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A Dynamic Behavior Analysis of a Rolling Mill's Main Drive System with Fractional Derivative and Stochastic Disturbance

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Abstract: Taking the random factors into account, a fractional main drive system of a rolling mill with Gaussian white noise is developed. First, the potential deterministic bifurcation is investigated by a linearized stability analysis. The results indicate that the fractional order changes the system from a stable point to a limit cycle with symmetric phase trajectories. Then, the stochastic response is obtained with the aid of the equivalent transformation of the fractional derivative and stochastic averaging methods. It is found that the joint stationary probability density function appears to have symmetric distribution. Finally, the influence of the fractional order and noise intensity on system dynamics behavior is discussed. The study is beneficial to understand the intrinsic mechanisms of vibration abatement.

Keywords: rolling mill system; bifurcation; fractional derivative; stochastic response; noise

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1. Introduction

The rolling mill system is a kind of complex system which combines mechanical, electrical, hydraulic and multiple nonlinear factors. For the convenience of analysis, researchers often highly abstract the system into a 'mass spring' system [1,2]. Yarita, I. et al. [3] are the first scholars to study the vibration problem of rolling mills. They analyzed the influence of process parameters and emulsion properties on vibration. Tlusty, J. et al. [4] propose that the vertical vibration of a rolling mill is a self-excited vibration caused by a negative damping effect when the phase difference between the rear tension fluctuation and rolling force fluctuation is 90°. In subsequent studies [5], the results all showed that the vibration of the rolling mills was caused by the dynamic change in the rolling mill's structure and the interaction of the rolling process, which caused the self-excited vibration. Therefore, the research focus was shifted to the theoretical modeling of the rolling mill structure and rolling process.

In rolling mill production, torsional vibration problems of complex rolling mill systems are inevitable [6,7]. For example, in the case of a sudden load (such as steel biting, steel throwing, etc.) [8,9] or a roll slipping, the static and stable state of the roller's connecting shaft torque is changed, resulting in a torsional vibration phenomenon of rolling mills. Therefore, it is very important and necessary to study the dynamics and responses of rolling mill systems [10,11].

In the past decades, fractional systems have attracted much attention and have been extensively studied in many scientific and engineering fields [12–14], such as bioengineering [15,16], automatic control [17], signal processing [18,19], quantum evolutionary complex systems [20], etc. Fractional systems have many better properties than integer-order differential systems. Because of this, some works have studied the effects of fractional order derivatives on the dynamic properties of rolling mill systems [21,22]. In 2014, Zhang [11] studied the dynamic properties of a class of rolling mill systems, and mainly analyzed the

Hopf analysis properties of the system. However, the influence of the fractional derivative on the dynamics of the system was ignored. Wang [23] analyzed the Hopf bifurcation control for the main drive delay system of rolling mills. However, the study did not consider the impact of noise on the system.

In addition, random factors are ubiquitous and non-negligible [24–29]. Actually, there are lots of random factors in rolling mill systems and rolling process [30,31]. However, there was little literature concerning the effect of stochastic excitations on the dynamics of the rolling mill system. Based on the above analysis, different from the previous studies on rolling mill systems, this paper considers the stochastic response of rolling mill systems. With the aid of the equivalent transformation of a fractional derivative and the stochastic averaging method, the effect of noise and the fractional derivative on the dynamics of the concerned system is indicated. Our results also provide a new perspective to studies on dynamical analysis of rolling mill systems. We end this part by highlighting the novelties and contributions of this work as follows:

- The fractional derivative and random factor are simultaneously introduced to the rolling mill's main drive system;
- Combining the equivalent transformation of the fractional derivative with the stochastic averaging method, we obtain the stochastic response of the proposed system;
- The influence of fractional derivative and noise intensity on system dynamics behavior is revealed.

The structure of this study is underscored as follows. In Section 2, the model of a rolling mill system with fractional damping and noise is designed. Simplification and an approximate analytical solution of the rolling mill model are presented in Section 3. In Section 4, deterministic bifurcation of the fractional rolling mill system is studied theoretically and numerically. Subsequently, the stochastic response of rolling mill system is investigated with varied fractional order and noise intensity in Section 5. In Section 6, we conclude this paper.

