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Stability of Two Kinds of Discretization Schemes for Nonhomogeneous Fractional Cauchy Problem

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Abstract: The full discrete approximation of solutions of nonhomogeneous fractional equations is considered in this paper. The methods of iteration, finite differences and projection are applied to obtain desired formulas of explicit- and implicit-difference schemes for discretization schemes. The stability of two difference schemes is also discussed using the Trotter–Kato theorem.

Keywords: fractional cauchy problem; fulldiscrete approximation; Trotter–Kato theorem; discretization scheme; iteration method; stability

MSC: 45L05; 45M10; 65J10

1. Introduction

Many results of the approximation theory to abstract differential equations in Banach spaces simplify the design of concrete numerical approaches. Thus, an approximation theory of differential equations has attracted much attention due to its wide application in recent years.

In [1], Guidetti, Karasözen and Piskarev investigated the general approximation theory for differential equations with first-order derivatives in Banach spaces. Using the approximation theory, they analyzed the numerical problems of homogeneous differential equations and semilinear differential equations, respectively. In [2,3], Li, Morozov and Piskarev considered the approximation theory for derivatives of integrated semigroups. For other papers on the approximation of first-order differential equations, we suggest that readers consult [4–9].

Recently, fractional Cauchy problems and their approximation have become an important topic due to their broad application in engineering, physics and biology. A large number of findings on this topic have been reported in the literature [10–33]. Among these, in [22], Liu, Li and Piskarev considered the fulldiscretization approximation for solutions of the following equation with fractional time derivative $\alpha \in (0,1)$

$$\begin{cases} D_t^{\alpha}u(t) = Au(t), \quad 0 < t \le L, \\ u(0) = u^0, \end{cases} \tag{1}$$

in abstract space E, by virtue of finite differences and projection methods. In the same year, by discussing the relations of compact convergence of resolvents and semidiscrete approximation, the authors [23] studied the semidiscretization approximation of semilinear fractional problems

$$\begin{cases} D_t^{\alpha}u(t) = Au(t) + J^{1-\alpha}f(t,u(t)), \quad 0 < t \le L, \\ u(0) = u^0, \end{cases} \tag{2}$$

where $0 < \alpha < 1$. They demonstrated that the semidiscrete approximation to the solution is convergent if the corresponding resolvents are compactly convergent. However,

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in [23], the authors did not consider the fulldiscretization of the nonlinear term $J^{1-\alpha}f(t,u(t))$. In [20], the authors discussed the well-posedness and maximal regularity of fractional semilinear differential equations in Hölder space, and derived the existence and stability of an implicit difference scheme for the fractional systems. We refer to [11,15,20,21,24,25,27,32] and the references therein for the approximation of various differential equations in Banach spaces.

Motivated by above papers, we investigate the fulldiscrete approximation of nonhomogeneous fractional equation

$$\begin{cases} D_t^\alpha u(t) = Au(t) + J^{1-\alpha}f(t), & 0 < t \le L, \\ u(0) = u^0, \end{cases} \tag{3}$$

in abstract space E, where operator A is the generator of C_0-semigroup $\exp(tA)$, $0 < \alpha \le 1$, f is a smooth enough function, the Caputo fractional-order derivative D_t^α with order α is defined by

$$D_t^\alpha u(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} u'(s)\,\mathrm{d}s$$

and the Riemann–Liouville fractional order integral $J^{1-\alpha}f(t)$ with order $1-\alpha$ is defined by

$$J^{1-\alpha}f(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s)\,\mathrm{d}s,$$

if the above two integrals exist.

The general discretization scheme for problem (3) in Banach space E_n is

$$\begin{cases} D_t^\alpha u_n(t) = A_n u_n(t) + J^{1-\alpha}f_n(t), & 0 < t \le L, \\ u_n(0) = u_n^0, \end{cases} \tag{4}$$

with a series of smooth enough functions $f_n(\cdot)$.

In this paper, we find new iteration formulas of solutions to the implicit scheme and explicit scheme for the nonhomogeneous Cauchy problem (3) using the methods of iteration, finite differences and projection. At the same time, we discuss the stability for the two schemes using the Trotter–Kato theorem.

Define E_n and E as Banach spaces, $p_n \in \mathscr{B}(E, E_n)$, $B_n \in \mathscr{B}(E_n)$ and $B \in \mathscr{B}(E)$ with $n \in \mathbb{N}$, where $\mathscr{B}(E, E_n)$ denotes the space of all continuous linear operators from E to E_n, $\mathscr{B}(E_n)$ denotes $\mathscr{B}(E_n, E_n)$. Now, we introduce some notations and definitions of approximation theory, as follows.

By [9], we always assume that $\{p_n\}$, $p_n \in \mathscr{B}(E, E_n)$, satisfies that $\|p_n x\|_{E_n}$ goes to $\|x\|_E$ when n tends to infinity for each $x \in E$.

Definition 1 ([8])**.** *The family $\{x_n\}$, $x_n \in E_n$, is $\mathcal{P}$-converging to x belonging to E if $\lim_{n\to\infty} \|x_n - p_n x\|_{E_n} = 0$. This can also be written as $x_n \xrightarrow{\mathcal{P}} x$.*

Definition 2 ([8])**.** *The family $\{B_n\}$, $B_n \in \mathscr{B}(E_n)$, is $\mathcal{PP}$-converging to B belonging to $\mathscr{B}(E)$ if $x_n \xrightarrow{\mathcal{P}} x$ implies $B_n x_n \xrightarrow{\mathcal{P}} Bx$ for any $x_n \in E_n$ and $x \in E$. It is also denoted as $B_n \xrightarrow{\mathcal{PP}} B$.*

Use $\mathcal{C}(E)$ to denote the space of all densely defined closed linear operators on E. One version of the Trotter–Kato theorem [1], which is essential in the investigation of the approximation theory for differential equations, is shown as follows.

