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Monomiality and a New Family of Hermite Polynomials

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Abstract: The monomiality principle is based on an abstract definition of the concept of derivative and multiplicative operators. This allows to treat different families of special polynomials as ordinary monomials. The procedure underlines a generalization of the Heisenberg–Weyl group, along with the relevant technicalities and symmetry properties. In this article, we go deeply into the formulation and meaning of the monomiality principle and employ it to study the properties of a set of polynomials, which, asymptotically, reduce to the ordinary two-variable Kampè dè Fèrièt family. We derive the relevant differential equations and discuss the associated orthogonality properties, along with the relevant generalized forms.

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1. Introduction

Significant efforts have been made in the past to provide a more efficient view to the theoretical foundations of special functions and polynomials. Most of the proposed methods trace back to the Lie algebraic technicalities, summarized in the Wigner Princeton lectures (see Refs. [1–3]). In this article we describe more recent developments, which yield interesting results worth being underscored.

The Heisenberg–Weyl group (HWG) has played a central role in the development of quantum mechanics (QM) [4–6], which represents a fairly direct way of embedding position and momentum operators to disclose invariance properties associated with the specific problems under study. At a more fundamental level, the HWG provides quite a natural environment to place questions related to the connection between QM and Fourier analysis [7–9]. The wealth of their properties allows the understanding of the transition from a classical to quantum mechanics phase space evolution [10], along with the relevant formulation in terms of Liouville or Von Neumann Equations [11]. Furthermore a non-secondary role is played in the study of special functions, and indeed the Hermite polynomials, and the associated orthogonal functions as well are directly associated with the HWG generators and the relevant exponentiation [12–16]. For a deeper insight on the previous points, see the recent papers reported in [17,18], where the interplay between HWGs and generalized Hermite functions has been studied in depth.

The differential realization of the HWG may occur through the use of operators not straightforwardly recognized as the ordinary position and momentum. The search for generalized forms of HWG generators has offered the possibility of exploiting new tools to construct families of orthogonal polynomials and study their properties to solve integro-differential equations, useful for the study of problems in optics, electromagnetism, and astrophysics [19].

The key element to accomplish this research program has been the formulation of the monomiality principle [19–21] and a revisitation of the umbral calculus [22] originally developed in [23–26]. Monomiality and umbrality are, within certain limits, complementary.

The former can be viewed as an abstract theory of the Heisenberg commutation bracket through non-trivial realizations of the derivative and position operators.

This point of view has been the main motivation in Ref. [19] and of the study of the Appèll/Sheffer polynomials [27] as images of ordinary monomials [28,29]. Monomiality is a modern formulation of a point of view, not only tracing back to Steffensen [30–32] but even to older studies by Jeffery (for a recent account see Ref. [33]), Boole [34], and other speculations developed almost two hundred years ago. These studies deepened their roots into the calculus of differences [35], and were the first to be recognized as amenable for a symbolic interpretation. The rules underlying monomiality are fairly simple and can be formulated as reported below [28,29,36,37].

The purpose of this article is to go through the theory of quasi-monomiality and exploit the associated formalism to construct derivative and multiplicative operators, which are used to define a new family of orthogonal Hermite-like operators.

Properties 1. $\forall x \in \mathbb{R}, \forall n \in \mathbb{N}$, *if a couple of operators* $\hat{P}, \hat{M}$ *are such that:*

(a) *They do exist along with a differential realization* [38,39];

(b) *They can be embedded to form Weyl algebra* [15,38,40], *namely, if the commutator is such that* $[\hat{P}, \hat{M}] = \hat{1}$;

(c) *It is possible to univocally define a polynomial set such that:*

$$(i)\ p_0(x) = 1, \qquad (ii)\ \hat{P}p_0(x) = 0, \qquad (iii)\ p_n(x) = \hat{M}^n 1, \tag{1}$$

then it follows that

(d)

$$\hat{M}p_n(x) = \hat{M}^{n+1}1 = p_{n+1}(x), \tag{2}$$

(e)

$$\hat{P}p_n(x) = \hat{P}\hat{M}^n 1 = np_{n-1}(x) \tag{3}$$

and the polynomials $p_n(x)$ *are said quasi-monomials.*

Proof. Equation (3) needs few lines of comment. We rearrange the operator product $\hat{P}\hat{M}^n$ as (we remind that $[\hat{P}, \hat{M}] = 1 \Rightarrow \hat{P}\hat{M} - \hat{M}\hat{P} = 1$) (see [41])

