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Raina's Function-Based Formulations of Right-Sided Simpson's and Newton's Inequalities for Generalized Coordinated Convex Functions

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Abstract: The main focus of this article is to derive some new counterparts to Simpson's and Newton's type inequalities involve a class of generalized coordinated convex mappings. This class contains several new and known classes of convexity as special cases. For further demonstration, we deploy the concept of right quantum derivatives to develop two new identities involving Raina's function. Moreover, by implementing these auxiliary results together with generalized convexity, we acquire a Holder-type inequality. We also acquire some applications of our main findings by making use of suitable substitutions in Raina's function.

Keywords: Raina; Simpson; Newton; convex; inequality

MSC: 05A30; 26A51; 26D07; 26D10; 26D15

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1. Introduction

Thomas Simpson, a pioneering mathematician from 1710–1761, made significant contributions to the field of numerical integration and the estimation of definite integrals, specifically through the development of Simpson's rule. A comparable approximation was employed by Kepler nearly a century earlier, leading to its alternate designation as Kepler's rule. Simpson's rule encompasses the three-point Newton–Cotes quadrature rule, giving rise to estimations that rely on a three-step quadratic kernel, which is occasionally referred to as Newton-type results.

(1) Simpson's quadrature formula (Simpson's 1/3 rule) is expressed as follows:

$$\int_{\pi_1}^{\pi_2} F(x)d(x) \approx \frac{\pi_2-\pi_1}{6}\left[F(\pi_1)+4F\left(\frac{\pi_1+\pi_2}{2}\right)+F(\pi_2)\right].$$

(1) Simpson's second formula, or the Newton–Cotes quadrature formula (Simpson's 3/8 rule), is expressed as follows:

$$\int_{\pi_1}^{\pi_2} F(x)d(x) \approx \frac{\pi_2-\pi_1}{8}\left[F(\pi_1)+3F\left(\frac{2\pi_1+\pi_2}{3}\right)+3F\left(\frac{\pi_1+2\pi_2}{3}\right)+F(\pi_2)\right].$$

A large variety of methods and techniques for estimations with these quadrature rules are available. One of them is Simpson's inequality, which can be stated in the following way:

Theorem 1. *Suppose that $F:[\pi_1,\pi_2]\to\mathbb{R}$ is a four times continuously differentiable mapping on (π_1,π_2), and let $\left\|F^{(4)}\right\|_\infty = \sup_{x\in(\pi_1,\pi_2)}\left|F^{(4)}(x)\right| < \infty$. Then, one has the inequality*

$$\left|\frac{1}{3}\left[\frac{F(\pi_1)+F(\pi_2)}{2}+2F\left(\frac{\pi_1+\pi_2}{2}\right)\right]-\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}F(x)d(x)\right|\leq\frac{1}{2880}\left\|F^{(4)}\right\|_{\infty}(\pi_2-\pi_1)^4.$$

The work of numerous mathematicians has been dedicated to the advancement of Simpson- and Newton-type results for convex functions. The theory of convexity has been instrumental in addressing various problems. Such a domain involves studying numerical integration quadrature formulas, where novel Simpson's inequalities and their far-reaching applications have been propounded by Dragomir et al. [1]. Furthermore, Alomari et al. [2] developed some new inequalities of Simpson-type s-convex functions. The classical concept of convexity also has a close connection with the concept of symmetry. In the literature, there exist numerous properties of symmetric convex sets. One fascinating aspect of this relationship is that we work on one and apply it to the other.

Mathematicians were inspired by Euler's groundbreaking work to establish a vital connection between physics and mathematics, paving the way for the development of quantum computing, quantum calculus, and quantitative computer mathematics. The impact of these innovations is wide-ranging, with implications for numerous mathematical disciplines such as combinatorics, number theory, polynomial orthogonal, and fundamental hypergeometric functions, as well as other scientific fields including mechanics, relativity theory, and quantum theory [3,4]. Quantum calculus leverages quantum information theory to address diverse applications spanning the domains of computer science, information theory, philosophy, and cryptography, thereby constituting a compelling interdisciplinary domain [5,6]. Subsequently, Jackson introduced q-calculus, which did not rely on limit calculus [7,8]. In 2013 the ${}_{\pi_1}D_q$difference operator and q_{π_1} integral were presented by Jessada Tariboon and Sotiris Ntouyas et al. [9]. After that, the expression of ${}^{\pi_2}D_q$ for the derivative and the q^{π_2}-integral were established by Bermudo et al. [10] in 2020. Sadjang et al. [11] generalized quantum calculus and post-quantum calculus and introduced new notions. Tunç and Göv introduced the post-quantum variant of the ${}_{\pi_1}D_q$ difference operator and q_{π_1} integral in [12]. The authors of [13] used the ${}_{\pi_1}D_q, {}^{\pi_2}D_q$ derivatives and q_{π_1}, q^{π_2} integrals in the proof of Hermite–Hadamard integral inequalities and left-right estimates for the coordinated convex and convex functions of these inequalities. The generalization of the results in [14] and the work on the proof of Hermite–Hadamard-type inequalities with the left estimates using a ${}_{\pi_1}D_{p,q}$ difference operator and $(p,q)_{\pi_1}$integral was carried out by Kunt et al. In [15] Noor has given quantum estimates for Hermite-Hadamard inequalities. Nwaeze et al. [16] proved some satisfying parameterized quantum integral inequalities for generalized qūasi-convex functions. The quantum Hermite–Hadamard inequality was proven by Khan et al. by using a green function [17]. In [18–21] authors have introduced the new quantum Simpson- and quantum Newton-type inequalities for convex and coordinated convex functions, respectively. The authors of [22–26] studied quantum Ostrowski inequalities for convex and coordinated convex functions. The notions of a ${}^{\pi_2}D_{p,q}$ difference operator and $(p,q)^{\pi_2}$-integral were employed by Latif and Chu in establishing their right estimates of Hermite–Hadamard-type inequalities [27]. These inequalities were used to prove Ostrowski's inequalities. For further interesting advancements in this field, see [28–33]. However, in [34] Latif gave q-analogues of Hermite-Hadamard inequality of functions of two variables on finite rectangles in the plane. In [35] Vivas-Cortez, introduced new quantum estimates of Trapezium-Type Inequalities for Generalized ϕ-Convex Functions.

Motivated by the above-mentioned works [20,35], in this study, we develop two new identities to obtain fresh error bounds for Simpson's one-third rule and Simpson's three-eighths rule named the Newton's rule, by coupling with the quantum integration convexity property of the functions over a rectangular domain. Our work presents a novel contribution by successfully integrating various existing and new findings from the literature, resulting in a significant unification of knowledge.

The rest of this study is structured as follows. Section 2 provides a concise overview of the history and applications of quantum calculus. Section 3 furnishes an abridged explanation of the terminologies and concepts underlying quantum calculus, as well as the relevant inequalities. Section 4 formulates the coordinated version of right quantum integral identities. Section 5 outlines the key findings, comprising coordinated versions of right quantum Simpson- and right quantum Newton-type inequalities, as well as related outcomes that were unearthed through appropriate fabrication. Section 6 presents the concluding remarks and highlights of this innovative research.

2. Quantum Calculus Preliminaries

In this section, we present some basic definitions and inequalities.

Definition 1 ([35]). *Let $\nu, \mu > 0$ and $\sigma = (\sigma(0), \ldots, \sigma(k), \ldots)$ be a bounded sequence of positive real numbers. A nonempty set Δ to be generalized convex if*

$$\pi_1 + t_1 \check{R}_{\mu,\nu,\sigma}(\pi_2 - \pi_1) \in \Delta, \quad \forall \pi_1, \pi_2 \in \Delta, \quad t \in [0,1].$$

Here, $\check{R}_{\mu,\nu,\sigma}(.)$ is Raina's function, which is as follows:

$$\check{R}_{\mu,\nu,\sigma}(\check{z}) = \check{R}_{\mu,\nu}^{\sigma(0),\sigma(1),\ldots}(\check{z}) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\mu + \nu)} \check{z}^{k}, \tag{1}$$

where $|\check{z}| < R$ and $\Gamma(.)$ is the well-known gamma function. For further details, see [36].

Definition 2 ([35]). *Let $\nu, \mu > 0$ and $\sigma = (\sigma(0), \ldots, \sigma(k), \ldots)$ be a bounded sequence of positive real numbers. A function $F : \Delta \to \mathbb{R}^+$ is said to be generalized convex if*

$$F(\pi_1 + t\check{R}_{\mu,\nu,\sigma}(\pi_2 - \pi_1) \leq (1-t)F(\pi_1) + tF(\pi_2), \quad \forall \pi_1, \pi_2 \in £, \quad t \in [0,1]. \tag{2}$$

Definition 3 ([37]). *For $\mu, \sigma > 0$ and $\sigma = (\sigma(0), \sigma(1), \ldots, \sigma(k), \ldots)$, this is assumed to be a bounded sequence of $\mathbb{R}^+$. A nonempty set Δ is known to be a coordinated generalized convex set*

$$\big(\pi_1 + t\check{R}_{\mu,\nu,\sigma}(\pi_2 - \pi_1), \pi_3 + s\check{R}_{\mu,\nu,\sigma}(\pi_4 - \pi_3)\big) \in \Delta \tag{3}$$

which holds for all $t, s \in [0,1]$, while $(\pi_1, \pi_3) \times (\pi_2, \pi_4) \in \Delta$ and $\check{R}^{\sigma}_{\mu,\nu}(.)$ denotes Raina's function.

Definition 4 ([37]). *For $\mu, \sigma > 0$ and $\sigma = (\sigma(0), \sigma(1), \ldots, \sigma(k), \ldots)$, this is assumed to be a bounded sequence of $\mathbb{R}^+$. A mapping $F : \Delta \to \mathbb{R}^+$ is said to be coordinated generalized convex if*

$$\begin{aligned} & F\big(\pi_1 + t\check{R}_{\mu,\nu,\sigma}(\pi_2 - \pi_1), \pi_3 + s\check{R}_{\mu,\nu,\sigma}(\pi_4 - \pi_3)\big) \\ \leq \; & tsF(\pi_2, \pi_4) + t(1-s)F(\pi_2, \pi_3) + (1-t)sF(\pi_1, \pi_3), \end{aligned} \tag{4}$$

holds for all $t, s \in [0,1]$ and $(\pi_1, \pi_3) \times (\pi_2, \pi_4) \in \Delta$.

Definition 5. *In [8], F. H. Jackson gave the q-Jackson integral from 0 to π_2 for $0 < q < 1$ as*

$$\int_0^{\pi_2} F(x) \; d_q(x) \; = (1-q)\pi_2 \sum_{n=0}^{\infty} q^n F(\pi_2 q^n), \tag{5}$$

provided that the sum converges absolutely. Moreover, he gave the q-Jackson integral in an arbitrary interval $[\pi_1, \pi_2]$ as follows:

$$\int_{\pi_1}^{\pi_2} F(x) \; d_q(x) \; = \int_0^{\pi_2} F(x) \; d_q(x) \; - \int_0^{\pi_1} F(x) \; d_q(x) \; .$$

Definition 6 ([9]). *For a continuous function $F : [\pi_1, \pi_2] \to \mathbb{R}$, the left quantum derivative (q_{π_1} − derivative) of F at $x \in [\pi_1, \pi_2]$ is as characterized below:*

$$_{\pi_1}D_q F(x) = \frac{F(x) - F(qx + (1-q)\pi_1)}{(1-q)(x - \pi_1)}, \; x \neq \pi_1. \tag{6}$$

For $x=\pi_1$, we state that $_{\pi_1}D_q F(\pi_1) = \lim_{x \to \pi_1} {}_{\pi_1}D_q F(x)$ if it exists and is finite.

Definition 7 ([10]). *For a continuous function $F : [\pi_1, \pi_2] \to \mathbb{R}$, the right quantum derivative (q^{π_2} − derivative) of F at $x \in [\pi_1, \pi_2]$ is characterized by the expression*

$$^{\pi_2}D_q F(x) = \frac{F(qx + (1-q)\pi_2) - F(x)}{(1-q)(\pi_2 - x)}, \; x \neq \pi_2. \tag{7}$$

For $x=\pi_2$, we state that $^{\pi_2}D_q F(\pi_2) = \lim_{x \to \pi_2} {}^{\pi_2}D_q F(x)$ if it exists and is finite.

Definition 8 ([9]). *Let $F : [\pi_1, \pi_2] \to \mathbb{R}$ be a continuous function. The left definite quantum integral (q_{π_1} − definite integral) on $[\pi_1, \pi_2]$ is defined as*

$$\begin{aligned} \int_{\pi_1}^{\pi_2} F(x) \; {}_{\pi_1}d_q(x) &= (1-q)(\pi_2 - \pi_1) \sum_{n=0}^{\infty} q^n F(q^n \pi_2 + (1-q^n)\pi_1) \\ &= (\pi_2 - \pi_1) \int_0^1 F((1-\tau)\pi_1 + \tau\pi_2) \; d_q(\tau) \; . \end{aligned} \tag{8}$$

On the other hand, S. Bermudo et al. [10] gave the following new definition:

Definition 9. *Let $F : [\pi_1, \pi_2] \to \mathbb{R}$ be a continuous function. The right definite quantum (q^{π_2} − definite integral) on $[\pi_1, \pi_2]$ is defined as*

$$\begin{aligned} \int_{\pi_1}^{\pi_2} F(x) \; {}^{\pi_2}d_q(x) &= (1-q)(\pi_2 - \pi_1) \sum_{n=0}^{\infty} q^n F(q^n \pi_1 + (1-q^n)\pi_2) \\ &= (\pi_2 - \pi_1) \int_0^1 F(\tau\pi_1 + (1-\tau)\pi_2) \; d_q(\tau) \; . \end{aligned} \tag{9}$$

For more details about q^{π_2} integrals and the corresponding inequalities, see [10].

Now, let us give the following notation, which will be used many times in the later sections (see [4]):

$$[n]_q = \frac{q^n - 1}{q - 1}.$$

Moreover, for our main results, we give the lemma

Lemma 1 ([9]). *We have the equality*

$$\int_{\pi_1}^{\pi_2} (x - \pi_1)^{\alpha} \; {}_{\pi_1}d_q(x) = \frac{(\pi_2 - \pi_1)^{\alpha+1}}{[\alpha + 1]_q}$$

for $\alpha \in \mathbb{R} \backslash \{-1\}$.

In [26], H. Budak et al. proved the following variant of the quantum Ostrowski inequality using the q_{π_1} and q^{π_2} integrals:

Theorem 2 ([26]). *Let $F : [\pi_1, \pi_2] \subset \mathbb{R} \to \mathbb{R}$ be a function and ${}^{\pi_2}D_q F$ and ${}_{\pi_1}D_q F$ be two continuous and integrable functions on $[\pi_1, \pi_2]$. If $\left| {}^{\pi_2}D_q \ F(\tau) \right| \le M$ and $\left| {}_{\pi_1}D_q \ F(\tau) \right| \le M$ for all $\tau \in [\pi_1, \pi_2]$, then we have the following quantum Ostrowski-type inequality for all $x \in [\pi_1, \pi_2]$:*

$$\left| F(x) - \frac{1}{\pi_2 - \pi_1} \left[\int_{\pi_1}^{x} F(\tau) \ {}_{\pi_1}d_q(\tau) \ + \int_{x}^{\pi_2} F(\tau) \ {}^{\pi_2}d_q(\tau) \ \right] \right| \le \frac{qM}{(\pi_2 - \pi_1)} \left[\frac{(x-\pi_1)^2 + (\pi_2 - x)^2}{[2]_q} \right] \tag{10}$$

where $0 < q < 1$.

In [38], H. Budak et al. gave the following definitions for the $q_{\pi_1\pi_3}$, $q_{\pi_1}^{\pi_4}$, $q_{\pi_2}^{\pi_3}$, and $q^{\pi_2\pi_4}$ integrals and the related inequalities of the quantum Hermite–Hadamard type:

Definition 10 ([34,38]). *Suppose that $F : [\pi_1, \pi_2] \times [\pi_3, \pi_4] \subset \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. Then, the following $q_{\pi_1\pi_3}$, $q_{\pi_1}^{\pi_4}$, $q_{\pi_3}^{\pi_2}$, and $q^{\pi_2\pi_4}$ integrals on $[\pi_1, \pi_2] \times [\pi_3, \pi_4]$ are defined by*

$$\begin{aligned} &\int_{\pi_1}^{x} \int_{\pi_3}^{y} F(\tau, s) \ {}_{\pi_3}d_{q_2}(s) \ \ {}_{\pi_1}d_{q_1}(\tau) \\ = \ & (1-q_1)(1-q_2)(x-\pi_1)(y-\pi_3) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m F(q_1^n x + (1-q_1^n)\pi_1, q_2^m y + (1-q_2^m)\pi_3); \end{aligned} \tag{11}$$

$$\begin{aligned} &\int_{\pi_1}^{x} \int_{y}^{\pi_4} F(\tau, s) \ \ d_{q_2}(s) \ \ {}_{\pi_1}d_{q_1}(\tau) \\ = \ & (1-q_1)(1-q_2)(x-\pi_1)(\pi_4 - y) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m F(q_1^n x + (1-q_1^n)\pi_1, q_2^m y + (1-q_2^m)\pi_4); \end{aligned} \tag{12}$$

$$\begin{aligned} &\int_{x}^{\pi_2} \int_{\pi_3}^{y} F(\tau, s) \ {}_{\pi_3}d_{q_2}(s) \ \ {}^{\pi_2}d_{q_1}(\tau) \\ = \ & (1-q_1)(1-q_2)(\pi_2 - x)(y-\pi_3) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m F(q_1^n x + (1-q_1^n)\pi_2, q_2^m y + (1-q_2^m)\pi_3); \end{aligned} \tag{13}$$

and

$$\begin{aligned} &\int_{x}^{\pi_2} \int_{y}^{\pi_4} F(\tau, s) \ \ d_{q_2}(s) \ \ {}^{\pi_2}d_{q_1}(\tau) \\ = \ & (1-q_1)(1-q_2)(\pi_2 - x)(\pi_4 - y) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m F(q_1^n x + (1-q_1^n)\pi_2, q_2^m y + (1-q_2^m)\pi_4). \end{aligned} \tag{14}$$

for $(x, y) \in [\pi_1, \pi_2] \times [\pi_3, \pi_4]$.