Theorem 1. *Assume that $A \in \mathcal{C}(E)$ and $A_n \in \mathcal{C}(E_n)$ are generators of C_0-semigroups, respectively. Then, the hypotheses (A) and (B) are equivalent to (C).*

(A). Coordination. There is one number $\lambda \in \rho(A) \bigcap \cap_n \rho(A_n)$ that satisfies $(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$.

(*B*). *Stability. There are two real numbers, ω and $M_1 \geq 1$, satisfying $\|\exp(tA_n)\| \leq M_1 \exp(\omega t)$ for each $t \geq 0$ and $n \in \mathbb{N}$, where ω and M_1 are independent of n.*

(*C*). *Convergence. For every $L > 0$, the relation*

$$\lim_{n\to\infty} \max_{t\in[0,L]} \| \exp(tA_n)u_n^0 - p_n \exp(tA)u^0\| = 0$$

holds if $u_n^0 \xrightarrow{\mathcal{P}} u^0$, $u_n^0 \in E_n$ and $u^0 \in E$.

2. Explicit and Implicit Schemes for the Approximation

The main purpose of the paper is to investigate the fulldiscrete approximation of the Equation (4). Therefore, the difference schemes for the general approximation to the problem (3) are needed.

Let $t_m = m\tau_n$, $m = 0, 1, 2, \ldots$; we approximate the fractional derivative $(D_t^\alpha x_n)(t_m)$ of functions $x_n : [0, L] \to E_n$ by the finite difference scheme $\triangle_{t_m}^\alpha x_n(\cdot)$, where

$$\begin{aligned}(D_t^\alpha x_n)(t_m) &= J^{1-\alpha}x_n'(t_m) \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_m} \frac{x_n'(t_m - s)}{s^\alpha} \mathrm{d}s \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \frac{x_n'(t_m - s)}{s^\alpha} \mathrm{d}s\end{aligned}$$

and

$$\triangle_{t_m}^\alpha x_n(\cdot) = \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{m-1} (t_{j+1}^{1-\alpha} - t_j^{1-\alpha}) \frac{x_n(t_{m-j}) - x_n(t_{m-j-1})}{\tau_n}.$$

In view of [24], the solution of the homogeneous equation of problem (4) can be expressed by $u_n(t) = S_\alpha(t, A_n)u_n^0$ for any smooth initial value $u_n^0 \in D(A_n^{l+1})$ with the smallest integer l, such that $(l+1)\alpha \geq 2$. In this situation, they proved the following relation regarding the order of convergence

$$\triangle_{t_m}^\alpha u_n(\cdot) - (D_t^\alpha u_n)(t_m) = O(\tau_n^\alpha).$$

On the other hand, we approximate $J^{1-\alpha} f(t_m)$ by $J_{t_m}^{1-\alpha} f_n(\cdot)$, where

$$\begin{aligned}J^{1-\alpha} f(t_m) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_m} \frac{f(t_m - s)}{s^\alpha} \mathrm{d}s \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \frac{f(t_m - s)}{s^\alpha} \mathrm{d}s\end{aligned}$$

and

$$J_{t_m}^{1-\alpha} f_n(\cdot) = \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{m-1} (t_{j+1}^{1-\alpha} - t_j^{1-\alpha}) f_n(t_{m-j}).$$

Now, we can approximate problem (3) using the implicit difference scheme

$$\begin{cases} \Delta_{t_m}^\alpha \overline{U}_n(\cdot) = A_n \overline{U}_n(t_m) + J_{t_m}^{1-\alpha} f_n(\cdot), \\ \overline{U}_n(0) = u_n^0, \end{cases} \tag{5}$$

and the explicit scheme

$$\begin{cases} \Delta_{t_m}^\alpha U_n(\cdot) = A_n U_n(t_{m-1}) + J_{t_m}^{1-\alpha} f_n(\cdot), \\ U_n(0) = u_n^0, \end{cases} \tag{6}$$

respectively.

3. Existence and Stability

Now, we present the proofs of the iteration formulas that solve the two difference schemes through the method of induction, and discuss the stability of the solutions under the condition (B) with $\omega = 0$ in the Trotter–Kato theorem.

Let $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$ in the sequel. The two iteration formulas of solutions for implicit and explicit difference schemes are presented as follows.

Theorem 2. *For the implicit scheme (5), we obtain the relation*

$$\overline{U}_n(m\tau_n) = \sum_{j=1}^{m} c_j^{(m)} R^j u_n^0 + \sum_{j=1}^{m} a_m^{(j)} R\tau_n f_n(j\tau_n), \tag{7}$$

where $R = (I_n - \Gamma(2-\alpha)\tau_n^\alpha A_n)^{-1}$, $\overline{U}_n(0) = u_n^0$, *and* $c_1^{(m)} = b_{m-1}$, $c_j^{(m)} = \sum_{i=1}^{m-j+1}(b_{i-1} - b_i)c_{j-1}^{(m-i)}$, $j = 2, ..., m$, $\sum_{j=1}^{m} c_j^{(m)} = 1$, $c_j^{(m)} > 0$, $j = 1, ..., m$, $a_m^{(1)} = b_{m-1} + R\sum_{i=1}^{m-1}(b_{i-1} - b_i)a_{m-i}^{(1)}$, $a_m^{(j)} = a_{m-j+1}^{(1)}$, $j = 2, ..., m$, $a_i^{(j)} = 0$, $j > i$.