2. The Rolling Mill's Main Drive System with Fractional Damping and Noise

In this work, the rolling mill's main drive system closely follows Ref. [11] and the dimensionless equation is given below in (1).

$$\ddot{\theta}(t) + \omega^2\theta(t) + k_1\dot{\theta}(t) + k_2\dot{\theta}^2(t) + k_3\dot{\theta}^3(t) = 0. \tag{1}$$

where θ stands for roll angle, and k_1, k_2, k_3 ω are system parameters. The specific meaning of the parameters can be seen in Ref. [11].

As the rolling mill's main drive system (1), there has been almost no consideration of the viscoelastic properties of the damping term and external disturbance of the system. To make the model more general, we adopt a model with fractional derivatives and external disturbance, and its kinetic equation is as follows:

$$\ddot{\theta}(t) + \omega^2\theta(t) + k_1\dot{\theta}(t) + k_2\dot{\theta}^2(t) + k_3\dot{\theta}^3(t) + D^\alpha\theta(t) {=} \xi(t), \tag{2}$$

where k_1, k_2, k_3 are constants.

The fractional order term is used to model the viscoelasticity of stick-slip friction between rolls and rolled parts and Gaussian white noise is adopted to represent the external stochastic disturbance.

$D^\alpha\eta$ represents the fractional derivative within the Captuo's definition:

$$D^\alpha\theta(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\dot{\theta}(t)d\tau, 0 < \alpha \leq 1, \tag{3}$$

and $\xi(t)$ represents the Gaussian white noise satisfying the following statistical characteristics:

$$\langle \xi(t) \rangle = 0, \langle \xi(t)\xi(t+h) \rangle = 2d\delta(h). \tag{4}$$

3. Equivalent Model and Theoretical Analysis

In consideration of $0 < \alpha \leq 1$, the term associated with the fractional derivative can be considered to contribute to both the damped term and the stiffness term [32,33].

$$D^{\alpha}\theta \approx \omega^{\alpha-1}\sin\frac{\alpha\pi}{2}\dot{\theta} + \omega^{\alpha}\cos\frac{\alpha\pi}{2}\theta. \tag{5}$$

Substituting (5) into (2):

$$\ddot{\theta}(t) + \omega_0^2\theta(t) + \left(k_1 + \omega^{\alpha-1}\sin\frac{\alpha\pi}{2}\right)\dot{\theta}(t) + k_2\dot{\theta}^2(t) + k_3\dot{\theta}^3(t) = \xi(t), \tag{6}$$

where the dot represents the derivative with respect to t.

$$\omega_0^2 = \omega^2 + \omega^{\alpha}\cos\frac{\alpha\pi}{2}.$$

The new variables transformation is introduced as follows:

$$\begin{aligned} \theta(t) &= a(t)\cos\phi, \\ \phi &= \omega_0 t + \varphi(t). \end{aligned} \tag{7}$$

To take the first derivative of (7), we have

$$\dot{\theta} = \dot{a}\cos\phi - a\omega_0\sin\phi - a\dot{\varphi}\sin\phi. \tag{8}$$

Under the assumption that damping and excitation terms are small, $a(t)$ and $\varphi(t)$ are two slowly varying processes, i.e., the amplitude and the phase will be slowly varying with respect to time. Equation (8) can be simplified as follows:

$$\dot{\theta}(t) = -a(t)\omega_0\sin\phi. \tag{9}$$

Then the potential energy $U(\theta)$ and the total energy H of the system are as follows:

$$\begin{aligned} U(\theta) &= \textstyle\int_0^{\theta}\omega_0^2 x dx = \frac{1}{2}\omega_0^2\theta^2, \\ H &= U(\theta) + \frac{1}{2}\dot{\theta}^2. \end{aligned} \tag{10}$$

By aid of (7) and (9), (6) can be rearranged as an equation within variables a and φ,

$$\begin{cases} \dot{a} = \frac{\sin\phi}{\omega_0}[f - \xi(t)], \\ \dot{\varphi} = \frac{\cos\phi}{a\omega_0}[f - \xi(t)], \end{cases} \tag{11}$$

where

$$f = -(k_1 + \omega^{\alpha-1}\sin\frac{\alpha\pi}{2})a\omega_0\sin\phi + k_2 a^2\omega_0^2\sin^2\phi - k_3 a^3\omega_0^3\sin^3\phi.$$