Definition 11 ([34]). *Let $F : [\pi_1, \pi_2] \times [\pi_3, \pi_4] \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a continuous function of two variables. The partial $q_1$derivatives, q_2 derivatives, and q_1q_2 derivatives at $(x, y) \in [\pi_1, \pi_2] \times [\pi_3, \pi_4]$ can be defined as*

$$
\begin{aligned}
\frac{{}_{\pi_1}\partial_{q_1} F(x,y)}{{}_{\pi_1}\partial_{q_1} x} &= \frac{F(q_1 x + (1-q_1)\pi_1, y) - F(x,y)}{(1-q_1)(x-\pi_1)},\ x \neq \pi_1; \\
\frac{{}_{\pi_3}\partial_{q_2} F(x,y)}{{}_{\pi_3}\partial_{q_2} y} &= \frac{F(x, q_2 y + (1-q_2)\pi_3) - F(x,y)}{(1-q_2)(y-\pi_3)},\ y \neq \pi_3; \\
\frac{{}_{\pi_1,\ \pi_3}\partial^2_{q_1,q_2} F(x,y)}{{}_{\pi_1}\partial_{q_1} x\ {}_{\pi_3}\partial_{q_2} y} &= \frac{1}{(x-\pi_1)(y-\pi_3)(1-q_1)(1-q_2)}[F(q_1 x + (1-q_1)\pi_1, q_2 y + (1-q_2)\pi_3) \\
&\quad - F(q_1 x + (1-q_1)\pi_1, y) - F(x, q_2 y + (1-q_2)\pi_3) + F(x,y)],\ x \neq \pi_1,\ y \neq \pi_3; \\
\frac{{}^{\pi_2}\partial_{q_1} F(x,y)}{{}^{\pi_2}\partial_{q_1} x} &= \frac{F(q_1 x + (1-q_1)\pi_2, y) - F(x,y)}{(1-q_1)(\pi_2 - x)},\ x \neq \pi_2; \\
\frac{{}^{\pi_4}\partial_{q_2} F(x,y)}{{}^{\pi_4}\partial_{q_2} y} &= \frac{F(x, q_2 y + (1-q_2)\pi_4) - F(x,y)}{(1-q_2)(\pi_4 - y)},\ y \neq \pi_4; \\
\frac{{}^{\pi_4}_{\pi_1}\partial^2_{q_1,q_2} F(x,y)}{{}_{\pi_1}\partial_{q_1} x\ {}^{\pi_4}\partial_{q_2} y} &= \frac{1}{(x-\pi_1)(\pi_4-y)(1-q_1)(1-q_2)}[F(q_1 x + (1-q_1)\pi_1, q_2 y + (1-q_2)\pi_4) \\
&\quad - F(q_1 x + (1-q_1)\pi_1, y) - F(x, q_2 y + (1-q_2)\pi_4) + F(x,y)],\ x \neq \pi_1,\ y \neq \pi_4; \\
\frac{{}^{\pi_2}_{\pi_3}\partial^2_{q_1,q_2} F(x,y)}{{}^{\pi_2}\partial_{q_1} x\ {}_{\pi_3}\partial_{q_2} y} &= \frac{1}{(\pi_2-x)(y-\pi_3)(1-q_1)(1-q_2)}[F(q_1 x + (1-q_1)\pi_2, q_2 y + (1-q_2)\pi_3) \\
&\quad - F(q_1 x + (1-q_1)\pi_2, y) - F(x, q_2 y + (1-q_2)\pi_3) + F(x,y)],\ x \neq \pi_2,\ y \neq \pi_3; \\
\frac{{}^{\pi_2,\ \pi_4}\partial^2_{q_1,q_2} F(x,y)}{{}^{\pi_2}\partial_{q_1} x\ {}^{\pi_4}\partial_{q_2} y} &= \frac{1}{(\pi_2-x)(\pi_4-y)(1-q_1)(1-q_2)}[F(q_1 x + (1-q_1)\pi_2, q_2 y + (1-q_2)\pi_4) \\
&\quad - F(q_1 x + (1-q_1)\pi_2, y) - F(x, q_2 y + (1-q_2)\pi_4) + F(x,y)],\ x \neq \pi_2,\ y \neq \pi_4.
\end{aligned}
$$

For clarity, we adopted the following notations to facilitate comprehension of this paper:

$$
\begin{aligned}
\check{R}^{\sigma_1}_{\mu_1,\nu_1}(\pi_1 - \pi_2) &= \acute{O}_1 \\
\check{R}^{\sigma_2}_{\mu_2,\nu_2}(\pi_3 - \pi_4) &= \acute{O}_2 \\
\check{R}^{\sigma_1}_{\mu_1,\nu_1}(\pi_2 - \pi_1) &= \grave{O}_1 \\
\check{R}^{\sigma_2}_{\mu_2,\nu_2}(\pi_4 - \pi_3) &= \grave{O}_2
\end{aligned} \tag{15}
$$

3. Identities

In this section, we prove two identities with the right quantum integral which are useful for the formulation of quantum Simpson- and quantum Newton-type inequalities.

Lemma 2. *Let $F : [\pi_2 + \acute{O}_1, \pi_2] \times [\pi_4 + \acute{O}_2, \pi_4] \subseteq \Delta \to \mathbb{R}$ be a twice partially $q_1 q_2$-differentiable function on $(\pi_2 + \acute{O}_1, \pi_2) \times (\pi_4 + \acute{O}_2, \pi_4)$ such that $\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2} F(t,s)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}$ is continuous and integrable. Then, we have the following identity:*

$$q_1q_2\grave{O}\grave{O}_2\int_0^1\int_0^1 \Lambda_{q_1}t\Lambda_{q_2}s\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+t\acute{O}_1\ ,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}\ t\ {}^{\pi_4}\partial_{q_2}s}\ {}^{\pi_2}d_{q_1}t\ {}^{\pi_4}d_{q_2}s \tag{16}$$

$$\begin{aligned}
=\ & \frac{1}{[6]_{q_1}[6]_{q_2}}\left[q_1q_2^2[4]_{q_2}F\left(\pi_2,\pi_4+\frac{\acute{O}_2}{[2]_{q_2}}\right)+q_1^2q_2^2[4]_{q_1}[4]_{q_2}F\left(\pi_2+\frac{\acute{O}_1}{[2]_{q_1}},\pi_4+\frac{\acute{O}_2}{[2]_{q2}}\right)\right.\\
& \left.+q_1^2[4]_{q_1}F\left(\pi_2+\frac{\acute{O}_1}{[2]_{q_1}},\pi_4+\acute{O}_2\right)+q_1^2q_2[4]_{q_1}F\left(\pi_2+\frac{\acute{O}_1}{[2]_{q_1}},\pi_4\right)+q_2^2[4]_{q_2}F\left(\pi_2+\acute{O}_1,\pi_4+\frac{\acute{O}_2}{[2]_{q_2}}\right)\right]\\
& +\frac{F(\pi_2+\acute{O}_1,\pi_4+\acute{O}_2)+q_1q_2F(\pi_2,\pi_4)+q_1F(\pi_2,\pi_4+\acute{O}_2)+q_2F(\pi_2+\acute{O}_1,\pi_4)}{[6]_{q_1}[6]_{q_2}}\\
& -\frac{1}{\grave{O}_1[6]_{q_2}}\int_{\pi_2+\acute{O}_1}^{\pi_2}\left[q_2F(x,\pi_4)+q_2^2[4]_{q_2}F\left(x,\pi_4+\frac{\acute{O}_2}{[2]_{q_2}}\right)+F(x,\pi_4+\acute{O}_2)\right]\ {}^{\pi_2}d_{q_1}(x)\\
& -\frac{1}{\grave{O}_2[6]_{q_1}}\int_{\pi_4+\acute{O}_2}^{\pi_4}\left[q_1F(\pi_2,y)+q_1^2[4]_{q_1}F\left(\pi_2+\frac{\acute{O}_1}{[2]_{q_1}},y\right)+F(\pi_2+\acute{O}_1,y)\right]\ {}^{\pi_4}d_{q_2}(y)\\
& +\frac{1}{\grave{O}_1\grave{O}_2}\int_{\pi_2+\acute{O}_1}^{\pi_2}\int_{\pi_4+\acute{O}_2}^{\pi_4}F(x,y)\ {}^{\pi_2}d_{q_1}x\ {}^{\pi_4}d_{q_2}(y),
\end{aligned}$$

where

$$\Lambda_{q_1}(t)=\begin{cases} t-\frac{1}{[6]_{q_1}}, & t\in\left[0,\frac{1}{[2]_{q_1}}\right),\\ t-\frac{[5]_{q_1}}{[6]_{q_1}}, & t\in\left[\frac{1}{[2]_{q_1}},1\right];\end{cases}\qquad \textit{and,}\qquad \Lambda_{q_2}(s)=\begin{cases} s-\frac{1}{[6]_{q_2}}, & s\in\left[0,\frac{1}{[2]_{q_2}}\right),\\ s-\frac{[5]_{q_2}}{[6]_{q_2}}, & s\in\left[\frac{1}{[2]_{q_2}},1\right].\end{cases}$$

Proof. By using the basic properties of the q integral and definitions $\Lambda_{q_1}(t)$ and $\Lambda_{q_2}(s)$, we have

$$\begin{aligned}
& \int_0^1\int_0^1 \Lambda_{q_1}(t)\Lambda_{q_2}(s)\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\ d_{q_1}(t)d_{q_2}(s)\\
=& \left(\frac{[5]_{q_2}-1}{[6]_{q_2}}\right)\left(\frac{[5]_{q_1}-1}{[6]_{q_1}}\right)\int_0^{\frac{1}{[2]_{q_1}}}\int_0^{\frac{1}{[2]_{q_2}}}\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}d_{q_2}(s)d_{q_1}(t)\\
& +\left(\frac{[5]_{q_2}-1}{[6]_{q_2}}\right)\int_0^{\frac{1}{[2]_{q_2}}}\int_0^1\left(t-\frac{[5]_{q_1}}{[6]_{q_1}}\right)\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}d_{q_2}(s)d_{q_1}(t)\\
& +\left(\frac{[5]_{q_1}-1}{[6]_{q_1}}\right)\int_0^{\frac{1}{[2]_{q_1}}}\int_0^1\left(s-\frac{[5]_{q_2}}{[6]_{q_2}}\right)\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}d_{q_2}(s)d_{q_1}(t)\\
& +\int_0^1\int_0^1\left(s-\frac{[5]_{q_2}}{[6]_{q_2}}\right)\left(t-\frac{[5]_{q_1}}{[6]_{q_1}}\right)\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}d_{q_2}(s)d_{q_1}(t)
\end{aligned}$$

$$\int_0^1\int_0^1 \Lambda_{q_1}(t)\Lambda_{q_2}(s)\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\ d_{q_1}(t)d_{q_2}(s)=I_1+I_2+I_3+I_4 \tag{17}$$

By using Definition 9 for the right quantum integrals, the quantum integral I_1 becomes

$$\begin{aligned}
I_1\ =\ & \frac{([5]_{q_1}-1)([5]_{q_2}-1)}{[6]_{q_1}[6]_{q_2}\grave{O}_1\grave{O}_2}\left[F\left(\pi_2+\frac{\acute{O}_1}{[2]_{q_1}},\pi_4+\frac{\acute{O}_2}{[2]_{q_2}}\right)+F(\pi_2,\pi_4)\right.\\
& \left.-F\left(\pi_2,\pi_4+\frac{\acute{O}_2}{[2]_{q_2}}\right)-F\left(\pi_2+\frac{\acute{O}_1}{[2]_{q_1}},\pi_4\right)\right].
\end{aligned}$$

Now, solving the quantum integrals involved in I_2 by using Definition 7 yields

$$\int_0^{\frac{1}{[2]_{q_2}}}\int_0^1 t\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+tÓ_1,\pi_4+sÓ_2)}{{}^{\pi_2}\partial_{q1}(t)\ {}^{\pi_4}\partial_{q_2}(s)}d_{q_1}sd_{q_2}t \tag{18}$$
$$= \frac{1}{Ò_1Ò_2}\left[\frac{1}{Ò_1q_1}\int_{\pi_2+Ó_1}^{\pi_2}F(x,\pi_4)\ {}^{\pi_2}d_{q_1}(x)+\frac{1}{q_1}F\left(\pi_2+Ó_1,\pi_4+\frac{Ó_2}{[2]_{q_2}}\right)\right.$$
$$\left.-\frac{1}{Ò_1q_1}\int_{\pi_2+Ó_1}^{\pi_2}F\left(x,\pi_4+\frac{Ó_2}{[2]_{q_2}}\right)\ {}^{\pi_2}d_{q_1}(x)-\frac{1}{q_1}F(\pi_2+Ó_1,\pi_4)\right].$$

By using similar arguments, we have

$$\int_0^{\frac{1}{[2]_{q_2}}}\int_0^1\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+tÓ_1,\pi_4+sÓ_2)}{{}^{\pi_2}\partial_{q1}(t)\ {}^{\pi_4}\partial_{q_2}(s)}d_{q_2}(s)d_{q_1}(t)=\frac{1}{Ò_1Ò_2}\left[F(\pi_2,\pi_4)-F\left(\pi_2,\pi_4+\frac{Ó_2}{[2]_{q_2}}\right)\right. \tag{19}$$
$$\left.-F(\pi_2+Ó_1,\pi_4)+F\left(\pi_2+Ó_1,\pi_4+\frac{Ó_2}{[2]_{q_2}}\right)\right].$$

By putting the values from Equations (18) and (19) into I_2, we have

$$I_2=\frac{[5]_{q_2}-1}{Ò_1\ {}_2R^{\sigma_2}_{u_2,v_2}(\pi_4,\pi_2)[6]_{q_2}}\left[\frac{1}{Ò_1q_1}\int_{\pi_2+Ó_1}^{\pi_2}F(x,\pi_4)\ {}^{\pi_2}d_{q_1}(x)-\frac{1}{q_1}F(\pi_2+Ó_1,\pi_4)\right.$$
$$\left.-\frac{1}{Ò_1q_1}\int_{\pi_2+Ó_1}^{\pi_2}F\left(x,\pi_4+\frac{Ó_2}{[2]_{q_2}}\right)^{\pi_2}d_{q_1}x+\frac{1}{q_1}F\left(\pi_2+Ó_1,\pi_4+\frac{Ó_2}{[2]_{q_2}}\right)\right]-\frac{[5]_{q_1}([5]_{q_2}-1)}{Ò_1Ò_2[6]_{q_1}[6]_{q_2}}$$
$$\times\left[F(\pi_2,\pi_4)-F\left(\pi_2,\pi_4+\frac{Ó_2}{[2]_{q_2}}\right)-F(\pi_2+Ó_1,\pi_4)+F\left(\pi_2+Ó_1,\pi_4+\frac{Ó_2}{[2]_{q_2}}\right)\right].$$

Similarly, I_3 is evaluated as

$$I_3=\frac{[5]_{q_1}-1}{Ò_1\ {}_2R^{\sigma_2}_{u_2,v_2}(\pi_4,\pi_2)[6]_{q_1}}\left[\frac{1}{Ò_2q_2}\int_{\pi_4+Ó_2}^{\pi_4}F(\pi_2,y)\ {}^{\pi_4}d_{q_2}(y)\right.$$
$$-\frac{F(\pi_2,\pi_4+Ó_2)}{q_2}-\frac{1}{Ò_2q_2}\int_{\pi_4+Ó_2}^{\pi_4}F\left(\pi_2+\frac{Ó_1}{[2]_{q_1}},y\right)\ {}^{\pi_4}d_{q_2}(y)$$
$$\left.+\frac{1}{q_2}F\left(\pi_2+\frac{Ó_1}{[2]_{q_1}},\pi_4+Ó_2\right)\right]-\frac{[5]_{q_2}([5]_{q_1}-1)}{Ò_1Ò_2[6]_{q_1}[6]_{q_2}}\left[F(\pi_2,\pi_4)\right.$$
$$\left.-F\left(\pi_2+\frac{Ó_1}{[2]_{q_1}},\pi_4\right)-F(\pi_2,\pi_4+Ó_2)+F\left(\pi_2+\frac{Ó_1}{[2]_{q_1}},\pi_4+Ó_2\right)\right].$$

Solving the quantum integrals in I_4 yields

$$\int_0^1\int_0^1\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+tÓ_1,\pi_4+sÓ_2)}{{}^{\pi_2}\partial_{q1}(t)\ {}^{\pi_4}\partial_{q_2}(s)}d_{q_2}(s)d_{q_1}(t)=\frac{1}{Ò_1Ò_2ts} \tag{20}$$

$$\left[F(\pi_2+Ó_1,\pi_4+Ó_2)+F(\pi_2,\pi_4)-F(\pi_2+Ó_1,\pi_4)\right.$$
$$\left.-F(\pi_2,\pi_4+Ó_2)\right]$$

$$\int_0^1\int_0^1 t\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+tÓ_1,\pi_4+sÓ_2)}{{}^{\pi_2}\partial_{q1}(t)\ {}^{\pi_4}\partial_{q_2}(s)}d_{q_2}(s)d_{q_1}(t)=\frac{1}{Ò_1Ò_2}$$

$$\times\left\{\frac{1}{\grave{O}_1\, q_1}\int_{\pi_2+\acute{O}_1}^{\pi_2} F(x,\pi_4)^{\pi_2}d_{q_1}(x)-\frac{F(\pi_2+\acute{O}_1,\pi_4)}{q_1}-\frac{1}{\grave{O}_1 q_1}\right.$$
$$\left.\times\int_{\pi_2+\acute{O}_1}^{\pi_2} F(x,\pi_4+\acute{O}_2)^{\pi_2}d_{q_1}(x)+\frac{F(\pi_2+\acute{O}_1,\pi_4+\acute{O}_2)}{q_1}\right\}$$

$$\int_0^1\int_0^1 s\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q1}(t)\;{}^{\pi_4}\partial_{q_2}(s)}d_{q_2}(s)d_{q_1}(t)=\frac{1}{\grave{O}_1\grave{O}_2} \tag{21}$$

$$\times\left\{\frac{1}{\grave{O}_2 q_2}\int_{\pi_4+\acute{O}_2}^{\pi_4} F(\pi_2,y)^{\pi_2}d_{q_2}(y)-\frac{1}{q_2}F(\pi_2,\pi_4+\acute{O}_2)-\frac{1}{\grave{O}_2 q_2}\right.$$
$$\left.\times\int_{\pi_4+\acute{O}_2}^{\pi_4} F(\pi_2+\acute{O}_1,y)^{\pi_2}d_{q_2}(y)+\frac{1}{q_2}F(\pi_2+\acute{O}_1,\pi_4+\acute{O}_2)\right\}.$$

In addition, we have

$$\int_0^1\int_0^1 ts\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q1}(t)\;{}^{\pi_4}\partial_{q_2}(s)}d_{q_2}(s)d_{q_1}(t)=\frac{1}{\grave{O}_1\grave{O}_2} \tag{22}$$

$$\times\left[\frac{1}{\grave{O}_1\grave{O}_2 q_1 q_2}\int_{\pi_2+\acute{O}_1}^{\pi_2}\int_{\pi_4+\acute{O}_2}^{\pi_4} F(x,y)\;{}^{\pi_2}d_{q_1}(x)\;{}^{\pi_4}d_{q_2}(y)-\frac{1}{q_1 q_2\grave{O}_1}\int_{\pi_2+\acute{O}_1}^{\pi_2} F(x,\pi_4+\acute{O}_2)^{\pi_2}d_{q_1}(x)\right.$$
$$\left.+\frac{1}{q_1q_2}F(\pi_2+\acute{O}_1,\pi_4+\acute{O}_2)-\frac{1}{q_1q_2\,\grave{O}_2}\int_{\pi_4+\acute{O}_2}^{\pi_4} F(\pi_2+\acute{O}_1,x)\;{}^{\pi_4}d_{q_2}(y)\right].$$