$$\begin{aligned}\hat{P}\hat{M}\hat{M}^{n-1} &= (\hat{M}\hat{P} + 1)\hat{M}^{n-1} = \hat{M}\hat{P}\hat{M}^{n-1} + \hat{M}^{n-1} \\ &= \hat{M}^2\hat{P}\hat{M}^{n-2} + 2\hat{M}^{n-1} = \cdots = \hat{M}^n\hat{P} + n\hat{M}^{n-1},\end{aligned} \tag{4}$$

which eventually yields

$$\hat{P}\hat{M}^n 1 = \left(\hat{M}^n\hat{P} + n\hat{M}^{n-1}\right)1 = \hat{M}^n\hat{P}\,1 + n\hat{M}^{n-1}1. \tag{5}$$

Being $\hat{M}^n\hat{P}\,1 = 0$ as a consequence of the (ii) of Equation (1), and using property (iii) too, we state the correctness of Equation (3). □

Remark 1. *The important point we like to convey is that the essence of the discussion on monomiality is the existence of the operators* $\hat{M}$ *(multiplicative), which univocally define the set of polynomials* $p_n(x)$ *(not vice versa), and* $\hat{P}$ *acting on the polynomials, as an ordinary derivative.*

According to the above statement, polynomial sets such as Appéll, Sheffer [42], and Boas Buck [43] can be ascribed to the monomial family, while others, e.g., Legendre, Chebyshev, and Jacobi [39,44], are not yet framed within such a context.

After these remarks, aimed at clarifying the environment in which we are going to develop our speculations, we remind that the $\hat{M}$ and $\hat{P}$ operators that define the Appèll family are specified by

$$\hat{M} = x + \frac{A'(\sigma)}{A(\sigma)} \,|_{\sigma=\partial_x}, \qquad \hat{P} = \partial_x \,, \tag{6}$$

where $A(\sigma)$ is assumed to be an analytic function.

According to our introductory remarks, the explicit form of the Appéll polynomials is obtained from the identity (property *iii*) of Equation (1))

$$a_n(x) = \left(x + \frac{A'(\partial_x)}{A(\partial_x)} \right)^n 1. \tag{7}$$

The use of standard operational rules allows us to cast Equation (7) in a more convenient form.

Corollary 1. *We note that* [45,46]

$$a_n(x) = \left(x + \frac{A'(\partial_x)}{A(\partial_x)} \right) \left(x + \frac{A'(\partial_x)}{A(\partial_x)} \right)^{n-1} 1 \tag{8}$$

and noting that

$$x + \frac{A'(\partial_x)}{A(\partial_x)} = A(\partial_x) x (A(\partial_x))^{-1}, \tag{9}$$

we can write, by iteration

$$a_n(x) = \left(A(\partial_x) x (A(\partial_x))^{-1} \right)^n = A(\partial_x) x^n \tag{10}$$

According to Equation (10)*, the generating function of Appèl polynomials reads*

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} a_n(x) = A(\partial_x) e^{tx} = A(t) e^{tx}. \tag{11}$$

It is evident that it consists of two contributions: the exponential term and $A(t)$, which will be defined as the "amplitude".

Corollary 2. *In the case of the two variable Hermite polynomials (HP), we have that the amplitude is specified by*

$$A(t) = e^{yt^2} \tag{12}$$

with the multiplicative operator being explicitly specified as

$$\hat{M} = x + 2y\partial_x \,. \tag{13}$$

The associated polynomial family is, accordingly, provided by [22]

$$H_n(x, y) = (x + 2y\partial_x)^n 1. \tag{14}$$

The use of the Crofton identity [47]

$$e^{y\partial_x^m} f(x) = f\left(x + m\, y\, \partial_x^{m-1} \right) e^{y\partial_x^m} \tag{15}$$

(or of identities (10) *and* (11) *as well) allows to cast Equation* (14) *in the form*

$$H_n(x, y) = e^{y\partial_x^2} x^n. \tag{16}$$

The expansion of the exponential operator in Equation (16), along with the relevant action on the monomial x^n, yields the explicit form of the two variable Hermite Kampè dè Fèrièt polynomials [27,38], namely,

$$H_n(x,y) = n! \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{x^{n-2r} y^r}{(n-2r)!r!}. \tag{17}$$

The operational identity in Equation (16) is particularly pregnant from the mathematical point of view. It states that the two variable Hermites (17) are solutions of the heat equation and can be used as a pivotal tool to prove the orthogonal properties of this polynomial family [22,38].

