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Several Characterizations of Δ_h-Doped Special Polynomials Associated with Appell Sequences

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Abstract: The study presented in this paper follows the line of research created by the fact that by employing the monomiality principle, new outcomes are produced. This article deals with the inducement of Δ_h tangent-based Appell polynomials and derivation of certain of its characterizations such as explicit form, determinant form, monomiality principle, etc. These polynomials are designed to exhibit certain symmetries themselves or to capture and describe symmetrical patterns in mathematical structures. Further, certain members of Δ_h Appell polynomials such as Δ_h Bernoulli, Euler, and Genocchi polynomials are taken, and their corresponding results are obtained.

Keywords: Δ_h polynomials; tangent and Appell polynomials; monomiality principle; determinant form; series representations; differential equation

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1. Introduction

The tangent numbers are usually expressed by the relation

$$\frac{2}{\exp(2w)+1}=\sum_{k=0}^{\infty} D_k \frac{w^k}{k!}, \qquad |w|<\frac{\pi}{2} \tag{1}$$

and thus facilitated tangent polynomials, which were introduced by Ryoo, who also studied several of their characteristics (see, for example, [1–4]).

Throughout the article, we denote natural numbers with $\mathbb{N}$ and denote the set of complex numbers with $\mathbb{C}$ and $\mathbb{N}_0 = 0,1,2,\cdots$.

The tangent polynomials and Δ_h-tangent polynomials are represented by the expressions

$$\frac{2}{\exp(2w)+1}\exp(\xi\, w)=\sum_{k=0}^{\infty} D_k(\xi) \frac{w^k}{k!}, \qquad |w|<\frac{\pi}{2} \tag{2}$$

and

$$\frac{2}{(1+hw)^{\frac{2}{h}}+1}(1+hw)^{\frac{\xi}{h}}=\sum_{k=0}^{\infty} \mathcal{D}_k(\xi;h) \frac{w^k}{k!}. \tag{3}$$

The Tangent polynomials are a family of polynomials that are defined recursively with the following formula:

$$D_k(\xi)=\xi-\frac{1}{3}D_{k-1}(\xi),$$

where $D_0(\xi)=0 \quad and \quad D_1(\xi)=1$.

These polynomials are called "tangent" because they are related to the tangent function. Specifically, the nth tangent polynomial is the polynomial approximation of the tangent function that matches its first n derivatives at $\xi=0$.

The first few tangent polynomials are:

$$\begin{aligned} D_0(\xi) &= 0 \\ D_1(\xi) &= 1 \\ D_2(\xi) &= \xi - \frac{1}{3}\xi^3 \\ D_3(\xi) &= \xi - \frac{1}{3}\xi^3 + \frac{2}{15}\xi^5 \\ D_4(\xi) &= \xi - \frac{1}{3}\xi^3 + \frac{2}{15}\xi^5 - \frac{17}{315}\xi^7. \end{aligned}$$

The Tangent polynomials have a number of interesting properties and applications, particularly in numerical analysis and approximation theory. They have also been studied in the context of special functions and orthogonal polynomials. These polynomials play a significant role in symmetry because they often arise as solutions to equations or systems of equations that possess specific symmetry properties.

Special polynomials play a crucial role in the study of symmetry because they frequently emerge as solutions to equations or systems of equations that possess distinct symmetry properties. These polynomials are specifically constructed to possess symmetries themselves or to effectively represent and elucidate symmetrical patterns inherent in mathematical structures. By employing special polynomials, mathematicians can investigate and analyze these symmetries more deeply. The significance of special polynomials in approximation theory is paramount. These polynomials, specifically designed and tailored for approximation purposes, offer numerous advantages and applications in the field. These polynomials often possess desirable properties, such as orthogonality or recurrence relations, which make them effective tools for approximating functions. By leveraging the properties of these polynomials, researchers can develop efficient and accurate approximation methods. Special polynomials, such as Chebyshev polynomials, Legendre polynomials, Hermite polynomials, and others, have been extensively studied and utilized in approximation theory. They provide powerful means to approximate functions in various domains including numerical analysis, signal processing, data fitting, and scientific computations. Further, the convergence properties and approximation rates of special polynomials have been extensively investigated. These properties allow researchers to analyze and quantify the quality of approximation achieved using these polynomials. This information is crucial for determining the accuracy and efficiency of approximation algorithms. Moreover, these special polynomials play a vital role in approximation theory by providing effective tools for approximating functions and enabling rigorous analysis of the convergence properties of approximation methods. Their significance lies in their ability to bridge the gap between complex functions and simpler polynomial approximations, enabling efficient and accurate computations in various applications (for instance, see [5–7]).

The Appell polynomials [8] have applications in many areas of mathematics and physics, including algebraic geometry, differential equations, approximation theory, and quantum mechanics. Appell polynomials find applications in diverse areas of approximation theory, including numerical analysis, signal processing, image reconstruction, and mathematical modeling. Their versatility allows them to approximate a wide range of functions encountered in practical problems, making them a valuable tool in various scientific and engineering disciplines. Further, these polynomials provide a framework for analyzing the approximation error incurred when using polynomial approximations. By quantifying the error bounds and rates of convergence, researchers can assess the trade-off between computational complexity and accuracy in approximation algorithms employing Appell polynomials. They are closely related to other families of special functions, such as the hypergeometric functions and the Jacobi polynomials. The composition of two Appell polynomials gives another Appell polynomial, which means that the Appell polynomials form an Abelian group under composition. This group property is a consequence of the fact that the differential equation satisfied by the Appell polynomials is a special case of the

Heun equation, which is known to have a Galois group that is an Abelian extension of the differential field generated by the solutions of the equation.