Proof. For the implicit difference scheme (5), i.e., for the scheme

$$\begin{aligned} &\frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{m-1} b_j \frac{\overline{U}_n((m-j)\tau_n) - \overline{U}_n((m-j-1)\tau_n)}{\tau_n{}^\alpha} \\ = \quad & A_n\overline{U}_n(t_m) + \frac{1}{\Gamma(2-\alpha)}\tau_n{}^{1-\alpha}\sum_{j=0}^{m-1} b_j f_n((m-j)\tau_n), \end{aligned}$$

it follows that

$$\begin{aligned} \overline{U}_n(m\tau_n) \quad = \quad & Rb_{m-1}u_n^0 + R\sum_{j=1}^{m-1}(b_{j-1} - b_j)\overline{U}_n((m-j)\tau_n) \\ & +R\sum_{j=0}^{m-1}\tau_n b_j f_n((m-j)\tau_n). \end{aligned}$$

We prove (7) by induction as follows.

For $m = 1$, $\overline{U}_n(\tau_n) = Ru_n^0 + R\tau_n f_n(\tau_n)$, $c_1^{(1)} = 1$, $a_1^{(1)} = b_0 = 1$.

For $m = 2$,

$$\begin{aligned} \overline{U}_n(2\tau_n) \quad &= \quad Rb_1u_n^0 + R(b_0 - b_1)\overline{U}_n(\tau_n) + R\tau_n f_n(2\tau_n) + Rb_1\tau_n f_n(\tau_n) \\ &= \quad Rb_1u_n^0 + R^2(1-b_1)u_n^0 + b_1R\tau_n f_n(\tau_n) \\ & \quad\quad +R^2(b_0 - b_1)\tau_n f_n(\tau_n) + R\tau_n f_n(2\tau_n) \\ &= \quad Rb_1u_n^0 + R^2(1-b_1)u_n^0 + [b_1 + R(b_0 - b_1)]R\tau_n f_n(\tau_n) \\ & \quad\quad +R\tau_n f_n(2\tau_n), \end{aligned}$$

where $c_1^{(2)} = b_1$, $c_2^{(2)} = 1 - b_1$, $c_1^{(2)} + c_2^{(2)} = 1$, $a_2^{(1)} = b_1 + R(b_0 - b_1)a_1^{(1)}$, $a_1^{(2)} = 0$ and $a_2^{(2)} = a_1^{(1)} = 1$.

Assume that (7) holds when $1 \le m \le M-1$. Then, for $m = M$, we deduce

$$\overline{U}_n(M\tau_n) \quad = \quad Rb_{M-1}u_n^0 + R\sum_{i=1}^{M-1}(b_{i-1} - b_i)\overline{U}_n((M-i)\tau_n)$$

$$
\begin{aligned}
&\quad +R\sum_{i=0}^{M-1}\tau_n b_i f_n((M-i)\tau_n)\\
&= Rb_{M-1}u_n^0+R\sum_{i=1}^{M-1}(b_{i-1}-b_i)\\
&\quad \cdot[\sum_{j=1}^{M-i}c_j^{(M-i)}R^j u_n^0+\sum_{j=1}^{M-i}a_{M-i}^{(j)}R\tau_n f_n(j\tau_n)]\\
&\quad +R\sum_{i=0}^{M-1}\tau_n b_i f_n((M-i)\tau_n)\\
&:= P_1+P_2,
\end{aligned}
$$

where

$$P_1=Rb_{M-1}u_n^0+R\sum_{i=1}^{M-1}(b_{i-1}-b_i)\sum_{j=1}^{M-i}c_j^{(M-i)}R^j u_n^0,$$

$$P_2=R\sum_{i=1}^{M-1}(b_{i-1}-b_i)\sum_{j=1}^{M-i}a_{M-i}^{(j)}R\tau_n f_n(j\tau_n)+R\sum_{i=0}^{M-1}\tau_n b_i f_n((M-i)\tau_n).$$

Next, we verify $P_1=\sum_{j=1}^{M}c_j^{(M)}R^j u_n^0$ and $P_2=\sum_{j=1}^{M}a_M^{(j)}R\tau_n f_n(j\tau_n)$ by induction, respectively.

In fact,

$$
\begin{aligned}
P_1 &= Rb_{M-1}u_n^0+R\sum_{i=1}^{M-1}(b_{i-1}-b_i)\sum_{j=1}^{M-i}c_j^{(M-i)}R^j u_n^0\\
&= Rb_{M-1}u_n^0+\sum_{j=1}^{M-1}\sum_{i=1}^{M-j}(b_{i-1}-b_i)c_j^{(M-i)}R^{j+1}u_n^0\\
&= Rb_{M-1}u_n^0+\sum_{j=2}^{M}\sum_{i=1}^{M-j+1}(b_{i-1}-b_i)c_{j-1}^{(M-i)}R^j u_n^0,
\end{aligned}
$$

where $c_1^{(M)}=b_{M-1}$, $c_j^{(M)}=\sum_{i=1}^{M-j+1}(b_{i-1}-b_i)c_{j-1}^{(M-i)}$, $j=2,\ldots,M$, and

$$
\begin{aligned}
\sum_{j=1}^{M}c_j^{(M)} &= b_{M-1}+\sum_{i=1}^{M-1}\sum_{j=1}^{M-i}(b_{i-1}-b_i)c_j^{(M-i)}\\
&= b_{M-1}+\sum_{i=1}^{M-1}(\sum_{j=1}^{M-i}c_j^{(M-i)})(b_{i-1}-b_i)\\
&= b_{M-1}+\sum_{i=1}^{M-1}(b_{i-1}-b_i)=1.
\end{aligned}
$$

Thus, $P_1=\sum_{j=1}^{M}c_j^{(M)}R^j u_n^0$.