In this article we consider the polynomial family generated by

$$A(p) = \left(1 + \frac{y}{N} p^2\right)^N, \qquad \forall N \in \mathbb{N}, \tag{18}$$

study the relevant properties, and look at the possibility of defining an associated orthogonal set.

2. Quasi-Hermite and Appéll Sequences

In this section, we exploit the general properties of the Appéll polynomials, discussed in the introductory remarks, to state the properties of the associated polynomials.

Definition 1. *Appéll polynomials with amplitude (18) are explicitly defined through the identity*

$$H_n(x,y;N) = \left(1 + \frac{y}{N} \partial_x^2\right)^N x^n \tag{19}$$

and they will be called quasi-Hermite polynomials (QHP) .

(Remark: According to the discussion of the previous section, it should be noted that)

Properties 2. *The relevant recurrences of QHP are obtained after noting that, for this specific case, we obtain*

$$\frac{A'(\partial_x)}{A(\partial_x)} = \frac{2\, y\, \partial_x}{\left(1 + \frac{y}{N}\partial_x^2\right)}, \tag{20}$$

so

$$\begin{aligned}
&(1)\ \partial_x H_n(x,y;N) = n\left(1 + \frac{y}{N}\partial_x^2\right)^N x^{n-1} = nH_{n-1}(x,y;N), \\
&(2)\ H_{n+1}(x,y;N) = \left(x + \frac{2\, y\, \partial_x}{1 + \frac{y}{N}\partial_x^2}\right) H_n(x,y;N), \\
&(3)\ H_{n+1}(x,y;N) - xH_n(x,y;N) - 2nyH_{n-1}(x,y;N) = \frac{y}{N} n(n-1)(xH_{n-2}(x,y;N) - H_{n-1}(x,y;N)).
\end{aligned} \tag{21}$$

Proof. Properties (1) and (2) are obtained from the realization of the derivative and multiplicative operators given in Equation (6), and the third one is the result of some algebraic steps (we simplify the writing for brevity by omitting the Hermite arguments):

(i) From property (2), we write

$$\left(1 + \frac{y}{N}\partial_x^2\right) H_{n+1} = \left(1 + \frac{y}{N}\partial_x^2\right) xH_n + 2y\partial_x H_n$$

which provides, from property (1),

(ii)

$$H_{n+1} + \frac{y}{N} n(n+1) H_{n-1} = xH_n + \frac{y}{N}(2nH_{n-1} + n(n-1)xH_{n-2}) + 2nyH_{n-1}$$

and finally

(iii)

$$
\begin{aligned}
H_{n+1} - xH_n - 2nyH_{n-1} &= \frac{y}{N}n((2-(n+1))H_{n-1} + (n-1)xH_{n-2}) \\
&= \frac{y}{N}n(n-1)(xH_{n-2} - H_{n-1}).
\end{aligned}
$$

□

Proposition 1. *The explicit form of the QHP is inferred from Equation (19), which yields*

$$
(a)\ H_n(x,y;N) = \sum_{r=0}^{\min\left[N,\,\lfloor\frac{n}{2}\rfloor\right]} \binom{N}{r}\left(\frac{y}{N}\right)^r \frac{n!}{(n-2r)!}x^{n-2r}, \quad \forall x,y \in \mathbb{R}, \forall n,N \in \mathbb{N} \tag{22}
$$

and the relevant differential equation is

$$
(b)\ \left(x + \frac{2\,y\,\partial_x}{1+\frac{y}{N}\partial_x^2}\right)\partial_x H_n(x,y;N) = nH_n(x,y;N). \tag{23}
$$

