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Fixed Points and λ-Weak Contractions

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Abstract: In this paper, we introduce a new type of contractions on a metric space (X, d) in which the distance $d(x, y)$ is replaced with a function, depending on a parameter λ, that is not symmetric in general. This function generalizes the usual case when $\lambda = 1/2$ and can take bigger values than $m_{1/2}$. We call these new types of contractions λ-weak contractions and we provide some of their properties. Moreover, we investigate cases when these contractions are Picard operators.

Keywords: fixed point; weak contraction; Picard operator

1. Introduction

Fixed point theory plays an important role in pure and applied mathematics. Among its applications, we mention nonlinear analysis, integral and differential equations, engineering, game theory, economics and so on.

Banach's famous theorem marks the beginning of the development of the metric fixed point theory. In the following, we recall some well-known results.

We let (X, d) be a metric space and $T : X \to X$ be a mapping. We recall that T is a Banach contraction if there exists $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \le \lambda d(x, y),\ (\forall) x, y \in X. \tag{1}$$

S. Banach [1] proved that every self-mapping T defined on a complete metric space satisfying (1) has a unique fixed point (i.e., $Tu = u$), and for every $x \in X$, sequence $\{T^n x\}$ converges to fixed point u. Due to its simplicity and wide range of applications, this result was generalized in various ways. See, for example, book [2] and recent papers [3–6].

Definition 1. *We assert that T is a Picard operator if T has a unique fixed point u in X and for any $x \in X$, sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to u (see [7,8] and book [2]) .*

Using this definition, the Banach theorem states the following: If (X, d) is a complete metric space, the Banach contraction $T : X \to X$ is a Picard operator.

After this remarkable result was obtained, a number of various generalizations appeared. We mention here one of the most cited results in the fixed point literature, obtained in 1969 by Meir and Keeler [9]. The authors introduced the notion of weakly uniformly strict contraction, which later became known as the Meir–Keeler contraction. Also, they extend Banach's metric fixed point theorem by replacing the contraction condition with this new type of contraction.

Definition 2. *We assert that T is a Meir–Keeler contraction if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$(\forall)\ x, y \in X,\ \varepsilon \le d(x, y) < \varepsilon + \delta \quad \textit{implies} \quad d(Tx, Ty) < \varepsilon.$$

Theorem 1. *(Meir, Keeler [9]) We let (X, d) be a complete metric space and T be a Meir–Keeler contraction. Then, T is a Picard operator.*

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New classes of Meir–Keeler contractions were obtained recently by the first author (see [10,11]).

In paper [12], S. Park and B.E. Rhoades provide fixed point results for weak Meir–Keeler contractions. As a particular case of their theorem, we have the following result. First, we denote

$$m(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

Theorem 2. *We let (X,d) be a complete metric space and T be a continuous mapping. We suppose T satisfies the following condition: for $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\varepsilon \le m(x,y) < \varepsilon + \delta \text{ implies } d(Tx,Ty) < \varepsilon. \tag{2}$$

Then, T is a Picard operator.

Another generalization of Meir–Keeler contractions is given in the following theorem. First, we remember the following definition:

Definition 3. *[13] We assert that T is a CJMP contraction (cf. [14–17]) if the following conditions hold:*

(a) T is contractive (i.e., the following inequality $d(Tx,Ty) < d(x,y)$ holds for $x,y \in X,\ x \ne y$);
(b) (The Matkowski–Węgrzyk condition [18]) for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$(\forall)\ x,y \in X, \varepsilon < d(x,y) < \varepsilon + \delta \Longrightarrow d(Tx,Ty) \le \varepsilon.$$

Lj. Ćirić [14] proved that the class of CJMP contractions contains the class of Meir–Keeler contractions. In paper [13], we provided a pedagogical proof for the following theorem:

Theorem 3. *(see [14–17])*
We let (X,d) be a complete metric space and T be a CJMP contraction on X. Then, T is a Picard operator.

