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Numerical Analysis of Nonlinear Fractional System of Jaulent–Miodek Equation

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Abstract: This paper presents the optimal auxiliary function method (OAFM) implementation to solve a nonlinear fractional system of the Jaulent–Miodek Equation with the Caputo operator. The OAFM is a vital method for solving different kinds of nonlinear equations. In this paper, the OAFM is applied to the fractional nonlinear system of the Jaulent–Miodek Equation, which describes the behavior of a physical system via a set of coupled nonlinear equations. The Caputo operator represents the fractional derivative in the equations, improving the system's accuracy and applicability to the real world. This study demonstrates the effectiveness and efficiency of the OAFM in solving the fractional nonlinear system of the Jaulent–Miedek equation with the Caputo operator. This study's findings provide important insights into the behavior of complex physical systems and may have practical applications in fields such as engineering, physics, and mathematics.

Keywords: optimal auxiliary function method; nonlinear system of Jaulent–Miodek equation; Caputo operator; fractional calculus

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1. Introduction

Fractional nonlinear systems of partial differential equations (PDEs) have attracted significant research attention due to their ability to model various phenomena in various fields, including physics, engineering, and biology. These systems include fractional derivatives, nonlocal operators considering the system's history, and nonlinearities. It has become clear that fractional calculus, which deals with fractional derivatives and integrals, is a crucial tool for simulating memory effects, long-range interactions, and anomalous diffusion in practical issues. Nonlinear dynamics investigates the behavior of complex systems that show nonlinear interactions between their constituent parts [1–3]. These two fields have been combined to create fractional nonlinear PDE systems, which can capture the complex dynamics of various physical and biological systems.

A rapidly expanding area of study, the study of fractional nonlinear systems of PDEs has produced a variety of intriguing findings and applications. Recent research has mainly analyzed the stability and bifurcations of solutions, as well as their existence and uniqueness, and numerically simulated these systems. Applications in fields like fluid dynamics, electrochemistry, and population dynamics have all been investigated in other studies [4–6].

Symmetry, a fundamental concept in various scientific disciplines, has proven to be a powerful tool for enhancing system performance and achieving the desired outcomes. Researchers have explored the application of symmetry in fields such as control theory, image processing, and mechanical systems. For instance, in satellite attitude maneuvers, Meng et al. (2019) demonstrate how the sum of squares method, leveraging symmetry, can design robust control strategies for precise maneuvering [7]. Similarly, symmetry considerations are crucial in ensuring optimal performance for unstable plants, as highlighted by Meng et al. (2018) [8]. In image processing, Sheng et al. (2023) utilize symmetry to develop a self-supervised super-resolution technique for light field imagery [9]. Xu et al. (2023) use symmetry in fault estimation for switched interconnected nonlinear systems [10].

Finally, Lu et al. (2020) exploit symmetry to effectively isolate vibrations in mechanical systems [11]. These studies collectively emphasize symmetry's broad applicability and importance in diverse scientific domains, enabling system stability, optimization, and overall performance advancements.

The Jaulent–Miodek equation is a nonlinear differential equation extensively studied in mathematical science and engineering. In 1988, Jaulent and Miodek first suggested this equation as a theoretical representation of a Josephson junction, a nonlinear superconducting device [12]. Since then, the equation has been applied to modeling many physical phenomena, such as the dynamics of semiconductor devices, the behavior of Bose–Einstein condensates, and the propagation of electromagnetic waves in nonlinear media. Fractional calculus research is a growing area of study, with applications in many branches of science and engineering. It has been demonstrated that fractional calculus is a potent modeling tool for complex systems that display memory effects, long-range interactions, and anomalous diffusion [13–17]. Fractional calculus deals with derivatives and integrals of non-integer order. Fractional calculus and the Jaulent–Miodek equation combine to create a new class of nonlinear fractional systems displaying complex and intriguing behaviors [18,19].

An important area of research in recent years has been analyzing the nonlinear fractional system of the Jaulent–Miodek equation. The Laplace transform, the Adomian decomposition technique, and numerical simulations are a few of the analytical and numerical methods developed to examine these systems' properties. These techniques have been applied to research the system's multi-stability, chaos, bifurcations, and stability. For some exceptional cases of the Jaulent–Miodek equation, exact solutions have been obtained using analytical techniques such as the Laplace transform. For the Jaulent–Miodek equation's nonlinear fractional system, the Adomian decomposition method has been used to obtain approximative analytical solutions [20,21]. As well as investigating the existence and stability of various attractors, numerical simulations have been used to study the system's behavior under different initial conditions.

Investigation of the nonlinear fractional system of the Jaulent–Miodek equation has also uncovered several intriguing phenomena. In contrast to the traditional Jaulent–Miodek equation, these include chaos, bifurcations, and multi-stability. Designing and managing physical systems that behave in nonlinear and fractional ways must consider the implications of the emergence of these phenomena.

The Adomian decomposition technique was used in a different study by Kumar and Singh (2019) to arrive at an analytical solution for the nonlinear fractional Jaulent–Miodek oscillator [20]. The study demonstrated that the approach successfully approximates the system's solutions. Additionally, scientists have been investigating the nonlinear fractional system of the Jaulent–Miodek equation's applications in various fields. The system, for instance, has been used to design and manage electronic circuits like oscillators and filters. The system has also been used to model the behavior of fractional-order viscoelastic materials and study the dynamics of Bose–Einstein condensates Abdeljawad et al., 2020 [21].