On the other hand,

$$
\begin{aligned}
P_2 &= R\sum_{i=1}^{M-1}\sum_{j=1}^{M-i}(b_{i-1}-b_i)a_{M-i}^{(j)}R\tau_n f_n(j\tau_n)+R\sum_{i=0}^{M-1}\tau_n b_i f_n((M-i)\tau_n)\\
&= R\sum_{j=1}^{M-1}\sum_{i=1}^{M-j}(b_{i-1}-b_i)a_{M-i}^{(j)}R\tau_n f_n(j\tau_n)+R\sum_{j=1}^{M}\tau_n b_{M-j}f_n(j\tau_n)
\end{aligned}
$$

$$\begin{aligned} &= R\sum_{j=1}^{M}\sum_{i=1}^{M-j}(b_{i-1}-b_i)a_{M-i}^{(j)}R\tau_n f_n(j\tau_n) + R\sum_{j=1}^{M}\tau_n b_{M-j} f_n(j\tau_n) \\ &= \sum_{j=1}^{M}\sum_{i=1}^{M-j}R(b_{i-1}-b_i)a_{M-i-j+1}^{(1)}R\tau_n f_n(j\tau_n) + R\sum_{j=1}^{M}\tau_n b_{M-j} f_n(j\tau_n). \end{aligned}$$

By assumption, $\sum_{i=1}^{M-j} R(b_{i-1}-b_i)a_{M-i-j+1}^{(1)} = a_{M-j+1}^{(1)} - b_{M-j}$. It follows that

$$\begin{aligned} P_2 &= \sum_{j=1}^{M}(a_{M-j+1}^{(1)} - b_{M-j})R\tau_n f_n(j\tau_n) + R\sum_{j=1}^{M}\tau_n b_{M-j} f_n(j\tau_n) \\ &= \sum_{j=1}^{M} a_{M-j+1}^{(1)} R\tau_n f_n(j\tau_n) \\ &= \sum_{j=1}^{M} a_M^{(j)} R\tau_n f_n(j\tau_n). \end{aligned}$$

Hence, $\overline{U}_n(M\tau_n) = \sum_{j=1}^{M} c_j^{(M)} R^j u_n^0 + \sum_{j=1}^{M} a_M^{(j)} R\tau_n f_n(j\tau_n)$. □

Theorem 3. *Considering the explicit difference scheme (6), the relation*

$$U_n(m\tau_n) = \sum_{j=0}^{m} \overline{c}_j^{(m)} \overline{R}^j u_n^0 + \sum_{j=1}^{m} \overline{a}_m^{(j)} \tau_n f_n(j\tau_n) \tag{8}$$

holds for $m \in \mathbb{N}$, where $\overline{R} = I_n + \frac{\Gamma(2-\alpha)}{1-b_1}{\tau_n}^{\alpha} A_n$ and
$\overline{c}_0^{(m)} = \sum_{i=2}^{m}(b_{i-1}-b_i)\overline{c}_0^{(m-i)} + b_m$,
$\overline{c}_j^{(m)} = (1-b_1)\overline{c}_{j-1}^{(m-1)} + \sum_{i=2}^{m-j}(b_{i-1}-b_i)\overline{c}_j^{(m-i)}$, $j = 1, ..., m-1$,
$\overline{c}_{m-1}^{(m)} = (1-b_1)\overline{c}_{m-2}^{(m-1)}$, $\overline{c}_m^{(m)} = (1-b_1)\overline{c}_{m-1}^{(m-1)}$,
$\overline{a}_m^{(1)} = b_{m-1} + \overline{R}(1-b_1)\overline{a}_{m-1}^{(1)} + \sum_{i=2}^{m-1}(b_{i-1}-b_i)\overline{a}_{m-i}^{(1)}$,
$\overline{a}_m^{(j)} = \overline{a}_{m-j+1}^{(1)}$, $j = 2, ..., m$, $\overline{a}_i^{(j)} = 0$, $j > i$,
and $\sum_{j=0}^{m} = \overline{c}_j^{(m)} = 1$.

Proof. From the explicit difference scheme (6), i.e.,

$$\frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{m-1} b_j \frac{U_n((m-j)\tau_n) - U_n((m-j-1)\tau_n)}{{\tau_n}^{\alpha}}$$

$$= A_n U_n((m-1)\tau_n) + \frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{m-1} {\tau_n}^{1-\alpha} b_j f_n((m-j)\tau_n),$$

we get

$$\begin{aligned} U_n(m\tau_n) &= (1-b_1)(I_n + \frac{\Gamma(2-\alpha)}{1-b_1}{\tau_n}^{\alpha} A_n)U_n((m-1)\tau_n) \\ &\quad + \sum_{j=2}^{m}(b_{j-1}-b_j)U_n((m-j)\tau_n) + b_m u_n^0 + \sum_{j=0}^{m-1}\tau_n b_j f_n((m-j)\tau_n) \\ &= (1-b_1)\overline{R}U_n((m-1)\tau_n) + \sum_{i=2}^{m}(b_{i-1}-b_i)U_n((m-i)\tau_n) \\ &\quad + b_m u_n^0 + \sum_{i=0}^{m-1}\tau_n b_i f_n((m-i)\tau_n). \end{aligned}$$

Next, we prove relation (8) by induction.

