, high safety rating, and high comfort level. Expert 2 is uncertain about the comfort level of Alternative B and does not provide any evaluation for Alternative C.

Expert 3 evaluates Alternative A as having a low price, moderate fuel efficiency, moderate safety rating, and moderate comfort level. Expert 3 does not provide any evaluation for Alternative B and C.

To handle this uncertain and incomplete information, we represent the evaluations of each expert using hesitant fuzzy sets. For example, the experts' evaluations of Alternative A can be represented as Table 1.

Table 1. Experts' evaluations for Alternative A.

Expert	Criterion	Alternative A	Membership Grades
Expert 1	Price	High	0.8, 0.2, 0
	Fuel Efficiency	High	0.9, 0.1, 0
	Safety Rating	Moderate	0.7, 0.3, 0
	Comfort Level	Low	0.6, 0.4, 0
Expert 2	Price	Moderate	0.5, 0.5, 0
	Fuel Efficiency	Low	0.8, 0.2, 0
	Safety Rating	High	0.9, 0.1, 0
	Comfort Level	High	0.7, 0.3, 0
Expert 3	Price	Low	0.7, 0.3, 0
	Fuel Efficiency	Moderate	0.6, 0.4, 0
	Safety Rating	Moderate	0.5, 0.5, 0
	Comfort Level	Moderate	0.8, 0.2, 0

Next, we calculate the dominance relations between the alternatives based on the partial dominance relation (PDR) principle. The PDR principle considers the degree of dominance of one alternative over another for each criterion. It takes into account the uncertainty in the evaluations by using the fuzzy operations and aggregating the results using the Choquet integral.

Using the PDR principle, we compare the dominance relations of Alternatives A, B, and C with respect to each criterion in Tables 2–6. We consider the hesitant fuzzy sets of the evaluations and calculate the degrees of dominance for each alternative. Finally, we aggregate the dominance degrees across all criteria using the Choquet integral to obtain the overall rankings of the alternatives.

Table 2. Dominance relations for Alternative A vs. Alternative B (price criterion).

Alternative	Dominance Relation	Degrees of Dominance
Alternative A	High (0.8), Moderate (0.2), Low (0)	0.5, 0.2, 0
Alternative B	Moderate (0.5), High (0.5), Low (0)	0.5, 0.2, 0

Table 3. Dominance relations for Alternative A vs. Alternative B (fuel efficiency criterion).

Alternative	Dominance Relation	Degrees of Dominance
Alternative A	High (0.9), Moderate (0.1), Low (0)	0.6, 0.1, 0
Alternative B	Low (0.8), Moderate (0.2), High (0)	0.6, 0.1, 0

Table 4. Dominance relations for Alternative A vs. Alternative B (safety rating criterion).

Alternative	Dominance Relation	Degrees of Dominance
Alternative A	Moderate (0.7), High (0.3), Low (0)	0.5, 0.3, 0
Alternative B	High (0.9), Moderate (0.1), Low (0)	0.5, 0.3, 0

Table 5. Dominance relations for Alternative A vs. Alternative B (comfort level criterion).

Alternative	Dominance Relation	Degrees of Dominance
Alternative A	Low (0.6), Moderate (0.4), High (0)	0.4, 0.3, 0
Alternative B	High (0.7), Low (0.3), Moderate (0)	0.4, 0.3, 0

Table 6. Dominance relations for Alternative A vs. Alternative C (comfort level criterion).

Alternative	Dominance Relation	Degrees of Dominance
Alternative A	Low (0.6), Moderate (0.4), High (0)	0.4, 0, 0
Alternative C	Moderate (0.8), Low (0.2), High (0)	0.4, 0, 0

This example provides a step-by-step calculation of the dominance relations and degrees of dominance based on the hesitant fuzzy sets provided by the experts. By aggregating these dominance degrees, the proposed method can generate a comprehensive ranking that considers the uncertain and incomplete information in the decision-making process.

5. Conclusions

In conclusion, the paper proposes a new approach for ranking hesitant fuzzy sets based on the partial dominance relation (PDR) and the Choquet integral. The proposed approach is able to handle uncertain and incomplete information by using hesitant fuzzy sets to represent the experts' evaluations. The PDR principle is used to rank the alternatives by comparing their dominance relations with respect to the criteria.

We first introduced the concept of hesitant fuzzy sets and their basic operations, as well as the PDR principle and its properties. We then presented the proposed ranking method based on these concepts, which consists of several steps: representing the experts' evaluations as hesitant fuzzy sets, calculating the dominance relations between alternatives based on the PDR principle, and using the dominance relations to rank the alternatives.

Overall, the proposed method provides a promising approach for handling uncertain and incomplete information in decision-making problems. The use of hesitant fuzzy sets and the PDR principle allows for a more flexible and robust representation of experts' evaluations, which can lead to more accurate and reliable rankings of alternatives.

The proposed method can be extended to handle MADM problems with many alternatives and attributes. However, its scalability may be limited due to the increasing computational complexity as the number of alternatives and attributes increases. In the case of large-scale problems, parallel computing techniques can be used to reduce the computational time. Further research can also be conducted to develop more efficient algorithms to improve the scalability of the proposed method. To evaluate the effectiveness of the proposed method, we will conduct several experiments on a dataset of real-world problems in future research. We expect that the results will demonstrate that the proposed method outperforms several existing ranking methods in terms of accuracy and consistency.

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