The optimal auxiliary function method (OAFM) is a recently developed technique for solving nonlinear differential equations. It is a powerful and efficient approach that offers analytical solutions to nonlinear issues that are frequently challenging to resolve using conventional methods. For the solution of nonlinear differential equations, Belendez and Hernandez (2010) first presented the OAFM [22]. The authors demonstrated the accuracy and dependability of the method by using it to solve the Duffing oscillator equation. Since then, the OAFM has been used to solve numerous other nonlinear physics and engineering problems. For instance, Belendez et al. (2012) used the OAFM to resolve the nonlinear Schrodinger equation when studying quantum mechanics [23]. The authors attained analytical solutions compared with numerical outcomes and demonstrated that the OAFM offers precise, accurate solutions. The Korteweg–de Vries equation is a nonlinear partial differential equation that appears in the study of fluid mechanics [24]. Fatoorehchi et al. (2017) used the OAFM to obtain analytical solutions for this equation in a different study [25].

The Jaulent–Miodek equation is a well-known mathematical model that describes the behavior of various physical systems, ranging from electrical circuits to biological processes. In recent years, there has been growing interest in extending the classical Jaulent–Miodek equation to include fractional calculus operators, which capture real-world systems' nonlocal and memory-dependent properties more accurately [26,27]. This paper presents a novel and significant advancement in the field by introducing a nonlinear fractional system of the Jaulent–Miodek equation. Including nonlinearity allows for a more realistic representation of complex dynamics and phenomena observed in many physical and engineering systems. The fractional calculus operators, characterized by the fractional order α, introduce a new dimension to the system dynamics, enabling a more comprehensive understanding of the underlying processes. The fractional-order nature of the equations offers a powerful tool to describe the long-range dependencies and memory effects present in the systems, which are often overlooked in traditional integer-order models.

Furthermore, the paper proposes using the optimal auxiliary function method (OAFM) as a numerical technique to obtain approximate solutions of the nonlinear fractional system. The OAFM method offers several advantages over conventional methods, such as its ability to handle complex nonlinearities and its accuracy in capturing the complex behavior of the system. By employing the OAFM approach, the paper investigates the solutions of the nonlinear fractional system for various values of the fractional order α and parameter ρ. The obtained solutions are then compared with the exact solutions to assess the accuracy and effectiveness of the proposed methodology. Overall, this research extends the classical Jaulent–Miodek equation to incorporate fractional calculus operators and introduces a powerful numerical technique to analyze and solve the resulting nonlinear fractional system. The combination of these innovations contributes to a deeper understanding of the complex dynamics exhibited by real-world systems and opens up new avenues for further exploration and application in various scientific and engineering domains.

An outline of this paper is as follows. In Section 2, we start by providing the basic definitions that are used in our study. The analysis of the optimal auxiliary function method (OAFM) is provided in Section 3. The implementation of OAFM to solve the nonlinear fractional system of the Jaulent–Miodek equation with the Caputo operator and the discussion of the results are presented in Section 4. Finally, Section 6 includes the conclusions of our study.

2. Preliminaries

Definition 1. *Fractional derivative of $f \in C^m_{-1}$ is given in the sense of Caputo as the following:*

$$D_t^\alpha U(\zeta,t) = \begin{cases} \frac{d^m U(\zeta,t)}{dt^m}, & \alpha = m \in \mathbb{N}, \\ \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-r)^{m-\alpha-1} U^{(m)}(\zeta,r)dr, & m-1 < \alpha < m, m \in \mathbb{N}. \end{cases} \tag{1}$$

Definition 2. *The formula for the Riemann fractional integral is as follows:*

$$J_t^\alpha U(\zeta,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} U(\zeta,r)dr. \tag{2}$$

Lemma 1. *For $m-1 < \alpha \le m$, $p > -1$, $t \ge 0$ and $\lambda \in R$, we have:*

1. $D_t^\alpha t^p = \frac{\Gamma(\alpha+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}$,
2. $D_t^\alpha \lambda = 0$,
3. $D_t^\alpha J_t^\alpha U(\zeta,t) = U(\zeta,t)$,
4. $J_t^\alpha D_t^\alpha U(\zeta,t) = U(\alpha,t) - \sum_{i=0}^{n-1} \partial^i U(\zeta,0) \frac{t^i}{i!}$.

3. Analysis of the Optimal Axillary Functions Method

In order to illustrate the fundamental concept of the optimal axillary functions method, let us analyze the general non-linear equation of the form [26,27]:

$$L(w(\zeta,t)) + N(w(\zeta,t)) + h(\zeta) = 0, \tag{3}$$

with the corresponding given initial/boundary conditions:

$$B\left(w(\zeta,t), \frac{\partial w(\zeta,t)}{\partial t}\right) = 0, \tag{4}$$

where t represents time, ζ represents variable function, L represents the linear term, N represents the non-linear term and h is the given function.