To derive the stochastic equations for $a(t)$ and $\varphi(t)$, we take the average of Equation (11) over one period base on the method of stochastic averaging [34,35].

$$\begin{cases} da = (F_1 + \frac{d}{2a\omega_0^2})dt + \sqrt{\frac{d}{\omega_0^2}}dW_0(t), \\ d\varphi = F_2 dt + \frac{1}{a}\sqrt{\frac{d}{\omega_0^2}}dW_1(t)\ , \end{cases} \tag{12}$$

where

$$\begin{aligned} F_1 &= \tfrac{1}{2\pi}\textstyle\int_0^{2\pi}\frac{f}{\omega_0}\sin\varphi d\varphi, \\ F_2 &= \tfrac{1}{2\pi}\textstyle\int_0^{2\pi}\frac{f}{a\omega_0}\cos\varphi d\varphi\ . \end{aligned}$$

The amplitude $a(t)$ and phase $\varphi(t)$ are decoupled to independent variables. Then, we can derive the first derivative moment and second derivative moment of amplitude $a(t)$ as follows:

$$\begin{aligned}\bar{a}_1 &= -\tfrac{1}{2}(k_1+\omega^{\alpha-1}\sin\tfrac{\alpha\pi}{2})a-\tfrac{3}{8}k_3\omega_0^2a^3+\tfrac{d}{2a\omega_0^2},\\ \bar{b}_{11} &= \tfrac{d}{\omega_0^2}.\end{aligned} \tag{13}$$

Then, the Fokker–Planck–Kolmogorov (FPK) equation of the transition probability density function complies with the following equation:

$$\frac{\partial p(a,t)}{\partial t}=-\frac{\partial}{\partial a}[\bar{a}_1p(a,t)]+\frac{1}{2}\frac{\partial}{\partial a^2}[\bar{b}_{11}p(a,t)]\ .$$

Letting $\frac{\partial p(a,t)}{\partial t}=0$, one ultimately derives the expression of the stationary density in (14).

$$P(a)=Na\exp\left[\frac{\omega_0^2}{2d}(k_1+\omega^{\alpha-1}\sin\frac{\alpha\pi}{2})a^2-\frac{3\omega_0^4}{16d}k_3a^4\right], \tag{14}$$

where N is normalization constant,

$$N=1\Bigg/\int_0^{+\infty}a\exp\left[\frac{\omega_0^2}{2d}(k_1+\omega^{\alpha-1}sin\frac{\alpha\pi}{2})a^2-\frac{3\omega_0^4}{16d}k_3a^4\right]da.$$

Meanwhile, the total energy $H=U(a)=\frac{1}{2}\omega_0^2a^2$, the stationary PDF of the total energy H can be obtained as follows:

$$P(H)=P(a)\left|\frac{da}{dH}\right|=\frac{P(a)}{\omega_0^2a}. \tag{15}$$

Then the joint PDF of the displacement θ and velocity $\dot{\theta}$ is as follows:

$$\begin{aligned}P(\theta,\dot{\theta}) &= \left.\tfrac{P(H)}{T(H)}\right|_{H=\frac{1}{2}\omega^2\theta^2+\frac{1}{2}\dot{\theta}^2}\\ &=N\exp\left[\tfrac{\omega_0^2}{2d}(k_1+\omega^{\alpha-1}\sin\tfrac{\alpha\pi}{2})(\theta^2+\tfrac{\dot{\theta}^2}{\omega_0^2})-\tfrac{3\omega_0^4}{16d}k_3(\theta^2+\tfrac{\dot{\theta}^2}{\omega_0^2})^2\right].\end{aligned} \tag{16}$$

In the above equation, $H=\frac{1}{2}\omega^2\theta^2+\frac{1}{2}\dot{\theta}^2$, $T(H)=\frac{2\pi}{\omega_0}$, N is normalization constant,

$$N=1\Bigg/\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp\left[\frac{\omega_0^2}{2d}(k_1+\omega^{\alpha-1}sin\frac{\alpha\pi}{2})(\theta^2+\frac{\dot{\theta}^2}{\omega_0^2})-\frac{3\omega_0^4}{16d}k_3(\theta^2+\frac{\dot{\theta}^2}{\omega_0^2})^2\right]d\theta d\dot{\theta}.$$