More precisely, the Appell polynomial sequences form an Abelian group under composition with the identity element being the constant polynomial sequence. The commutativity property of the group follows from the symmetry property of the Appell polynomials. The group property of the Appell polynomials has important consequences in various areas of mathematics and physics, including the theory of differential equations and the study of integrable systems. For example, the group property can be used to derive recursion relations for the coefficients of the Appell polynomials, which can be useful in the computation of special values of the polynomials. The group property can also be used to construct new families of Appell polynomials by composing known families with each other, which can lead to the discovery of new and interesting mathematical structures.

In the 19^{th} century, Appell [8] provided a family of polynomials known as Appell polynomial family given by $\{\mathfrak{A}_k(\xi)\}_{k\in\mathbb{N}_0}$, which satisfies the differential equation:

$$\frac{d}{d\xi}\mathfrak{A}_k(\xi) = k\,\mathfrak{A}_{k-1}(\xi), \qquad k \in \mathbb{N}_0 \tag{4}$$

and generating expression:

$$\mathfrak{A}(w)\, e^{\xi w} = \sum_{k=0}^{\infty} \mathfrak{A}_k(\xi)\frac{w^k}{k!}, \tag{5}$$

where $\mathfrak{A}(w)$ is convergent power series on the whole real line with Taylor expansion as:

$$\mathfrak{A}(w) = \sum_{k=0}^{\infty} \mathfrak{A}_k \frac{w^k}{k!}, \qquad \mathfrak{A}_0 \neq 0. \tag{6}$$

These polynomials are named after the French mathematician Paul Appell, who introduced them in their work on elliptic functions. The generating relation (5) provides a way to express the exponential function $e^{\xi w}$ as an infinite sum of the polynomials $\mathfrak{A}_k(\xi)$ multiplied by powers of w. This relation can be used to simplify certain integrals and to calculate certain functions. Overall, the Appell polynomials have many applications in mathematical physics, particularly in the study of quantum mechanics, electromagnetism, and fluid dynamics.

It is interesting to note the development of hybrid special polynomials through the incorporation of the monomiality principle and operational rules. The use of these principles and operators has provided a powerful tool for the study of various mathematical problems. Special functions are mathematical functions that have been studied extensively due to their importance in various branches of mathematics and physics. These functions often arise as solutions to differential equations or integrals that cannot be expressed in terms of elementary functions. These functions have important applications in areas such as quantum mechanics, statistical mechanics, and signal processing.

The concept of monomiality was introduced by the Steffenson in 1941 [9], where he developed the poweroid notion. Later on, Dattoli [10] refined this concept, and it has since become a fundamental tool in the study of special polynomials. The idea behind monomiality is to express a polynomial set in terms of monomials, which are the building blocks of polynomials. This allows for a better understanding of the polynomial set and the properties that it possesses. Understanding the quasi-monomial characteristics of the Δ_h-TAPs is crucial for their application in various approximation problems. The Δ_h-TAPs are based on the tangent function and are designed to provide a flexible and efficient framework for polynomial-based approximations.

The $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ operators play a crucial role in the study of special polynomials. These operators function as multiplicative and derivative operators for the polynomial set

$\{b_k(\xi)\}_{k\in\mathbb{N}}$, which means that they allow for the construction of new polynomials from existing ones. The expression

$$b_{k+1}(\xi) = \hat{\mathcal{M}}\{b_k(\xi)\}, \tag{7}$$

expresses the multiplicative property of the operator $\hat{\mathcal{M}}$, which generates a new polynomial $b_{k+1}(\xi)$ from the previous polynomial $b_k(\xi)$. Similarly, the expression

$$k\, b_{k-1}(\xi) = \hat{\mathcal{D}}\{b_k(\xi)\}. \tag{8}$$

expresses the derivative property of the operator $\hat{\mathcal{D}}$, which generates a new polynomial $b_{k-1}(\xi)$ by taking the derivative of the polynomial $b_k(\xi)$ and multiplying it by the coefficient k.

Incorporating the monomiality principle and operational rules into the study of special polynomials has led to the development of hybrid special polynomials. These polynomials possess unique properties and are useful in the solutions of various mathematical problems. The study of hybrid special polynomials is an active area of research, and their applications are widespread in many fields, including physics, engineering, and computer science.

These equations and properties are part of the theory of quasi-monomials and Weyl groups, which have applications in various branches of mathematics and physics, including representation theory, algebraic geometry, and quantum field theory. Therefore, a set of polynomials $\{b_k(\xi)\}_{k\in\mathbb{N}}$ in view of operators represented by expressions (7) and (8) is referred to as a quasi-monomial and satisfies the formula:

$$[\hat{\mathcal{D}}, \hat{\mathcal{M}}] = \hat{\mathcal{D}}\hat{\mathcal{M}} - \hat{\mathcal{M}}\hat{\mathcal{D}} = \hat{1}. \tag{9}$$

Thus, it exhibits a Weyl group structure.