The approximate solution of Equation (3) can be written as:

$$w^*(\zeta,t,C_i) = w_0(\zeta,t) + w_1(\zeta,t,C_n), \;\; n = 1,2,3,4\ldots s. \tag{5}$$

To acquire the initial and first approximate solution of Equation (3) we place Equation (5) in Equation (3) to obtain:

$$L(w_0(\zeta,t) + w_1(\zeta,t,C_n)) + N(w_0(\zeta,t) + w_1(\zeta,t,C_n)) + h(\zeta) = 0. \tag{6}$$

The initial approximation $w_0(\zeta,t)$ can be obtained from the linear term:

$$L(w_0(\zeta,t)) + h(\zeta) = 0, \;\; B(w_0, \frac{dw_0}{\zeta}) = 0. \tag{7}$$

The linear operator L depends on the given initial/boundary condition and the function $h(\zeta,t)$ is not fixed. Taking the initial approximation and non-linear differential equation with the given corresponding initial/boundary conditions into consideration to find the value of the first approximation $w_1(\zeta,t)$

$$L(w_1(\zeta,t,C_n)) + N(w_0(\zeta,t) + w_1(\zeta,t,C_n)) = 0, \tag{8}$$

with the corresponding initail/boundary condition

$$B\left(w_1(\zeta,t,C_n), \frac{\partial w_1(\zeta,t,C_n)}{\partial t}\right) = 0. \tag{9}$$

The non-linear term in the last equation can be expanded in the form

$$N(w_0 + w_1) = N(w_0) + \sum_{k=1}^{\infty} \frac{w_1^{(k)}}{k!} N^{(k)}(w_0(w)). \tag{10}$$

Equation (10) can be stated in the algorithmic sequence to achieve the limit solution. To control all the challenges that occur while solving the non-linear differential of Equation (6) and to accelerate the convergence of the first approximation $w_1(\zeta,t,C_n)$, we use an alternate expression that represents Equation (8):

$$L(w_1(\zeta,t,C_n)) + A_1(w_0(\zeta,t),C_n)N(w_0(\zeta,t))))A_2(w_0(\zeta,t),C_m) = 0, \tag{11}$$

$$B(w_1(\zeta,t,C_n), \frac{dw_1(\zeta,t,C_n)}{t}) = 0, \tag{12}$$

Remark 1. *A_1 and A_2 are the axillary functions, which depend on the $w_0(\zeta,t)$ and unknown parameters C_n and C_m, where $n = 1,2,3,\cdots s$ and $m = s+1, s+2, s+3, \cdots q$.*

Remark 2. *A_1 and A_2 are not fixed. They may be $w_0(\zeta,t)$ or $N(w_0(\zeta,t))$ and can be the combination of both $w_0(\zeta,t)$ and $N(w_0(\zeta,t))$.*

Remark 3. *If $w_0(\zeta,t)$ or $N(w_0(\zeta,t))$ are polynomial functions, then $A_1(w_0(\zeta,t,C_n))$ and $A_2(w_0(\zeta,t,C_m))$ are the sum polynomial function. When $w_0(\zeta,t)$ or $N(w_0(\zeta,t))$ are trigonometric functions, then $A_1(w_0(\zeta,t,C_n))$ and $A_2(w_0(\zeta,t,C_m))$ are the sum trigonometric function. Similarly, when $w_0(\zeta,t)$ or $N(w_0(\zeta,t))$ are exponential functions, then $A_1(w_0(\zeta,t,C_n))$ and $A_2(w_0(\zeta,t,C_m))$ are the sum exponential function. If $N(w_0(\zeta,t)) = 0$, which is the special case, then $w_0(\zeta,t)$ is the exact solution.*

Remark 4. *To find the values of the unknown parameters C_n and C_m, we use different methods, either the Ritz method, collocation method, least square method or Galerkin's method.*

4. Implementation of OAFM

The coupled fractional-order Jaulent–Miodek equations represent a significant advancement in mathematical modeling by incorporating fractional calculus operators into a system of interdependent equations. These equations describe the behavior of interconnected physical or engineering systems and provide a more accurate representation of their dynamics by accounting for memory-dependent and nonlocal effects. The coupling between the equations allows for investigations of how the variables interact and influence each other within the system, offering valuable insights into the complex relationships and phenomena that occur in real-world scenarios [26,27]. Studying these coupled fractional-order Jaulent–Miodek equations advances our understanding of diverse fields, including electrical circuits, biological systems, and other areas where interdependencies and fractional calculus are crucial ft accurate modeling.

Consider the coupled fractional-order Jaulent–Miodek equations [28]:

$$\begin{aligned}
&D_t^{\alpha}U(\zeta,t) + \frac{\partial^3 U(\zeta,t)}{\partial\zeta^3} + \frac{3}{2}V(\zeta,t)\frac{\partial^3 V(\zeta,t)}{\partial\zeta^3} + \frac{9}{2}\frac{\partial V(\zeta,t)}{\partial\zeta}\frac{\partial^2 V(\zeta,t)}{\partial\zeta^2} - 6U(\zeta,t)\frac{\partial U(\zeta,t)}{\partial\zeta} \\
&-6U(\zeta,t)V(\zeta,t)\frac{\partial V(\zeta,t)}{\partial\zeta} - \frac{3}{2}V^2(\zeta,t)\frac{\partial U(\zeta,t)}{\partial\zeta} = 0, \\
&D_t^{\alpha}V(\zeta,t) + \frac{\partial^3 V(\zeta,t)}{\partial\zeta^3} - 6\frac{\partial U(\zeta,t)}{\partial\zeta}V(\zeta,t) - 6U(\zeta,t)\frac{\partial V(\zeta,t)}{\partial\zeta} - \frac{15}{2}\frac{\partial V(\zeta,t)}{\partial\zeta}V^2(\zeta,t) = 0, \\
&\text{where} \quad 0 < \alpha \le 1,
\end{aligned} \tag{13}$$