4. Deterministic Case

In this section, we will investigate the potential bifurcation phenomenon of the rolling mill's main drive system without stochastic disturbance ($d=0$). Then, (6) reduces to the following equation:

$$\ddot{\theta}(t)+\omega_0\theta(t)+\left(k_1+\omega^{\alpha-1}\sin\frac{\alpha\pi}{2}\right)\dot{\theta}(t)+k_2\dot{\theta}^2(t)+k_3\dot{\theta}^3(t)=0. \tag{17}$$

The eigenvalues of the Jacobian can be obtained in virtue of linearizing Equation (11) at $(\theta,\dot{\theta})=(0,0)$

$$\lambda_{12}=\frac{1}{2}\left[-(k_1+\omega^{\alpha-1}\sin\frac{\alpha\pi}{2})\pm\sqrt{(k_1+\omega^{\alpha-1}\sin\frac{\alpha\pi}{2})^2-4\omega_0^2}\right], \tag{18}$$

which yields the Hopf bifurcation condition as follows:

$$k_1 + \omega^{\alpha-1} \sin\frac{\alpha\pi}{2} = 0. \tag{19}$$

Next, the details of the bifurcation with the variation in the fractional order α will be explored. The parameters $k_2 = 0.01,\ k_3 = 0.05$ are fixed. The bifurcation diagram in parameter plane $k_1 - \alpha$ can be found and is shown in Figure 1 based on Equation (19).

Figure 1 shows that the red curve (the edge of the Hopf bifurcation) divides the parametric space into two regions. Subsequently, we fix $k_1 = -0.95$ and investigate the bifurcation on the fractional order q that vary along the horizontal dotted line in Figure 1. When $\alpha < 0.797$, the rolling mill's main drive system yields a stable limit cycle. When $\alpha > 0.797$, one yields a stable steady state.

The phase diagrams with fractional order of 0.7, 0.75, 0.85 are depicted in Figure 2. The time history diagram with fractional orders of 0.7, 0.75, 0.85 are depicted in Figure 3. In Figures 2 and 3, the same representative initial condition $I_1 = (\theta, \dot{\theta}) = (0.1, 0), I_2 = (\theta, \dot{\theta}) = (1, 0)$ are selected. A scrutiny of Figures 2 and 3 indicates that the phase diagram the rolling mill's main drive system changes from a limit cycle to a stable steady state along with the increase in the fractional order. This confirms the validity of our research results.

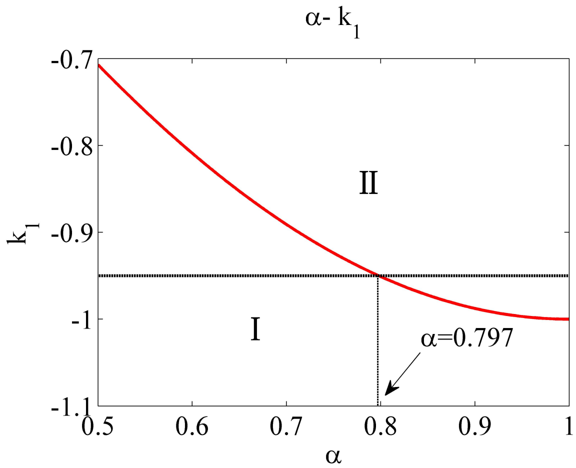

Figure 1. Bifurcation diagram of the deterministic system for $k_2 = 0.01,\ k_3 = 0.05$; The red curve denotes the edge of the Hopf bifurcation.

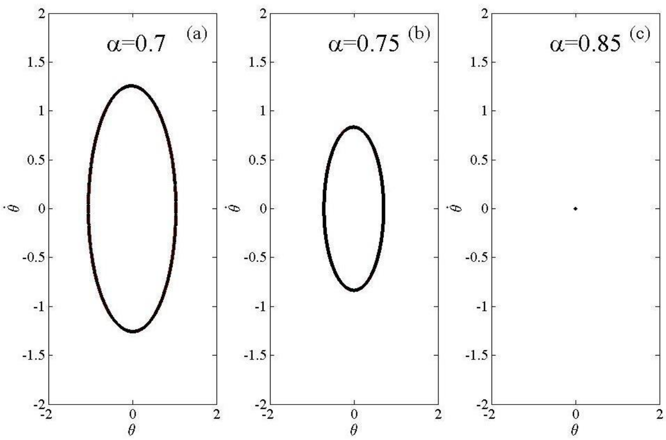

Figure 2. Phase planes of the deterministic system for different fractional order: (**a**) $\alpha = 0.7$, the system yields a large limit cycle; (**b**) $\alpha = 0.75$, the system yields a small limit cycle; (**c**) $\alpha = 0.85$, the system yields a stable steady state.