Proof. (a) $\forall x,y \in \mathbb{R}, \forall n,N \in \mathbb{N}$, we use binomial Newton to write

$$
H_n(x,y;N) = \sum_{r=0}^{N}\binom{N}{r}\left(\frac{y}{N}\right)^r \partial_x^{2r}x^n = \sum_{r=0}^{\min\left[N,\,\lfloor\frac{n}{2}\rfloor\right]}\binom{N}{r}\left(\frac{y}{N}\right)^r \frac{n!}{(n-2r)!}x^{n-2r}.
$$

(b) The relevant differential equation is easily obtained by applying Equation (21) in Properties 2. □

Corollary 3. *After a few algebraic manipulations, Equation (23) can be reduced to the following third-order ODE*

$$
\frac{y}{N}xz''' + y\left(2 - \frac{n-2}{N}\right)z'' + xz' = nz, \qquad z = H_n(x,y;N) \tag{24}
$$

which evidently tends to the ordinary (two variables) Hermite equation, producing large N values.

Proof. By starting from Equation (23), we proceed as outlined below

$$
\begin{aligned}
&\left(\left(1+\frac{y}{N}\partial_x^2\right)x + 2y\partial_x\right)\partial_x z = \left(1+\frac{y}{N}\partial_x^2\right)nz \\
\to\ & x\partial_x z + \frac{y}{N}\partial_x^2 x\partial_x z + 2y\partial_x^2 z = nz + \frac{y}{N}\partial_x^2 nz \\
\to\ & xz' + \frac{y}{N}\partial_x(z' + xz'') + 2yz'' - \frac{y}{N}nz'' = nz \\
\to\ & \frac{y}{N}xz''' + y\left(2 - \frac{n-2}{N}\right)z'' + xz' = nz
\end{aligned}
$$

□

Corollary 4. *The PDE satisfied by the QHP (expected to be an extension of the heat equation) is obtained by keeping the partial derivative with respect to y of both sides of Equation (19), namely,*

$$
\partial_y H_n(x,y;N) = \partial_x^2\left(1+\frac{y}{N}\partial_x^2\right)^{N-1}x^n. \tag{25}
$$

Example 1. *Equation (25) can eventually be written as*

$$\begin{cases} \partial_y H_n(x,y;N) = \dfrac{\partial_x^2}{1+\frac{y}{N}\partial_x^2} H_n(x,y;N) \\ H_n(x,0;N) = x^n \end{cases} \tag{26}$$

The relevant (formal) solution can be obtained as

$$H_n(x,y;N) = \hat{U}_{y,N}\, x^n, \qquad \hat{U}_{y,N} = \exp\left\{ \int_0^y \frac{\partial_x^2}{1+\frac{\xi}{N}\partial_x^2}\, d\xi \right\}, \tag{27}$$

where $\hat{U}$ is a kind of evolution operator. To be eventually written as in Equation (19), after explicitly working out the integral in the exponent of Equation (27), we find

$$\hat{U}_{y,N} = \exp\left\{ N \log\left(1+\frac{y}{N}\partial_x^2\right)\right\} = \left(1+\frac{y}{N}\partial_x^2\right)^N. \tag{28}$$

According to the previous definition, the QHP satisfies the composition rule

$$\hat{U}_{y,N}\hat{U}_{z,N} = \left(1+\frac{y+z}{N}\partial_x^2 + \frac{yz}{N^2}\partial_x^4\right)^N. \tag{29}$$

Therefore, unlike the two variable HP specified by an amplitude that is an exponential , the composition property $\hat{U}_{y,N}\hat{U}_{z,N} = \hat{U}_{y+z,N}$ does not hold; therefore,

$$\hat{U}_{y,N}\hat{U}_{z,N} \neq \hat{U}_{y+z;N}. \tag{30}$$

An important (albeit naive) consequence of Equation (29) is the following composition rule

$$\hat{U}_{-y,N}\hat{U}_{y,N}x^n = \left(1-\frac{y^2}{N^2}\partial_x^4\right)^N x^n, \tag{31}$$

which suggests the necessity of a suitable extension of QHP, possibly involving higher-order forms, as discussed in the forthcoming section.

Observation 1. *The non exponential nature of the QHP amplitude determines the further worth to be noted as a consequence*

$$\hat{U}_{-y,N} \neq \hat{U}_{y,N}^{-1} = \frac{1}{\Gamma(N)} \int_0^\infty s^{N-1} e^{-s\left(1+\frac{y}{N}\partial_x^2\right)} ds, \tag{32}$$

where the r.h.s. has been obtained after exploiting standard Laplace transform methods.