Also, in paper [13], we obtained two general theorems concerning the existence of the Picard operators on complete metric spaces and some applications.

In this this paper, we obtain new classes of Picard operators on a complete metric space (X,d), by replacing distance $d(x,y)$ with a non-symmetric function. Many results in the literature are obtained from our results by taking $\lambda = 1/2$. Our function is given by

$$m_\lambda(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \lambda d(y,Tx) + (1-\lambda)d(x,Ty)\}$$

and is used here for the first time in the context of fixed point theory. The reason for the introduction of this function is the fact that m_λ can take bigger values than $m_{\frac{1}{2}} = m$.

We consider that our results can be applied in the study of Ulam's type stability and in the theory of integral equations.

2. Main Results

In this paper, we introduce and investigate a new type of contraction named λ-weak contraction. First, we denote for $0 < \lambda < 1$ and for all $x,y \in X$

$$m_\lambda(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \lambda d(y,Tx) + (1-\lambda)d(x,Ty)\}.$$

Definition 4. *We assert that T is a λ-weak contraction if the following conditions hold:*

(C_1) *$d(Tx,Ty) < m_\lambda(x,y)$, if $m_\lambda(x,y) > 0$ (T is λ-weak contractive)*
(C_2) $(\forall)\ \varepsilon > 0,\ (\exists)\ \delta = \delta(\varepsilon) > 0$ *such that*

$$(\forall)\ x,y \in X,\ \varepsilon < m_\lambda(x,y) < \varepsilon + \delta \Longrightarrow d(Tx,Ty) \le \varepsilon.$$

We remark that function m_λ is not symmetric in general. It is a symmetric function if and only if $\lambda = \frac{1}{2}$. Another motivation for the introduction of this function is the fact that m_λ can take bigger values than $m_{\frac{1}{2}} = m$.

We provide an example inspired by paper [19] that justifies the introduction of these new types of contractions.

Example 1. *We consider $M = [0,1]$, $0 < \lambda < 1$ and mapping $T : M \to M$ is defined as follows:*

$$Tx = \begin{cases} \lambda x, x \neq 0 \\ 1, x = 0 \end{cases}.$$

M is a complete metric space with the usual metric. In this case,

$$m_\lambda(x,0) = \max\{1, \lambda^2 x + (1-\lambda)(1-x)\} = \begin{cases} \lambda^2 x + (1-\lambda)(1-x),\ \textit{if}\ \dfrac{\sqrt{5}-1}{2} < \lambda < 1 \\ 1,\ \textit{otherwise}. \end{cases}$$

We observe that for $\dfrac{\sqrt{5}-1}{2} < \lambda < 1$, $m_\lambda(x,0) > m_{1/2}(x,0)$.

In 1975, J. Matkowski [16] proved that if T is $\frac{1}{2}$-weak contraction on a complete metric space and T is continuous or given $\varepsilon > 0$, $(\exists)\mu$, $0 < \mu < \varepsilon$ such that for all $x,y \in X$

$$\left.\begin{array}{r} 0 < d(Tx,x), \dfrac{d(Tx,y)+d(x,Ty)}{2} \le \varepsilon \\ 0 < d(x,y), d(y,Ty) < \mu \end{array}\right\} \Longrightarrow d(Tx,Ty) < \varepsilon - \mu,$$

then T is a Picard operator.

In the following, we provide some properties of λ-weak contractions and we prove that if T is a λ-weak contraction and T is continuous or verifies the condition

$$(C_3)\ \text{given}\ \varepsilon > 0,\ \text{there exists a}\ \mu, 0 < \mu \le \varepsilon\ \text{such that for}\ x,y \in X,$$

$$\left.\begin{array}{r} 0 < d(Tx,x), \lambda d(y,Tx) + (1-\lambda)d(x,Ty) \le \varepsilon \\ 0 < d(x,y), d(y,Ty) < \mu \end{array}\right\} \Longrightarrow d(Tx,Ty) < \varepsilon - \mu.$$

Then, T is a Picard operator.