Using (20)–(22), we obtain

$$I_4=\frac{1}{\grave{O}_1\grave{O}_2}\left[\left\{\frac{1}{q_1q_2\grave{O}_1\grave{O}_2}\int_{\pi_2+\acute{O}_1}^{\pi_2}\int_{\pi_4+\acute{O}_2}^{\pi_4} F(x,y)^{\pi_2}d_{q_1}(x)\;{}^{\pi_4}d_{q_2}(y)-\frac{1}{q_1q_2\grave{O}_1}\int_{\pi_2+\acute{O}_1}^{\pi_2}\right.\right.$$
$$\left.\times F(x,\pi_4+\acute{O}_2)^{\pi_2}d_{q_1}(x)+\frac{F(\pi_2+\acute{O}_1,\pi_4+\acute{O}_2)}{q_1q_2}-\frac{1}{\grave{O}_2}\int_{\pi_4+\acute{O}_2}^{\pi_4} F(\pi_2+\acute{O}_1,y)^{\pi_2}d_{q_2}(y)\right\}$$
$$-\frac{[5]_{q_2}}{[6]_{q_2}}\left\{\frac{1}{\grave{O}_1 q_1}\int_{\pi_4+\acute{O}_2}^{\pi_4} F(\pi_2,y)\;{}^{\pi_4}d_{q_2}(y)-\frac{1}{\grave{O}_2 q_2}\int_{\pi_4+\acute{O}_2}^{\pi_4} F(\pi_2+\acute{O}_1,y)^{\pi_4}d_{q_2}(y)\right.$$
$$\left.+\frac{1}{q_2}F(\pi_2+\acute{O}_1,\pi_4+\acute{O}_2)-\frac{1}{q_2}F(\pi_2,\pi_4+\acute{O}_2)\right\}-\frac{[5]_{q_2}}{[6]_{q_2}}\left\{\frac{1}{\acute{O}_1 q_1}\int_{\pi_2+\acute{O}_1}^{\pi_2} F(x,\pi_4)^{\pi_2}d_{q_1}(x)\right.$$
$$\left.-\frac{1}{q_1}F(\pi_2+\acute{O}_1,\pi_4)+\frac{1}{q_1}F(\pi_2+\acute{O}_1,\pi_4+\acute{O}_2)-\frac{1}{\acute{O}_1 q_1}\int_{\pi_2+\acute{O}_1}^{\pi_2} F(x,\pi_4+\acute{O}_2)^{\pi_2}d_{q_1}(x)\right\}$$
$$\left.+\frac{[5]_{q_1}[5]_{q_2}}{\grave{O}_1\grave{O}_2[6]_{q_1}[6]_{q_2}}F(\pi_2,\pi_4)-F(\pi_2+\acute{O}_1,\pi_4)-F(\pi_2,\pi_4+\acute{O}_2)+F(\pi_2+\acute{O}_1,\pi_4+\acute{O}_2)\right].$$

By using integrals (I_1-I_4) in Equation (17) and multiplying $\grave{O}_1\grave{O}_2$ on both sides, and after some simplification, we obtain the required result. □

Remark 1. *If we set* $\grave{O}_1=\pi_2-\pi_1$, $\acute{O}_1=\pi_1-\pi_2$, $\acute{O}_2=(\pi_3-\pi_4)$, *and* $\grave{O}_2=\pi_4-\pi_3$ *in Lemma 2, then we have an identity:*

$$q_1q_2(\pi_2-\pi_1)(\pi_4-\pi_3)\int_0^1\int_0^1\Lambda_{q_1}(t)\Lambda_{q_2}(s)\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+t(\pi_1-\pi_2),\pi_4+s(\pi_3-\pi_4))}{{}^{\pi_2}\partial_{q1}(t)\ {}^{\pi_4}\partial_{q_2}(s)}d_{q_1}(t)d_{q_2}(s)=\frac{1}{[6]_{q_1}[6]_{q_2}}$$
$$\Bigg[q_1^2q_2[4]_{q_1}F\left(\pi_2+\frac{\pi_1-\pi_2}{[2]_{q_1}},\pi_4\right)+q_1q_2^2[4]_{q_2}F\left(\pi_2,\pi_4+\frac{\pi_3-\pi_4}{[2]_{q_2}}\right)+q_2^2[4]_{q_2}F\left(\pi_1,\pi_4+\frac{\pi_3-\pi_4}{[2]_{q_2}}\right)$$
$$+q_1^2q_2^2[4]_{q_1}[4]_{q_2}F\left(\pi_2+\frac{\pi_1-\pi_2}{[2]_{q_1}},\pi_4+\frac{\pi_3-\pi_4}{[2]_{q2}}\right)+q_1^2[4]_{q_1}F\left(\pi_2+\frac{\pi_1-\pi_2}{[2]_{q_1}},\pi_3\right)\Bigg]$$
$$-\frac{1}{[6]_{q_2}(\pi_2-\pi_1)}\int_{\pi_1}^{\pi_2}\left[q_2F(x,\pi_4)+q_2^2[4]_{q_2}F\left(x,\pi_4+\frac{\pi_3-\pi_4}{[2]_{q_2}}\right)+F(x,\pi_3)\right]^{\pi_2}d_{q_1}(x)$$
$$-\frac{1}{[6]_{q_1}(\pi_2-\pi_1)}\int_{\pi_3}^{\pi_4}\left[q_1F(\pi_2,y)+q_1^2[4]_{q_1}F\left(\pi_2+\frac{\pi_1-\pi_2}{[2]_{q1}},y\right)+F(\pi_1,y)\right]^{\pi_2}d_{q_2}(y)$$
$$+\frac{q_1F(\pi_2,\pi_3)+F(\pi_1,\pi_3))+q_2F(\pi_1,\pi_4)+q_1q_2F(\pi_2,\pi_4)}{[6]_{q_1}[6]_{q_2}}+\frac{\int_{\pi_1}^{\pi_2}\int_{\pi_3}^{\pi_4}F(x,y)^{\pi_2}d_{q_1}(x)^{\pi_4}d_{q_2}(y)}{(\pi_2-\pi_1)(\pi_4-\pi_3)},$$

where

$$\Lambda_{q_1}(t)=\begin{cases}t-\frac{1}{[6]_{q_1}}, & t\in\left[0,\frac{1}{[2]_{q_1}}\right),\\ t-\frac{[5]_{q_1}}{[6]_{q_1}}, & t\in\left[\frac{1}{[2]_{q_1}},1\right];\end{cases}\quad \textit{and,}\quad \Lambda_{q_2}(s)=\begin{cases}s-\frac{1}{[6]_{q_2}}, & s\in\left[0,\frac{1}{[2]_{q_2}}\right),\\ s-\frac{[5]_{q_2}}{[6]_{q_2}}, & s\in\left[\frac{1}{[2]_{q_2}},1\right].\end{cases}$$

Lemma 3. *Let $F:[\pi_2+Ó_1,\pi_2]\times[\pi_4+Ó_2,\pi_4]\subseteq\Delta\to\mathbb{R}$ be a twice partially q_1q_2-differentiable function on $(\pi_2+Ó_1,\pi_2)\times(\pi_4+Ó_2,\pi_4)$ such that the partial q_1q_2 derivative $\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(t,s)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}$ is continuous and integrable on $[\pi_2+Ó_1,\pi_2]\times[\pi_4+Ó_2,\pi_4]$, where $0\le q_1<1$ and $0\le q_1<1$. Then, the following identity holds:*

$$q_1q_2Ò_1Ò_2\int_0^1\int_0^1\Lambda_{q_1}(t)\Lambda_{q_2}(s)\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1q_2}F(\pi_2+tÓ_1,\pi_4+sÓ_2)}{\partial_{q_1}(t)\partial_{q_2}(s)}\ {}^{\pi_1}d_{q_1}(t)\ {}^{\pi_2}d_{q_2}(s) \tag{23}$$
$$=\frac{1}{Ò_1\,Ò_2}\int_{\pi_2+\eta_1}^{\pi_2}\int_{\pi_4+\eta_2}^{\pi_4}F(x,y)\ {}^{\pi_2}d_{q_1}(x)\ {}^{\pi_4}d_{q_2}(y)-\frac{1}{Ò_1[8]_{q_2}}\int_{\pi_2+\eta_1}^{\pi_2}\big[F(x,\pi_4+Ó_2)$$
$$+\frac{q_2^2[6]_{q_2}}{[2]_{q_2}}F\left(x,\pi_4+\frac{[2]_{q_2}Ó_2}{[3]_{q_2}}\right)+q_2F(x,\pi_4)+\frac{q_2^3[6]_{q_2}}{[2]_{q_2}}F\left(x,\pi_4+\frac{Ó_2}{[3]_{q_2}}\right)\Bigg]\ {}^{\pi_2}d_{q_1}(x)$$
$$-\frac{1}{Ò_2[8]_{q_1}}\int_{\pi_4+\eta_2}^{\pi_4}\Bigg[\frac{q_1^3[6]_{q_1}}{[2]_{q_1}}F\left(\pi_2+\frac{Ó_1}{[3]_{q_1}},y\right)+F(\pi_2+Ó_2,y)+q_1F(\pi_2,y)+\frac{q_1^2[6]_{q_1}}{[2]_{q_1}}$$
$$\times F\left(\pi_2+\frac{[2]_{q_1}Ó_1}{[3]_{q_1}},y\right)d_{q_2}(y)\Bigg]\ {}^{\pi_4}d_{q_2}(y)+\frac{1}{[8]_{q_2}[8]_{q_1}}\Bigg[\frac{q_2^3[6]_{q_2}}{[2]_{q_2}}F\left(\pi_2,\pi_4+\frac{Ó_2}{[3]_{q_2}}\right)+q_2F(\pi_2,\pi_4)$$
$$+F(\pi_2,\pi_4+Ó_2)+\frac{q_2^2[6]_{q_2}}{[2]_{q_2}}F\left(\pi_2,\pi_4+\frac{[2]_{q_2}Ó_2}{[3]_{q_2}}\right)\Bigg]+\frac{1}{[8]_{q_1}[8]_{q_2}}\big[F(\pi_2+Ó_1,\pi_4)$$
$$+\frac{q_1^3[6]_{q_1}}{[2]_{q_1}}F\left(\pi_2+\frac{Ó_1}{[3]_{q_1}},\pi_4\right)+\frac{q_1^2[6]_{q1}}{[2]_{q_1}}F\left(\pi_2+\frac{[2]_{q_1}Ó_1}{[3]_{q_1}},\pi_4\right)+F(\pi_2+Ó_1,\pi_4+Ó_2)\Big],$$

where

$$\Lambda_{q_1}(t)=\begin{cases} t-\frac{1}{[8]_{q_1}}, & if \quad t\in\left[0,\frac{1}{[3]_{q_1}}\right),\\ t-\frac{1}{[2]_{q_1}}, & if \quad t\in\left[\frac{1}{[3]_{q_1}},\frac{[2]_{q_1}}{[3]_{q_1}}\right),\\ t-\frac{[7]_{q_1}}{[8_{q_1}}, & if \quad t\in\left[\frac{[2]_{q_1}}{[3]_{q_1}},1\right]; \end{cases} \quad and, \quad \Lambda_{q_2}(s)=\begin{cases} t-\frac{1}{[8]_{q_2}}, & if \quad s\in\left[0,\frac{1}{[3]_{q_2}}\right),\\ t-\frac{1}{[2]_{q_2}}, & if \quad s\in\left[\frac{1}{[3]_{q_2}},\frac{[2]_{q_2}}{[3]_{q_2}}\right),\\ t-\frac{[7]_{q_2}}{[8_{q_2}}, & if \quad s\in\left[\frac{[2]_{q_2}}{[3]_{q_2}},1\right]. \end{cases}$$

Proof. Through the fundamental properties of q integrals and $\Lambda_{q_2}(s)$, $\Lambda_{q_1}(t)$, we obtain that

$$\begin{aligned}
&\int_0^1\int_0^1 \Lambda_{q_1}(t)\Lambda_{q_2}(s)\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}d_{q_1}(t)d_{q_2}(s) \qquad (24)\\
=&\frac{[8]_{q_2}-[2]_{q_2}}{[8]_{q_2}[2]_{q_2}}\int_0^{\frac{1}{[3]_{q_2}}}\left[\frac{[8]_{q_1}-[2]_{q_1}}{[8]_{q_1}[2]_{q_1}}\int_0^{\frac{1}{[3]_{q_1}}}\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}d_{q_1}(t)\right.\\
&+\frac{[7]_{q_1}[2]_{q_1}-[8]_{q_1}}{[8]_{q_1}[2]_{q_1}}\int_0^{\frac{[2]_{q_1}}{[3]_{q_1}}}\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}d_{q_1}(t)\\
&\left.+\int_0^1\left(t-\frac{[7]_{q_1}}{[8]_{q_1}}\right)\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}d_{q_1}(t)\right]d_{q_2}(s)\\
&+\frac{[7]_{q_2}[2]_{q_2}-[8]_{q_2}}{[8]_{q_2}[2]_{q_2}}\int_0^{\frac{[2]_{q_2}}{[3]_{q_2}}}\left[\frac{[8]_{q_1}-[2]_{q_1}}{[8]_{q_1}[2]_{q_1}}\int_0^{\frac{1}{[3]_{q_1}}}\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}d_{q_1}(t)\right.\\
&+\frac{[7]_{q_1}[2]_{q_1}-[8]_{q_1}}{[8]_{q_1}[2]_{q_1}}\int_0^{\frac{[2]_{q_1}}{[3]_{q_1}}}\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}d_{q_1}(t)\\
&\left.+\int_0^1\left(t-\frac{[7]_{q_1}}{[8]_{q_1}}\right)\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}d_{q_1}(t)\right]d_{q_2}(s)\\
&+\int_0^1\left(s-\frac{[7]_{q_2}}{[8]_{q_2}}\right)\left[\frac{[8]_{q_1}-[2]_{q_1}}{[8]_{q_1}[2]_{q_1}}\int_0^{\frac{1}{[3]_{q_1}}}\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}d_{q_1}(t)\right.\\
&+\frac{[7]_{q_1}[2]_{q_1}-[8]_{q_1}}{[8]_{q_1}[2]_{q_1}}\int_0^{\frac{[2]_{q_1}}{[3]_{q_1}}}\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}d_{q_1}(t)\\
&\left.+\int_0^1\left(t-\frac{[7]_{q_1}}{[8]_{q_1}}\right)\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}d_{q_1}(t)\right]d_{q_2}(s).
\end{aligned}$$

By following the arguments similar to those which were used in the proof of Lemma 2 and multiplying with $\grave{O}_1\grave{O}_2$ on both sides, we have the required identity. □

Remark 2. *If we replace* $\grave{O}_1{=}\pi_2-\pi_1$, $\acute{O}_1{=}\pi_1-\pi_2$, $\acute{O}_2{=}\pi_3-\pi_4$, *and* $\grave{O}_2{=}\pi_4-\pi_3$ *in Lemma 3, then we have*

$$\begin{aligned}
&q_1q_2(\pi_2-\pi_1)(\pi_4-\pi_3)\int_0^1\int_0^1\Lambda_{q_1}(t)\Lambda_{q_2}(s)\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}d_{q_1}(t)d_{q_2}(s)\\
=&\frac{1}{(\pi_2-\pi_1)(\pi_4-\pi_3)}\int_{\pi_1}^{\pi_2}\int_{\pi_3}^{\pi_4}F(x,y)\,{}^{\pi_2}d_{q_1}(x)\,{}^{\pi_4}d_{q_2}(y)-\frac{1}{(\pi_2-\pi_1)[8]_{q_2}}\int_{\pi_1}^{\pi_2}\left[\frac{q_2^3[6]_{q_2}}{[2]_{q_2}}F\left(x,\pi_4+\frac{(\pi_3-\pi_4)}{[3]_{q_2}}\right)\right.\\
&\left.+q_2F(x,\pi_4)+\frac{q_2^2[6]_{q_2}}{[2]_{q_2}}F\left(x,\pi_4+\frac{[2]_{q_2}(\pi_3-\pi_4)}{[3]_{q_2}}\right)d_{q_1}(x)+F(x,\pi_3)\right]{}^{\pi_2}d_{q_1}(x)-\frac{1}{(\pi_4-\pi_3)[8]_{q_1}}\int_{\pi_4+\eta_2}^{\pi_4}
\end{aligned}$$

$$\times\left[F(\pi_3,y)+\frac{q_1^3[6]_{q_1}}{[2]_{q_1}}F\left(\pi_2+\frac{(\pi_1-\pi_2)}{[3]_{q_1}},y\right)+\frac{q_1^2[6]_{q_1}}{[2]_{q_1}}F\left(\pi_2+\frac{[2]_{q_1}(\pi_3-\pi_4)}{[3]_{q_1}},y\right)d_{q_2}(y)+q_1F(\pi_2,y)\right]{}^{\pi_4}d_{q_2}(y)$$

$$+\frac{1}{[8]_{q_2}[8]_{q_1}}\left[F(\pi_2,\pi_3)+\frac{q_2^3[6]_{q_2}}{[2]_{q_2}}F\left(\pi_2,\pi_4+\frac{(\pi_3-\pi_4)}{[3]_{q_2}}\right)+\frac{q_2^2[6]_{q_2}}{[2]_{q_2}}F\left(\pi_2,\pi_4+\frac{[2]_{q_2}(\pi_3-\pi_4)}{[3]_{q_2}}\right)+q_2F(\pi_2,\pi_4)\right]$$

$$+\frac{1}{[8]_{q_1}[8]_{q_2}}\left[F(\pi_1,\pi_4)+\frac{q_1^3[6]_{q_1}}{[2]_{q_1}}F\left(\pi_2+\frac{(\pi_1-\pi_2)}{[3]_{q_1}},\pi_4\right)+\frac{q_1^2[6]_{q1}}{[2]_{q_1}}F\left(\pi_2+\frac{[2]_{q_1}(\pi_1-\pi_2)}{[3]_{q_1}},\pi_4\right)+F(\pi_1,\pi_3)\right]$$

4. Main Results

Within this section, we present fresh inequalities of the Simpson and Newton types for generalized coordinated convex functions by utilizing Lemmas 2 and 3, respectively. We shall commence this segment by introducing certain concepts that will be useful in verifying our outcomes:

$$A_1(q)=\frac{2q^2[2]_q^2+[6]_q^2\left([6]_q-[3]_q\right)}{[2]_q^3[3]_q[6]_q^3}, \tag{25}$$

$$B_1(q)=2\frac{q[3]_q[6]_q-q^2}{[2]_q[3]_q[6]_q^3}+\frac{1}{[2]_q^3}\left(\frac{q+q^2}{[3]_q}-\frac{q^2+2q}{[6]_q}\right), \tag{26}$$

$$A_2(q)=\frac{2q^2[5]_q^3}{[2]_q[3]_q[6]_q^3}+\frac{[6]_q\left(1+[2]_q^3\right)-[3]_q[5]_q\left(1+[2]_q^2\right)}{[2]_q^3[3]_q[6]_q}, \tag{27}$$

$$B_2(q)=2\frac{q[5]_q^2[6]_q[3]_q-q^2[5]_q^3}{[2]_q[3]_q[6]_q^3}+\frac{q^2}{[2]_q[3]_q}-\frac{q[5]_q}{[2]_q[6]_q}-\frac{1}{[2]_q^3}\left[\frac{[5]_q(q^2+2q)}{[6]_q}-\frac{q+q^2}{[3]_q}\right], \tag{28}$$

$$A_3(q)=\frac{2q^2[3]_q^3+[8]_q^2\left([8]_q[2]_q-[3]_q^2\right)}{[2]_q[3]_q^4[8]_q^3}, \tag{29}$$

$$B_3(q)=2\frac{q[8]_q[3]_q-q^2}{[2]_q[3]_q[8]_q^3}+\frac{[3]_q^2-[2]_q}{[2]_q[3]_q^4}+\frac{1-[2]_q[3]_q}{[2]_q[3]_q^2[2]_q}, \tag{30}$$

$$A_4(q)=\frac{2q^2}{[2]_q^4[3]_q}+\frac{[2]_q^2(1+[2]_q^3)-[3]_q^2(1+[2]_q^2)}{[2]_q^2[3]_q^4}, \tag{31}$$

$$B_4(q)=\frac{2q}{[2]_q^3}-\frac{q}{[3]_q^2}-\frac{q^2}{[3]_q^2}-A_4(q), \tag{32}$$

$$A_5(q)=\frac{2q^2[7]_q^3}{[2]_q[3]_q[8]_q^3}+\frac{[8]_q[2]_q\left([3]_q^3+[2]_q^3\right)-[7]_q[3]_q^2\left([2]_q^2+[3]_q^2\right)}{[2]_q[3]_q^4[8]_q}, \tag{33}$$

$$B_5(q)=2\frac{q[8]_q[7]_q^2[3]_q-q^2[7]_q^3}{[2]_q[3]_q[8]_q^3}+\frac{q^2}{[2]_q[3]_q}-\frac{q[7]_q}{[2]_q[8]_q}+\frac{[2]_q([3]_q^2-[2]_q^2)}{[3]_q^4}-\frac{(q^2+q)[2]_q[7]_q}{[8]_q[3]_q^2}. \tag{34}$$