with the initial conditions:

$$\begin{aligned}
U(\zeta,0) &= \frac{\rho^2}{8}\left(1 - \operatorname{sech}^2\left(\frac{\rho\zeta}{2}\right)\right), \\
V(\zeta,0) &= \rho\operatorname{sech}\left(\frac{\rho\zeta}{2}\right).
\end{aligned} \tag{14}$$

We write the linear terms in Equation (13) as follows:

$$\begin{aligned}
L(u) &= \frac{\partial^{\alpha} U_0(\zeta,t)}{\partial t^{\alpha}}, \\
L(v) &= \frac{\partial^{\alpha} V_0(\zeta,t)}{\partial t^{\alpha}}.
\end{aligned} \tag{15}$$

We write the nonlinear terms in Equation (13) as follows:

$$N(U)=\frac{\partial^3 U(\zeta,t)}{\partial\zeta^3}+\frac{3}{2}V(\zeta,t)\frac{\partial^3 V(\zeta,t)}{\partial\zeta^3}+\frac{9}{2}\frac{\partial V(\zeta,t)}{\partial\zeta}\frac{\partial^2 V(\zeta,t)}{\partial\zeta^2}-6U(\zeta,t)\frac{\partial U(\zeta,t)}{\partial\zeta}$$
$$-6U(\zeta,t)V(\zeta,t)\frac{\partial V(\zeta,t)}{\partial\zeta}-\frac{3}{2}V^2(\zeta,t)\frac{\partial U(\zeta,t)}{\partial\zeta}, \quad (16)$$

$$N(V)=\frac{\partial^3 V(\zeta,t)}{\partial\zeta^3}-6\frac{\partial U(\zeta,t)}{\partial\zeta}V(\zeta,t)-6U(\zeta,t)\frac{\partial V(\zeta,t)}{\partial\zeta}-\frac{15}{2}\frac{\partial V(\zeta,t)}{\partial\zeta}V^2(\zeta,t).$$

The zero-order equations are:

$$\frac{\partial^\alpha U_0(\zeta,t)}{\partial t^\alpha}=0,$$
$$\frac{\partial^\alpha V_0(\zeta,t)}{\partial t^\alpha}=0. \quad (17)$$

By making use of the inverse operator, we obtain the following solutions:

$$U_0(\zeta,t)=\frac{\rho^2}{8}\Big(1-\operatorname{sech}^2\Big(\frac{\rho\zeta}{2}\Big)\Big),$$
$$V_0(\zeta,t)=\rho\operatorname{sech}\Big(\frac{\rho\zeta}{2}\Big)\Big). \quad (18)$$

We substitute Equation (18) into Equation (16) to obtain:

$$N[U_0(y,t)]=\frac{1}{128}\rho^5\operatorname{sech}^7\Big(\frac{\zeta\rho}{2}\Big)\Big(794\sinh\Big(\frac{\zeta\rho}{2}\Big)-165\sinh\Big(\frac{3\zeta\rho}{2}\Big)+\sinh\Big(\frac{5\zeta\rho}{2}\Big)\Big),$$
$$N[V_0(y,t)]=-\frac{1}{32}\rho^4\operatorname{sech}^6\Big(\frac{\zeta\rho}{2}\Big)\tanh\Big(\frac{\zeta\rho}{2}\Big)\Big(-189+52\cosh(\zeta\rho)+\cosh\Big(\frac{\zeta\rho}{2}\Big)\Big). \quad (19)$$

We choose the auxiliary functions A_1, A_2, A_3 and A_4 as follows :

$$A_1=C_1\Big(\frac{\rho^2}{8}\Big(1-\operatorname{sech}^2\Big(\frac{\rho\zeta}{2}\Big)\Big)\Big)^3,$$
$$A_2=C_2\Big(\rho\operatorname{sech}^2\Big(\frac{\rho\zeta}{2}\Big)\Big)^5,$$
$$A_1=C_3\Big(\rho\operatorname{sech}^2\Big(\frac{\rho\zeta}{2}\Big)\Big)^6,$$
$$A_2=C_4\Big(\rho\operatorname{sech}^2\Big(\frac{\rho\zeta}{2}\Big)\Big)^7. \quad (20)$$

where $C_1=109.0725, C_2=-2.7562, C_3=-9.5065$ and $C_4=0.1739$

The first-order equations according to the OAFM procedure discussed in Section 3 are:

$$\frac{\partial^\alpha U_1(\zeta,t)}{\partial t^\alpha}=-\big(A_1N[U_0(\zeta,t)]+A_2\big),$$
$$\frac{\partial^\alpha V_1(\zeta,t)}{\partial t^\alpha}=-\big(A_3N[V_0(\zeta,t)]+A_4\big). \quad (21)$$