We will see in the following that Equation (32) is of pivotal importance for the definition of the orthogonal properties of the *QHP*.

Observation 2. *Before closing this section, we notice that Equation (23) can be generalized $\forall m \in \mathbb{N}$ such that*

$$\left(x + \frac{m\, y\, \partial_x^{m-1}}{1+\frac{y}{N}\partial_x^m}\right)\partial_x H_n^{(m)}(x,y;N) = n H_n^{(m)}(x,y;N). \tag{33}$$

and by following the same procedure provided in Corollary 3, it is possible to deduce the relative differential equation.

3. Multivariable QHP

Higher-order Hermite polynomials (also called *Lacunary HP*) are defined through the operational rule [38,47]

$$H_n^{(m)}(x,y) = e^{y\partial_x^m} x^n, \qquad \forall m \in \mathbb{N} \tag{34}$$

and, in analogy, the higher-order QHPs are specified by$H_n(x,y;N)$ and the notation $H_n^{(2)}(x,y;N)$; however, we drop the superscript for $m = 2$ and add it whenever ambiguities arise.)

$$H_n^{(m)}(x,y;N) = \hat{U}_{y,N}^{(m)}\, x^n, \qquad \hat{U}_{y,N}^{(m)} = \left(1 + \frac{y}{N}\partial_x^m\right)^N. \tag{35}$$

Example 2. *According to Equation (31), we find*

$$\hat{U}_{-y,N}\hat{U}_{y,N}\, x^n = \hat{U}_{-y^2,N}^{(4)}\, x^n = H_n^{(4)}\left(x, -y^2; N\right) \tag{36}$$

and, more in general,

$$\hat{U}_{-y,N}^{(m)}\hat{U}_{y,N}^{(m)} x^n = \hat{U}_{-y^2,N}^{(2m)}\, x^n = H_n^{(2m)}\left(x, -y^2; N\right) \tag{37}$$

Example 3. *Before going further, we consider the definition of the QHP of order one, which will be referred to as quasi-binomial polynomials (QBP), namely,*

$$H_n^{(1)}(x,y;N) = \left(1 + \frac{y}{N}\partial_x\right)^N x^n. \tag{38}$$

For large N, they reduce to $(x+y)^n$, hence the name. The explicit form of this family of polynomials can be written as

$$H_n^{(1)}(x,y;N) = \sum_{r=0}^{N} \binom{N}{r}\left(\frac{y}{N}\right)^r \partial_x^r x^n = \sum_{r=0}^{\min[N,\,n]} \binom{N}{r}\left(\frac{y}{N}\right)^r \frac{n!}{(n-r)!} x^{n-r}. \tag{39}$$

The same strategy adopted in Corollary 3, by exploiting Equation (33), yields for the QBP, the ODE

$$\frac{y}{N} x z'' + \left[(x+y) - (n-1)\frac{y}{N}\right] z' = nz, \qquad z = H_n^{(1)}(x,y;N) \tag{40}$$

and the PDE

$$\begin{cases} \partial_y F(x,y) = \dfrac{\partial_x}{1 + \frac{y}{N}\partial_x} F(x,y) \\ F(x,0) = x^n \end{cases}. \tag{41}$$

The last identity can also be cast in the integro-differential form

$$\partial_y F(x,y) = \partial_x \int_0^\infty e^{-s} F\left(x - \frac{y}{N}s,\ y\right) ds \tag{42}$$

where the Laplace transform provides the integral representation

$$\frac{1}{1 + \frac{y}{N}\partial_x} = \int_0^\infty e^{-s\left(1 + \frac{y}{N}\partial_x\right)} ds \tag{43}$$

which when inserted into Equation (41) yields

$$\partial_y F(x,y) = \partial_x \int_0^\infty e^{-s} e^{-\frac{ys}{N}\partial_x} ds F(x,y) \tag{44}$$

and after exploiting the shift operator identity $e^{a\partial_x} f(x) = f(x+a)$ [38], we obtain Equation (42).