4.1. Simpson-Type Inequalities

Theorem 3. *We assume that the conditions of Lemma 2 hold. If* $\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|$ *is a generalized convex function and integrable on* Δ*, then the following inequality holds for the right quantum integrals:*

$$\Bigg|\frac{1}{[6]_{q_1}[6]_{q_2}}\Bigg[q_1^2[4]_{q_1}F\left(\pi_2+\frac{\acute{O}_1}{[2]_{q_1}},\pi_4+\acute{O}_2\right)+q_1^2q_2[4]_{q_1}F\left(\pi_2+\frac{\acute{O}_1}{[2]_{q_1}},\pi_4\right)$$
$$+q_1^2q_2^2[4]_{q_1}[4]_{q_2}F\left(\pi_2+\frac{\acute{O}_1}{[2]_{q_1}},\pi_4+\frac{\acute{O}_2}{[2]_{q2}}\right)+q_1q_2^2[4]_{q_2}F\left(\pi_2,\pi_4+\frac{\acute{O}_2}{[2]_{q_2}}\right)+q_2^2[4]_{q_2}$$
$$\times F\left(\pi_2+\acute{O}_1,\pi_4+\frac{\acute{O}_2}{[2]_{q2}}\right)\Bigg]+\frac{1}{[6]_{q_1}[6]_{q_2}}\left[\,q_1F(\pi_2,\pi_4+\acute{O}_2)+q_2F(\pi_2+\acute{O}_1,\pi_4)\right.$$
$$\left.+F(\pi_2+\acute{O}_1,\pi_4+\acute{O}_2)+q_1q_2F(\pi_2,\pi_4)\,\right]-\frac{1}{[6]_{q_2}\acute{O}_1}\int_{\pi_2+\acute{O}_1}^{\pi_2}\Big[q_2F(x,\pi_4)$$
$$+F(x,\pi_4+\acute{O}_2)+q_2^2[4]_{q_2}F\left(x,\pi_4+\frac{\acute{O}_2}{[2]_{q_2}}\right)\Bigg]\,{}^{\pi_2}d_{q_1}(x)-\frac{1}{[6]_{q_1}\grave{O}_2}\int_{\pi_4+\acute{O}_2}^{\pi_4}$$
$$\times\left[q_1F(\pi_2,y)+q_1^2[4]_{q_1}F\left(\pi_2+\frac{\acute{O}_1}{[2]_{q1}},y\right)+F(\pi_2+\acute{O}_1,y)\right]{}^{\pi_4}d_{q_2}(y)$$
$$+\frac{1}{\grave{O}_1\grave{O}_2}\int_{\pi_2+\acute{O}_1}^{\pi_2}\int_{\pi_4+\acute{O}_2}^{\pi_4}F(x,y)^{\pi_2}d_{q_1}(x)^{\pi_4}d_{q_2}(y)\Bigg|$$

$$\leqslant q_1q_2\grave{O}_1\grave{O}_2\Bigg[(A_1(q_1)+A_2(q_1))(A_1(q_2)+A_2(q_2))\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|$$
$$+(B_1(q_2)+B_2(q_2))(A_1(q_1)+A_2(q_1))\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|+(B_1(q_1)+B_2(q_1))(A_1(q_2)+A_2(q_2))$$
$$\times\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|+(B_1(q_1)+B_2(q_1))(B_1(q_2)+B_2(q_2))\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|\Bigg].$$

Proof. By taking the modulus on the right side of the inequality in Lemma 2 and using the properties of the modulus along with the definition of pre-invex, we have the inequality

$$\left|q_1q_2\grave{O}_1\grave{O}_2\int_0^1\int_0^1\Lambda_{q_1}(t)\Lambda_{q_2}(s)\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}d_{q_1}(t)d_{q_2}(s)\right| \tag{35}$$
$$\leqslant q_1q_2\grave{O}_1\grave{O}_2\int_0^1\Lambda_{q_2}(s)\left[\int_0^1\Lambda_{q_1}(t)t\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|\right.$$
$$\left.+(1-t)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|d_{q_1}(t)\right]d_{q_2}(s).$$

Using the definitions of $\Lambda_{q_1}(t)$ and $\Lambda_{q_2}(s)$, the integral on the right side of Equation (35) yields

$$\int_0^1 \Lambda_{q_2}(s)\left[\{A_1(q_1)+A_2(q_1)\}\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|+\{B_1(q_1)+B_2(q_1)\}\right.$$
$$\times\left.\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|\right]d_{q_2}(s)\leqslant A_1(q_2)\left[\{A_1(q_1)+A_2(q_1)\}\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|\right.$$
$$\left.+\{B_1(q_1)+B_2(q_1)\}\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|\right]+B_1(q_2)\left[\{(A_1(q_1)+A_2(q_1))\}\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|\right.$$
$$\left.+\{B_1(q_1)+B_2(q_1)\}\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|\right]+A_2(q_2)\left[\{A_1(q_1)+A_2(q_1)\}\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|\right.$$
$$\left.+\{B_1(q_1)+B_2(q_1)\}\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|\right]+B_2(q_2)\left[\{A_1(q_1)+A_2(q_1)\}\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|\right.$$
$$\left.+\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|\{B_1(q_1)+B_2(q_1)\}\right],$$

Multiplying $q_1q_2\grave{O}_1\grave{O}_2$ on both sides and some simple shuffling gives the required result. □

Corollary 1. *If we have $q\to 1^-$, then Theorem 3 gives us the following quantum Simpson-type inequality:*

$$\left|\frac{1}{9}\left[F\left(\pi_2+\tfrac{\acute{O}_1}{2},\pi_4+\acute{O}_2\right)+F\left(\pi_2+\acute{O}_1,\pi_4+\tfrac{\acute{O}_2}{2}\right)+4F\left(\pi_2+\tfrac{\acute{O}_1}{2},\pi_4+\tfrac{\acute{O}_2}{2}\right)\right.\right.$$
$$\left.+F\left(\pi_2,\pi_4+\tfrac{\acute{O}_2}{2}\right)+F\left(\pi_2+\tfrac{\acute{O}_1}{2},\pi_4\right)\right]+\frac{1}{36}\left[F(\pi_2,\pi_4+\acute{O}_2)+F(\pi_2+\acute{O}_1,\pi_4+\acute{O}_2)\right.$$
$$\left.+F(\pi_2+\acute{O}_1,\pi_4)+F(\pi_2,\pi_4)\right]-\frac{1}{6\eta_1(\pi_1,\pi_2)}\int_{\pi_2+\acute{O}_1}^{\pi_2}\left[F(x,\pi_4)+4F\left(x,\pi_4+\tfrac{\acute{O}_2}{2}\right)\right.$$
$$\left.+F(x,\pi_4+\acute{O}_2)\right]\,{}^{\pi_2}d_{q_1}(x)-\frac{1}{6\eta_2(\pi_1,\pi_2)}\int_{\pi_4+\acute{O}_2}^{\pi_4}\left[+4F\left(\pi_2+\tfrac{\acute{O}_1}{2},y\right)+F(\pi_2+\acute{O}_1,y)\right.$$
$$\left.\left.+F(\pi_2,y)\right]\,{}^{\pi_4}d_{q_2}(y)+\tfrac{1}{\grave{O}_1\grave{O}_2}\int_{\pi_2+\acute{O}_1}^{\pi_2}\int_{\pi_4+\acute{O}_2}^{\pi_4}F(x,y)\,{}^{\pi_2}d_{q_1}(x)\,{}^{\pi_4}d_{q_2}(y)\right|\leqslant\grave{O}_1\ \grave{O}_2$$
$$\times\frac{25}{5184}\left[\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3))}{{}^{\pi_2}\partial(t)\,{}^{\pi_4}\partial(s)}\right|+\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4))}{{}^{\pi_2}\partial(t)\,{}^{\pi_4}\partial(s)}\right|+\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3))}{{}^{\pi_2}\partial(t)\,{}^{\pi_4}\partial(s)}\right|+\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4))}{{}^{\pi_2}\partial(t)\,{}^{\pi_4}\partial(s)}\right|\right]$$

Remark 3. *If we replace $\grave{O}_1{=}\pi_2-\pi_1$, $\acute{O}_1{=}\pi_1-\pi_2$, $\acute{O}_2{=}\pi_3-\pi_4$, and $\grave{O}_2{=}\pi_4-\pi_3$ in Theorem 3, then we have the identity*

$$\left|\frac{1}{[6]_{q_1}[6]_{q_2}}\left[q_1^2[4]_{q_1}F\left(\tfrac{q_1\pi_2+\pi_1}{[2]_{q1}},\pi_3\right)+q_1^2q_2^2[4]_{q_1}[4]_{q_2}F\left(\tfrac{q_1\pi_2+\pi_1}{[2]_{q1}},\tfrac{q_2\pi_4+\pi_3}{[2]_{q_2}}\right)+q_1q_2^2[4]_{q_2}F\left(\pi_2,\tfrac{q_2\pi_4+\pi_3}{[2]_{q_2}}\right)\right.\right.$$
$$\left.+q_1^2q_2[4]_{q_1}F\left(\tfrac{q_1\pi_2+\pi_1}{[2]_{q1}},\pi_4\right)+q_2^2[4]_{q_2}F\left(\pi_1,\tfrac{q_2\pi_4+\pi_3}{[2]_{q_2}}\right)\right]-\tfrac{1}{[6]_{q_2}(\pi_3-\pi_4)}\int_{\pi_1}^{\pi_2}\left[q_2F(x,\pi_4)+q_2^2[4]_{q_2}F\left(x,\tfrac{q_2\pi_4+\pi_3}{[2]_{q_2}}\right)\right.$$

$$+F(x,\pi_3)]\ {}^{\pi_2}d_{q_1}(x)+\frac{q_1F(\pi_2,\pi_3)+F(\pi_1,\pi_3)+q_2F(\pi_1,\pi_4)+q_1q_2F(\pi_2,\pi_4)}{[6]_{q_1}[6]_{q_2}}-\frac{1}{[6]_{q_1}(\pi_2-\pi_1)}\int_{\pi_3}^{\pi_4}[q_1F(\pi_2,y)$$

$$+q_1^2[4]_{q_1}F\left(\tfrac{q_1\pi_2+\pi_1}{[2]_{q1}},y\right)+F(\pi_1,y)\Big]\ {}^{\pi_4}d_{q_2}(y)+\frac{1}{(\pi_2-\pi_1)(\pi_4-\pi_3)}\int_{\pi_1}^{\pi_2}\int_{\pi_3}^{\pi_4}F(x,y)^{\pi_2}d_{q_1}(x)^{\pi_4}d_{q_2}(y)\Bigg|$$

$$\leqslant(\pi_2-\pi_1)(\pi_4-\pi_3)\Bigg[(A_1(q_1)+A_2(q_1))(A_1(q_2)+A_2(q_2))\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3))}{{}^{\pi_2}\partial_{q_1}\ (t)^{\pi_4}\partial_{q_2}(s)}\right|$$

$$+(B_1(q_2)+B_2(q_2))(A_1(q_1)+A_2(q_1))\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4))}{{}^{\pi_2}\partial_{q_1}\ (t)^{\pi_4}\partial_{q_2}(s)}\right|+(B_1(q_1)+B_2(q_1))(A_1(q_2)+A_2(q_2))$$

$$\times\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3))}{{}^{\pi_2}\partial_{q_1}\ (t)^{\pi_4}\partial_{q_2}(s)}\right|+(B_1(q_1)+B_2(q_1))(B_1(q_2)+B_2(q_2))\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4))}{{}^{\pi_2}\partial_{q_1}\ (t)^{\pi_4}\partial_{q_2}(s)}\right|\Bigg].$$

Remark 4. *If we replace* $\grave{O}_1=\pi_2-\pi_1$, $\acute{O}_1=\pi_1-\pi_2$, $\acute{O}_2=\pi_3-\pi_4$, *and* $\grave{O}_2=\pi_4-\pi_3$ *in Theorem 3 along with* $q_1,q_2\to 1^-$, *then we have the identity*

$$\Bigg|\frac{1}{9}\left[F\left(\tfrac{\pi_1-\pi_2}{2},\pi_3\right)+F\left(\tfrac{\pi_1-\pi_2}{2},\pi_4\right)+4F\left(\tfrac{\pi_1-\pi_2}{2},\tfrac{\pi_3-\pi_4}{2}\right)+F\left(\pi_2,\tfrac{\pi_3-\pi_4}{2}\right)+F\left(\pi_1,\tfrac{\pi_3-\pi_4}{2}\right)\right]$$

$$+\frac{1}{36}[F(\pi_2,\pi_3)+F(\pi_1,\pi_3)+F(\pi_1,\pi_4)+F(\pi_2,\pi_4)\,]-\frac{1}{6(\pi_2-\pi_1)}\int_{\pi_1}^{\pi_2}\Big[4F\left(x,\tfrac{\pi_3+\pi_4}{2}\right)$$

$$+F(x,\pi_4)+F(x,\pi_3)\,]\ {}^{\pi_2}d(x)-\frac{1}{6(\pi_2-\pi_1)}\int_{\pi_3}^{\pi_4}\left[F(\pi_2,y)+4F\left(\tfrac{\pi_1+\pi_2}{2},y\right)+F(\pi_1,y)\,\right]\ {}^{\pi_4}d(y)$$

$$+\frac{1}{(\pi_2-\pi_1)(\pi_4-\pi_3)}\int_{\pi_1}^{\pi_2}\int_{\pi_3}^{\pi_4}F(x,y)\ {}^{\pi_2}d(x)^{\pi_4}d(y)\Bigg|\leqslant\frac{25}{5184}(\pi_2-\pi_1)(\pi_4-\pi_3)\Bigg[\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial(t)^{\pi_4}\partial(s)}\right|$$

$$+\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial(t)^{\pi_4}\partial(s)}\right|+\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial(t)^{\pi_4}\partial(s)}\right|+\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial(t)^{\pi_4}\partial(s)}\right|\Bigg].$$

Theorem 4. *We assume that the conditions of Lemma 2 hold. If* $\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}\ (t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}$ *is a generalized convex function and integrable on* Δ, *where* $p_1>1$ *and* $\dfrac{1}{r_1}+\dfrac{1}{p_1}=1$, *then we have the following inequality:*

$$\Bigg|\frac{1}{[6]_{q_1}[6]_{q_2}}\Big[q_1^2[4]_{q_1}F\left(\pi_2+\tfrac{\acute{O}_1}{[2]_{q_1}},\pi_4+\acute{O}_2\right)+q_2^2[4]_{q_2}F\left(\pi_2+\acute{O}_1,\pi_4+\tfrac{\acute{O}_2}{[2]_{q2}}\right)+q_1^2q_2^2[4]_{q_1}[4]_{q_2} \tag{36}$$

$$\times F\left(\pi_2+\tfrac{\acute{O}_1}{[2]_{q_1}},\pi_4+\tfrac{\acute{O}_2}{[2]_{q2}}\right)+q_1^2q_2[4]_{q_1}F\left(\pi_2+\tfrac{\acute{O}_1}{[2]_{q_1}},\pi_4\right)+q_1q_2^2[4]_{q_2}F\left(\pi_2,\pi_4+\tfrac{\acute{O}_2}{[2]_{q_2}}\right)\Big]$$

$$+\frac{q_1F(\pi_2,\pi_4+\acute{O}_2)+q_1q_2F(\pi_2,\pi_4)+F(\pi_2+\acute{O}_1,\pi_4+\acute{O}_2)+q_2F(\pi_2+\acute{O}_1,\pi_4)}{[6]_{q_1}[6]_{q_2}}$$

$$-\frac{1}{[6]_{q_2}\grave{O}_1}\int_{\pi_2+\acute{O}_1}^{\pi_2}\left[q_2F(x,\pi_4)+q_2^2[4]_{q_2}F\left(x,\pi_4+\tfrac{\acute{O}_2}{[2]_{q_2}}\right)+F(x,\pi_4+\acute{O}_2)\right]^{\pi_2}d_{q_1}(x)$$

$$-\frac{1}{[6]_{q_1}\eta_2(\pi_2,\pi_1)}\int_{\pi_4+\acute{O}_2}^{\pi_4}\left[q_1F(\pi_2,y)+q_1^2[4]_{q_1}F\left(\pi_2+\tfrac{\acute{O}_1}{[2]_{q^1}},y\right)+F(\pi_2+\acute{O}_1,y)\right]^{\pi_4}d_{q_2}(y)$$

$$+\frac{1}{\grave{O}_1\grave{O}_2}\int\limits_{\pi_2+\acute{O}_1}^{\pi_2}\int\limits_{\pi_4+\acute{O}_2}^{\pi_4}F(x,y)^{\pi_2}d_{q_1}(x)^{\pi_4}d_{q_2}(y)\Bigg|\le q_1q_2\grave{O}_1\grave{O}_2$$

$$\times\left(\int\limits_0^1\int\limits_0^1\left|\Lambda_{q_2}(s)\Lambda_{q_1}(t)\right|^{r_1\pi_4}d_{q_2}(s)^{\pi_2}d_{q_1}(t)\right)^{\frac{1}{r_1}}\left[\frac{1}{[2]_{q_1}[2]_{q_2}}\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+\frac{q_2}{[2]_{q_1}[2]_{q_2}}\right.$$

$$\left.\times\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+\frac{q_1}{[2]_{q_1}[2]_{q_2}}\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+\frac{q_1q_2}{[2]_{q_1}[2]_{q_2}}\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right]^{\frac{1}{p_1}},$$

where $0<q_1<1$ *and* $0<q_2<1$.

