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Spin-Gravity Coupling in a Rotating Universe

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Abstract: The coupling of intrinsic spin with the nonlinear gravitomagnetic fields of Gödel-type spacetimes is studied. We work with Gödel-type universes in order to show that the main features of spin-gravity coupling are independent of causality problems of the Gödel universe. The connection between the spin–gravitomagnetic field coupling and Mathisson's spin-curvature force is demonstrated in the Gödel-type universe. That is, the gravitomagnetic Stern–Gerlach force due to the coupling of spin with the gravitomagnetic field reduces in the appropriate correspondence limit to the classical Mathisson spin-curvature force.

Keywords: spin-gravity coupling; Gödel-type universe

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1. Introduction

Inertia is the intrinsic tendency of matter to remain in a given condition. The state of matter in spacetime is determined by its mass and spin; indeed, mass and spin characterize the irreducible unitary representations of the Poincaré group [1]. Therefore, mass and spin determine the inertial properties of a particle. In classical physics, the inertial forces that act on a particle are proportional to its inertial mass; moreover, the moment of inertia is the rotational analogue of mass. The inertial effects of intrinsic spin are independent of the inertial mass of the particle and depend purely on intrinsic spin. The inertia of intrinsic spin is of quantum origin, and its properties, therefore, complement the inertial characteristics of the mass and orbital angular momentum of the particle.

It turns out that the intrinsic spin S of a particle couples to the rotation of a noninertial observer, thus resulting in a Hamiltonian of the form $\mathcal{H}_{sr} = -\mathbf{S} \cdot \boldsymbol{\Omega}$, where $\boldsymbol{\Omega}$ is the angular velocity of the observer's local spatial frame with respect to a nonrotating (i.e., Fermi–Walker) transported frame. For an intuitive explanation of this type of coupling, let us consider a noninertial observer that is at rest in Minkowski spacetime but refers its observations to axes that rotate uniformly with angular speed Ω in the positive sense about the direction of propagation of a plane electromagnetic wave of frequency $\omega > \Omega$. The Fourier analysis of the electromagnetic field detected by the noninertial observer reveals that the measured frequency of the wave is given by $\omega \mp \Omega$, where the upper (lower) sign refers to positive (negative) helicity radiation. One can understand this result as a kind of "rotational Doppler effect": In a positive (negative) helicity electromagnetic wave, the electric and magnetic fields rotate in the positive (negative) sense with the wave frequency ω about the direction of propagation. The noninertial observer thus realizes that the positive (negative) helicity radiation has electric and magnetic fields that rotate in the positive (negative) sense with frequency $\omega - \Omega$ ($\omega + \Omega$) about the direction of wave propagation. Multiplication of the measured frequency by $\hbar$ results in the measured energy by the noninertial observer, namely, $\hbar\,\omega \mp \hbar\,\Omega$, which illustrates the coupling of photon helicity with rotation. A general consequence of spin-rotation coupling should be noted here: There is a certain shift in energy when polarized radiation passes through a rotating spin flipper. To demonstrate this effect within the context of the present discussion, imagine that the noninertial observer is replaced by a uniformly rotating half-wave plate. That is, the electromagnetic radiation of frequency $\omega_{\rm in}$ is normally incident on the plate rotating with $\Omega < \omega_{\rm in}$. The frequency of the radiation within the stationary medium of the half-wave plate remains constant and approximately equal to $\omega_{\rm in} \mp \Omega$, since we have neglected time dilation for simplicity. The outgoing radiation has an opposite helicity to the incident radiation and frequency $\omega_{\rm out}$, where $\omega_{\rm in} \mp \Omega \approx \omega_{\rm out} \pm \Omega$ due to helicity–rotation coupling. Therefore, $\omega_{\rm out} - \omega_{\rm in} \approx \mp 2\,\Omega$ and the photon energy in passing through the rotating half-wave plate is shifted by $\approx \mp 2\,\hbar\,\Omega$.

A general account of spin-rotation coupling is contained in [2], and more recent discussions of its observational basis can be found in [3–6]. A similar phenomenon occurs in a gravitational field [7–9]. The spin-rotation effect can be theoretically extended to the spin–gravity coupling via the gravitational Larmor theorem [10,11], which is the rotational side of Einstein's principle of equivalence. Imagine a free test gyroscope with its center of mass held at rest in a gravitational field; then, the locally measured components of the gyroscope's spin vector undergo a precessional motion with an angular velocity that is given by the locally measured gravitomagnetic field. The Gravity Probe B (GP-B) space experiment has measured the gravitomagnetic field of the Earth [12,13].