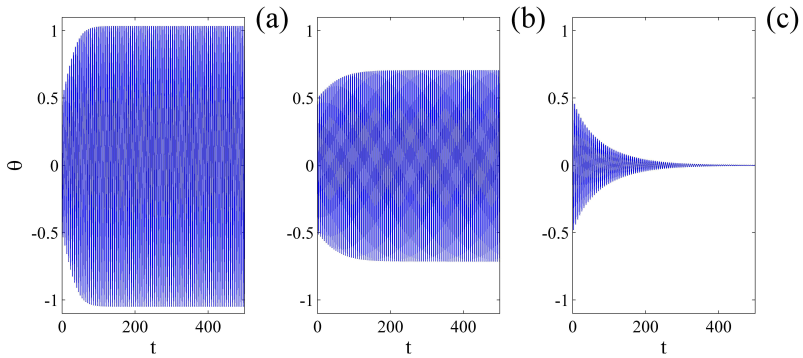

Figure 3. The time history diagram of θ in the deterministic system: (**a**) $\alpha = 0.7$; (**b**) $\alpha = 0.75$; (**c**) $\alpha = 0.85$.

5. Stochastic Case

As is well known, noise is omnipresent in many dynamics systems. Therefore, it is important to study the response of the the rolling mill's main drive system in the presence of noise. Subsequently, the effect of noise intensity and fractional order will be investigated in the rolling mill's main drive system. The parameters $k_1 = -0.95,\ k_2 = 0.01, k_3 = 0.05$ are fixed.

5.1. Effect of Noise Intensity

The effects of noise intensity d on the rolling mill's main drive system will be studied in this part. The theoretical results and numerical results of the stationary probability density function (PDF) $P(a)$ and joint stationary probability density function $P(\theta, \dot{\theta})$ are obtained and shown in Figures 4 and 5.

As can be seen from Figure 4, the stationary PDF $P(a)$ for different noise intensity showed a unimodal shape. Firstly, for noise intensity $d = 0.02$, the peak of the stationary PDF $P(a)$ corresponds to a smaller amplitude (see curve 1). For noise intensity $d = 0.06$, the amplitude corresponding to the peak of the stationary PDF becomes larger(see curve 2). With the noise intensity further increase ($d = 0.12$), the amplitude corresponding to the peak of the PDF still increases further (see curve 3). This implies that the system response is concentrated near a certain amplitude in the presence of noise and increases gradually with the monotonically increasing of noise intensity.

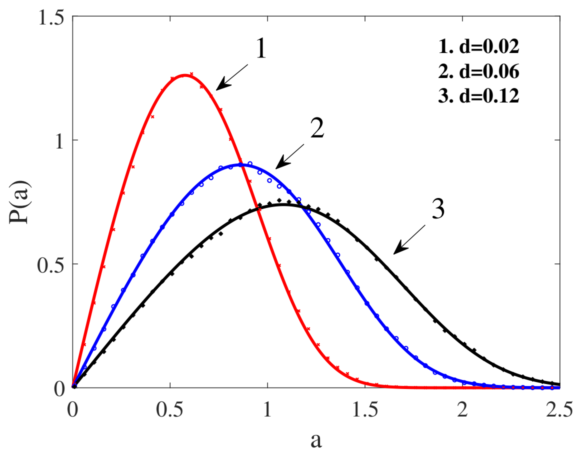

Figure 4. The stationary probability density function $P(a)$ of the amplitude for different noise intensity d with $k_1 = -0.95,\ k_2 = 0.01,\ k_3 = 0.05,\ \alpha = 0.9$. The lines denote the analytical results, whereas dots represent the numerical results.