Proof. From the integrals on the right side of Equation (35), and by applying the well-known quantum Hölders integral inequality, it is found that

$$\left|q_1q_2\grave{O}_1\grave{O}_2\int\limits_0^1\int\limits_0^1\Lambda_{q_1}(t)\Lambda_{q_2}(s)\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}d_{q_1}(t)d_{q_2}(s)\right| \tag{37}$$

$$\le q_1q_2\grave{O}_1\grave{O}_2\left(\int\limits_0^1\int\limits_0^1\left|\Lambda_{q_2}(s)\Lambda_{q_1}(t)\right|^{r_1}d_{q_2}(s)d_{q_1}(t)\right)^{\frac{1}{r_1}}\left[\int\limits_0^1\int\limits_0^1\left(ts\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right.\right.$$

$$+(1-t)s\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+(1-t)(1-s)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}$$

$$\left.\left.+t(1-s)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)d_{q_1}(t)d_{q_2}(s)\right]^{\frac{1}{p_1}}.$$

By applying the technique of Lemma 1 when $\alpha=0$ to the above integrals, we have the following results:

$$\int_0^1\int_0^1 ts\,d_{q_1}(t)\,d_{q_2}(s)=\left(\int_0^1 td_{q_1}(t)\right)\left(\int_0^1 sd_{q_2}(s)\right)=\frac{1}{[2]_{q_1}[2]_{q_2}}, \tag{38}$$

$$\int_0^1\int_0^1 t(1-s)d_{q_1}(t)d_{q_2}(s)=\frac{[2]_{q_2}-1}{[2]_{q_1}[2]_{q_2}}, \tag{39}$$

$$\int_0^1\int_0^1(1-t)sd_{q_1}(t)d_{q_2}(s)=\frac{[2]_{q_1}-1}{[2]_{q_1}[2]_{q_2}}, \tag{40}$$

$$\int_0^1\int_0^1(1-t)(1-s)d_{q_1}(t)d_{q_2}(s)=\frac{\left([2]_{q_1}-1\right)\left([2]_{q_2}-1\right)}{[2]_{q_1}[2]_{q_2}}. \tag{41}$$

By using all these values in Equation (37), we find the required result. □

Corollary 2. *By replacing* $\acute{O}_1=\pi_1-\pi_2$, $\grave{O}_1=\pi_2-\pi_1$, $\acute{O}_2=\pi_3-\pi_4$, *and* $\grave{O}_2=\pi_4-\pi_3$ *in Theorem 4, we obtain the following inequality:*

$$\left|\frac{1}{[6]_{q_1}[6]_{q_2}}\left[q_1^2[4]_{q_1}F\left(\frac{q_1\pi_2+\pi_1}{[2]_{q_1}},\pi_3\right)+q_2^2[4]_{q_2}F\left(\pi_1,\frac{q_2\pi_4+\pi_3}{[2]_{q2}}\right)+q_1^2q_2^2[4]_{q_1}[4]_{q_2}\right.\right.$$
$$\times F\left(\frac{q_1\pi_2+\pi_1}{[2]_{q_1}},\frac{q_2\pi_4+\pi_3}{[2]_{q2}}\right)+q_1^2q_2[4]_{q_1}F\left(\frac{q_1\pi_2+\pi_1}{[2]_{q_1}},\pi_4\right)+q_1q_2^2[4]_{q_2}$$
$$\left.\times F\left(\pi_2,\frac{q_2\pi_4+\pi_3}{[2]_{q2}}\right)\right]+\frac{q_1F(\pi_2,\pi_4)+F(\pi_1,\pi_4)+q_2F(\pi_1,\pi_4)+q_1q_2F(\pi_2,\pi_4)}{[6]_{q_1}[6]_{q_2}}-\frac{1}{[6]_{q_2}(\pi_2-\pi_1)}$$
$$\times\int_{\pi_1}^{\pi_2}\left[q_2^2[4]_{q_2}F\left(x,\frac{q_2\pi_4+\pi_3}{[2]_{q2}}\right)+F(x,\pi_3)+q_2F(x,\pi_4)\right]{}^{\pi_2}d_{q_1}(x)+\int_{\pi_1}^{\pi_2}\int_{\pi_3}^{\pi_4}\frac{F(x,y)^{\pi_2}d_{q_1}(x)^{\pi_4}d_{q_2}(y)}{(\pi_2-\pi_1)(\pi_4-\pi_3)}$$
$$\left.-\frac{1}{[6]_{q_1}(\pi_2-\pi_1)}\int_{\pi_3}^{\pi_4}\left[q_1F(\pi_2,y)+q_1^2[4]_{q_1}F\left(\frac{q_1\pi_2+\pi_1}{[2]_{q_1}},y\right)+F(\pi_1,y)\right]{}^{\pi_4}d_{q_2}(y)\right|$$
$$\leq q_1q_2(\pi_2-\pi_1)(\pi_4-\pi_3)\left(\int_0^1\int_0^1|\Lambda_{q_2}(s)\Lambda_{q_1}(t)|^{r_1}d_{q_2}(s)d_{q_1}(t)\right)^{\frac{1}{r_1}}\left[\frac{1}{[2]_{q_1}[2]_{q_2}}\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+\frac{q_2}{[2]_{q_1}[2]_{q_2}}\right.$$
$$\left.\times\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+\frac{q_1}{[2]_{q_1}[2]_{q_2}}\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+\frac{q_1q_2}{[2]_{q_1}[2]_{q_2}}\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right]^{\frac{1}{p_1}}.$$

Theorem 5. *We assume that the conditions of Lemma 2 hold. If* $\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}$ *is generalized convex and an integrable function on* Δ*, where* $p_1>1$*, then*

$$\left|\frac{1}{[6]_{q_1}[6]_{q_2}}\left[q_1^2[4]_{q_1}F\left(\pi_2+\frac{\acute{O}_1}{[2]_{q_1}},\pi_4+\acute{O}_2\right)+q_2^2[4]_{q_2}F\left(\pi_2+\acute{O}_1,\pi_4+\frac{\acute{O}_2}{[2]_{q2}}\right)+q_1^2q_2^2[4]_{q_1}[4]_{q_2}\right.\right.$$
$$\left.\times F\left(\pi_2+\frac{\acute{O}_1}{[2]_{q_1}},\pi_4+\frac{\acute{O}_2}{[2]_{q2}}\right)+q_1q_2^2[4]_{q_2}F\left(\pi_2,\pi_4+\frac{\acute{O}_2}{[2]_{q_2}}\right)+q_1^2q_2[4]_{q_1}F\left(\pi_2+\frac{\acute{O}_1}{[2]_{q_1}},\pi_4\right)\right]$$
$$+\frac{q_1F(\pi_2,\pi_4+\acute{O}_2)+q_2F(\pi_2+\acute{O}_1,\pi_4)+F(\pi_2+\acute{O}_1,\pi_4+\acute{O}_2)+q_1q_2F(\pi_2,\pi_4)}{[6]_{q_1}[6]_{q_2}}$$
$$-\frac{1}{[6]_{q_2}\grave{O}_1}\int_{\pi_2+\acute{O}_1}^{\pi_2}\left[q_2F(x,\pi_4)+F(x,\pi_4+\acute{O}_2)+q_2^2[4]_{q_2}F\left(x,\pi_4+\frac{\acute{O}_2}{[2]_{q_2}}\right)\right]{}^{\pi_2}d_{q_1}(x)$$
$$-\frac{1}{[6]_{q_1}\grave{O}_2}\int_{\pi_4+\acute{O}_2}^{\pi_4}\left[q_1F(\pi_2,y)+q_1^2[4]_{q_1}F\left(\pi_2+\frac{\acute{O}_1}{[2]_{q1}},y\right)+F(\pi_2+\acute{O}_1,y)\right]{}^{\pi_4}d_{q_2}(y)$$
$$\left.+\frac{1}{\grave{O}_1\grave{O}_2}\int_{\pi_2+\acute{O}_1}^{\pi_2}\int_{\pi_4+\acute{O}_2}^{\pi_4}F(x,y)^{\pi_2}d_{q_1}(x)^{\pi_4}d_{q_2}(y)\right|\leq q_1q_2\grave{O}_1\grave{O}_2$$
$$\times\left[\left(\frac{2q_1}{[2]_{q_1}[6]^2_{q_1}}+\frac{q_1^3[3]_{q_1}-q_1}{[6]_{q_1}[2]^3_{q_1}}\right)\left(\frac{2q_2}{[2]_{q_2}[6]^2_{q_2}}+\frac{q_2^3[3]_{q_2}-q_2}{[6]_{q_2}[2]^3_{q_2}}\right)\right]^{1-\frac{1}{p_1}}\left[A_1(q_1)\left(A_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_1(q_2)\right.\right.$$
$$\left.\left.\times\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4))}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)+B_1(q_1)\left(A_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\right]^{\frac{1}{p_1}}$$
$$+q_1q_2\grave{O}_1\grave{O}_2\left[\left(\frac{2q_1}{[2]_{q_1}[6]^2_{q_1}}+\frac{q_1^3[3]_{q_1}-q_1}{[6]_{q_1}[2]^3_{q_1}}\right)\left(\frac{2q_2[5]^2_{q_2}}{[2]_{q_2}[6]^2_{q_2}}+\frac{1}{[2]_{q_2}}-\frac{[5]_{q_2}}{[6]_{q_2}}-\frac{[2]^2_{q_2}[5]_{q_2}-[6]_{q_2}}{[6]_{q_2}[2]^3_{q_2}}\right)\right]^{1-\frac{1}{p_1}}$$

$$\times\left[A_1(q_1)\left(A_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}\ (t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}\ (t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)+B_1(q_1)\right.$$

$$\left.\times\left(A_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}\ (t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}\ (t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\right]^{\frac{1}{p_1}}+q_1q_2\grave{O}_1\grave{O}_2$$

$$\times\left[\left(\frac{2q_1[5]^2_{q_1}}{[2]_{q_1}[6]^2_{q_1}}+\frac{1}{[2]_{q_1}}-\frac{[5]_{q_1}}{[6]_{q_1}}-\frac{[2]^2_{q_1}[5]_{q_1}-[6]_{q_1}}{[6]_{q_1}[2]^3_{q_1}}\right)\left(\frac{2q_2}{[2]_{q_2}[6]^2_{q_2}}+\frac{q_1^3[3]_{q_2}-q_2}{[6]_{q_2}[2]^3_{q_2}}\right)\right]^{1-\frac{1}{p_1}}$$

$$\times\left[A_2(q_1)\left(A_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}\ (t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}\ (t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\right.$$

$$\left.+B_2(q_1)\left(A_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}\ (t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}\ (t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\right]^{\frac{1}{p_1}}+q_1q_2\grave{O}_1\grave{O}_2$$

$$\times\left[\left(\frac{2q_1[5]^2_{q_1}}{[2]_{q_1}[6]^2_{q_1}}+\frac{1}{[2]_{q_1}}-\frac{[5]_{q_1}}{[6]_{q_1}}-\frac{[2]^2_{q_1}[5]_{q_1}-[6]_{q_1}}{[6]_{q_1}[2]^3_{q_1}}\right)\left(\frac{2q_2[5]^2_{q_2}}{[2]_{q_2}[6]^2_{q_2}}+\frac{1}{[2]_{q_2}}-\frac{[5]_{q_2}}{[6]_{q_2}}-\frac{[2]^2_{q_2}[5]_{q_2}-[6]_{q_2}}{[6]_{q_2}[2]^3_{q_2}}\right)\right]^{1-\frac{1}{p_1}}$$

$$\times\left[A_2(q_1)\left(A_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}\ (t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}\ (t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\right.$$

$$\left.+B_2(q_1)\left(A_2(q_2)\left|\frac{{}^{\pi_2,d}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}\ (t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}\ (t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\right]^{\frac{1}{p_1}}$$

where $0<q_1<1$, $0<q_2<1$, *and* $A_1(q_1), A_2(q_1), A_3(q_1), A_4(q_1), B_1(q_2), B_2(q_2), B_3(q_2)$, *and* $B_4(q_2)$ *are given in Equations* (25)–(30).

Proof. By applying the power mean inequality for integrals on the right side of Equation (35) (used in the proof of Theorem 3) along with definition of preinvexity of $\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}\ (t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}$, we have

$$\left|q_1q_2\grave{O}_1\grave{O}_2\int_0^1\int_0^1\Lambda_{q_1}(t)\Lambda_{q_2}(s)\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}\ (t)^{\pi_4}\partial_{q_2}(s)}\,{}^{\pi_1}d_{q_1}(t)\,{}^{\pi_2}d_{q_2}(s)\right| \tag{42}$$

$$\leq q_1q_2\grave{O}_1\grave{O}_2\left\{\left(\int_0^{\frac{1}{[2]_{q_1}}}\left|t-\frac{1}{[6]_{q_1}}\right|d_{q_1}(t)\right)\left(\int_0^{\frac{1}{[2]_{q_2}}}\left|s-\frac{1}{[6]_{q_2}}\right|d_{q_2}(s)\right)\right\}^{1-\frac{1}{p_1}}\left[\int_0^{\frac{1}{[2]_{q_2}}}\left|s-\frac{1}{[6]_{q_2}}\right|\right.$$

$$\times\left\{\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}\ (t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\left(\int_0^{\frac{1}{[2]_{q_1}}}t\left|t-\frac{1}{[6]_{q_1}}\right|d_{q_1}(t)\right)+\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}\ (t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right.$$

$$\left.\left.\times\left(\int_0^{\frac{1}{[2]_{q_1}}}(1-t)\left|t-\frac{1}{[6]_{q_1}}\right|d_{q_1}(t)\right)\right\}d_{q_2}(s)\right]^{\frac{1}{p_1}}+q_1q_2\grave{O}_1\grave{O}_2\left\{\left(\int_0^{\frac{1}{[2]_{q_1}}}\left|t-\frac{1}{[6]_{q_1}}\right|d_{q_1}(t)\right)\right.$$

$$\left.\times\left(\int_{\frac{1}{[2]_{q_2}}}^1\left|s-\frac{[5]_{q_2}}{[6]_{q_2}}\right|d_{q_2}(s)\right)\right\}^{1-\frac{1}{p_1}}\left[\int_{\frac{1}{[2]_{q_2}}}^1\left|s-\frac{[5]_{q_2}}{[6]_{q_2}}\right|\left\{\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}\ (t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right.\right.$$

$$\times\left(\int_0^{\frac{1}{[2]_{q_1}}} t\left|t-\frac{1}{[6]_{q_1}}\right|d_{q_1}(t)\right)+\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\left(\int_0^{\frac{1}{[2]_{q_1}}}(1-t)\left|t-\frac{1}{[6]_{q_1}}\right|d_{q_1}(t)\right)\Bigg\}d_{q_2}(s)\Bigg]^{\frac{1}{p_1}}$$

$$+q_1q_2\grave{O}_1\grave{O}_2\Bigg\{\left(\int_{\frac{1}{[2]_{q_1}}}^1\left|t-\frac{[5]_{q_1}}{[6]_{q_1}}\right|d_{q_1}(t)\right)\left(\int_0^{\frac{1}{[2]_{q_2}}}\left|s-\frac{1}{[6]_{q_2}}\right|d_{q_2}(s)\right)\Bigg\}^{1-\frac{1}{p_1}}\Bigg[\int_0^{\frac{1}{[2]_{q_2}}}\left|s-\frac{1}{[6]_{q_2}}\right|$$

$$\times\Bigg\{\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\left(\int_{\frac{1}{[2]_{q_1}}}^1 t\left|t-\frac{[5]_{q_1}}{[6]_{q_1}}\right|d_{q_1}(t)\right)+\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}$$

$$\times\left(\int_{\frac{1}{[2]_{q_1}}}^1(1-t)\left|t-\frac{[5]_{q_1}}{[6]_{q_1}}\right|d_{q_1}(t)\right)\Bigg\}d_{q_2}(s)\Bigg]^{\frac{1}{p_1}}+q_1q_2\grave{O}_1\grave{O}_2\Bigg\{\left(\int_{\frac{1}{[2]_{q_1}}}^1\left|t-\frac{[5]_{q_1}}{[6]_{q_1}}\right|d_{q_1}(t)\right)$$

$$\times\left(\int_{\frac{1}{[2]_{q_2}}}^1\left|s-\frac{[5]_{q_2}}{[6]_{q_2}}\right|d_{q_2}(s)\right)\Bigg\}^{1-\frac{1}{p_1}}\Bigg[\int_{\frac{1}{[2]_{q_2}}}^1\left|s-\frac{[5]_{q_2}}{[6]_{q_2}}\right|\Bigg\{\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}$$

$$\times\left(\int_{\frac{1}{[2]_{q_1}}}^1 t\left|t-\frac{[5]_{q_1}}{[6]_{q_1}}\right|d_{q_1}(t)\right)+\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\left(\int_{\frac{1}{[2]_{q_1}}}^1(1-t)\left|t-\frac{[5]_{q_1}}{[6]_{q_1}}\right|d_{q_1}(t)\right)\Bigg\}d_{q_2}(s)\Bigg]^{\frac{1}{p_1}}$$

$$\leq q_1q_2\grave{O}_1\grave{O}_2\Bigg\{\left(\frac{2q_1}{[2]_{q_1}[6]^2_{q_1}}+\frac{q_1^3[3]_{q_1}-q_1}{[6]_{q_1}[2]^3_{q_1}}\right)\left(\frac{2q_2}{[2]_{q_2}[6]^2_{q_2}}+\frac{q_2^3[3]_{q_2}-q_2}{[6]_{q_2}[2]^3_{q_2}}\right)\Bigg\}^{1-\frac{1}{p_1}}$$

$$\times\Bigg[A_1(q_1)\Bigg(\int_0^{\frac{1}{[2]_{q_2}}} s\left|s-\frac{1}{[6]_{q_2}}\right|d_{q_2}(s)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+\int_0^{\frac{1}{[2]_{q_2}}}(1-s)\left|s-\frac{1}{[6]_{q_2}}\right|d_{q_2}(s)$$

$$\times\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\Bigg)+B_1(q_1)\Bigg(\int_0^{\frac{1}{[2]_{q_2}}} s\left|s-\frac{1}{[6]_{q_2}}\right|\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}d_{q_2}(s)$$

$$+\int_0^{\frac{1}{[2]_{q_2}}}(1-s)\left|s-\tfrac{1}{[6]_{q_2}}\right|d_{q_2}(s)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\Bigg)\Bigg]^{\frac{1}{p_1}}+q_1q_2\grave{O}_1\grave{O}_2$$

$$\times\Bigg\{\left(\frac{2q_1}{[2]_{q_1}[6]^2_{q_1}}+\frac{q_1^3[3]_{q_1}-q_1}{[6]_{q_1}[2]^3_{q_1}}\right)\left(\frac{2q_2[5]^2_{q_2}}{[2]_{q_2}[6]^2_{q_2}}+\frac{1}{[2]_{q_2}}-\frac{[5]_{q_2}}{[6]_{q_2}}-\frac{[2]^2_{q_2}[5]_{q_2}-[6]_{q_2}}{[6]_{q_2}[2]^3_{q_2}}\right)\Bigg\}^{1-\frac{1}{p_1}}\Bigg[A_2(q_1)\int_{\frac{1}{[2]_{q_2}}}^1$$

$$\times\Bigg(s\left|s-\frac{[5]_{q_2}}{[6]_{q_2}}\right|\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}d_{q_2}(s)+\int_{\frac{1}{[2]_{q_2}}}^1(1-s)\left|s-\frac{[5]_{q_2}}{[6]_{q_2}}\right|d_{q_2}(s)$$

$$\times\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\Bigg)+B_1(q_1)\Bigg(\int_{\frac{1}{[2]_{q_2}}}^1 s\left|s-\tfrac{1}{[6]_{q_2}}\right|\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}d_{q_2}(s)$$

$$+\int_{\frac{1}{[2]_{q_2}}}^1(1-s)\left|s-\frac{1}{[6]_{q_2}}\right|d_{q_2}(s)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\Bigg)\Bigg]^{\frac{1}{p_1}}+q_1q_2\grave{O}_1\grave{O}_2$$