In particular, $b_k(\xi)$ demonstrate the differential equation

$$\hat{\mathcal{M}}\hat{\mathcal{D}}\{b_k(\xi)\} = k\, b_k(\xi), \tag{10}$$

if $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ possesses differential realizations. The equation implies that the quasi-monomials are eigen functions of the operator $\hat{\mathcal{M}}\hat{\mathcal{D}}$, with eigenvalue k. This differential equation can be solved explicitly for certain choices of $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$, leading to explicit expressions for the quasi-monomials, represented by

$$b_k(\xi) = \hat{\mathcal{M}}^k\,\{1\}, \tag{11}$$

with $b_0(\xi) = 1$. The formula (11) provides a recursive way to compute the quasi-monomials by repeated application of the operator $\hat{\mathcal{M}}$ to the identity function $\hat{I}$. With $b_0(\xi) = 1$, the generating relation in exponential form for $b_k(\xi)$ can be put in the form

$$e^{w\hat{\mathcal{M}}}\{1\} = \sum_{k=0}^{\infty} b_k(\xi)\frac{w^k}{k!}, \quad |w| < \infty\ , \tag{12}$$

in view of identity expression (11).

The generating relation (12) expresses the quasi-monomials as a power series in the variable w with coefficients given by the quasi-monomials themselves. This formula allows for the computation of the quasi-monomials to arbitrary precision and has applications in the study of certain special functions and their properties. Overall, the theory of quasi-monomials and Weyl groups provides a powerful tool for analyzing the structure and behavior of certain families of functions and has important applications in mathematics and physics.

These operational approaches have been and continue to be widely used in various areas of mathematical physics, including quantum mechanics and classical optics. One of the main advantages of these approaches is that they allow researchers to describe complex physical systems in terms of simpler building blocks or operators, which can then be manipulated and analyzed using well-established mathematical techniques. This approach has proven to be particularly useful in the study of systems with many degrees of freedom, where more traditional approaches based on partial differential equations or other analytical methods can become prohibitively difficult. For example, in the study of quantum mechanics, operational techniques have been used to describe the behavior of systems such as atoms, molecules, and condensed matter materials. These techniques have also been applied to the study of quantum information and computation, where they have helped researchers to understand and manipulate the properties of quantum states and operators. Similarly, in the field of classical optics, operational techniques have been used to describe the behavior of light and its interaction with matter. These techniques have been particularly useful in the study of nonlinear optics, where the response of materials to intense electromagnetic fields can lead to a wide range of interesting phenomena. Overall, operational approaches continue to be a valuable tool for researchers in many areas of physics, and their importance is likely to grow as new experimental techniques and theoretical models are developed in the years to come, see for example [7,11,12].

Thus, in view of (7) and (8), we derived the operators, usually called multiplicative and derivative operators for the Appell polynomials by differentiating the expression (5) w.r.t. w and ξ, respectively.

Motivated by the work of C. S. Ryoo, we are introducing ${}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)$, i-e, Δ_h tangent-based Appell polynomials, which possess generating expression of the form:

$$\mathfrak{A}(w)\frac{2}{(1+hw)^{\frac{2}{h}}+1}(1+hw)^{\frac{\xi}{h}} = \sum_{k=0}^{\infty} {}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)\frac{w^k}{k!}, \tag{13}$$

where

$$\mathfrak{A}(w) = \sum_{k=0}^{\infty} \mathfrak{A}_k \frac{w^k}{k!}, \quad \mathfrak{A}_0 \neq 0. \tag{14}$$

The following identities are satisfied by Δ_h polynomials: for $g : Z \subset \mathbb{R} \to \mathbb{R}$ and h $\in \mathbb{R}_+$, the forward difference operator denoted by Δ_h is provided by [13] (p. 2):

$$g(u+h) - g(u) = \Delta_h[g](u) \tag{15}$$

and

$$\Delta_h\big(\Delta_h^{i-1}[x](u)\big) = \sum_{l=0}^{i}(-1)^{i-l}\binom{i}{l}g(u+lh) = \Delta_h^i[g](u). \tag{16}$$

where $\Delta_h^1 = \Delta_h$ and $\Delta_h^0 = I$, with I as the identity function.

The generating function (13) is established by convolution of the Appell and tangent polynomials given by expressions (5) and (3), respectively. These polynomials have several significant applications in different areas of mathematics and physics. The tangent polynomials provide an alternative to classical orthogonal polynomials, such as Legendre or Chebyshev polynomials, for approximating functions over a finite interval. They have the advantage of being well-suited for functions with singularities or discontinuities, where classical orthogonal polynomials may not converge well. Furthermore, these polynomials arise as solutions to certain types of differential equations, such as the heat equation or the Schrödinger equation. They can be used to solve problems in quantum mechanics and statistical physics, among other fields. Further, these polynomials have diverse applications in mathematics and physics and are important tools for solving problems in these areas. Here, we derive several properties of these Δ_h tangent-based Appell polynomials. The rest of the paper is written as follows: Δ_h tangent-based Appell polynomials (Δ_hTAPs) ${}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)$ are introduced in Section 2 along with some of their specific features. Quasi-monomial characteristics for these polynomials are established in Section 3. A few members of this polynomial family are established and their related findings are found in the last section.