For $m = 1$,

$$\begin{aligned} U_n(\tau_n) &= (1-b_1)(I_n + \frac{\Gamma(2-\alpha)}{1-b_1}\tau_n{}^{\alpha} A_n)u_n^0 + b_1 u_n^0 + \tau_n f_n(\tau_n) \\ &= (1-b_1)\overline{R}u_n^0 + b_1 u_n^0 + \tau_n f_n(\tau_n), \end{aligned}$$

where $\overline{c}_0^{(1)} = b_1 > 0$, $\overline{c}_1^{(1)} = 1 - b_1 > 0$, $\overline{c}_0^{(1)} + \overline{c}_1^{(1)} = 1$ and $\overline{a}_1^{(1)} = b_0 = 1$.

For $m = 2$,

$$\begin{aligned} U_n(2\tau_n) &= (1-b_1)\overline{R}U_n(\tau_n) + (b_1 - b_2)U_n(0) + b_2 U_n(0) \\ &\quad + \tau_n f_n(2\tau_n) + \tau_n b_1 f_n(\tau_n) \\ &= (1-b_1)^2\overline{R}^2 u_n^0 + b_1(1-b_1)\overline{R}u_n^0 + b_1 u_n^0 \\ &\quad + (1-b_1)\overline{R}\tau_n f_n(\tau_n) + \tau_n f_n(2\tau_n) + \tau_n b_1 f_n(\tau_n), \end{aligned}$$

where $\overline{c}_0^{(2)} = b_1 > 0$, $\overline{c}_1^{(2)} = b_1(1-b_1) > 0$, $\overline{c}_2^{(2)} = (1-b_1)^2 > 0$, $\overline{c}_0^{(2)} + \overline{c}_1^{(2)} + \overline{c}_2^{(2)} = 1$, $\overline{a}_1^{(1)} = b_0$, $\overline{a}_1^{(2)} = 0$, $\overline{a}_2^{(1)} = b_1 + (1-b_1)\overline{R} = b_1 + \overline{R}(1-b_1)\overline{a}_1^{(1)}$, $\overline{a}_2^{(2)} = 1 = \overline{a}_1^{(1)}$.

Assume the relation (8) holds for $1 \le m \le M-1$. Then,

$$\begin{aligned} &U_n(M\tau_n) \\ = \;& (1-b_1)\overline{R}U_n((M-1)\tau_n) + \sum_{i=2}^{M}(b_{i-1} - b_i)U_n((M-i)\tau_n) \\ & + b_M U_n(0) + \sum_{i=0}^{M-1}\tau_n b_i f_n((M-i)\tau_n) \\ = \;& (1-b_1)\overline{R}[\sum_{j=0}^{M-1}\overline{c}_j^{(M-1)}\overline{R}^j u_n^0 + \sum_{j=1}^{M-1}\overline{a}_{M-1}^{(j)}\tau_n f_n(j\tau_n)] \\ & + \sum_{i=2}^{M}(b_{i-1} - b_i)[\sum_{j=0}^{M-i}\overline{c}_j^{(M-i)}\overline{R}^j u_n^0 + \sum_{j=1}^{M-i}\overline{a}_{M-i}^{(j)}\tau_n f_n(j\tau_n)] \\ & + b_M u_n^0 + \sum_{i=0}^{M-1}\tau_n b_i f_n((M-i)\tau_n) \\ := \;& Q_1 + Q_2, \end{aligned}$$

where

$$Q_1 = (1-b_1)\overline{R}\sum_{j=0}^{M-1}\overline{c}_j^{(M-1)}\overline{R}^j u_n^0 + \sum_{i=2}^{M}(b_{i-1} - b_i)\sum_{j=0}^{M-i}\overline{c}_j^{(M-i)}\overline{R}^j u_n^0 + b_M u_n^0,$$

$$\begin{aligned} Q_2 &= (1-b_1)\overline{R}\sum_{j=1}^{M-1}\overline{a}_{M-1}^{(j)}\tau_n f_n(j\tau_n) + \sum_{i=2}^{M}(b_{i-1} - b_i)\sum_{j=1}^{M-i}\overline{a}_{M-i}^{(j)}\tau_n f_n(j\tau_n) \\ &\quad + \sum_{i=0}^{M-1}\tau_n b_i f_n((M-i)\tau_n). \end{aligned}$$