$$\times\Bigg\{\left(\frac{2q_1[5]^2_{q_1}}{[2]_{q_1}[6]^2_{q_1}}+\frac{1}{[2]_{q_1}}-\frac{[5]_{q_1}}{[6]_{q_1}}-\frac{[2]^2_{q_1}[5]_{q_1}-[6]_{q_1}}{[6]_{q_1}[2]^3_{q_1}}\right)\left(\frac{2q_2}{[2]_{q_2}[6]^2_{q_2}}+\frac{q_1^3[3]_{q_2}-q_2}{[6]_{q_2}[2]^3_{q_2}}\right)\Bigg\}^{1-\frac{1}{p_1}}\Bigg[A_2(q_1)\int_{\frac{1}{[2]_{q_2}}}^1$$

$$\times\left(s\left|s-\frac{1}{[6]_{q_2}}\right|d_{q_2}(s)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+\int_{\frac{1}{[2]_{q_2}}}^{1}(1-s)\left|s-\frac{1}{[6]_{q_2}}\right|d_{q_2}(s)\right.$$
$$\left.\times\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)+B_2(q_1)\left(\int_{\frac{1}{[2]_{q_2}}}^{1}s\left|s-\frac{1}{[6]_{q_2}}\right|d_{q_2}(s)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right.$$
$$\left.\left.+\int_{\frac{1}{[2]_{q_2}}}^{1}(1-s)\left|s-\frac{1}{[6]_{q_2}}\right|d_{q_2}(s)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\right]^{\frac{1}{p_1}}+q_1q_2\grave{O}_1\grave{O}_2$$
$$\times\left\{\left(\frac{2q_1[5]^2_{q_1}}{[2]_{q_1}[6]^2_{q_1}}+\frac{1}{[2]_{q_1}}-\frac{[5]_{q_1}}{[6]_{q_1}}-\frac{[2]^2_{q_1}[5]_{q_1}-[6]_{q_1}}{[6]_{q_1}[2]^3_{q_1}}\right)\left(\frac{2q_2[5]^2_{q_2}}{[2]_{q_2}[6]^2_{q_2}}+\frac{1}{[2]_{q_2}}-\frac{[5]_{q_2}}{[6]_{q_2}}-\frac{[2]^2_{q_2}[5]_{q_2}-[6]_{q_2}}{[6]_{q_2}[2]^3_{q_2}}\right)\right\}^{1-\frac{1}{p_1}}$$
$$\times\left[A_2(q_1)\left(\int_{\frac{1}{[2]_{q_2}}}^{1}s\left|s-\frac{[5]_{q_2}}{[6]_{q_2}}\right|\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}d_{q_2}(s)+\int_{\frac{1}{[2]_{q_2}}}^{1}(1-s)\left|s-\frac{[5]_{q_2}}{[6]_{q_2}}\right|d_{q_2}(s)\right.\right.$$
$$\left.\times\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)+B_2(q_1)\left(\int_{\frac{1}{[2]_{q_2}}}^{1}s\left|s-\frac{[5]_{q_2}}{[6]_{q_2}}\right|d_{q_2}(s)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right.$$
$$\left.\left.+\int_{\frac{1}{[2]_{q_2}}}^{1}(1-s)\left|s-\frac{[5]_{q_2}}{[6]_{q_2}}\right|d_{q_2}(s)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\right]^{\frac{1}{p_1}}.$$

□

Corollary 3. *If we take the limit $q_1,q_2\to1^-$ in Theorem 5, then we have the following inequality:*

$$\left|\frac{1}{9}\left[F\left(\pi_2+\frac{\acute{O}_1}{2},\pi_4+\acute{O}_2\right)+F\left(\pi_2+\frac{\acute{O}_1}{2},\pi_4\right)+4F\left(\pi_2+\frac{\acute{O}_1}{2},\pi_4+\frac{\acute{O}_2}{2}\right)\right.\right.$$
$$\left.+F\left(\pi_2+\acute{O}_1,\pi_4+\frac{\acute{O}_2}{2}\right)+F\left(\pi_2,\pi_4+\frac{\acute{O}_2}{2}\right)\right]+\frac{1}{\grave{O}_1\grave{O}_2}\int_{\pi_2+\acute{O}_1}^{\pi_2}\int_{\pi_4+\acute{O}_2}^{\pi_4}$$
$$\times F(x,y)^{\pi_2}d(x)^{\pi_4}d(y)-\frac{1}{6\eta_2(\pi_1,\pi_2)}\int_{\pi_4+\acute{O}_2}^{\pi_4}\left[4F\left(\pi_2+\frac{\acute{O}_1}{2},y\right)+F(\pi_2+\acute{O}_1,y)\right.$$
$$\left.+F(\pi_2,y)\right]{}^{\pi_4}d(y)-\frac{1}{6\eta_1(\pi_1,\pi_2)}\int_{\pi_2+\acute{O}_1}^{\pi_2}\left[F(x,\pi_4+\acute{O}_2)+F(x,\pi_4)+4F\left(x,\pi_4+\frac{\acute{O}_2}{2}\right)\right]{}^{\pi_2}d(x)$$
$$\left.+\frac{F(\pi_2,\pi_4+\acute{O}_2)+F(\pi_2+\acute{O}_1,\pi_4)+F(\pi_2+\acute{O}_1,\pi_4+\acute{O}_2)+F(\pi_2,\pi_4)}{36}\right|$$
$$\leq\grave{O}_1\grave{O}_2\left[\tfrac{25}{5184}\right]^{1-\frac{1}{p_1}}\left[\tfrac{29}{1296}\left(\tfrac{29}{1296}\left|\frac{\partial^2F(\pi_1,\pi_3)}{\partial(t)\partial(s)}\right|^{p_1}+\tfrac{61}{1296}\left|\frac{\partial^2F(\pi_1,\pi_4))}{\partial(t)\partial(s)}\right|^{p_1}\right)+\tfrac{61}{1296}\right.$$
$$\left.\times\left(\tfrac{29}{1296}\left|\frac{\partial^2F(\pi_2,\pi_3)}{\partial(t)\partial(s)}\right|^{p_1}+\tfrac{61}{1296}\left|\frac{\partial^2F(\pi_2,\pi_4)}{\partial(t)\partial(s)}\right|^{p_1}\right)\right]^{\frac{1}{p_1}}+\left[\tfrac{29}{1296}\left(\tfrac{61}{1296}\left|\frac{\partial^2F(\pi_1,\pi_3)}{\partial(t)\partial(s)}\right|^{p_1}+\tfrac{29}{1296}\left|\frac{\partial^2F(\pi_1,\pi_4)}{\partial(t)\partial(s)}\right|^{p_1}\right.\right.$$
$$\left.+\tfrac{61}{1296}\left(\tfrac{61}{1296}\left|\frac{\partial^2F(\pi_2,\pi_3)}{\partial(t)\partial(s)}\right|^{p_1}+\tfrac{29}{1296}\left|\frac{\partial^2F(\pi_2,\pi_4)}{\partial(t)\partial(s)}\right|^{p_1}\right)\right]^{\frac{1}{p_1}}+\left[\tfrac{61}{1296}\left(\tfrac{29}{1296}\left|\frac{\partial^2F(\pi_1,\pi_3)}{{}^{\pi_2}\partial(t)^{\pi_4}\partial(s)}\right|^{p_1}+\tfrac{61}{1296}\right.\right.$$
$$\left.\times\left|\frac{\partial^2F(\pi_1,\pi_4)}{\partial(t)\partial(s)}\right|^{p_1}\right)+\tfrac{29}{1296}\left(\tfrac{29}{1296}\left|\frac{\partial^2F(\pi_2,\pi_3)}{\partial(t)\partial(s)}\right|^{p_1}+\tfrac{61}{1296}\left|\frac{\partial^2F(\pi_2,\pi_4)}{\partial(t)\partial(s)}\right|^{p_1}\right)\right]^{\frac{1}{p_1}}+\left[\tfrac{61}{1296}\left(\tfrac{61}{1296}\right.\right.$$
$$\left.\left.\times\left|\frac{\partial^2F(\pi_1,\pi_3)}{\partial(t)\partial(s)}\right|^{p_1}+\tfrac{29}{1296}\left|\frac{\partial^2F(\pi_1,\pi_4)}{\partial(t)\partial(s)}\right|^{p_1}\right)+\tfrac{29}{1296}\left(\tfrac{61}{1296}\left|\frac{\partial^2F(\pi_2,\pi_3)}{\partial(t)\partial(s)}\right|^{p_1}+\tfrac{29}{1296}\left|\frac{\partial^2F(\pi_2,\pi_4)}{\partial(t)\partial(s)}\right|^{p_1}\right)\right]^{\frac{1}{p_1}}$$

Remark 5. *By replacing* $\acute{O}_1=\pi_1-\pi_2$, $\grave{O}_1=\pi_2-\pi_1$, $\acute{O}_2=(\pi_3-\pi_4)$, *and* $\grave{O}_2=\pi_4-\pi_3$ *in Theorem 5, we have the following inequality:*

$$\begin{aligned}
&\Bigg|\frac{1}{[6]_{q_1}[6]_{q_2}}\Bigg[q_1^2[4]_{q_1}F\left(\pi_2+\frac{\pi_1-\pi_2}{[2]_{q_1}},\pi_3\right)+q_1^2q_2^2[4]_{q_1}[4]_{q_2}F\left(\pi_2+\frac{(\pi_1-\pi_2}{[2]_{q_1}}),\pi_4+\frac{\pi_3-\pi_4}{[2]_{q2}}\right)\\
&+q_1^2q_2[4]_{q_1}F\left(\pi_2+\frac{\pi_1-\pi_2}{[2]_{q_1}},\pi_4\right)+q_1q_2^2[4]_{q_2}F\left(\pi_2,\pi_4+\frac{\pi_3-\pi_4}{[2]_{q_2}}\right)+q_2^2[4]_{q_2}F\left(\pi_1,\pi_4+\frac{\pi_3-\pi_4}{[2]_{q2}}\right)\Bigg]\\
&+\frac{q_1F(\pi_2,\pi_3)+F(\pi_1,\pi_3)+q_2F(\pi_1,\pi_4)+q_1q_2F(\pi_2,\pi_4)}{[6]_{q_1}[6]_{q_2}}+\int_{\pi_1}^{\pi_2}\int_{\pi_3}^{\pi_4}\frac{F(x,y)\ {}^{\pi_2}d_{q_1}(x)\ {}^{\pi_4}d_{q_2}(y)}{(\pi_2-\pi_1)(\pi_4-\pi_3)}\\
&-\frac{1}{[6]_{q_2}(\pi_2-\pi_1)}\int_{\pi_1}^{\pi_2}\left[F(x,\pi_3)+q_2F(x,\pi_4)+q_2^2[4]_{q_2}F\left(x,\pi_4+\frac{\pi_3-\pi_4}{[2]_{q_2}}\right)\right]{}^{\pi_2}d_{q_1}(x)\\
&-\frac{1}{[6]_{q_1}(\pi_4-\pi_3)}\int_{\pi_3}^{\pi_4}\left[q_1F(\pi_2,y)+q_1^2[4]_{q_1}F\left(\pi_2+\frac{(\pi_1-\pi_2)}{[2]_{q1}},y\right)+F(\pi_1,y)\right]{}^{\pi_4}d_{q_2}(y)\Bigg|\\
&\leq q_1q_2(\pi_2-\pi_1)(\pi_4-\pi_3)\left[\left\{\left(\frac{2q_1}{[2]_{q_1}[6]_{q_1}^2}+\frac{q_1^3[3]_{q_1}-q_1}{[6]_{q_1}[2]_{q_1}^3}\right)\left(\frac{2q_2}{[2]_{q_2}[6]_{q_2}^2}+\frac{q_2^3[3]_{q_2}-q_2}{[6]_{q_2}[2]_{q_2}^3}\right)\right\}^{1-\frac{1}{p_1}}\{A_1(q_1)(A_1(q_2)\right.\\
&\times\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4))}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\Bigg)+B_1(q_1)\Bigg(A_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\\
&+B_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\Bigg)\Bigg\}^{\frac{1}{p_1}}+\Bigg\{\left(\frac{2q_2[5]_{q_2}^2}{[2]_{q_2}[6]_{q_2}^2}+\frac{1}{[2]_{q_2}}-\frac{[5]_{q_2}}{[6]_{q_2}}-\frac{[2]_{q_2}^2[5]_{q_2}-[6]_{q_2}}{[6]_{q_2}[2]_{q_2}^3}\right)\\
&\times\left(\frac{2q_1}{[2]_{q_1}[6]_{q_1}^2}+\frac{q_1^3[3]_{q_1}-q_1}{[6]_{q_1}[2]_{q_1}^3}\right)\Bigg\}^{1-\frac{1}{p_1}}\Bigg\{A_1(q_1)\Bigg(A_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\Bigg)\\
&+B_1(q_1)\Bigg(A_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\Bigg)\Bigg\}^{\frac{1}{p_1}}\\
&+\Bigg\{\left(\frac{2q_1[5]_{q_1}^2}{[2]_{q_1}[6]_{q_1}^2}+\frac{1}{[2]_{q_1}}-\frac{[5]_{q_1}}{[6]_{q_1}}-\frac{[2]_{q_1}^2[5]_{q_1}-[6]_{q_1}}{[6]_{q_1}[2]_{q_1}^3}\right)\left(\frac{2q_2}{[2]_{q_2}[6]_{q_2}^2}+\frac{q_1^3[3]_{q_2}-q_2}{[6]_{q_2}[2]_{q_2}^3}\right)\Bigg\}^{1-\frac{1}{p_1}}\\
&\times\Bigg\{A_2(q_1)\Bigg(A_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\Bigg)\\
&+B_2(q_1)\Bigg(A_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\Bigg)\Bigg\}^{\frac{1}{p_1}}\\
&+\Bigg\{\left(2q_1\frac{[5]_{q_1}^2}{[2]_{q_1}[6]_{q_1}^2}+\frac{1}{[2]_{q_1}}-\frac{[5]_{q_1}}{[6]_{q_1}}-\frac{[2]_{q_1}^2[5]_{q_1}-[6]_{q_1}}{[6]_{q_1}[2]_{q_1}^3}\right)\\
&\times\left(2q_2\frac{[5]_{q_2}^2}{[2]_{q_2}[6]_{q_2}^2}+\frac{1}{[2]_{q_2}}-\frac{[5]_{q_2}}{[6]_{q_2}}-\frac{[2]_{q_2}^2[5]_{q_2}-[6]_{q_2}}{[6]_{q_2}[2]_{q_2}^3}\right)\Bigg\}^{1-\frac{1}{p_1}}\\
&\times\Bigg\{A_2(q_1)\Bigg(A_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\Bigg)\\
&\left.+B_2(q_1)\Bigg(A_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\Bigg)\Bigg\}\right]^{\frac{1}{p_1}}.
\end{aligned}$$

4.2. Newton-Type Inequalities

Theorem 6. *It is assumed that the assumptions of Lemma 3 hold. If* $\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|$ *is a generalized convex function and integrable on* Δ, *then the following inequality holds for the right quantum integrals:*

$$\begin{aligned}
&\left|\frac{1}{\grave{O}_1\grave{O}_2}\int_{\pi_2+\acute{O}_1}^{\pi_2}\int_{\pi_4+\acute{O}_2}^{\pi_4}F(x,y)\,{}^{\pi_2}d_{q_1}(x)\,{}^{\pi_4}d_{q_2}(y)-\frac{1}{\grave{O}_1\,[8]_{q_2}}\int_{\pi_2+\acute{O}_1}^{\pi_2}\left[q_2F(x,\pi_4)+\frac{q_2^3[6]_{q_2}}{[2]_{q_2}}F\left(x,\pi_4+\frac{\acute{O}_2}{[3]_{q_2}}\right)\right.\right. \\
&+F(x,\pi_4+\acute{O}_2)+\frac{q_2^2[6]_{q_2}}{[2]_{q_2}}F\left(x,\pi_4+\frac{[2]_{q_2}\acute{O}_2}{[3]_{q_2}}\right)\Bigg]\,{}^{\pi_2}d_{q_1}(x)-\frac{1}{\grave{O}_2[8]_{q_1}}\int_{\pi_4+\acute{O}_2}^{\pi_4}\left[\frac{q_1^3[6]_{q_1}}{[2]_{q_1}}F\left(\pi_2+\frac{\acute{O}_1}{[3]_{q_1}},y\right)\right. \\
&+F(\pi_2+\acute{O}_2,y)+\frac{q_1^2[6]_{q_1}}{[2]_{q_1}}F\left(\pi_2+\frac{[2]_{q_1}\acute{O}_1}{[3]_{q_1}},y\right)+q_1F(\pi_2,y)\Bigg]\,{}^{\pi_4}d_{q_2}(y)+\frac{1}{[8]_{q_2}[8]_{q_1}} \\
&\times\left[F(\pi_2,\pi_4+\acute{O}_2)+\frac{q_2^2[6]_{q_2}}{[2]_{q_2}}F\left(\pi_2,\pi_4+\frac{[2]_{q_2}\acute{O}_2}{[3]_{q_2}}\right)+q_2F(\pi_2,\pi_4)+\frac{q_2^3[6]_{q_2}}{[2]_{q_2}}\right. \\
&\times F\left(\pi_2,\pi_4+\frac{\acute{O}_2}{[3]_{q_2}}\right)\Bigg]+\frac{1}{[8]_{q_1}[8]_{q_2}}\left[F(\pi_2+\acute{O}_1,\pi_4)+\frac{q_1^3[6]_{q_1}}{[2]_{q_1}}F\left(\pi_2+\frac{\acute{O}_1}{[3]_{q_1}},\pi_4\right)\right. \\
&\left.\left.+\frac{q_1^2[6]_{q1}}{[2]_{q_1}}F\left(\pi_2+\frac{[2]_{q_1}\acute{O}_1}{[3]_{q_1}},\pi_4\right)+F(\pi_2+\acute{O}_1,\pi_4+\acute{O}_2)\right]\right| \\
&\leq q_1q_2\acute{O}_1\acute{O}_2(A_3(q_1)+A_4(q_1)+A_5(q_1))\left[(A_3(q_2)+A_4(q_2)+A_5(q_2))\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|\right. \\
&\left.+(B_3(q_2)+B_4(q_2)+B_5(q_2))\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4))}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|\right]+q_1q_2\acute{O}_1\acute{O}_2(B_3(q_1)+B_4(q_1)+B_5(q_1)) \\
&\times\left[(A_3(q_2)+A_4(q_2)+A_5(q_2))\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|+(B_3(q_2)+B_4(q_2)+B_5(q_2))\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4))}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|\right],
\end{aligned}\tag{43}$$

where $0<q_1<1$, $0<q_2<1$, *and* $A_3(q_2)$, $A_4(q_2)$, $A_5(q_2)$, $A_3(q_1)$, $A_4(q_1)$, $A_5(q_1)$, $B_3(q_1)$, $B_4(q_1)$, $B_5(q_1)$, $B_3(q_2)$, $B_4(q_2)$, *and* $B_5(q_2)$ *are from Equations (25)–(30).*

Proof. Following the same arguments used in the proof of Theorem 3, and by using Lemma 3, we have the required inequality in Equation (43). □