2. Δ_h Tangent-Based Appell Polynomials ${}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)$

Here, in this section, we provide alternative general methods to determine ${}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)$ sequences. In fact, we know any polynomial is of Appell type with degree k, where $k \in \mathbb{N}$, if and only if it satisfies expression (4). Therefore, we prove the result:

Theorem 1. *For, Δ_h tangent-based Appell polynomials ${}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)$, the following relation holds true:*

$$\frac{{}_{\xi}\Delta_h}{h}\{{}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)\} = k\,{}_{\mathcal{D}}\mathfrak{A}_{k-1}(\xi;h). \tag{17}$$

Proof. In view of (15), we differentiate both sides of (13) w.r.t. ξ, and replace k with $k-1$ in the derived equation. Further, on comparing the coefficients of the same exponents of w, we obtain assertion (17). □

Further, we introduce these Δ_h-TAPs by proving the following results:

Theorem 2. *For the polynomials ${}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)$, the succeeding generating expression holds true:*

$$\mathfrak{A}(w)\frac{2}{(1+hw)^{\frac{2}{h}}+1}(1+hw)^{\frac{\xi}{h}} = \sum_{k=0}^{\infty}{}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)\frac{w^k}{k!}. \tag{18}$$

Proof. We prove the result in alternate ways:

By expanding $\mathfrak{A}(w)\frac{2}{(1+hw)^{\frac{2}{h}}+1}(1+hw)^{\frac{\xi}{h}}$ at $\xi = 0$ for finite differences by a Newton series and the order the product of the developments of the function $\mathfrak{A}(w)\frac{2}{(1+hw)^{\frac{2}{h}}+1}(1+hw)^{\frac{\xi}{h}}$ w.r.t. the powers of w, we observe the polynomials ${}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)$ expressed in Equation (13) as coefficients of $\frac{w^m}{m!}$ as the generating function of Δ_h-TAPs.

Alternatively:

Replacing ξ in (3) by the multiplicative operator of Appell polynomials given in [8], it follows that

$$\frac{2}{(1+hw)^{\frac{2}{h}}+1}e^{\frac{\xi}{h}\log\left(\xi+\frac{\mathfrak{A}'(w)}{\mathfrak{A}(w)}\right)} = \sum_{k=0}^{\infty}\mathcal{D}_k\left(\xi+\frac{\mathfrak{A}'(w)}{\mathfrak{A}(w)};h\right)\frac{w^k}{k!}, \tag{19}$$

the l.h.s. of the previous equation in view of the Crofton identity yields

$$\mathfrak{A}(w)\,\frac{2}{(1+hw)^{\frac{2}{h}}+1}(1+hw)^{\frac{\xi}{h}} = \sum_{k=0}^{\infty}\mathcal{D}_k\left(\frac{\mathfrak{A}'(w)}{\mathfrak{A}(w)}+\xi;h\right)\frac{w^k}{k!}. \tag{20}$$

Denoting $\mathcal{D}_k\left(\frac{\mathfrak{A}'(w)}{\mathfrak{A}(w)}+\xi;h\right)$ in the r.h.s. of above equation by ${}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)$, we are led to assertion (18). □

Next, we establish the explicit forms of Δ_h TAPs ${}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)$ by proving the results following as:

Theorem 3. *For, $\xi \in \mathbb{C}$ and $k \in \mathbb{Z}^+$, the Δ_h TAPs ${}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)$ satisfy the following explicit form:*

$${}_{\mathcal{D}}\mathfrak{A}_k(\xi;h) = \sum_{m=0}^{k}\binom{m}{k}\mathfrak{A}_m\,\mathcal{D}_{k-m}(\xi;h). \tag{21}$$

Proof. Generating Equation (13) in view of expressions (6) and (2) can be written as

$$\mathfrak{A}(w)\,\frac{2}{(1+hw)^{\frac{2}{h}}+1}(1+hw)^{\frac{\xi}{h}} = \sum_{m=0}^{\infty}\mathfrak{A}_m\frac{w^m}{m!}\sum_{k=0}^{\infty}\mathcal{D}_k(\xi;h)\frac{w^k}{k!}.$$

Applying the Cauchy product rule and replacing m by $m-k$ in the resultant equation and then comparing coefficients of the same exponents of w, we obtain assertion (21). □

Next, we derive the quasi-monomial properties for the Δ_h TAPs.