We substitute Equations (19) and (20) into Equation (21) to get:

$$\frac{\partial^{\alpha} U_1(\zeta,t)}{\partial t^{\alpha}} = \frac{1}{8192}\rho^6\left(1-4\,\mathrm{sech}^2\left(\frac{\zeta\rho}{2}\right)\right)^2\left(16C_2\left(-1+4\,\mathrm{sech}^2\left(\frac{\zeta\rho}{2}\right)\right)-\right.$$
$$\left.C_1\rho^3\,\mathrm{sech}^7\left(\frac{\zeta\rho}{2}\right)\left(794\sinh\left(\frac{\zeta\rho}{2}\right)-165\sinh\left(\frac{3\zeta\rho}{2}\right)+\sinh\left(\frac{5\zeta\rho}{2}\right)\right)\right),$$
$$\frac{\partial^{\alpha} V_1(\zeta,t)}{\partial t^{\alpha}} = -C_4\rho^7\sec^{14}\left(\frac{\zeta\rho}{2}\right)+\frac{1}{32}C_3\rho^{10}\Big(-189+52\cosh(\zeta\rho)+$$
$$\cosh\left(2\zeta\rho\right)\Big)\;\mathrm{sech}^{18}\left(\frac{\zeta\rho}{2}\right)\tanh\left(\frac{\zeta\rho}{2}\right). \tag{22}$$

Now, by applying the inverse operator to Equation (22), we can obtain:

$$U_1(\zeta,t) = \frac{t^{\alpha}}{8192\Gamma(\alpha+1)}\left(\rho^6\left(1-4\,\mathrm{sech}^2\left(\frac{\zeta\rho}{2}\right)\right)^2\left(16C_2\left(-1+4\,\mathrm{sech}^2\left(\frac{\zeta\rho}{2}\right)\right)-\right.\right.$$
$$\left.\left.C_1\rho^3\,\mathrm{sech}^7\left(\frac{\zeta\rho}{2}\right)\left(794\sinh\left(\frac{\zeta\rho}{2}\right)-165\sinh\left(\frac{3\zeta\rho}{2}\right)+\sinh\left(\frac{5\zeta\rho}{2}\right)\right)\right)\right),$$
$$V_1(\zeta,t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big(-C_4\rho^7\sec^{14}\left(\frac{\zeta\rho}{2}\right)+\frac{1}{32}C_3\rho^{10}\Big(-189+52\cosh(\zeta\rho)+$$
$$\cosh(2\zeta\rho)\Big)\;\mathrm{sech}^{18}\left(\frac{\zeta\rho}{2}\right)\tanh\left(\frac{\zeta\rho}{2}\right)\Big). \tag{23}$$

According to the OAFM procedure, the approximation solutions of U and V has two components, which can be written as follows:

$$\begin{aligned} U(\zeta,t) &= U_0(\zeta,t)+U_1(\zeta,t),\\ V(\zeta,t) &= V_0(\zeta,t)+V_1(\zeta,t). \end{aligned} \tag{24}$$

Substituting Equations (18) and (23) into Equation (24) leads to the following approximation solutions of the system in Equation (13).

$$U(\zeta,t) = \frac{\rho^2}{8}\left(1-\mathrm{sech}^2\left(\frac{\rho\zeta}{2}\right)\right)+\frac{t^{\alpha}}{8192\Gamma(\alpha+1)}\left(\rho^6\left(1-4\,\mathrm{sech}^2\left(\frac{\zeta\rho}{2}\right)\right)^2\left(16C_2\left(-1\right.\right.\right.$$
$$\left.\left.\left.+4\,\mathrm{sech}^2\left(\frac{\zeta\rho}{2}\right)\right)-C_1\rho^3\,\mathrm{sech}^7\left(\frac{\zeta\rho}{2}\right)\left(794\sinh\left(\frac{\zeta\rho}{2}\right)-165\sinh\left(\frac{3\zeta\rho}{2}\right)+\sinh\left(\frac{5\zeta\rho}{2}\right)\right)\right)\right),$$
$$V(\zeta,t) = \rho\,\mathrm{sech}\left(\frac{\rho\zeta}{2}\right)+\frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big(-C_4\rho^7\sec^{14}\left(\frac{\zeta\rho}{2}\right)+\frac{1}{32}C_3\rho^{10}\Big(-189+52\cosh(\zeta\rho)$$
$$+\cosh(2\zeta\rho)\Big)\;\mathrm{sech}^{18}\left(\frac{\zeta\rho}{2}\right)\tanh\left(\frac{\zeta\rho}{2}\right)\Big). \tag{25}$$

The exact result is given as [28]:

$$\begin{aligned} U(\zeta,t) &= \frac{\rho^2}{8}\left(1-\mathrm{sech}^2\left(\frac{\rho}{2}\left(\zeta-\frac{\rho^2 t}{2}\right)\right)\right),\\ V(\zeta,t) &= \rho\,\mathrm{sech}\left(\frac{\rho}{2}\left(\zeta-\frac{\rho^2 t}{2}\right)\right). \end{aligned} \tag{26}$$

5. Results and Discussion

Figure 1 showcases the approximate solutions of $U(\zeta,t)$ obtained using OAFM at two different values of ρ. Subfigure (a) demonstrates the solution at $\rho = 1.5$, while subfigure (b) presents the solution at $\rho = 2$. The figure visually represents the behavior of the solutions, allowing for a qualitative comparison between the two cases. By observing the graphs, how the change in ρ affects the solution becomes evident, enabling a deeper understanding of the impact of this parameter on the system dynamics. Figure 2a,b represent the approximate solution of $U(\zeta, t)$ using OAFM and the exact solution at $\alpha = 1$, respectively. The approximate solution of $V(\zeta, t)$ using OAFM and the exact solution at $\alpha = 1$ and $\rho = 0.2$ are shown in Figure 2c,d, respectively.