Also, we prove that T is a Picard operator if T verifies conditions (C_2) and

$$(C_4)\ \text{given}\ \varepsilon > 0,\ (\exists)\ \mu,\ 0 < \mu < \varepsilon\ \text{such that if}\ m_\lambda(x,y) = \varepsilon, \text{then}\ d(Tx,Ty) \le \varepsilon - \mu.$$

Proposition 1. *If T is λ-weak contractive and $Tx \neq T^2x$, then*

$$d(Tx,T^2x) < d(x,Tx).$$

Proof. We have

$$m_\lambda(x,Tx) = \max\{d(x,Tx), d(Tx,T^2x), (1-\lambda)d(x,T^2x)\},$$

hence

$$m_\lambda(x,Tx) \geq d(Tx,T^2x) > 0.$$

In the following, we use the proof by contradiction to prove that $d(Tx,T^2x) < d(x,Tx)$. We suppose that $d(Tx,T^2x) \geq d(x,Tx)$. We obtain

$$\begin{aligned} d(x,T^2x) &\leq d(x,Tx) + d(Tx,T^2x) \\ &\leq 2d(Tx,T^2x), \end{aligned}$$

hence

$$m_\lambda(x,Tx) \leq \max\{1, 2(1-\lambda)\}d(Tx,T^2x) = d(Tx,T^2x)$$

if $\lambda \geq \dfrac{1}{2}$. It follows that

$$d(Tx,T^2x) < m_\lambda(x,Tx) = d(Tx,T^2x),$$

which is absurd. Hence, for $\lambda \geq \dfrac{1}{2}$, we have $d(Tx,T^2x) < d(x,Tx)$.

Let $\lambda < \dfrac{1}{2}$. We suppose that $d(Tx,T^2x) \geq d(x,Tx)$. Then,

$$m_\lambda(y,x) = \max\{d(x,y), d(y,Ty), d(x,Tx), \lambda d(x,Ty) + (1-\lambda)d(y,Tx)\},$$

$$m_\lambda(Tx,x) = \max\{d(x,Tx), d(Tx,T^2x), \lambda d(x,T^2x)\}.$$

Because $d(Tx,T^2x) \geq d(x,Tx)$, it follows that

$$d(x,T^2x) \leq d(x,Tx) + d(Tx,T^2x) \leq 2\lambda d(Tx,T^2x).$$

We obtain

$$\begin{aligned} m_\lambda(Tx,x) &\leq \max\{d(Tx,T^2x), 2\lambda d(Tx,T^2x)\} \\ &= d(Tx,T^2x). \end{aligned}$$

But T is λ-weak contractive, hence

$$d(T^2x,Tx) < m_\lambda(Tx,x) = d(Tx,T^2x),$$

which is absurd. □

Corollary 1. *If T is λ-weak contractive and $x \in X$ is such that $Tx \neq T^2x$, then*

- *if $\lambda \geq \dfrac{1}{2}$, it follows that $m_\lambda(x,Tx) = d(x,Tx)$;*
- *if $0 < \lambda < \dfrac{1}{2}$, it follows that $m_\lambda(Tx,x) = d(x,Tx)$.*

Proof. ∘ If $\lambda \geq \dfrac{1}{2}$, it follows that

$$m_\lambda(x,Tx) = \{d(x,Tx), d(Tx,T^2x), (1-\lambda)d(x,T^2x)\}.$$

From Proposition 1, it follows that

$$d(x,T^2x) \leq d(x,Tx) + d(Tx,T^2x) \leq 2d(x,Tx).$$

Hence,

$$m_\lambda(x,Tx) \leq \max\{d(x,Tx), 2(1-\lambda)\} \leq d(x,Tx),$$

because $2(1-\lambda) \le 1 \iff \lambda \ge \frac{1}{2}$.