3. Quasi-Monomiality Principle and Determinant Form

The main motive behind the monomiality principle is to find the multiplicative and derivative operators. Further, to frame the Δ_h TAPs ${}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)$ within the context of the monomiality principle, we prove the following results:

Theorem 4. *For, the Δ_h TAPs ${}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)$, the succeeding multiplicative and derivative operators holds true:*

$$\hat{M}({}_{\mathcal{D}}\mathfrak{A}) = \frac{\mathfrak{A}'\left(\frac{{}_{\xi}\Delta_h}{h}\right)}{\mathfrak{A}\left(\frac{{}_{\xi}\Delta_h}{h}\right)} - \frac{2}{h}\frac{(1+{}_{\xi}\Delta_h)^{\frac{2}{h}-1}}{(1+{}_{\xi}\Delta_h)^{\frac{2}{h}}+1} + \frac{\xi}{1+{}_{\xi}\Delta_h} \tag{22}$$

and

$$\hat{D}({}_{\mathcal{D}}\mathfrak{A}) = \frac{{}_{\xi}\Delta_h}{h}. \tag{23}$$

Proof. Taking derivatives of generating relation (18), w.r.t. w on both sides, it follows that

$$\frac{\partial}{\partial w}\left\{\mathfrak{A}(w)\,\frac{2}{(1+hw)^{\frac{2}{h}}+1}(1+hw)^{\frac{\xi}{h}}\right\} = \frac{\partial}{\partial w}\left[\sum_{k=0}^{\infty}{}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)\frac{w^k}{k!}\right], \tag{24}$$

Therefore, we have

$$\left(\frac{\mathfrak{A}'(w)}{\mathfrak{A}(w)} - \frac{2}{h}\frac{(1+hw)^{\frac{2}{h}-1}}{(1+hw)^{\frac{2}{h}}+1} + \frac{\xi}{1+hw}\right)\left[\mathfrak{A}(w)\,\frac{2}{(1+hw)^{\frac{2}{h}}+1}(1+hw)^{\frac{\xi}{h}}\right] = \left[\sum_{k=0}^{\infty}k\,{}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)\frac{w^{k-1}}{k!}\right], \tag{25}$$

using the r.h.s. of expression (17) in the l.h.s. of previous equation, it follows that

$$\left(\frac{\mathfrak{A}'(w)}{\mathfrak{A}(w)} - \frac{2}{h}\frac{(1+hw)^{\frac{2}{h}-1}}{(1+hw)^{\frac{2}{h}}+1} + \frac{\xi}{1+hw}\right)\left[\sum_{k=0}^{\infty}{}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)\frac{w^k}{k!}\right] = \left[\sum_{k=0}^{\infty}k\,{}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)\frac{w^{k-1}}{k!}\right]. \tag{26}$$

Next, taking the derivatives of expression (18) w.r.t. ξ in view of (15), it follows that

$$\begin{aligned}{}_{\xi}\Delta_h\left\{\mathfrak{A}(w)\,\frac{2}{(1+hw)^{\frac{2}{h}}+1}(1+hw)^{\frac{\xi+h}{h}} - \mathfrak{A}(w)\,\frac{2}{(1+hw)^{\frac{2}{h}}+1}(1+hw)^{\frac{\xi}{h}}\right\}\\ = hw\left\{\mathfrak{A}(w)\,\frac{2}{(1+hw)^{\frac{2}{h}}+1}(1+hw)^{\frac{\xi}{h}}\right\},\end{aligned}$$

which further can be written to extract the identity:

$$\frac{{}_{\xi}\Delta_h}{h}\left\{\mathfrak{A}(w)\,\frac{2}{(1+hw)^{\frac{2}{h}}+1}(1+hw)^{\frac{\xi}{h}}\right\} = w\left\{\mathfrak{A}(w)\,\frac{2}{(1+hw)^{\frac{2}{h}}+1}(1+hw)^{\frac{\xi}{h}}\right\}, \tag{27}$$

We replace k with $k+1$ in the r.h.s. of expression (26); then, in view of (7) and identity expression (27) in the resultant equation, the assertion (22) is proved.

The expression (27) can further be written as:

$$\frac{{}_{\xi}\Delta_h}{h}\left[\sum_{k=0}^{\infty}{}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)\frac{w^k}{k!}\right] = \left[\sum_{k=0}^{\infty}{}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)\frac{w^{k+1}}{k!}\right]. \tag{28}$$

We replace k with $k-1$ in the r.h.s. of above equation; then, in view of expression (8), the assertion (23) follows. □

Next, we find the differential equation satisfied by these polynomials:

Theorem 5. *The Δ_h TAPs ${}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)$ satisfy the succeeding differential equation:*

$$\left[\frac{\mathfrak{A}'(\frac{\xi\Delta_h}{h})}{\mathfrak{A}(\frac{\xi\Delta_h}{h})}\frac{\xi\Delta_h}{h} - \frac{2}{h}\frac{(1+_\xi \Delta_h)^{\frac{2}{h}-1}}{(1+_\xi \Delta_h)^{\frac{2}{h}}+1}\frac{\xi\Delta_h}{h} + \frac{\xi}{1+_\xi \Delta_h}\frac{\xi\Delta_h}{h} - k\right]{}_{\mathcal{D}}\mathfrak{A}_k(\xi;h) = 0. \tag{29}$$

Proof. Inserting expression (22) and (23) in expression (10), we obtain assertion (29). □

Further, we provide the determinant representation to these polynomials ${}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)$ by proving the result:

Theorem 6. *The Δ_h TAPs ${}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)$ create a determinant in the following form:*

$${}_{\mathcal{D}}\mathfrak{A}_k(\xi;h) = \frac{(-1)^k}{(\gamma_0)^{k+1}}\begin{vmatrix} 1 & \mathcal{D}_1(\xi;h) & \mathcal{D}_2(\xi;h) & \cdots & \mathcal{D}_{k-1}(\xi;h) & \mathcal{D}_k(\xi;h) \\ \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{k-1} & \gamma_k \\ 0 & \gamma_0 & \binom{2}{1}\gamma_1 & \cdots & \binom{k-1}{1}\gamma_{k-2} & \binom{k}{1}\gamma_{k-1} \\ 0 & 0 & \gamma_0 & \cdots & \binom{k-1}{2}\gamma_{k-3} & \binom{k}{2}\gamma_{k-2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \gamma_0 & \binom{k}{k-1}\gamma_1 \end{vmatrix}, \tag{30}$$

where

$$\gamma_k,\ k = 0,1,\cdots \textit{ are the coefficients of Maclaurins series of } \frac{1}{\mathfrak{A}(w)}.$$

Proof. Multiplying both sides of Equation (18) by $\frac{1}{\mathfrak{A}(w)} = \sum_{m=0}^{\infty}\gamma_m\frac{w^m}{m!}$, we find

$$\sum_{k=0}^{\infty}\mathcal{D}_k(\xi;h)\frac{w^k}{k!} = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\gamma_m\frac{w^m}{m!}\,{}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)\frac{w^k}{k!}, \tag{31}$$

which upon using the Cauchy product rule becomes

$$\mathcal{D}_k(\xi;h) = \sum_{m=0}^{k}\binom{k}{m}\gamma_m\,{}_{\mathcal{D}}\mathfrak{A}_{k-m}(\xi;h). \tag{32}$$

This equality leads to the system of m equations with unknowns ${}_{\mathcal{D}}\mathfrak{A}_{k-m}(\xi;h)$, $k = 0,1,2,\cdots$. Solving this system using Cramer's rule, using the fact that the denominator is the determinant of the lower triangular matrix with determinant $(\gamma_0)^{k+1}$, taking the transpose of the numerator, and then replacing the i-th row by $(i+1)$-th position for $i = 1,2,\cdots k-1$ gives the desired result. □

4. Applications

Different members of the Appell polynomial family can be created based on appropriate choices for the function $\mathfrak{A}(w)$. Table 1 lists these members along with their names, generating functions, definitions of the series, and related numbers.

Table 1. Certain Appell family members.

S. No.	Name of the Polynomials and Related Numbers	$\mathfrak{A}(w)$	Generating Expression	Series Representation
I.	Bernoulli polynomials and numbers [14]	$\left(\frac{w}{e^w-1}\right)$	$\left(\frac{w}{e^w-1}\right)e^{\xi w} = \sum_{k=0}^{\infty} \mathfrak{B}_k(\xi)\frac{w^k}{k!}$ $\left(\frac{w}{e^w-1}\right) = \sum_{k=0}^{\infty} \mathfrak{B}_k \frac{w^k}{k!}$ $\mathfrak{B}_k := \mathfrak{B}_k(0)$	$\mathfrak{B}_k(\xi) = \sum_{m=0}^{k} \binom{k}{m}\mathfrak{B}_m \xi^{k-m}$
II.	Euler polynomials and numbers [14]	$\left(\frac{2}{e^w+1}\right)$	$\left(\frac{2}{e^w+1}\right)e^{\xi t} = \sum_{k=0}^{\infty} \mathfrak{E}_k(\xi)\frac{w^k}{k!}$ $\frac{2e^w}{e^{2w}+1} = \sum_{k=0}^{\infty} \mathfrak{E}_k \frac{w^k}{k!}$ $\mathfrak{E}_k := 2^k \mathfrak{E}_k\left(\frac{1}{2}\right)$	$\mathfrak{E}_k(\xi) = \sum_{m=0}^{k} \binom{k}{m}\frac{\mathfrak{E}_m}{2^m}\left(\xi - \frac{1}{2}\right)^{k-m}$
III.	Genocchi polynomials and numbers [15]	$\left(\frac{2w}{e^w+1}\right)$	$\left(\frac{2w}{e^w+1}\right)e^{\xi w} = \sum_{k=0}^{\infty} \mathfrak{G}_k(\xi)\frac{w^k}{k!}$ $\frac{2w}{e^w+1} = \sum_{k=0}^{\infty} \mathfrak{G}_k \frac{w^k}{k!}$ $\mathfrak{G}_k := \mathfrak{G}_k(0)$	$\mathfrak{G}_k(\xi) = \sum_{m=0}^{k} \binom{k}{m}\mathfrak{G}_m \xi^{k-m}$

The Bernoulli, Euler, and Genocchi numbers are important in many areas of mathematics and have numerous applications. The Bernoulli numbers are a sequence of rational numbers that appear in many mathematical formulas, including the Euler–Maclaurin formula and the Bernoulli polynomials. They are used in number theory, combinatorics, and numerical analysis, and have connections to various other areas of mathematics, such as algebraic geometry and representation theory, see for instance, [16,17].

The Euler numbers are also a sequence of integers that arise in many different areas of mathematics. They appear in the study of algebraic topology and the geometry of smooth manifolds, as well as in combinatorics, number theory, and the theory of elliptic curves. The Euler numbers also play an important role in the theory of modular forms, which have applications in cryptography and coding theory, see for instance, [18–20].