Now, our aim is to deduce $Q_1 = \sum_{j=0}^{M}\overline{c}_j^{(M)}\overline{R}^j u_n^0$ and $Q_2 = \sum_{j=1}^{M}\overline{a}_M^j\tau_n f_n(j\tau_n)$ by induction, respectively. As a matter of fact,

$$Q_1 = (1-b_1)\overline{R}\sum_{j=0}^{M-1}\overline{c}_j^{(M-1)}\overline{R}^j u_n^0 + \sum_{i=2}^{M}(b_{i-1} - b_i)\sum_{j=0}^{M-i}\overline{c}_j^{(M-i)}\overline{R}^j u_n^0 + b_M u_n^0$$

$$
\begin{aligned}
&= (1-b_1)\sum_{j=1}^{M}\overline{c}_{j-1}^{(M-1)}\overline{R}^j u_n^0 + \sum_{j=0}^{M-2}\sum_{i=2}^{M-j}(b_{i-1}-b_i)\overline{c}_j^{(M-i)}\overline{R}^j u_n^0 + b_M u_n^0 \\
&= \sum_{j=0}^{M}\overline{c}_j^{(M)}\overline{R}^j u_n^0.
\end{aligned}
$$

where
$\overline{c}_0^{(M)} = \sum_{i=2}^{M}(b_{i-1}-b_i)\overline{c}_0^{(M-i)} + b_M > 0,$
$\overline{c}_j^{(M)} = (1-b_1)\overline{c}_{j-1}^{(M-1)} + \sum_{i=2}^{M-j}(b_{i-1}-b_i)\overline{c}_j^{(M-i)} > 0, j = 1, ..., M-2,$
$\overline{c}_{M-1}^{(M)} = (1-b_1)\overline{c}_{M-2}^{(M-1)} > 0, \overline{c}_M^{(M)} = (1-b_1)\overline{c}_{M-1}^{(M-1)} > 0.$
Meanwhile, we can obtain

$$
\begin{aligned}
\sum_{j=0}^{M}\overline{c}_j^{(M)} &= (1-b_1)\sum_{j=0}^{M-1}\overline{c}_j^{(M-1)} + \sum_{i=2}^{M}(b_{i-1}-b_i)\sum_{j=0}^{M-i}\overline{c}_j^{(M-i)} + b_M \\
&= (1-b_1)\sum_{i=2}^{M}(b_{i-1}-b_i) + b_M \\
&= 1.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
Q_2 &= (1-b_1)\overline{R}\sum_{j=1}^{M-1}\overline{a}_{M-1}^{(j)}\tau_n f_n(j\tau_n) \\
&\quad + \sum_{i=2}^{M}(b_{i-1}-b_i)\sum_{j=1}^{M-i}\overline{a}_{M-i}^{(j)}\tau_n f_n(j\tau_n) + \sum_{i=0}^{M-1}\tau_n b_i f_n((M-i)\tau_n) \\
&= (1-b_1)\overline{R}\sum_{j=1}^{M}\overline{a}_{M-1}^{(j)}\tau_n f_n(j\tau_n) \\
&\quad + \sum_{j=1}^{M-2}\sum_{i=2}^{M-j}(b_{i-1}-b_i)\overline{a}_{M-i-j+1}^{(1)}\tau_n f_n(j\tau_n) + \sum_{j=1}^{M}\tau_n b_{M-j} f_n(j\tau_n) \\
&= (1-b_1)\overline{R}\sum_{j=1}^{M}\overline{a}_{M-j}^{(1)}\tau_n f_n(j\tau_n) \\
&\quad + \sum_{j=1}^{M}\sum_{i=2}^{M-j}(b_{i-1}-b_i)\overline{a}_{M-i-j+1}^{(1)}\tau_n f_n(j\tau_n) + \sum_{j=1}^{M}\tau_n b_{M-j} f_n(j\tau_n) \\
&= \sum_{j=1}^{M}[b_{M-j} + \overline{R}(1-b_1)\overline{a}_{M-j}^{(1)} + \sum_{i=2}^{M-j}(b_{i-1}-b_i)\overline{a}_{M-i-j+1}^{(1)}]\tau_n f_n(j\tau_n) \\
&= \sum_{j=1}^{M}\overline{a}_{M-j+1}^{(1)}\tau_n f_n(j\tau_n) \\
&= \sum_{j=1}^{M}\overline{a}_M^{(j)}\tau_n f_n(j\tau_n),
\end{aligned}
$$

where $\overline{a}_M^{(1)} = b_{M-1} + \overline{R}(1-b_1)\overline{a}_{M-1}^{(1)} + \sum_{i=2}^{M-1}(b_{i-1}-b_i)\overline{a}_{M-i}^{(1)}$, $\overline{a}_i^{(j)} = 0$, $j > i$, and

$$
\begin{aligned}
\overline{a}_M^{(j)} &= \overline{R}(1-b_1)\overline{a}_{M-j}^{(1)} + \sum_{i=2}^{M-j}(b_{i-1}-b_i)\overline{a}_{M-i-j+1}^{(1)} + b_{M-j} \\
&= \overline{a}_{M-j+1}^{(1)}, \quad j = 2, ..., M.
\end{aligned}
$$

Consequently,

$$U_n(M\tau_n) = \sum_{j=0}^{M} \overline{c}_j^{(M)} \overline{R}^j u_n^0 + \sum_{j=1}^{M} \overline{a}_M^{(j)} \tau_n f_n(j\tau_n).$$

□

On account of the above two relations, we now can establish the proof of stability to the solutions, under the following conditions.

Theorem 4. *Suppose condition (B) holds, with $\omega = 0$. Then, the implicit difference scheme (5) is stable, i.e.,*

$$\|\overline{U}_n(m\tau_n)\| \le \overline{M}\|u_n^0\| + \overline{M}m\tau_n \sup_{1\le j\le m} \|f_n(j\tau_n)\|, \tag{9}$$

where $\overline{M} = \max\{1, M_1\}$, $m\tau_n \in [0, L]$.