Remark 6. *If we replace* $\grave{O}_1=\pi_2-\pi_1$, $\grave{O}_2=\pi_4-\pi_3$, $\acute{O}_1=\pi_1-\pi_2$, *and* $\acute{O}_2=\pi_3-\pi_4$ *in Theorem 6, then we have the following inequality:*

$$\begin{aligned}
&\left|\int_{\pi_1}^{\pi_2}\int_{\pi_3}^{\pi_4}\frac{F(x,y)\,{}^{\pi_2}d_{q_1}(x)\,{}^{\pi_4}d_{q_2}(y)}{(\pi_2-\pi_1)(\pi_4-\pi_3)}-\frac{1}{(\pi_2-\pi_1)[8]_{q_2}}\int_{\pi_1}^{\pi_2}\left[\frac{q_2^3[6]_{q_2}}{[2]_{q_2}}F\left(x,\frac{[3]_{q_2}\pi_4+(\pi_3-\pi_4)}{[3]_{q_2}}\right)\right.\right. \\
&+\frac{q_2^2[6]_{q_2}}{[2]_{q_2}}F\left(x,\frac{[3]_{q_2}\pi_4+[2]_{q_2}(\pi_3-\pi_4)}{[3]_{q_2}}\right)+q_2F(x,\pi_4)+F(x,\pi_3)\Bigg]\,{}^{\pi_2}d_{q_1}(x)-\frac{1}{(\pi_4-\pi_3)[8]_{q_1}} \\
&\times\int_{\pi_3}^{\pi_4}\left[\frac{q_1^3[6]_{q_1}}{[2]_{q_1}}F\left(\frac{[3]_{q_1}\pi_2+(\pi_1-\pi_2)}{[3]_{q_1}},y\right)+\frac{q_1^2[6]_{q_1}}{[2]_{q_1}}F\left(\frac{[3]_{q_1}\pi_2+[2]_{q_1}(\pi_3-\pi_4)}{[3]_{q_1}},y\right)\right. \\
&+F(\pi_3,y)+q_1F(\pi_2,y)\Big]\,{}^{\pi_4}d_{q_2}(y)+\frac{1}{[8]_{q_2}[8]_{q_1}}\left[F(\pi_2,\pi_3)+\frac{q_2^3[6]_{q_2}}{[2]_{q_2}}F\left(\pi_2,\frac{[3]_{q_2}\pi_4+(\pi_3-\pi_4)}{[3]_{q_2}}\right)\right.
\end{aligned}$$

$$+\frac{q_2^2[6]_{q_2}}{[2]_{q_2}}F\left(\pi_2,\frac{[3]_{q_2}\pi_4+[2]_{q_2}(\pi_3-\pi_4)}{[3]_{q_2}}\right)+q_2F(\pi_2,\pi_4)\Bigg]+\frac{1}{[8]_{q_1}[8]_{q_2}}\Bigg[F(\pi_1,\pi_4)+F(\pi_1,\pi_3)+\frac{q_1^3[6]_{q_1}}{[2]_{q_1}}$$
$$\times F\left(\frac{[3]_{q_1}\pi_2+(\pi_1-\pi_2)}{[3]_{q_1}},\pi_4\right)+\frac{q_1^2[6]_{q1}}{[2]_{q_1}}F\left(\frac{[3]_{q_1}\pi_2+[2]_{q_1}(\pi_1-\pi_2)}{[3]_{q_1}},\pi_4\right)\Bigg]\Bigg|$$
$$\leq q_1q_2(\pi_2-\pi_1)(\pi_4-\pi_3)\Bigg[(A_3(q_1)+A_4(q_1)+A_5(q_1))\Bigg\{(A_3(q_2)+A_4(q_2)+A_5(q_2))\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|$$
$$+(B_3(q_2)+B_4(q_2)+B_5(q_2))\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4))}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|\Bigg\}+\{(B_3(q_1)+B_4(q_1)+B_5(q_1))$$
$$(A_3(q_2)+A_4(q_2)+A_5(q_2))\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|+(B_3(q_2)+B_4(q_2)+B_5(q_2))\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4))}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|\Bigg\}\Bigg].$$

Corollary 4. *If we replace $\grave{O}_1{=}\pi_2-\pi_1$, $\grave{O}_2{=}\pi_4-\pi_3$, $\acute{O}_1{=}\pi_1-\pi_2$, and $\acute{O}_2{=}(\pi_3-\pi_4)$ along with the limits $q_1\to1^-$ and $q_2\to1^-$ in Theorem 6, then we have the following inequality:*

$$\Bigg|\frac{1}{(\pi_2-\pi_1)(\pi_4-\pi_3)}\int_{\pi_1}^{\pi_2}\int_{\pi_3}^{\pi_4}F(x,y)\,{}^{\pi_2}d(x)\,{}^{\pi_2}d(y)-\frac{1}{8(\pi_2-\pi_1)}\int_{\pi_1}^{\pi_2}\Bigg[F(x,\pi_4)+\frac{6}{2}F\left(x,\frac{\pi_3+2\pi_4}{3}\right)$$
$$+F(x,\pi_3)+\frac{6}{2}F\left(x,\frac{2\pi_3+\pi_4}{3}\right)\Bigg]{}^{\pi_2}d(x)-\frac{1}{8(\pi_4-\pi_3)}\int_{\pi_3}^{\pi_4}\Bigg[F(\pi_3,y)+\frac{6}{2}F\left(\frac{\pi_1+2\pi_2}{3},y\right)+F(\pi_2,y)$$
$$+\frac{6}{2}F\left(\frac{2\pi_1+\pi_2}{3},y\right)\Bigg]{}^{\pi_4}d(y)+\frac{1}{64}\Bigg[F(\pi_2,\pi_3)+\frac{6}{2}F\left(\pi_2,\frac{\pi_3+2\pi_4}{3}\right)+F(\pi_2,\pi_4)+\frac{6}{2}F\left(\pi_2,\frac{2\pi_3+\pi_4}{3}\right)\Bigg]$$
$$+\frac{1}{64}\Bigg[F(\pi_1,\pi_4)+\frac{6}{2}F\left(\frac{\pi_1+2\pi_2}{3},\pi_4\right)+\frac{6}{2}F\left(\frac{2\pi_1+\pi_2}{3},\pi_4\right)+F(\pi_1,\pi_3)\Bigg]\leq\grave{O}_1\grave{O}_2\left(\frac{25}{576}\right)^2$$
$$\left|\frac{{}^{\pi_2,\pi_4}\partial^2F(\pi_1,\pi_3)}{{}^{\pi_2}\partial(t)^{\pi_4}\partial(s)}\right|+\left|\frac{{}^{\pi_2,\pi_4}\partial^2F(\pi_1,\pi_4))}{{}^{\pi_2}\partial(t)^{\pi_4}\partial(s)}\right|+\left|\frac{{}^{\pi_2,\pi_4}\partial^2F(\pi_2,\pi_3)}{{}^{\pi_2}\partial(t)^{\pi_4}\partial(s)}\right|+\left|\frac{{}^{\pi_2,\pi_4}\partial^2F(\pi_2,\pi_4))}{{}^{\pi_2}\partial(t)^{\pi_4}\partial(s)}\right|\Bigg].$$

Theorem 7. *We assume that the conditions of Lemma 3 hold. If $\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}$ is a generalized convex function and integrable on Δ, where $p_1>1$ and $\frac{1}{r_1}+\frac{1}{p_1}{=}1$, then we have the following inequality:*

$$\Bigg|\frac{1}{\grave{O}_1\,\grave{O}_2}\int_{\pi_2+\acute{O}_1}^{\pi_2}\int_{\pi_4+\acute{O}_2}^{\pi_4}F(x,y)\,{}^{\pi_2}d_{q_1}(x)\,{}^{\pi_4}d_{q_2}(y)-\frac{1}{\grave{O}_1\,[8]_{q_2}}$$
$$\int_{\pi_2+\acute{O}_1}^{\pi_2}\Bigg[q_2F(x,\pi_4)+\frac{q_2^3[6]_{q_2}}{[2]_{q_2}}F\left(x,\pi_4+\frac{\acute{O}_2}{[3]_{q_2}}\right)+F(x,\pi_4+\acute{O}_2)+\frac{q_2^2[6]_{q_2}}{[2]_{q_2}}$$
$$F\left(x,\pi_4+\frac{[2]_{q_2}\acute{O}_2}{[3]_{q_2}}\right)\Bigg]{}^{\pi_2}d_{q_1}(x)-\frac{1}{\grave{O}_2[8]_{q_1}}\int_{\pi_4+\acute{O}_2}^{\pi_4}\big[F(\pi_2+\acute{O}_2,y)$$
$$+\frac{q_1^3[6]_{q_1}}{[2]_{q_1}}F\left(\pi_2+\frac{\acute{O}_1}{[3]_{q_1}},y\right)+\frac{q_1^2[6]_{q_1}}{[2]_{q_1}}F\left(\pi_2+\frac{[2]_{q_1}\acute{O}_1}{[3]_{q_1}},y\right)+q_1F(\pi_2,y)\Bigg]{}^{\pi_4}d_{q_2}(y)$$
$$+\frac{1}{[8]_{q_2}[8]_{q_1}}\Bigg[\frac{q_2^3[6]_{q_2}}{[2]_{q_2}}F\left(\pi_2,\pi_4+\frac{\acute{O}_2}{[3]_{q_2}}\right)+\frac{q_2^2[6]_{q_2}}{[2]_{q_2}}F\left(\pi_2,\pi_4+\frac{[2]_{q_2}\acute{O}_2}{[3]_{q_2}}\right)$$
$$+q_2F(\pi_2,\pi_4)+F(\pi_2,\pi_4+\acute{O}_2)\big]+\frac{1}{[8]_{q_1}[8]_{q_2}}\Bigg[F(\pi_2+\acute{O}_1,\pi_4)+\frac{q_1^3[6]_{q_1}}{[2]_{q_1}}$$

$$\times F\left(\pi_2+\frac{\acute{O}_1}{[3]_{q_1}},\pi_4\right)+\frac{q_1^2[6]_{q1}}{[2]_{q_1}}F\left(\pi_2+\frac{[2]_{q_1}\acute{O}_1}{[3]_{q_1}},\pi_4\right)+F\left(\pi_2+\acute{O}_1,\pi_4+\acute{O}_2\right)\Bigg]\Bigg|$$

$$\leq q_1q_2\grave{O}_1\ \ \grave{O}_2[\bar{A}_1(q_1)]^{\frac{1}{r_1}}[\bar{A}_1(q_2)]^{\frac{1}{r_1}}$$

$$\times\Bigg[\check{D}_1(q_1)\left(\check{D}_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+\check{D}_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)$$

$$+\check{D}_2(q_1)\left(\check{D}_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+\check{D}_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\Bigg]^{\frac{1}{p_1}}$$

$$+q_1q_2\grave{O}_1\ \ \grave{O}_2[\bar{A}_1(q_1)]^{\frac{1}{r_1}}[\bar{A}_2(q_2)]^{\frac{1}{r_1}}$$

$$\times\Bigg[\check{D}_1(q_1)\left(\check{E}_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+\check{E}_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)$$

$$+\check{D}_2(q_1)\left(\check{E}_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+\check{E}_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\Bigg]^{\frac{1}{p_1}}$$

$$+q_1q_2\grave{O}_1\ \ \grave{O}_2[\bar{A}_1(q_1)]^{\frac{1}{r_1}}[\bar{A}_3(q_2)]^{\frac{1}{r_1}}$$

$$\times\Bigg[\check{D}_1(q_1)\left(\check{G}_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+\check{G}_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)$$

$$+\check{D}_2(q_1)\left(\check{G}_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+\check{G}_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\Bigg]^{\frac{1}{p_1}}$$

$$+q_1q_2\grave{O}_1\ \ \grave{O}_2[\bar{A}_2(q_1)]^{\frac{1}{r_1}}[\bar{A}_1(q_2)]^{\frac{1}{r_1}}$$

$$\times\Bigg[\check{E}_1(q_1)\left(\check{D}_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+\check{D}_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)$$

$$+\check{E}_2(q_1)\left(\check{D}_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+\check{D}_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\Bigg]^{\frac{1}{p_1}}$$

$$+q_1q_2\grave{O}_1\ \ \grave{O}_2[\bar{A}_2(q_1)]^{\frac{1}{r_1}}[\bar{A}_2(q_2)]^{\frac{1}{r_1}}$$

$$\times\Bigg[\check{E}_1(q_1)\left(\check{E}_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+\check{E}_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)$$

$$+\check{E}_2(q_1)\left(\check{E}_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+\check{E}_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\Bigg]^{\frac{1}{p_1}}$$

$$+q_1q_2\grave{O}_1\ \ \grave{O}_2[\bar{A}_2(q_1)]^{\frac{1}{r_1}}[\bar{A}_3(q_2)]^{\frac{1}{r_1}}$$

$$\times\Bigg[\check{E}_1(q_1)\left(\check{G}_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+\check{G}_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)$$

$$+\check{E}_2(q_1)\left(\check{G}_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+\check{G}_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\Bigg]^{\frac{1}{p_1}}$$

$$+q_1q_2\grave{O}_1\grave{O}_2[\bar{A}_3(q_1)]^{\frac{1}{r_1}}[\bar{A}_1(q_2)]^{\frac{1}{r_1}}$$

$$\times\Bigg[\check{G}_1(q_1)\left(\check{D}_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+\check{D}_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)$$

$$+\breve{G}_2(q_1)\left(\breve{D}_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+\breve{D}_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\right]^{\frac{1}{p_1}}$$
$$+q_1q_2\grave{O}_1\grave{O}_2[\bar{A}_3(q_1)]^{\frac{1}{r_1}}[\bar{A}_2(q_2)]^{\frac{1}{r_1}}$$
$$\times\left[\breve{G}_1(q_1)\left(\breve{E}_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+\breve{E}_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\right.$$
$$+\breve{G}_2(q_1)\left(\breve{E}_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+\breve{E}_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\right]^{\frac{1}{p_1}}$$
$$+q_1q_2\grave{O}_1\grave{O}_2[\bar{A}_3(q_1)]^{\frac{1}{r_1}}[\bar{A}_3(q_2)]^{\frac{1}{r_1}}$$
$$\times\left[\breve{G}_1(q_1)\left(\breve{G}_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+\breve{G}_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\right.$$
$$+\breve{G}_2(q_1)\left(\breve{G}_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+\breve{G}_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\right]^{\frac{1}{p_1}},$$

For $i\in\{1,2\}$ *we have*

$$\bar{A}_1(q_i)=\frac{q_i^{3r_1}[5]^{r_1}_{q_i}}{[3]^{r_1+1}_{q_i}[2]^{r_1}_{q_i}},\qquad \breve{D}_1(q_i)=\frac{1}{[3]^2_{q_i}[2]_{q_i}},\qquad \breve{D}_2(q_i)=\frac{[3]_{q_i}[2]_{q_i}-1}{[3]^2_{q_i}[2]_{q_i}},$$
$$\bar{A}_2(q_i)=\frac{q_i^{r_1}[2]_{q_i}-q_i^{2r_1}}{[3]^{r_1+1}_{q_i}[2]^{r_1}_{q_i}},\qquad \breve{E}_1(q_i)=\frac{q_i^2+2}{[3]^2_{q_i}[2]_{q_i}},\qquad \breve{E}_2(q_i)=\frac{q_2[3]_{q_i}[2]_{q_i}-(q_i^2+2q_i)}{[3]^2_{q_i}[2]_{q_i}},$$
$$\bar{A}_3(q_i)=\frac{q_i^{7r_1}}{[8]^{r_1}_{q_i}}-\frac{[2]_{q_i}\left([7]_{q_i}[3]_{q_i}-[8]_{q_i}[2]_{q_i}\right)}{[8]^{r_1}_{q_i}[3]^{r_1+1}_{q_i}},\qquad \breve{G}_1(q_i)=\frac{[3]^2_{q_i}-[2]^2_{q_i}}{[3]^2_{q_i}[2]_{q_i}},\qquad \breve{G}_2(q_i)=\frac{q_i^2[3]_{q_i}[2]_{q_i}+[2]^2_{q_i}-[3]^2_{q_i}}{[3]^2_{q_i}[2]_{q_i}}.$$

where $0<q_i<1$.

Theorem 8. *We assume that the conditions of Lemma 3 hold. If* $\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}$, *where* $p_1>1$ *and* $\frac{1}{r_1}+\frac{1}{p_1}=1$, *is a generalized convex function and integrable on* Δ, *then we have the following inequality:*

$$\left|\frac{1}{\grave{O}_1\grave{O}_2}\int_{\pi_2+\acute{O}_1}^{\pi_2}\int_{\pi_4+\acute{O}_2}^{\pi_4}F(x,y)\,{}^{\pi_2}d_{q_1}(x)\,{}^{\pi_4}d_{q_2}(y)-\frac{1}{\grave{O}_1[8]_{q_2}}\int_{\pi_2+\acute{O}_1}^{\pi_2}\right. \tag{44}$$
$$\times\left[q_2F(x,\pi_4)+\frac{q_2^3[6]_{q_2}}{[2]_{q_2}}F\left(x,\pi_4+\frac{\acute{O}_2}{[3]_{q_2}}\right)+\frac{q_2^2[6]_{q_2}}{[2]_{q_2}}F\left(x,\pi_4+\frac{[2]_{q_2}\acute{O}_2}{[3]_{q_2}}\right)+F\left(x,\pi_4+\acute{O}_2\right)\right]d_{q_1}(x)$$
$$-\frac{1}{\grave{O}_2[8]_{q_1}}\int_{\pi_4+\acute{O}_2}^{\pi_4}\left[F(\pi_2+\acute{O}_2,y)+\frac{q_1^3[6]_{q_1}}{[2]_{q_1}}F\left(\pi_2+\frac{\acute{O}_1}{[3]_{q_1}},y\right)+q_1F(\pi_2,y)\right.$$
$$\left.+\frac{q_1^2[6]_{q_1}}{[2]_{q_1}}F\left(\pi_2+\frac{[2]_{q_1}\acute{O}_1}{[3]_{q_1}},y\right)\right]{}^{\pi_4}d_{q_2}(y)+\frac{1}{[8]_{q_2}[8]_{q_1}}\left[\frac{q_2^3[6]_{q_2}}{[2]_{q_2}}F\left(\pi_2,\pi_4+\frac{\acute{O}_2}{[3]_{q_2}}\right)+\frac{q_2^2[6]_{q_2}}{[2]_{q_2}}\right.$$
$$\left.\times F\left(\pi_2,\pi_4+\frac{[2]_{q_2}\acute{O}_2}{[3]_{q_2}}\right)+q_2F(\pi_2,\pi_4)+F\left(\pi_2,\pi_4+\acute{O}_2\right)\right]+\frac{1}{[8]_{q_1}[8]_{q_2}}\left[F\left(\pi_2+\acute{O}_1,\pi_4\right)\right.$$
$$\left.\left.+\frac{q_1^2[6]_{q1}}{[2]_{q_1}}F\left(\pi_2+\frac{[2]_{q_1}\acute{O}_1}{[3]_{q_1}},\pi_4\right)+\frac{q_1^3[6]_{q_1}}{[2]_{q_1}}F\left(\pi_2+\frac{\acute{O}_1}{[3]_{q_1}},\pi_4\right)+F\left(\pi_2+\acute{O}_1,\pi_4+\acute{O}_2\right)\right]\right|$$

$$\leq q_1 q_2 \grave{O}_1\ \grave{O}_2\big(\hat{S}_1(q_1)\big)^{1-\frac{1}{p_1}}\big(\hat{S}_1(q_2)\big)^{1-\frac{1}{p_1}}\left[A_3(q_1)\left(A_3(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_3(q_2)\right.\right.$$

$$\left.\left.\times\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)+B_3(q_1)\left(A_3(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_3(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\right]^{\frac{1}{p_1}}$$