- If $0 < \lambda < \dfrac{1}{2}$, we have

$$m_\lambda(Ty, y) = \max\{d(Ty, y), d(Ty, T^2y), \lambda(y, T^2y)\}.$$

From Proposition 1,

$$d(Ty, T^2y) < d(y, Ty).$$

Using the triangle inequality, we obtain

$$\lambda d(y, T^2y)\lambda d(y, Ty) + \lambda d(Ty, T^2y) \le 2\lambda d(y, Ty).$$

Hence,

$$m_\lambda(Ty, y) = \max\{d(Ty, y), 2\lambda d(y, Ty)\} = d(Ty, y)$$

if $1 < 2\lambda \iff \lambda < \dfrac{1}{2}$. □

Proposition 2. *We let $T : X \to X$ be a λ-weak contraction as in Definition 4 and*

$$\eta = \eta(\varepsilon) \le \frac{\delta(\varepsilon)}{2} = \frac{\delta}{2}.$$

If $d(x, Tx) < \eta$, $d(y, Ty) < \eta$ and $d(x, y) \le \eta + \varepsilon$, then $d(Tx, Ty) \le \varepsilon$.

Proof. If $x = y$, it is obvious. If $x \ne y$, then $m_\lambda(x, y) > 0$ and

$$d(y, Tx) \le d(y, x) + d(x, Tx) \le \varepsilon + 2\eta,$$

and, respectively,

$$d(x, Ty) \le d(x, y) + d(y, Ty) \le \varepsilon + 2\eta.$$

We have

$$\lambda d(y, Tx) + (1-\lambda)d(x, Ty) \le \varepsilon + 2\eta,$$

Hence,

$$m_\lambda(x, y) \le \varepsilon + 2\eta < \varepsilon + \delta.$$

If $m_\lambda(x, y) \le \varepsilon$, from (C_1), it follows that $d(Tx, Ty) < m_\lambda(x, y) \le \varepsilon$.
If $m_\lambda(x, y) > \varepsilon$, from (C_2), it follows that $d(Tx, Ty) \le \varepsilon$. □

Proposition 3. *We let $T : X \to X$ be an arbitrary mapping. If*

$$\lambda d(x, y) + (1-\lambda)d(x, Ty) \le (1-\lambda)d(x, Tx),$$

then $m_\lambda(x, y) = d(x, Tx)$.

Proof. From the definition of m_λ, it follows that $d(x, Tx) \le m_\lambda(x, y)$. Using the triangle inequality, we have

$$d(y, Tx) \le d(y, x) + d(x, Tx);$$

hence,

$$\begin{aligned} \lambda d(y, Tx) + (1-\lambda)d(x, Ty) &\le \lambda d(x, y) + \lambda d(x, Tx) + (1-\lambda)d(x, Ty) \\ &\le (1-\lambda)d(x, Tx) + \lambda d(x, Tx) = d(x, Tx). \end{aligned}$$

□

In the following Theorem, we provide a generalization of the theorem of Matkowski [16] (see also [15]) by taking, instead of

$$\frac{d(y,Tx)+d(x,Ty)}{2},$$

a convex combination of $d(y,Tx)+d(x,Ty)$, i.e.,

$$\lambda d(y,Tx)+(1-\lambda)d(x,Ty).$$

Also, we assign new conditions (C_2) and (C_4) for T to be a Picard operator.

Theorem 4. *We let (X,d) be a complete metric space and $T:X\to X$ be a mapping. We suppose that T verifies one of the conditions:*

(1) T is a λ-weak contraction and T is continuous;
(2) T verifies conditions (C_2) and (C_4);
(3) T is a λ-weak contraction and verifies condition (C_3).

Then, T is a Picard operator.

Proof. *Step I.* We prove that in sequence $\{x_n\}$ defined by $x_{n+1}=T^n x_n,\ n\in\mathbb{N}$ has the limit of 0.

We can suppose that
$$d(x_n,x_{n+1})>0,\ (\forall)n\in\mathbb{N}.$$

Indeed, if there exists n such that $d(x_n,x_{n+1})=0$, it follows that $x_n=Tx_n$, hence x_n is a fixed point.