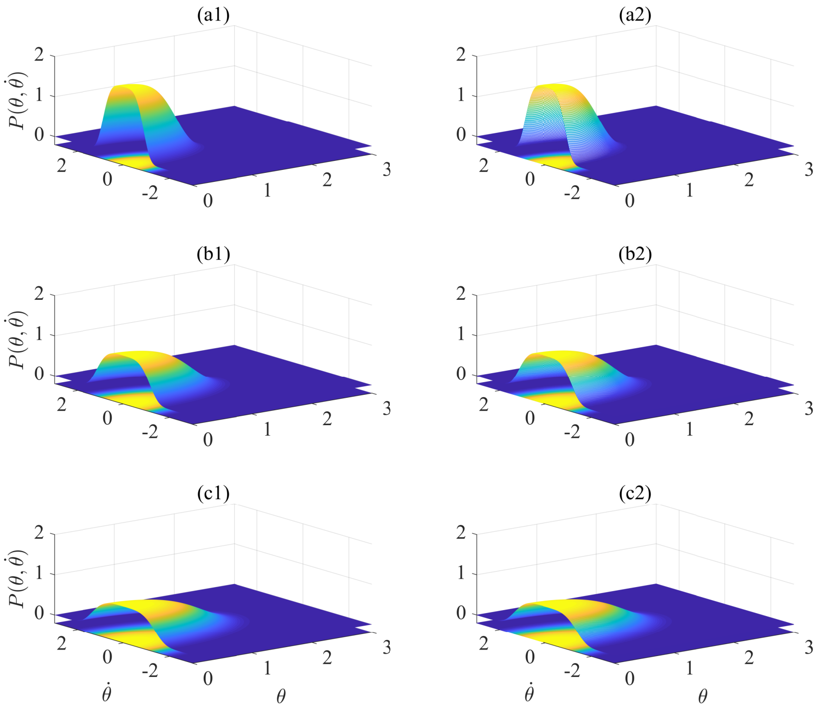

Figure 5. The joint stationary probability density function $P(\theta, \dot{\theta})$ with different noise intensity d. The the left side of the figure represent the analytic results and the right side denotes the numerical results. (**a1**, **a2**) $d = 0.002$, (**b1**, **b2**) $d = 0.06$, (**c1**, **c2**) $d = 0.12$.

5.2. Effect of Fractional Order

The effects of fractional order on the rolling mill's main drive system have been studied in this subsection. The theoretical results and numerical results of the stationary PDF $P(a)$ and joint stationary PDF $P(\theta, \dot{\theta})$ are obtained and shown Figures 6 and 7.

It reflects that both the stationary PDF $P(a)$ for different noise intensities showed a unimodal shape in Figure 6. Firstly, for fractional order $\alpha = 0.6$, the peak of the stationary PDF $P(a)$ corresponds to a larger amplitude (see curve 1). For fractional order $\alpha = 0.7$, the amplitude corresponding to the peak of the stationary PDF $P(a)$ becomes smaller (see curve 2). With the fractional order further increasing ($\alpha = 0.95$, see curve 3), the amplitude corresponding to the peak of the stationary PDF $P(a)$ still decreases further. This implies that the system response is concentrated near a certain amplitude in the presence of noise and decreases gradually with the monotonically increasing of fractional order.

To conclude, all of the above results mirror that noise intensity and fractional order can modulate the amplitude corresponding to the peak of the stationary PDF's left shift or right shift. The evolution of the response with the monotonic increasing of noise intensity and fractional order indicate that the noise intensity is conducive to modulate a larger amplitude. In contrast, the fractional order is conducive to induce a small amplitude.

It is worth pointing out that the response of the rolling mill's main drive system for different fractional orders yields a limit cycle or a stable fixed point in the absence of noise. Nevertheless, both the stationary probability density functions $P(a)$ for different system parameters (noise intensity and fractional order) showed a unimodal shape in the presence of noise. The theoretical and numerical results of the stationary probability density function $P(a)$ and joint stationary probability density function $P(\theta, \dot{\theta})$ verify the validity of the conclusion.