Example 4. *We can combine the various definitions given before to introduce three variables QHP as*

$$\begin{aligned} H_n^{(1,2)}(x, y_1, y_2; N) &:= \left(\hat{U}^{(1)}_{y_1,N}\, \hat{U}^{(2)}_{y_2,N}\right) x^n = \left(1 + \frac{y_1}{N}\partial_x\right)^N \left(1 + \frac{y_2}{N}\partial_x^2\right)^N x^n \\ &= \left(1 + \frac{y_1}{N}\partial_x\right)^N H_n(x, y_2; N) \\ &= n! \sum_{r=0}^{\min[N,n]} \binom{N}{r} \left(\frac{y_1}{N}\right)^r \frac{H_{n-r}(x, y_2; N)}{(n-r)!}. \end{aligned} \tag{45}$$

Further generalizations can easily be obtained. For example, the m-th variable extension reads

$$H_n^{(1,\ldots,m)}(x, y_1, \ldots, y_m; N) = \left(\prod_{s=1}^{n} \hat{U}^{(s)}_{y_s,N}\right) x^n. \tag{46}$$

The examples we have just touched on in this section yield an idea of the possible extensions of this family of polynomials, which will be more carefully discussed in forthcoming research.

4. Final Comments

We have already mentioned the possible orthogonal nature of the QHP, and in this section we address the problem by the use of the techniques developed in Refs. [48,49].

Proposition 2. *We assume that a given function $f(x)$ can be expanded in terms of QHP, according to the identity*

$$f(x) = \sum_{n=0}^{\infty} a_n H_n(x, - \mid y \mid; N) \tag{47}$$

which can be inverted, thus yielding

$$\frac{1}{\left(1 - \frac{\mid y \mid}{N}\partial_x^2\right)^N} f(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{48}$$

(Note: The reasons of "$- \mid y \mid$" will be clarified below.)
The use of Equation (32) allows to cast the l.h.s. of Equation (48) in the form

$$\frac{1}{\Gamma(N)} \int_0^{\infty} s^{N-1} e^{-s\left(1 - \frac{|y|}{N}\partial_x^2\right)} ds\, f(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{49}$$

Corollary 5. *Equation (49) can be further elaborated:*

1. *We apply the Gauss–Weierstrass transform [22] to write*

$$\begin{aligned} e^{s\frac{|y|}{N}\partial_x^2} f(x) &= \frac{1}{2\sqrt{\pi\, s\frac{|y|}{N}}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x-\xi)^2}{4\, s\frac{|y|}{N}}\right\} f(\xi) d\xi \\ &= \frac{1}{2\sqrt{\pi\, s\frac{|y|}{N}}} \int_{-\infty}^{\infty} e^{-\frac{1}{4\, s\frac{|y|}{N}}\xi^2}\, e^{\frac{x}{2s\frac{|y|}{N}}\xi}\, e^{-\frac{x^2}{4s\frac{|y|}{N}}} d\xi. \end{aligned} \tag{50}$$

It holds for $s\frac{|y|}{N} \geq 0$ (hence the choice of the sign in the expansion (47)).

2. *We use the two variable Hermite-generating functions (we have* $\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x,y) = e^{xt+yt^2}$*)* [38] *to write*

$$e^{s\frac{|y|}{N}\partial_x^2} f(x) = \frac{1}{2\sqrt{\pi\, s\frac{|y|}{N}}} \sum_{n=0}^{\infty} \frac{x^n}{n!} \int_{-\infty}^{\infty} H_n\left(\frac{\xi}{2s\frac{|y|}{N}}, -\frac{1}{4s\frac{|y|}{N}}\right) e^{-\frac{\xi^2}{4\,s\frac{|y|}{N}}} f(\xi) d\xi. \tag{51}$$

3. *We insert the result of Equation* (51) *into Equation* (49) *and compare the similar x powers, thus eventually finding*

$$\begin{aligned} a_n &= \frac{1}{\Gamma(N) n!\, 2\sqrt{\pi \frac{|\,y\,|}{N}}} \int_0^{\infty} s^{N-\frac{3}{2}} e^{-s}\, {}_nG_{y,N}(s) ds, \\ {}_nG_{y,N}(s) &= \int_{-\infty}^{\infty} H_n\left(\frac{\xi}{2s\frac{|y|}{N}}, -\frac{1}{4s\frac{|y|}{N}}\right) e^{-\frac{\xi^2}{4\,s\frac{|y|}{N}}} f(\xi) d\xi. \end{aligned} \tag{52}$$