From Proposition 1, it follows that $\{d(x_n,x_{n+1})\}$ is strictly decreasing and bounded below by 0. Hence, $\{d(x_n,x_{n+1})\}$ is convergent.

We denote by $\varepsilon_0=\lim\limits_{n\to\infty}d(x_n,x_{n+1})$ and we show that $\varepsilon_0=0$. We assume that $\varepsilon_0>0$.

Because T is a λ-weak contraction, there exists $\delta=\delta(\varepsilon_0)>0$ such that

$$\varepsilon_0<m_\lambda(x,y)<\varepsilon_0+\delta\Longrightarrow d(Tx,Ty)\le\varepsilon_0.$$

We have $\varepsilon_0<d(x_n,x_{n+1})<\varepsilon_0+\delta$, for $n\ge n_0$. From Corollary 1, we have

$$d(x_n,x_{n+1})=m_\lambda(x_n,Tx_n),$$

hence
$$\varepsilon_0<m_\lambda(x_n,Tx_n)<\varepsilon_0+\delta.$$

From (C_2), it follows that
$$d(Tx_n,T^2x_n)\le\varepsilon_0,$$

hence
$$d(x_{n+1},x_{n+2})\le\varepsilon_0$$

for $n\ge n_0$, which is absurd.

*Step II.*We prove that $\{x_n\}$ is a Cauchy sequence.

From *Step I*, we have that for all $\varepsilon>0$, $n_1=n_1(\varepsilon)$ exists such that

$$d(x_{n-1},x_n)<\gamma:=min\left\{\varepsilon,\frac{\delta}{2}\right\},\ n\ge n_1.$$

We use induction to prove that $d(x_n, x_{n+p}) \leq \varepsilon$, $p \in \mathbb{N}$. We suppose that

$$d(x_n, x_{n+p}) \leq \varepsilon$$

and we prove that $d(x_n, x_{n+p+1}) \leq \varepsilon$.

For $p = 1$, we have $d(x_n, x_{n+1}) < d(x_{n-1}, x_n) < \gamma \leq \varepsilon$. If the induction hypothesis is true, it follows that

$$d(x_{n-1}, x_{n+p}) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+p}) < \gamma + \varepsilon.$$

From Proposition 2, it follows that

$$d(Tx_{n-1}, Tx_{n+p}) \leq \varepsilon,\ n \geq n_1.$$

Step III. We prove that T is a Picard operator.

From *Step II*, we have that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complet metric space, it follows that $\{x_n\}$ is convergent. We denote $p = \lim_{n\to\infty} x_n$.

- If T is continuous, we have $\lim_{n\to\infty} Tx_n = Tp$.
  From the uniqueness of the limit, we obtain $p = Tp$; hence, p is a fixed point.
- If T verifies conditions (C_2) and (C_4), it is obvious that T is a λ-weak contraction. In this case, we prove also that $p = Tp$. If $p \neq Tp$,

  $$m_\lambda(p, x_n) = d(p, Tp) = \varepsilon_1.$$

  From (C_4) it follows that $d(Tp, Tx_n) \leq \varepsilon_1 - \mu < \varepsilon_1$. By taking the limit as n moves to infinity, we obtain $d(Tp, p) \leq \varepsilon_1 - \mu < \varepsilon_1$, which is a contradiction.
- If T is a weak contraction and verifies (C_3), we suppose that $Tp \neq p$.
  We denote by $\varepsilon = d(p, Tp) > 0$. Since $x_n \to p$, we have the following inequality:

  $$\lambda d(p, x_n) + (1 - \lambda) d(p, x_{n+1}) \leq (1 - \lambda)\mu$$

  for a large enough n. Then,

  $$\begin{aligned} \lambda d(Tp, x_n) + (1 - \lambda) d(p, Tx_n) &\leq \lambda[d(Tp, p) + d(p, x_n)] + (1 - \lambda) d(p, x_{n+1}) \\ &\leq \lambda d(Tp, p) + [\lambda d(p, x_n) + (1 - \lambda) d(p, x_{n+1})] \\ &\leq \lambda\varepsilon + (1 - \lambda)\mu < \lambda\varepsilon + (1 - \lambda)\varepsilon = \varepsilon. \end{aligned}$$