Figure 3 illustrates the approximate solutions of $U(\zeta, t)$ using OAFM at three distinct values of the fractional order α. Subfigure (a) corresponds to $\alpha = 0.5$, subfigure (b) represents $\alpha = 0.8$, and subfigure (c) displays $\alpha = 1$. This figure provides an excellent opportunity to study the impact of changing the fractional order on the system's response.

Figure 4 additionally investigates the influence of the fractional order on the approximate solutions of $V(\zeta, t)$ using OAFM. Subfigure (a) showcases the solution at $\alpha = 0.5$, subfigure (b) exhibits the solution at $\alpha = 0.8$, and subfigure (c) displays the solution at $\alpha = 1$. The fractional order parameter significantly affects the dynamics and characteristics of the system. By altering the fractional order, one can observe variations in the stability, oscillatory behavior, and convergence properties of the Jaulent–Miodek Equation. The fractional order is a critical factor in determining the system's response, making it a crucial parameter to consider when analyzing and modeling the system's behavior.

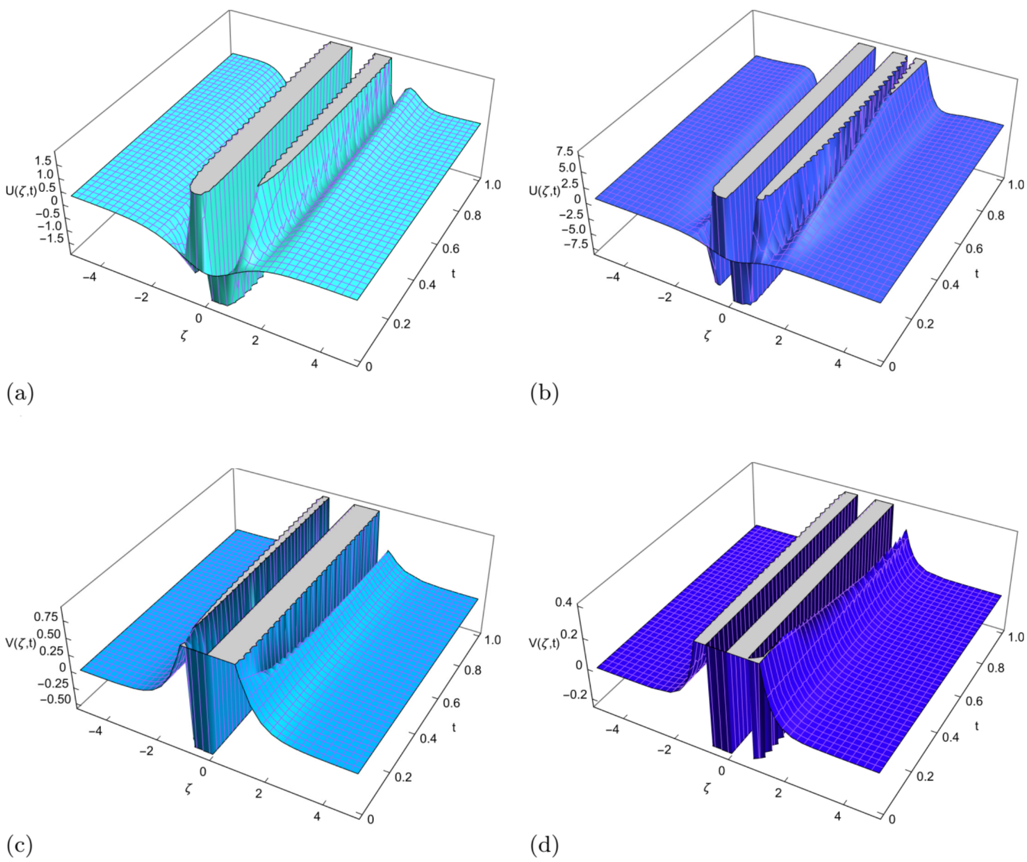


Figure 1. The (**a**) Approximate solution of $U(\zeta, t)$ using OAFM at $\rho = 1.5$, (**b**) Approximate solution of $U(\zeta, t)$ using OAFM at $\rho = 2$, (**c**) Approximate solution of $V(\zeta, t)$ using OAFM at $\rho = 1.5$ and (**d**) Approximate solution of $V(\zeta, t)$ using OAFM at $\rho = 2$ of fractional order $\alpha = 1$.