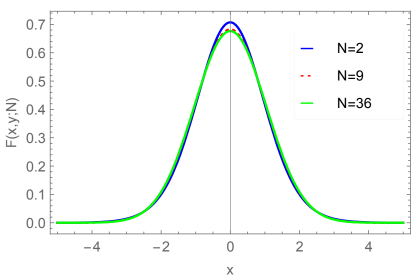
According to the above results, the expansion holds only if the integrals appearing in Equation (52) *are converging. In order to provide an example we consider the generalization of the Glaisher formula* [47]*. Namely, Equation* (48) *for* $f(x) = e^{-x^2}$ *becomes*

$$\begin{aligned} F(x,y;N) &= \frac{1}{\left(1 - \frac{|\,y\,|}{N}\partial_x^2\right)^N} f(x) = \frac{1}{\Gamma(N)} \int_0^{\infty} s^{N-1} e^{-s\left(1-\frac{|y|}{N}\partial_x^2\right)} e^{-x^2} ds \\ &= \frac{1}{\Gamma(N)} \int_0^{\infty} s^{N-1} e^{-s} \frac{e^{-\frac{x^2}{1+4\frac{|y|}{N}s}}}{\sqrt{1+4\frac{|\,y\,|}{N}s}} ds. \end{aligned} \tag{53}$$

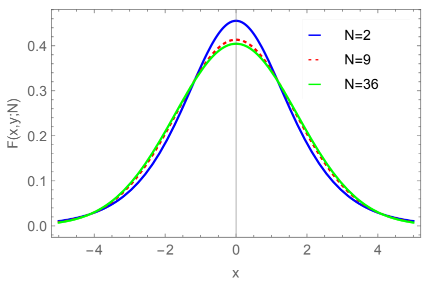
For very large N, $\frac{1}{\left(1-\frac{|y|}{N}\partial_x^2\right)^N} e^{-x^2}$ *reduces to the ordinary Glaisher identity*

$$\lim_{N\to\infty} F(x,y;N) = e^{|y|\partial_x^2} e^{-x^2} = \frac{1}{\sqrt{1+4y}} e^{-\frac{x^2}{1+4y}}. \tag{54}$$

In Figure 1, we have reported $F(x,y;N)$ *vs. x for different values of N and y.*



(a)



(b)

Figure 1. $F(x,y;N)$ vs. x for different values of N and y. (**a**) $y = 0.3$, (**b**) $y = 1.3$.

The definition of higher-order QHPs is not unique, and another possibility is offered by the relation

$$H_n^{(q,p)}(x,y,z;N) = \left(1 + \frac{y}{N}\partial_x^q + \frac{z}{N}\partial_x^p\right)^N x^n \tag{55}$$

where $q < p$ are relatively prime integers. The definition in Equation (55) allows to write the composition identity (29) as

$$\hat{U}_{y,N}\hat{U}_{z,N}x^n = H_n^{(2,4)}(x, y+z, yz; N)^N\ x^n. \tag{56}$$

In this article we have looked at the properties of polynomials using the monomiality principle and the classification in terms of Appèll/Sheffer polynomials [42,50–52]. In this respect, Laguerre polynomials cannot be considered members of this family, and notwithstanding that they can be treated using the formalism of quasi-monomials, the associated derivative and multiplicative operators are realized in terms of differential and integral operators [29], namely,

$$\hat{P} = \frac{d}{dx}x\frac{d}{dx}, \qquad \hat{M} = \left(\frac{d}{dx}\right)^{-1} \tag{57}$$

where $\hat{M}$ is an integral operator such that

$$\hat{M} \circ f(x) = \int_0^x f(\xi)d\xi, \qquad \hat{M}^n \circ f(x) = \frac{1}{(n-1)!}\int_0^x f(\xi)(x-\xi)^{n-1}d\xi. \tag{58}$$

As easily checked, the operators (57) and (58) realize the generators of the HWG, and what is remarkable is that it allows the generalization of new families of Hermite polynomials and of Hermite functions defined no longer by ordinary differential equations, with non-constant coefficients, but by integral Equations [53].

In a forthcoming investigation we will reconsider the concepts associated with the monomiality realization in terms of integral operators to provide a new generalization of the HWG and draw further consequences on new families of Laguerre-type polynomials.

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