  Hence, from (C_3), it follows $d(Tp, Tx_n) < \varepsilon - \mu$ and so

  $$d(Tp, p) \leq d(Tp, Tx_n) + d(Tx_n, p) < \varepsilon - \mu + \mu = \varepsilon,$$

  which is a contradiction; therefore, $Tp = p$.

  Now, we prove that the fixed point is unique in each case.

We suppose that there exists another fixed point q such that $Tq = q$, with $p \neq q$. Since T is λ-weak contractive, we obtain

$$d(Tp, Tq) < m_\lambda(p, q).$$

But we also have $d(Tp, Tq) = d(p, q)$ and

$$\begin{aligned} m_\lambda(p, q) &= \max\{d(p, q), d(p, Tp), d(q, Tq), \lambda d(q, Tp) + (1 - \lambda) d(p, Tq)\} \\ &= \max\{d(p, q), d(p, q), d(q, q), \lambda d(q, p) + (1 - \lambda) d(p, q)\} \\ &= \max\{d(p, q), d(p, q)\} = d(p, q). \end{aligned}$$

It follows that $d(p,q) < d(p,q)$, which is absurd. $\square$

We apply Theorem 4 to obtain new fixed point theorems, generalizing the idea from paper [13].

Definition 5. *([13]) We let E, F be two real functions defined on $(0,\infty)$. We assert that (E,F) is a compatible pair of functions if the following conditions hold:*

(E_1) for $t,s \in (0,\infty)$, $t \le s \Rightarrow E(t) < F(s)$;

(E_2) given $t > 0$ and $\{t_n\}_{n\in\mathbb{N}} \subset (t,\infty)$, a sequence with $\lim\limits_{n\to\infty} t_n = t$, for any sequence $\{s_n\}_{n\in\mathbb{N}}$, $t < s_n < t_n$, $n \in \mathbb{N}$, we have

$$\limsup_{n\to\infty}(F(s_n) - E(t_n)) > 0.$$

Here, we introduce a new type of contraction called (λ, E, F)-weak contraction.

Definition 6. *We assert that T is a (λ, E, F)-weak contraction if (E,F) is a compatible pair of functions such that*

$$Tx \neq Ty \Rightarrow F(d(Tx,Ty)) \le E(m_\lambda(x,y)). \tag{3}$$

Theorem 5. *We let (X,d) be a complete metric space and $T : X \to X$ be a (λ, E, F)-weak contraction. Moreover, we suppose that one of the following conditions holds:*

(i) T is continuous;

(ii) T verifies condition (C_3);

(iii) T verifies condition (C_4).

Then, T is a Picard operator.

Proof. First, we prove that T is λ-contractive, i.e.,

$$d(Tx,Ty) < m_\lambda(x,y), \text{ if } m_\lambda(x,y) > 0.$$

If $Tx = Ty$, the above inequality is obvious.
If $Tx \neq Ty$, we suppose that $d(Tx,Ty) \ge m_\lambda(x,y)$. By condition (E_1), we have

$$F(d(Tx,Ty)) > E(m_\lambda(x,y)),$$

which is in contradiction with (3).