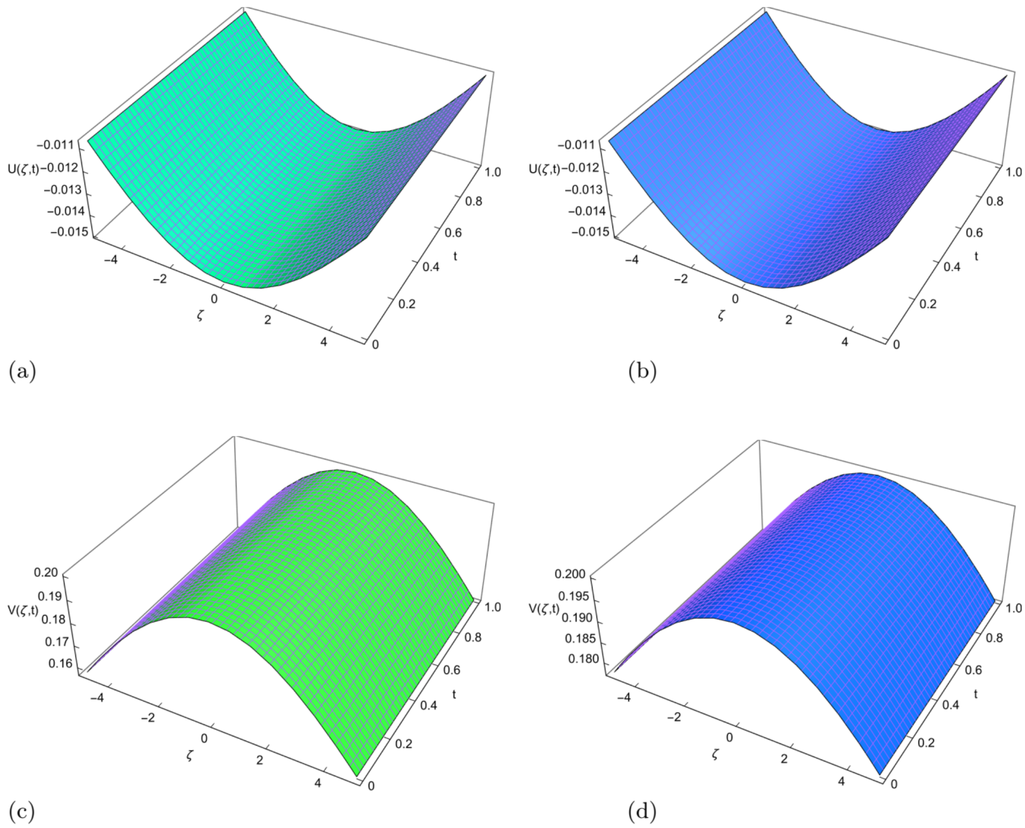
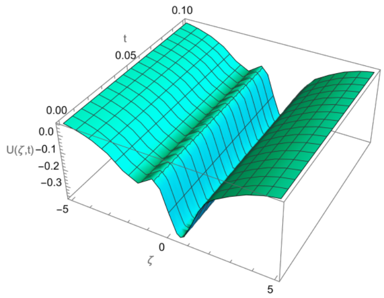
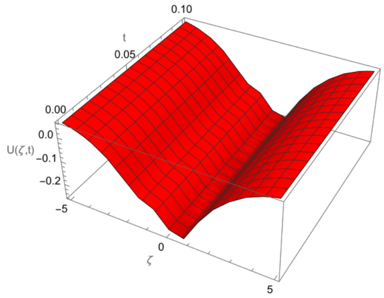


Figure 2. (**a**) Approximate solution of $U(\zeta,t)$ using OAFM (**b**) Exaxt solution of $U(\zeta,t)$, (**c**) Approximate solution of $V(\zeta,t)$ using OAFM and (**d**) Exaxt solution of $V(\zeta,t)$ at $\alpha = 1$ and $\rho = 0.2$

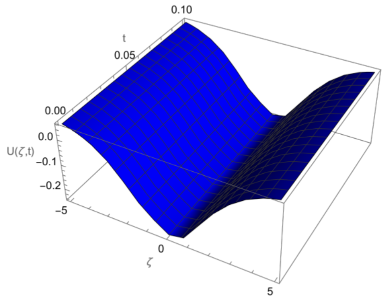
To complement the graphical analysis, Tables 1 and 2 provide a quantitative comparison between the approximate solutions obtained using OAFM and the exact solutions of $U(\zeta,t)$ and $V(\zeta,t)$ at two different fractional orders, $\alpha = 0.8$ and $\alpha = 1$. These tables present a comprehensive overview of the absolute errors associated with the approximate solutions, allowing for a detailed assessment of the accuracy of the OAFM approach. By examining the values in the tables, one can determine the reliability of the OAFM method for different fractional orders and make informed decisions about its application. The graphical discussion presented in Figures 1–4 and Tables 1 and 2 provides an in-depth analysis of the approximate solutions of the coupled fractional-order Jaulent–Miodek equations using the optimal auxiliary function method (OAFM). These figures and tables offer valuable insights into the behavior and accuracy of the OAFM approach under various conditions and parameters. In summary, the graphical discussion and tables comprehensively analyze the approximate solutions of the coupled fractional-order Jaulent–Miodek equations using OAFM. These visualizations and quantitative comparisons shed light on the system's behavior under various conditions and parameters, facilitating a deeper understanding of the dynamics and accuracy of the OAFM approach.