Proof. By condition (B), we have $\|e^{tA_n}\| \le M_1$ for any $t \ge 0$. Thus,

$$\begin{aligned}
\|R^j\| &= \|(I_n - \Gamma(2-\alpha)\tau_n{}^\alpha A_n)^{-j}\| \\
&= \|(\Gamma(2-\alpha)\tau_n{}^\alpha)^{-j}(\frac{I_n}{\Gamma(2-\alpha)\tau_n{}^\alpha} - A_n)^{-j}\| \\
&\le (\Gamma(2-\alpha)\tau_n{}^\alpha)^{-j}\frac{M_1}{(\Gamma(2-\alpha)\tau_n{}^\alpha)^{-j}} \\
&= M_1.
\end{aligned}$$

Next, we prove the inequality

$$\|a_j^{(1)}\| \le \overline{M}, \|Ra_j^{(1)}\| \le \overline{M}, j = 1, 2, ..., m, \tag{10}$$

by induction.

For $m = 1$, $\|a_1^{(1)}\| = b_0 \le \overline{M}$.

For $m = 2$, $\|a_2^{(1)}\| = \|b_1 + R(b_0 - b_1)a_1^{(1)}\| \le \overline{M}b_1 + \overline{M}(b_0 - b_1) = \overline{M}$.

Suppose the relation (10) holds for every $1 \le m \le M - 1$. Then, for $m = M$, we obtain

$$\begin{aligned}
\|a_M^{(1)}\| &= \|b_{M-1} + R\sum_{i=1}^{M-1}(b_{i-1} - b_i)a_{M-i}^{(1)}\| \\
&\le \overline{M}b_{M-1} + \overline{M}\sum_{i=1}^{M-1}(b_{i-1} - b_i) \\
&= \overline{M}.
\end{aligned}$$

From the above proof, one can also obtain that

$$\|Ra_j^{(1)}\| \le \overline{M}, j = 1, 2, ..., m.$$

Consequently, using Theorem 2, we obtain

$$\begin{aligned}
&\|\overline{U}_n(M\tau_n)\| \\
\le\ & \sum_{j=1}^{M} c_j^{(M)}\|R^j u_n^0\| + \sum_{j=1}^{M}\|a_M^{(j)}R\|\tau_n\|f_n(j\tau_n)\| \\
\le\ & \sum_{j=1}^{M} c_j^{(M)} M_1\|u_n^0\| + \sum_{j=1}^{M}\|a_{M-j+1}^{(1)}R\|\tau_n \sup_{1\le j\le M}\|f_n(j\tau_n)\|
\end{aligned}$$

$$\begin{aligned} &\leq \overline{M}\|u_n^0\| + \overline{M}\sum_{j=1}^{M}\tau_n \sup_{1\leq j\leq M}\|f_n(j\tau_n)\| \\ &= \overline{M}\|u_n^0\| + \overline{M}M\tau_n \sup_{1\leq j\leq M}\|f_n(j\tau_n)\|. \end{aligned}$$

□

Theorem 5. *Let $\frac{1}{2} < \alpha \leq 1$. Suppose condition (B) holds with $\omega = 0$ and $\|\tau_n{}^{2\alpha-1}A_n{}^2\| \leq c$, where c is a constant. Then, the explicit scheme (6) is stable, i.e.,*

$$\|U_n(m\tau_n)\| \leq \widetilde{M}\exp\{\frac{c\Gamma^2(2-\alpha)}{(1-b_1)^2}m\tau_n\}\|u_n^0\| + \widetilde{M}m\tau_n \sup_{1\leq j\leq m}\|f_n(j\tau_n)\|, \quad (11)$$

where $\widetilde{M} = \max\{1, M_1(1 + \frac{c\Gamma^2(2-\alpha)}{(1-b_1)^2}\tau_n)\}$, c is independent of n and $m\tau_n \in [0, L]$.

Proof. By condition (B), we have $\|e^{tA_n}\| \leq M_1$ for any $t \geq 0$. Then, we have $\|(I_n - \frac{\Gamma(2-\alpha)}{1-b_1}\tau_n{}^{\alpha}A_n)^{-j}\| \leq M_1$. Thus,

$$\begin{aligned} \|\overline{R}^j\| &= \|(I_n + \frac{\Gamma(2-\alpha)}{1-b_1}\tau_n{}^{\alpha}A_n)^j\| \\ &= \|(I_n - \frac{\Gamma(2-\alpha)}{1-b_1}\tau_n{}^{\alpha}A_n)^{-j}(I_n - \frac{\Gamma^2(2-\alpha)}{(1-b_1)^2}\tau_n{}^{2\alpha}A_n{}^2)^j\| \\ &\leq M_1(1 + \|\frac{\Gamma^2(2-\alpha)}{(1-b_1)^2}\tau_n{}^{2\alpha-1}A_n{}^2\|\tau_n)^j \\ &\leq M_1(1 + \frac{c\Gamma^2(2-\alpha)}{(1-b_1)^2}\tau_n)^j. \end{aligned}$$

Next, we prove

$$\|\overline{a}_j^{(1)}\| \leq \widetilde{M}, j = 1, 2, ..., h, \quad (12)$$

by induction.

For $m = 1$, $\|\overline{a}_1^{(1)}\| = b_0 \leq \widetilde{M}$.

For $m = 2$, $\|\overline{a}_2^{(1)}\| = \|b_1 + \overline{R}(b_0 - b_1)\overline{a}_1^{(1)}\| \leq \widetilde{M}b_1 + \widetilde{M}(b_0 - b_1) = \widetilde{M}$.