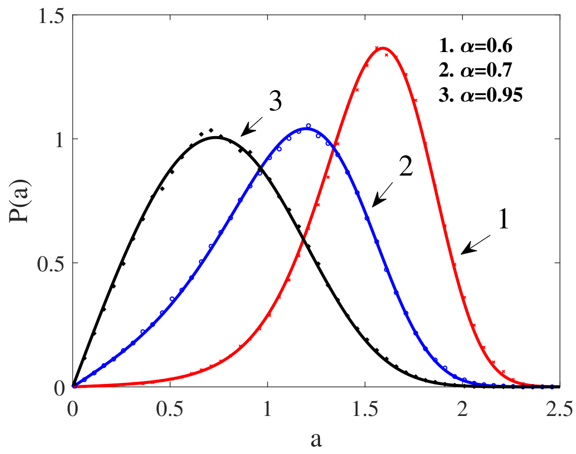

Figure 6. The stationary probability density function $P(a)$ of the amplitude for different fractional order α with $k_1 = -0.95,\ k_2 = 0.01, k_3 = 0.05,\ d = 0.04$. The lines denote the analytical results, whereas dots represent the numerical results.

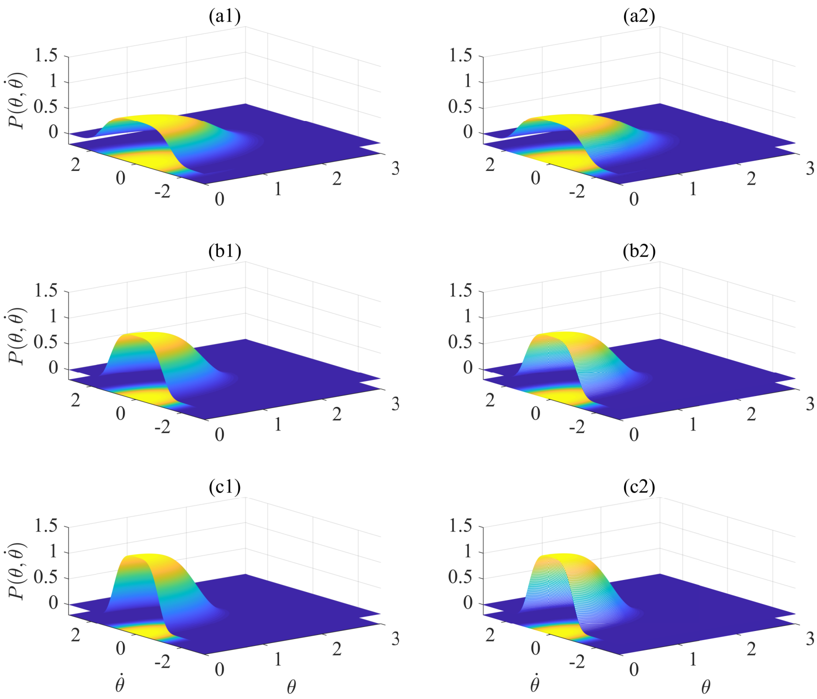

Figure 7. The joint stationary probability density function $P(\theta, \dot{\theta})$ with distinct values of α. The left side of the figure represent the analytic results and the right side denotes the numerical results. (**a1**, **a2**) $\alpha = 0.6$, (**b1**, **b2**) $\alpha = 0.7$, (**c1**, **c2**) $\alpha = 0.95$.

6. Conclusions and Discussion

In a summary, the rolling mill's main drive system with a fractional order derivative and stochastic disturbance was considered. The dynamics of the rolling mill's main drive system was investigated both in the absence and in the presence of stochastic disturbance.

For the absence of stochastic disturbance, the deterministic bifurcations induced by fractional order were explored based on the linearization method and a numerical simulation for the rolling mill's main drive system. The results indicated that fractional order can change the system from a stable point to a limit cycle.

For the presence of stochastic disturbance, the response of the rolling mill's main drive system was investigated with varying the fractional order and noise intensity. The evolution of the response with the monotonic increase in noise intensity and fractional order implied that the noise intensity was conducive to modulate a larger amplitude. In contrast, the fractional order was conducive to induce a small amplitude. Therefore, it provides an efficient strategy to control the system so that the amplitude of the vibration was small enough when the vibration occurs, which is going to be our later work.

In this paper, the rolling mill's main drive system with fractional order and stochastic disturbance is considered and the dynamic response is investigated both in the absence and presence of stochastic disturbance. We mainly focused on the impact of Gaussian white noise and fractional order on the dynamic behavior of the system. The impact of other types of noise and time delays on the dynamic behavior of the system is also a problem that needs further research.

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