We prove that T verifies condition (C_2). On the contrary, there is $\varepsilon_0 > 0$ such that for any $\delta > 0$, there are $x_\delta, y_\delta \in X$ such that

$$\varepsilon_0 < m_\lambda(x,y) < \varepsilon_0 + \delta \text{ and } d(Tx_\delta, Ty_\delta) > \varepsilon_0.$$

We consider $\{\gamma_n\}_{n\in\mathbb{N}}$ a sequence of strict positive numbers such that $\lim\limits_{n\to\infty} \gamma_n = 0$. For $n \in \mathbb{N}$, we take $\delta = \gamma_n$. Then, there are two sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \subset X$ such that

$$\varepsilon_0 < m_\lambda(x_n,y_n) < \varepsilon_0 + \gamma_n \tag{4}$$

and

$$d(Tx_n, Ty_n) > \varepsilon_0,\ n \in \mathbb{N}. \tag{5}$$

From the above relations, with notations

$$t_n = m_\lambda(x_n,y_n),\ s_n = d(Tx_n,Ty_n),\ n \in \mathbb{N},$$

we obtain that

$$\{s_n\}_{n\in\mathbb{N}}, \{t_n\}_{n\in\mathbb{N}} \subset (\varepsilon_0, \infty);$$

hence, $\lim_{n\to\infty} t_n = \varepsilon_0$, and since T is λ-contractive, we also have $s_n < t_n$, $n \in \mathbb{N}$.

From condition (E_2), we have

$$\limsup(F(s_n) - E(s_n)) > 0.$$

From relation (3), we have $F(s_n) \leq E(t_n)$, $n \in \mathbb{N}$, so

$$\limsup_{n\to\infty}(F(s_n) - E(t_n)) \leq 0,$$

which is a contradiction. □

3. Applications

In the following, we apply Theorem 5 for the case when

$$E(t) = F(t) - \tau,$$

where $\tau > 0$ is a constant and F is a nondecreasing function. Conditions (E_1) and (E_2) take place. Indeed,

If $t \leq s$, $F(s) - E(t) = F(s) - F(t) + \tau \geq \tau > 0$, so condition (E_1) is verified. To verify condition (E_2), we observe that

$$\limsup(F(s_n) - E(t_n)) = F(t+0) - F(t+0) + \tau = \tau > 0,$$

where $(t_n)_{n\in\mathbb{N}} \subset (t,\infty)$ is a sequence with $\lim_{n\to\infty} t_n = t$ and

$$(s_n)_{n\in\mathbb{N}},\ t < s_n < t_n,\ n \in \mathbb{N}.$$

We obtain the following result, comparable with the main result of the paper [20]:

Theorem 6. *We let (X,d) be a complete metric space and $T : X \to X$ be a mapping. We let $\tau > 0$ and $F : (0,\infty) \to \mathbb{R}$ be a nondecreasing mapping. We suppose that*

$$\tau + F(d(Tx,Ty)) \leq F(m_\lambda(x,y))$$

for $x, y \in X$ with $d(Tx,Ty) > 0$. If T verifies one of the following conditions,

(i) T is continuous;
(ii) T verifies condition (C_3);
(iii) T verifies condition (C_4);

Then, T is a Picard operator.

In the following, we provide a fixed point theorem for λ-weak Meir–Keeler contractions, which is a generalization of Theorem 2.

Definition 7. *We assert that T is a λ-weak Meir–Keeler contraction if $(\forall)\varepsilon > 0$, $(\exists)\delta = \delta(\varepsilon) > 0$ such that*

$$(\forall)\ x,y \in X,\ \varepsilon \leq m_\lambda(x,y) < \varepsilon + \delta$$

implies

$$d(Tx,Ty) < \varepsilon.$$

It is clear that every λ-weak Meir–Keeler contraction is a λ-weak contraction. Hence, we have the following result:

Theorem 7. *We let (X, d) be a complete metric space and $T : X \to X$. If T is a λ-weak Meir–Keeler contraction and T is continuous or T verifies condition (C_3) or condition (C_4), then T is a Picard operator.*

4. Conclusions

In this this paper, we obtained new classes of Picard operators on a complete metric space (X, d). These classes are provided by a weak type contraction by replacing distance $d(x, y)$ with a non-symmetric function. This function is given by

$$m_\lambda(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \lambda d(y, Tx) + (1 - \lambda)d(x, Ty)\}$$

and is used here for the first time in the context of fixed point theory.

We consider that our results can be applied in the study of Ulam's type stability and in the theory of integral equations.

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