Suppose the relation (12) holds for every $1 \leq m \leq M-1$. Then, for $m = M$, we obtain

$$\begin{aligned} \|\overline{a}_M^{(1)}\| &= \|b_{M-1} + \overline{R}(b_0 - b_1)\overline{a}_{M-1}^{(1)} + \sum_{i=2}^{M-1}(b_{i-1} - b_i)\overline{a}_{M-i}^{(1)}\| \\ &\leq \widetilde{M}b_{M-1} + \widetilde{M}(b_0 - b_1) + \widetilde{M}\sum_{i=2}^{M-1}(b_{i-1} - b_i) \\ &= \widetilde{M}. \end{aligned}$$

Consequently, we have the following estimate

$$\begin{aligned} &\|U_n(M\tau_n)\| \\ \leq &\sum_{j=0}^{M}\overline{c}_j^{(M)}\|\overline{R}^j\|\|u_n^0\| + \sum_{j=1}^{M}\|\overline{a}_M^{(j)}\|\tau_n \sup_{1\leq j\leq M}\|f_n(j\tau_n)\| \\ = &\sum_{j=0}^{M}\overline{c}_j^{(M)}\|\overline{R}^j\|\|u_n^0\| + \sum_{j=1}^{M}\|\overline{a}_{M-j+1}^{(1)}\|\tau_n \sup_{1\leq j\leq M}\|f_n(j\tau_n)\| \\ \leq &\sum_{j=0}^{M}\overline{c}_j^{(M)}M_1(1 + \frac{c\Gamma^2(2-\alpha)}{(1-b_1)^2}\tau_n)^j\|u_n^0\| + \widetilde{M}\sum_{j=1}^{M}\tau_n \sup_{1\leq j\leq M}\|f_n(j\tau_n)\| \end{aligned}$$

$$\leq \quad \widetilde{M}\exp\{\frac{c\Gamma^2(2-\alpha)}{(1-b_1)^2}M\tau_n\}\|u_n^0\| + \widetilde{M}M\tau_n \sup_{1\leq j\leq M}\|f_n(j\tau_n)\|.$$

□

Remark 1. *Our results generalize Proposition 1, Proposition 2, Theorem 2 and Theorem 7 in [22], where the authors consider the existence and stability of homogeneous fractional equations. Our contribution in the present paper is that we find the new iteration formulas of solutions for the implicit scheme and explicit scheme of the nonhomogeneous Cauchy problem (3) and obtain the stability results for these two schemes.*

4. Numerical Example

In this section, we provide a numerical example in one-dimensional space to show the validity of our results. We consider the following differential equation

$$\begin{cases} D_t^\alpha u(t) = -u(t) + J^{1-\alpha}\sin t, & 0 < t \leq 20, \\ u(0) = 0.1, \end{cases} \tag{13}$$

in Euclidean space $\mathbb{R}$, when $\tau_n = 0.2$ and α equals 0.5, 0.25, 0.7, respectively.

According to Figures 1–3, one can see that the solutions of implicit schemes are stable. Therefore, Theorem 4 is valid by means of these Figures. On the other hand, one can see that the solutions of explicit schemes are unstable in Figures 1 and 2. The solution of explicit scheme is shown to be stable in Figure 3. Thus, Theorem 5 is also valid, since α must be greater than 0.5 in this theorem.

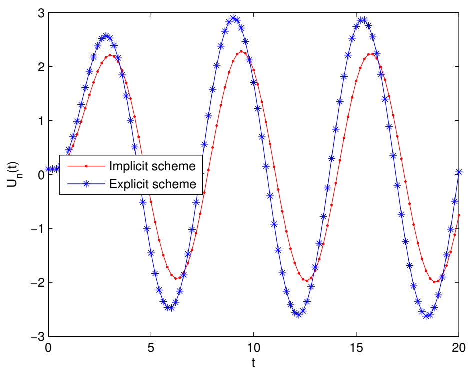

Figure 1. $\alpha = 0.5$.

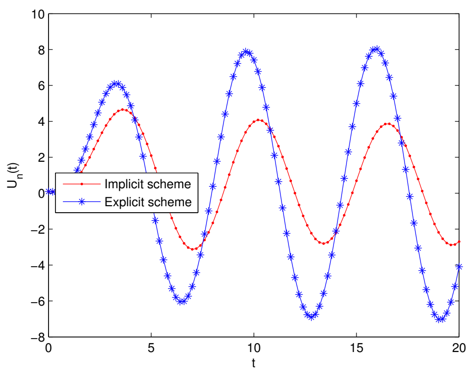

Figure 2. $\alpha = 0.25$.

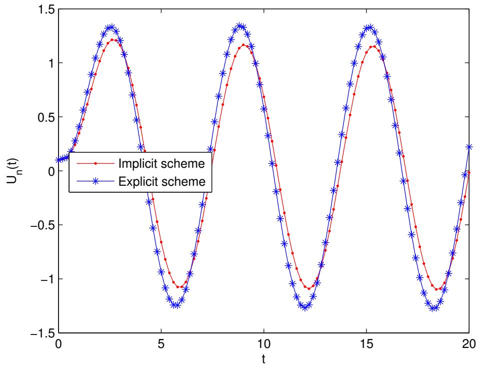

Figure 3. $\alpha = 0.75$.

5. Conclusions

In this work, the existence and stability of two difference schemes for nonhomogeneous fractional Cauchy problem are obtained in the space $C(E_n)$ using of the methods of numerical analysis and functional analysis. These approaches are efficient, simple and can be applied to analogous problems. In the near future, we will investigate the order of convergence of difference schemes and stability for problem (3) in suitable spaces.

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