$$+q_1 q_2 \grave{O}_1\grave{O}_2\big(\hat{S}_1(q_1)\big)^{1-\frac{1}{p_1}}\big(\hat{S}_2(q_2)\big)^{1-\frac{1}{p_1}}\left[A_3(q_1)\left(A_4(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_4(q_2)\right.\right.$$

$$\left.\left.\times\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)+B_3(q_1)\left(A_4(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_4(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\right]^{\frac{1}{p_1}}$$

$$+q_1 q_2 \grave{O}_1\grave{O}_2\big(\hat{S}_1(q_1)\big)^{1-\frac{1}{p_1}}\big(\hat{S}_3(q_3)\big)^{1-\frac{1}{p_1}}\left[A_3(q_1)\left(A_5(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_5(q_2)\right.\right.$$

$$\left.\left.\times\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)+B_3(q_1)\left(A_5(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_5(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\right]^{\frac{1}{p_1}}$$

$$+q_1 q_2 \grave{O}_1\ \grave{O}_2\big(\hat{S}_2(q_1)\big)^{1-\frac{1}{p_1}}\big(\hat{S}_1(q_2)\big)^{1-\frac{1}{p_1}}\left[A_4(q_1)\left(A_3(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_3(q_2)\right.\right.$$

$$\left.\left.\times\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)+B_4(q_1)\left(A_3(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_3(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\right]^{\frac{1}{p_1}}$$

$$+q_1 q_2 \grave{O}_1\grave{O}_2\big(\hat{S}_2(q_1)\big)^{1-\frac{1}{p_1}}\big(\hat{S}_2(q_2)\big)^{1-\frac{1}{p_1}}\left[A_4(q_1)\left(A_4(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_4(q_2)\right.\right.$$

$$\left.\left.\times\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)+B_4(q_1)\left(A_4(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_4(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\right]^{\frac{1}{p_1}}$$

$$+q_1 q_2 \grave{O}_1\ \grave{O}_2\big(\hat{S}_2(q_1)\big)^{1-\frac{1}{p_1}}\big(\hat{S}_3(q_2)\big)^{1-\frac{1}{p_1}}\left[A_4(q_1)\left(A_5(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_5(q_2)\right.\right.$$

$$\left.\left.\times\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)+B_4(q_1)\left(A_5(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_5(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\right]^{\frac{1}{p_1}}$$

$$+q_1 q_2 \grave{O}_1\grave{O}_2\big(\hat{S}_3(q_1)\big)^{1-\frac{1}{p_1}}\big(\hat{S}_1(q_2)\big)^{1-\frac{1}{p_1}}\left[A_5(q_1)\left(A_3(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_3(q_2)\right.\right.$$

$$\left.\left.\times\left|\frac{{}^{\pi_2,d}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)+B_5(q_1)\left(A_3(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_3(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\right]^{\frac{1}{p_1}}$$

$$+q_1 q_2 \grave{O}_1\grave{O}_2\big(\hat{S}_3(q_1)\big)^{1-\frac{1}{p_1}}\big(\hat{S}_1(q_2)\big)^{1-\frac{1}{p_1}}\left[A_5(q_1)\left(A_4(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_4(q_2)\right.\right.$$

$$\left.\left.\times\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)+B_5(q_1)\left(A_4(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_4(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\right)\right]^{\frac{1}{p_1}}$$

$$+q_1 q_2 \grave{O}_1\grave{O}_2\big(\hat{S}_3(q_1)\big)^{1-\frac{1}{p_1}}\big(\hat{S}_3(q_2)\big)^{1-\frac{1}{p_1}}\left[A_5(q_1)\left(A_5(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_5(q_2)\right.\right.$$

$$\times\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\Bigg)+B_5(q_1)\Bigg(A_5(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}+B_5(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|^{p_1}\Bigg)\Bigg]^{\frac{1}{p_1}}$$

where for $i \in \{1,2\}$, we have

$$\hat{S}_1(q_i)=\frac{2q_i}{[8]^2_{q_i}[2]_{q_i}}+\frac{[8]_{q_i}-[3]_{q_i}[2]_{q_i}}{[3]^2_{q_i}[2]_{q_i}[8]_{q_i}}, \qquad \hat{S}_2(q_i)=\frac{2q_i}{[2]^3_{q_i}}+\frac{q_i}{[3]^2_{q_i}[2]_{q_i}}+\frac{1-[3]_{q_i}[2]_{q_i}}{[3]^2_{q_i}[2]_{q_i}},$$

$$\hat{S}_3(q_i)=2\frac{q_i[7]^2_{q_i}}{[8]^2_{q_i}[2]_{q_i}}+\frac{[3]^2_{q_i}+[2]^2_{q_i}}{[2]_{q_i}[3]^2_{q_i}}-\frac{[7]_{q_i}\left([3]_{q_i}+[2]_{q_i}\right)}{[8]_{q_i}[3]_{q_i}}.$$

Proof. Following the same arguments used in proof of Theorem 6, and by using Lemma 3, we have the required inequality (Equation (44)). □

5. Applications

In this section, we discuss some applications of our main findings.

5.1. Applications to Hypergeometric Functions

Now, we derive some more inequalities pertaining to hypergeometric functions and Mittag–Leffler functions.

For Equation (1), if we set $\mu = 1, \nu = 0$, and $\sigma(\c{k}) = \left((\alpha)_{\c{k}}(\beta)_{\c{k}}\right)/\left((\gamma)_{\c{k}}\right)$ for $\c{k}$= 0, 1, 2, 3, . . . , where α, β, and γ are parameters that may be real or complex values, provided that $\gamma \neq 0, -1, -2, \ldots$ and $(m)_{\c{k}}$ is defined as

$$(m)_{\c{k}} = \frac{\Gamma(m+\c{k})}{\Gamma(m)} = m(m+1)\ldots(m+\c{k}+1), \quad \c{k} = 0, 1, \ldots,$$

and restrict the domain to $|x| \leq 1$ (with $x \in \mathbb{C}$), then we find the following classical hypergeometric functions [30,32]:

$$\check{R}_{\mu,\nu,\sigma}(\check{z}) = \check{R}(\alpha,\beta;\gamma;\check{z}) = \sum_{\c{k}=0}^{\infty}\frac{(\alpha)_{\c{k}}(\beta)_{\c{k}}}{\Gamma(\gamma)_{\c{k}}\c{k}!}\check{z}^{\c{k}}, \tag{45}$$

Using these substitutions and the assumption in our Riana functions will transform them into hypergeometric functions, and thus Equation (15) will be replaced by follows:

$$\begin{aligned}\check{R}(\alpha,\beta;\gamma;\pi_1-\pi_2) &= Ħ_1, \qquad \check{R}(\alpha,\beta;\gamma;\pi_2-\pi_1) = {}_1Ħ,\\ \check{R}(\alpha,\beta;\gamma;\pi_3-\pi_4) &= Ħ_2, \qquad \check{R}(\alpha,\beta;\gamma;\pi_4-\pi_3) = {}_2Ħ.\end{aligned}$$

Therefore, our Theorem 3 can be changed as follows:

Theorem 9. *We assume that the conditions of Lemma 2 hold. If $\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1q_2}F(\pi_2+tÓ_1,\pi_4+sÓ_2)}{{}^{\pi_2}\partial_{q_1}(t)^{\pi_4}\partial_{q_2}(s)}\right|$ is a generalized convex function and integrable on Δ, then the following inequality holds for the right quantum integrals:*

$$\Bigg|\frac{1}{[6]_{q_1}[6]_{q_2}}\Bigg[q_1^2[4]_{q_1}F\left(\pi_2+\frac{Ħ_1}{[2]_{q_1}},\pi_4+Ħ_2\right)+q_1^2q_2[4]_{q_1}F\left(\pi_2+\frac{Ħ_1}{[2]_{q_1}},\pi_4\right)+q_1^2q_2^2[4]_{q_1}[4]_{q_2}F\left(\pi_2+\frac{Ħ_1}{[2]_{q_1}},\pi_4+\frac{Ħ_2}{[2]_{q2}}\right)$$
$$+q_1q_2^2[4]_{q_2}F\left(\pi_2,\pi_4+\frac{Ħ_2}{[2]_{q_2}}\right)+q_2^2[4]_{q_2}F\left(\pi_2+Ħ_1,\pi_4+\frac{Ħ_2}{[2]_{q2}}\right)\Bigg]+\frac{1}{[6]_{q_1}[6]_{q_2}}[q_1F(\pi_2,\pi_4+Ħ_2)+q_2F(\pi_2+Ħ_1,\pi_4)$$

$$+F(\pi_2+\text{Ħ}_1,\pi_4+\text{Ħ}_2)+q_1q_2F(\pi_2,\pi_4)]-\int_{\pi_2+\text{Ħ}_1}^{\pi_2}\frac{\left[q_2F(x,\pi_4)+F(x,\pi_4+\text{Ħ}_2)+q_2^2[4]_{q_2}F\left(x,\pi_4+\frac{\text{Ħ}_2}{[2]_{q_2}}\right)\right]}{{}_1\text{Ħ}[6]_{q_2}}\,{}^{\pi_2}d_{q_1}(x)$$
$$-\int_{\pi_4+\text{Ħ}_2}^{\pi_4}\frac{\left[q_1F(\pi_2,y)+q_1^2[4]_{q_1}F\left(\pi_2+\frac{\text{Ħ}_1}{[2]_{q1}},y\right)+F(\pi_2+\text{Ħ}_1,y)\right]}{{}_2\text{Ħ}[6]_{q_1}}\,{}^{\pi_4}d_{q_2}(y)+\int_{\pi_2+\text{Ħ}_1}^{\pi_2}\int_{\pi_4+\text{Ħ}_2}^{\pi_4}\frac{F(x,y)}{{}_1\text{Ħ}\,{}_2\text{Ħ}}\,{}^{\pi_2}d_{q_1}(x)\,{}^{\pi_4}d_{q_2}(y)\Bigg|$$

$$\leqslant {}_1\text{Ħ}\,{}_2\text{Ħ}q_1q_2\Bigg[(A_1(q_1)+A_2(q_1))(A_1(q_2)+A_2(q_2))\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|$$
$$+(B_1(q_2)+B_2(q_2))(A_1(q_1)+A_2(q_1))\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|+(B_1(q_1)+B_2(q_1))(A_1(q_2)+A_2(q_2))$$
$$\times\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|+(B_1(q_1)+B_2(q_1))(B_1(q_2)+B_2(q_2))\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|\Bigg].$$

Similarly, one can find all inequalities of Sections 4.1 and 4.2 involving a hypergeometric function.

5.2. Applications to Mittag–Leffler Functions

Furthermore, by considering $\sigma=(1,1,1,\ldots),\nu=1$, and $\mu=\Phi$ with $\check{R}(\Phi)>0$ in Equation (1), we arrive at the widely recognized Mittag–Leffler function:

$$\check{R}_\Phi(\check{z})=\sum_{\text{ķ}=0}^{\infty}\frac{1}{\Gamma(1+\Phi\text{ķ})}\check{z}^{\text{ķ}},$$

Therefore, Theorem 3 can be changed as follows:

$$\check{R}_\Phi(\pi_1-\pi_2)=\text{Ļ}_1,\qquad \check{R}_\Phi(\pi_2-\pi_1)={}_1\text{Ļ},$$
$$\check{R}_\Phi(\pi_3-\pi_4)=\text{Ļ}_2,\qquad \check{R}_\Phi(\pi_4-\pi_3)={}_2\text{Ļ}.$$

Theorem 10. *We assume that the conditions of Lemma 2 hold. If* $\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1q_2}F(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|$ *is a generalized convex function and integrable on* Δ*, then the following inequality holds for the right quantum integrals:*

$$\Bigg|\frac{1}{[6]_{q_1}[6]_{q_2}}\left[q_1^2[4]_{q_1}F\left(\pi_2+\frac{\text{Ļ}_1}{[2]_{q_1}},\pi_4+\text{Ļ}_2\right)+q_1^2q_2[4]_{q_1}F\left(\pi_2+\frac{\text{Ļ}_1}{[2]_{q_1}},\pi_4\right)+q_1^2q_2^2[4]_{q_1}[4]_{q_2}F\left(\pi_2+\frac{\text{Ļ}_1}{[2]_{q_1}},\pi_4+\frac{\text{Ļ}_2}{[2]_{q2}}\right)\right.$$
$$\left.+q_1q_2^2[4]_{q_2}F\left(\pi_2,\pi_4+\frac{\text{Ļ}_2}{[2]_{q_2}}\right)+q_2^2[4]_{q_2}F\left(\pi_2+\text{Ļ}_1,\pi_4+\frac{\text{Ļ}_2}{[2]_{q2}}\right)\right]+\frac{1}{[6]_{q_1}[6]_{q_2}}[q_1F(\pi_2,\pi_4+\text{Ļ}_2)+q_2F(\pi_2+\text{Ļ}_1,\pi_4)$$
$$+F(\pi_2+\text{Ļ}_1,\pi_4+\text{Ļ}_2)+q_1q_2F(\pi_2,\pi_4)]-\int_{\pi_2+\text{Ļ}_1}^{\pi_2}\frac{\left[q_2F(x,\pi_4)+F(x,\pi_4+\text{Ļ}_2)+q_2^2[4]_{q_2}F\left(x,\pi_4+\frac{\text{Ļ}_2}{[2]_{q_2}}\right)\right]}{{}_1\text{Ļ}[6]_{q_2}}\,{}^{\pi_2}d_{q_1}(x)$$
$$-\int_{\pi_4+\text{Ļ}_2}^{\pi_4}\frac{\left[q_1F(\pi_2,y)+q_1^2[4]_{q_1}F\left(\pi_2+\frac{\text{Ļ}_1}{[2]_{q1}},y\right)+F(\pi_2+\text{Ļ}_1,y)\right]}{[6]_{q_1}\,{}_2\text{Ļ}}\,{}^{\pi_4}d_{q_2}(y)+\int_{\pi_2+\text{Ļ}_1}^{\pi_2}\int_{\pi_4+\text{Ļ}_2}^{\pi_4}\frac{F(x,y)}{{}_1\text{Ļ}\,{}_2\text{Ļ}}\,{}^{\pi_2}d_{q_1}(x)\,{}^{\pi_4}d_{q_2}(y)\Bigg|$$

$$\leqslant {}_1\text{Ļ}_2\text{Ļ}q_1q_2\Bigg[(A_1(q_1)+A_2(q_1))(A_1(q_2)+A_2(q_2))\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|$$
$$+(B_1(q_2)+B_2(q_2))(A_1(q_1)+A_2(q_1))\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|+(B_1(q_1)+B_2(q_1))(A_1(q_2)+A_2(q_2))$$
$$\times\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|+(B_1(q_1)+B_2(q_1))(B_1(q_2)+B_2(q_2))\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right|\Bigg].$$

Similarly, one can find all inequalities of Sections 4.1 and 4.2 involving Mittag–Leffler Functions.

Applications to Bounded Functions

We assume that the given conditions hold true:

$$\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right| \leq M_1, \qquad \left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right| \leq M_2,$$
$$\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right| \leq M_3, \qquad \left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}F(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}(t)\,{}^{\pi_4}\partial_{q_2}(s)}\right| \leq M_4.$$

In addition, $M_1, M_2, M_3, M_4 \leq M$, where F is a function that is twice (q_1, q_2) differentiable and bounded by the positive real number M.

Proposition 1. *Under the conditions of Theorem 3, we have*

$$\begin{aligned}
&\Bigg|\frac{1}{[6]_{q_1}[6]_{q_2}}\Bigg[q_1^2[4]_{q_1}F\left(\pi_2+\frac{\acute{O}_1}{[2]_{q_1}},\pi_4+\acute{O}_2\right)+q_1^2q_2[4]_{q_1}F\left(\pi_2+\frac{\acute{O}_1}{[2]_{q_1}},\pi_4\right)+q_1^2q_2^2[4]_{q_1}[4]_{q_2}\\
&\times F\left(\pi_2+\frac{\acute{O}_1}{[2]_{q_1}},\pi_4+\frac{\acute{O}_2}{[2]_{q2}}\right)+q_1q_2^2[4]_{q_2}F\left(\pi_2,\pi_4+\frac{\acute{O}_2}{[2]_{q_2}}\right)+q_2^2[4]_{q_2}F\left(\pi_2+\acute{O}_1,\pi_4+\frac{\acute{O}_2}{[2]_{q2}}\right)\Bigg]\\
&+\frac{[q_1F(\pi_2,\pi_4+\acute{O}_2)+q_2F(\pi_2+\acute{O}_1,\pi_4)+F(\pi_2+\acute{O}_1,\pi_4+\acute{O}_2)+q_1q_2F(\pi_2,\pi_4)\,]}{[6]_{q_1}[6]_{q_2}}\\
&-\int_{\pi_2+\acute{O}_1}^{\pi_2}\frac{q_2F(x,\pi_4)+F(x,\pi_4+\acute{O}_2)+q_2^2[4]_{q_2}F\left(x,\pi_4+\frac{\acute{O}_2}{[2]_{q_2}}\right)}{[6]_{q_2}\acute{O}_1}\,{}^{\pi_2}d_{q_1}(x)\\
&-\int_{\pi_4+\acute{O}_2}^{\pi_4}\frac{q_1F(\pi_2,y)+q_1^2[4]_{q_1}F\left(\pi_2+\frac{\acute{O}_1}{[2]_{q1}},y\right)+F(\pi_2+\acute{O}_1,y)}{[6]_{q_1}\,\grave{O}_2}\,{}^{\pi_4}d_{q_2}(y)\\
&+\frac{1}{\grave{O}_1\grave{O}_2}\int_{\pi_2+\acute{O}_1}^{\pi_2}\int_{\pi_4+\acute{O}_2}^{\pi_4}F(x,y)\,{}^{\pi_2}d_{q_1}(x)\,{}^{\pi_4}d_{q_2}(y)\Bigg|\\
&\leqslant q_1q_2\grave{O}_1\grave{O}_2[(A_1(q_1)+A_2(q_1))(A_1(q_2)+A_2(q_2))M+(B_1(q_2)+B_2(q_2))(A_1(q_1)+A_2(q_1))M\\
&+(B_1(q_1)+B_2(q_1))(A_1(q_2)+A_2(q_2))M+(B_1(q_1)+B_2(q_1))(B_1(q_2)+B_2(q_2))M].
\end{aligned}$$

6. Conclusions

The theory of inequalities has flourished at a fast rate in recent years due to its diverse range of applications. Undoubtedly, convexity and its general representation are the main tools for developing fundamental inequalities. Numerous inequalities involving convex functions have been retained in the literature. Various generalizations of Simpson-type inequalities have been proposed in the literature through quantum calculus, fractional calculus, and post-quantum calculus. To summarize, our work has established quantum integral identities and introduced new inequalities of the Simpson and Newton types for coordinated generalized convex functions in the context of right quantum integrals. By performing appropriate substitutions of $i\check{R}\mu_i, \nu_i{}^{\sigma_i}$ and using the limit $q_i \to 1$, where $i \in \{1, 2\}$, we also obtained additional results. These coordinated formulations of novel and intriguing problems can prove useful for future research on identifying analogous inequalities for diverse forms of generalized convexity, which have numerous applications in various mathematical fields. In the future, we will develop some new variants of integral inequalities involving new generalizations of quantum calculus, fractional calculus, and fractional quantum calculus involving different kinds of convexity. We hope that this paper will open a new arena for research.

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