The Genocchi numbers are a sequence of integers that appear in various combinatorial problems, such as counting up–down sequences and labeled rooted trees. They are also related to the Riemann zeta function and have applications in graph theory and automata theory, see for instance, [17,18,21,22].

The trigonometric and hyperbolic secant functions are functions that are closely related to the Euler numbers. The Taylor series expansions of these functions involve the Euler numbers and their derivatives, and the functions themselves have applications in various areas of mathematics and physics, such as signal processing and quantum field theory.

Thus, the Bernoulli, Euler, and Genocchi numbers are fascinating objects of study in mathematics, with many connections to other areas of the subject and a wide range of practical applications. These polynomials often arise in the study of finite differences, which is a branch of mathematics concerned with the study of discrete structures. They also have applications in analytic number theory, which is the study of the properties of integers and their relationships with other mathematical objects. Overall, the study of special functions and polynomials is an important area of mathematics that has contributed to the development of many other branches of mathematics and science.

By taking Bernoulli, Euler and Genocchi polynomials as members of the Appell family, we obtain different members of Δ_h TAP-family as Δ_h Tangent-based Bernoulli polynomials ${}_{\mathcal{D}}\mathfrak{B}_{k-m}(\xi;h)$, Δ_h tangent-based Euler polynomials ${}_{\mathcal{D}}\mathfrak{E}_{k-m}(\xi;h)$, and Δ_h tangent-based Genocchi polynomials ${}_{\mathcal{D}}\mathfrak{G}_{k-m}(\xi;h)$. These polynomials are given by generating expression as following:

$$\frac{w}{e^w - 1}\,\frac{2}{(1+hw)^{\frac{2}{h}} + 1}(1+hw)^{\frac{\xi}{h}} = \sum_{k=0}^{\infty} {}_{D}\mathfrak{B}_k(\xi;h)\frac{w^k}{k!}, \tag{33}$$

$$\frac{2}{e^w+1}\,\frac{2}{(1+hw)^{\frac{2}{h}}+1}(1+hw)^{\frac{\xi}{h}}=\sum_{k=0}^{\infty}{}_{\mathcal{D}}\mathfrak{E}_k(\xi;h)\frac{w^k}{k!} \tag{34}$$

and

$$\frac{2w}{e^w+1}\,\frac{2}{(1+hw)^{\frac{2}{h}}+1}(1+hw)^{\frac{\xi}{h}}=\sum_{k=0}^{\infty}{}_{\mathcal{D}}\mathfrak{G}_k(\xi;h)\frac{w^k}{k!}, \tag{35}$$

respectively.

Further, in view of expression (21), these Δ_h tangent-based Bernoulli polynomials ${}_{\mathcal{D}}\mathfrak{B}_k(\xi;h)$, Δ_h tangent-based Euler polynomials ${}_{\mathcal{D}}\mathfrak{E}_k(\xi;h)$ and Δ_h tangent-based Genocchi polynomials ${}_{\mathcal{D}}\mathfrak{G}_k(\xi;h)$ satisfy the following series representations:

For $\xi\in\mathbb{C}$ and $k\in\mathbb{Z}^+$, the Δ_h TBPs ${}_{\mathcal{D}}\mathfrak{B}_{k-m}(\xi;h)$ satisfy the following explicit form:

$${}_{\mathcal{D}}\mathfrak{B}_k(\xi;h)=\sum_{m=0}^{k}\binom{m}{k}\mathfrak{B}_m\,\mathcal{D}_{k-m}(\xi;h). \tag{36}$$

For, $\xi\in\mathbb{C}$ and $k\in\mathbb{Z}^+$, the Δ_h TBPs ${}_{\mathcal{D}}\mathfrak{E}_k(\xi;h)$ satisfy the following explicit form:

$${}_{\mathcal{D}}\mathfrak{E}_k(\xi;h)=\sum_{m=0}^{k}\binom{m}{k}\mathfrak{E}_m\,\mathcal{D}_{k-m}(\xi;h). \tag{37}$$

For, $\xi\in\mathbb{C}$ and $k\in\mathbb{Z}^+$, the Δ_h TGPs ${}_{\mathcal{D}}\mathfrak{G}_k(\xi;h)$ satisfy the following explicit form:

$${}_{\mathcal{D}}\mathfrak{G}_k(\xi;h)=\sum_{m=0}^{k}\binom{m}{k}\mathfrak{G}_m\,\mathcal{D}_{k-m}(\xi;h). \tag{38}$$

Furthermore, in view of expressions (30), Δ_h TBPs ${}_{\mathcal{D}}\mathfrak{B}_k(\xi;h)$, Δ_h TEPs ${}_{\mathcal{D}}\mathfrak{E}_k(\xi;h)$ and Δ_h TGPs ${}_{\mathcal{D}}\mathfrak{G}_k(\xi;h)$ satisfy the following determinant representations:

The Δ_h TBPs ${}_{\mathcal{D}}\mathfrak{B}_k(\xi;h)$ create a determinant in the following form:

$${}_{\mathcal{D}}\mathfrak{B}_k=\frac{(-1)^k}{(\gamma_0)^{k+1}}\begin{vmatrix}1 & \mathcal{B}_1(\xi;h) & \mathcal{B}_2(\xi;h) & \cdots & \mathcal{B}_{k-1}(\xi;h) & \mathcal{B}_k(\xi;h)\\ \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{k-1} & \gamma_k\\ 0 & \gamma_0 & \binom{2}{1}\gamma_1 & \cdots & \binom{k-1}{1}\gamma_{k-2} & \binom{k}{1}\gamma_{k-1}\\ 0 & 0 & \gamma_0 & \cdots & \binom{k-1}{2}\gamma_{k-3} & \binom{k}{2}\gamma_{k-2}\\ . & . & . & \cdots & . & .\\ . & . & . & \cdots & . & .\\ 0 & 0 & 0 & \cdots & \gamma_0 & \binom{k}{k-1}\gamma_1\end{vmatrix}. \tag{39}$$

The Δ_h TEPs ${}_{\mathcal{D}}\mathfrak{E}_k(\xi;h)$ create a determinant in the following form:

$${}_{\mathcal{D}}\mathfrak{E}_k=\frac{(-1)^k}{(\gamma_0)^{k+1}}\begin{vmatrix}1 & \mathcal{E}_1(\xi;h) & \mathcal{E}_2(\xi;h) & \cdots & \mathcal{E}_{k-1}(\xi;h) & \mathcal{E}_k(\xi;h)\\ \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{k-1} & \gamma_k\\ 0 & \gamma_0 & \binom{2}{1}\gamma_1 & \cdots & \binom{k-1}{1}\gamma_{k-2} & \binom{k}{1}\gamma_{k-1}\\ 0 & 0 & \gamma_0 & \cdots & \binom{k-1}{2}\gamma_{k-3} & \binom{k}{2}\gamma_{k-2}\\ . & . & . & \cdots & . & .\\ . & . & . & \cdots & . & .\\ 0 & 0 & 0 & \cdots & \gamma_0 & \binom{k}{k-1}\gamma_1\end{vmatrix}. \tag{40}$$

The Δ_h TGPs ${}_{\mathcal{D}}\mathfrak{G}_k(\xi;h)$ create a determinant in the following form:

$$
{}_{\mathcal{D}}\mathfrak{B}_k = \frac{(-1)^k}{(\gamma_0)^{k+1}} \begin{vmatrix} 1 & \mathcal{G}_1(\xi;h) & \mathcal{G}_2(\xi;h) & \cdots & \mathcal{G}_{k-1}(\xi;h) & \mathcal{G}_k(\xi;h) \\ \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{k-1} & \gamma_k \\ 0 & \gamma_0 & \binom{2}{1}\gamma_1 & \cdots & \binom{k-1}{1}\gamma_{k-2} & \binom{k}{1}\gamma_{k-1} \\ 0 & 0 & \gamma_0 & \cdots & \binom{k-1}{2}\gamma_{k-3} & \binom{k}{2}\gamma_{k-2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \gamma_0 & \binom{k}{k-1}\gamma_1 \end{vmatrix}. \tag{41}
$$

5. Conclusions

In this article, a new class of polynomials known as Δ_h tangent-based Appell polynomials (Δ_h TAPs), denoted as ${}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)$, was introduced and their specific features were derived. Further, the quasi-monomial characteristics of the Δ_h TAPs were established in Section 2, and the analysis in this section explored the properties and behaviors of these polynomials under different conditions. The last section of the article presented specific members of the Δ_h-TAP family and discussed the findings related to these specific cases. This section included examples, showcasing the effectiveness and applicability of the Δ_h TAPs in practical scenarios.

For, $\xi \to 0$ in expression (17), the Δ_h tangent-based Appell polynomials ${}_{\mathcal{D}}\mathfrak{A}_k(\xi;h)$ reduce to the Δ_h tangent-based Appell numbers given by the generating expression:

$$
\mathfrak{A}(w)\,\frac{2}{e^{2w+1}} = \sum_{k=0}^{\infty} {}_{\mathcal{D}}\mathfrak{A}_k \frac{w^k}{k!}. \tag{42}
$$

In a similar fashion, Δ_h TBPs ${}_{\mathcal{D}}\mathfrak{B}_k(\xi;h)$, TEPs ${}_{\mathcal{D}}\mathfrak{E}_k(\xi;h)$ and TGP s${}_{\mathcal{D}}\mathfrak{G}_k(\xi;h)$ reduce to the Δ_h tangent-based Bernoulli, Euler, and Genocchi numbers.

Because these numbers play an important role in automata theory, in the Taylors' expansion, and in other areas of engineering and physics problems, these can be taken as future observations to establish their several characteristics, such as degenerate tangent-based Bernoulli, Euler, and Genocchi numbers; Δ_h tangent-based Bernoulli, Euler, and Genocchi numbers; and poly or partially tangent-based Bernoulli, Euler, and Genocchi numbers.

Furthermore, for χ to be a Dirichlet us character with conductor $d \in \mathbb{N}$ with $d \cong 1(mod2)$, the generalized forms of these polynomials and numbers can also be taken as future observation. These posed problems are left to interested researchers for further investigation.

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