(a)

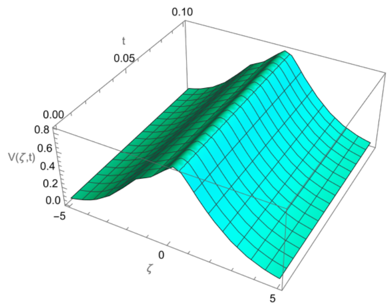


(b)

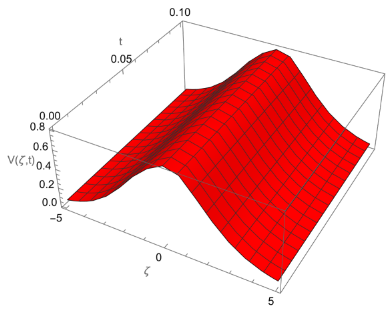


(c)

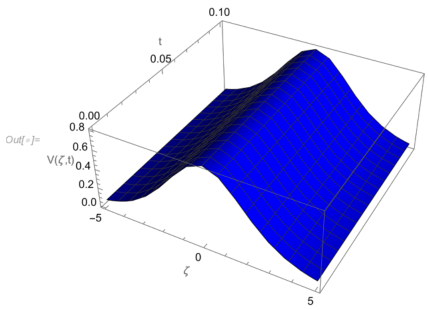
Figure 3. (**a**) Approximate solution of $U(\zeta, t)$ using OAFM at $\alpha = 0.$, (**b**) Approximate solution of $U(\zeta, t)$ using OAFM at $\alpha = 0.8$ and (**c**) Approximate solution of $U(\zeta, t)$ using OAFM at $\alpha = 1$ using $\rho = 0.8$.



(a)



(b)



(c)

Figure 4. (**a**) Approximate solution of $V(\zeta, t)$ using OAFM at $\alpha = 0.$, (**b**) Approximate solution of $V(\zeta, t)$ using OAFM at $\alpha = 0.8$ and (**c**) Approximate solution of $V(\zeta, t)$ using OAFM at $\alpha = 1$ using $\rho = 0.8$.

Table 1. Comparison between the approximate solution using OAFM and the exact solution of $U(\zeta, t)$ with their absolute error at fractional order $\alpha = 0.8$ and $\alpha = 1$; and $t = 0.5$ and $\rho = 0.2$.

	ζ	OAFM	Exact	Absolute Error
$\alpha = 0.8$	0.1	−0.015	−0.015	2.72275×10^{-6}
	0.2	−0.01499	−0.01499	2.78848×10^{-6}
	0.3	−0.01498	−0.01498	2.85155×10^{-6}
	0.4	−0.01497	−0.01497	2.91172×10^{-6}
	0.5	−0.01495	−0.01495	2.96874×10^{-6}
	0.6	−0.01493	−0.01493	3.02238×10^{-6}
$\alpha = 1$	0.1	−0.015	−0.015	2.05729×10^{-7}
	0.2	−0.01499	−0.01499	2.252×10^{-7}
	0.3	−0.01498	−0.01498	2.44461×10^{-7}
	0.4	−0.01497	−0.01497	2.63483×10^{-7}
	0.5	−0.01495	−0.01495	2.82237×10^{-7}
	0.6	−0.01493	−0.01493	3.00695×10^{-7}
	0.7	−0.0149	−0.0149	3.18831×10^{-7}

Table 2. Comparison between the approximate solution using OAFM and the exact solution of $V(\zeta, t)$ with their absolute error at fractional order $\alpha = 0.8$ and $\alpha = 1$; and $t = 0.5$ and $\rho = 0.2$.

	ζ	OAFM	Exact	Absolute Error
$\alpha = 0.8$	0.1	0.19998	0.19999	1.02×10^{-5}
	0.2	0.19992	0.19996	4.01×10^{-5}
	0.3	0.19982	0.19991	9×10^{-5}
	0.4	0.19968	0.19984	1.6×10^{-4}
	0.5	0.199501	0.19975	2.49×10^{-4}
	0.6	0.199281	0.19964	3.59×10^{-4}
$\alpha = 1$	0.1	0.19998	0.19999	1.0×10^{-5}
	0.2	0.19992	0.19996	3.99×10^{-5}
	0.3	0.19982	0.19991	8.98×10^{-5}
	0.4	0.19968	0.19984	1.6×10^{-4}
	0.5	0.199501	0.19975	2.49×10^{-4}
	0.6	0.199282	0.19964	3.58×10^{-4}
	0.7	0.199023	0.19951	4.87×10^{-4}

6. Conclusions

In conclusion, we successfully solved the fractional nonlinear system of the Jaulent–Miodek equation with the Caputo operator using the optimal auxiliary function method (OAFM). This approach has proven to be effective for tackling challenging fractional calculus math problems. The findings of this study show how effective and precise the OAFM is at resolving these kinds of issues. There is a notable agreement between the approximate solutions obtained by OAFM with the exact solutions, as shown in Figure 2 and Tables 1 and 2. The approach can be improved and used with additional nonlinear fractional differential equations. As a result, the OAFM is a promising method for handling various fractional calculus-based problems in science and engineering.

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