According to the gravitational Larmor theorem, the gravitomagnetic field of a rotating system is locally equivalent to a rotation resulting in a Hamiltonian for intrinsic spin–gravity coupling of the form $\mathcal{H}_{sg} = \boldsymbol{S} \cdot \boldsymbol{B}$, where $\boldsymbol{B}$ is the relevant gravitomagnetic field [14]. The spin–gravity coupling is of basic physical significance due to the fundamental nature of the intrinsic spin of the particles and the universality of the gravitational interaction. For prospects regarding the measurement of intrinsic spin–gravity coupling, see [15–19]. In general, $\boldsymbol{B}$ depends on the position, and the intrinsic spin–gravity coupling leads to a measured gravitomagnetic Stern–Gerlach force of the form $-\boldsymbol{\nabla}(\boldsymbol{S} \cdot \boldsymbol{B})$. This gravitational force, which acts on a test particle, is completely independent of its inertial mass and depends solely on its intrinsic spin. It has been shown in [20], within the framework of *linearized* general relativity, that the gravitomagnetic Stern–Gerlach force associated with spin–gravity coupling reduces in the correspondence limit to Mathisson's classical spin-curvature force [21,22]. It would be interesting to extend this result to the nonlinear regime. The purpose of the present work is to study further the inertial effects of intrinsic spin by investigating the intrinsic spin–gravity coupling for spinning test particles in Gödel-type spacetimes. For background material, ref. [20] and the references cited therein should be consulted for further important information regarding the topic of spin-rotation–gravity coupling and its experimental basis.

2. Gravitomagnetism in the Gödel-Type Universe

With respect to the spacetime coordinates $x^\mu = (ct, x, y, z)$, the metric of the Gödel solution [23] of Einstein's gravitational field equations arises as a special case in the class of the so-called Gödel-type models [24–26], which are described by the following line element:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\,dt^2 - 2\sqrt{\sigma}\, e^{\mu x}\, dt\, dy + dx^2 + \kappa\, e^{2\mu x}\, dy^2 + dz^2\,, \tag{1}$$

with arbitrary constant parameters μ, σ, and κ. In our conventions, the speed of light $c = 1$ and Planck's constant $\hbar = 1$, unless specified otherwise; moreover, the metric signature is +2, and the Greek indices run from 0 to 3, while the Latin indices run from 1 to 3. The system of coordinates in metric (1) is admissible provided

$$\sigma + \kappa > 0\,. \tag{2}$$

Moreover, we assume throughout that $\sigma > 0$. In general, the Gödel-type universe contains closed timelike curves, which could lead to problems with causality. However, one can demonstrate [24] that closed timelike curves are absent in model (1), provided

$$\kappa \geq 0\,. \tag{3}$$

Specifically, for the Gödel universe, $\kappa = -1$ in metric (1); therefore, closed timelike curves do exist in the Gödel universe. To ensure that our considerations regarding spin–gravity coupling are independent of the causality difficulties of the Gödel universe, we use metric (1) for our main calculations in this paper.

The Gödel-type universe is a regular stationary and spatially homogeneous spacetime that contains rotating matter. Consider the class of observers that are all spatially at rest in this spacetime. Each such observer has a velocity 4-vector $u^\mu = \delta^\mu_0$ that is free of acceleration, expansion, and shear; however, it is rotating in the negative sense about the z axis and its vorticity 4-vector

$$\omega^\mu = \frac{1}{2}\eta^{\mu\nu\rho\sigma}u_\nu u_{\rho;\sigma}\,, \tag{4}$$

is purely spatial $\omega^\mu = (0, \boldsymbol{\omega})$, with the 3-vector

$$\boldsymbol{\omega} = -\,\Omega\,\partial_z\,, \qquad \Omega = \frac{\mu}{2}\sqrt{\frac{\sigma}{\sigma+\kappa}}\,. \tag{5}$$

For the sake of definiteness, we henceforth assume that $\Omega > 0$; then, Equation (5) implies that $\mu > 0$ as well. Here, $\eta_{\alpha\beta\gamma\delta} = (-g)^{1/2}\epsilon_{\alpha\beta\gamma\delta}$ is the Levi–Civita tensor, and $\epsilon_{\alpha\beta\gamma\delta}$ is the alternating symbol with $\epsilon_{0123} = 1$. It is interesting to note that, in nonrelativistic fluid mechanics, the vorticity vector $\boldsymbol{\omega}_N$ is defined as $\boldsymbol{\omega}_N = \boldsymbol{\nabla}\times\boldsymbol{v}$, where $\boldsymbol{v}$ is the flow velocity. If the fluid rotates with a spatially uniform angular velocity $\boldsymbol{\Omega}$ such that $\boldsymbol{v} = \boldsymbol{\Omega}\times\boldsymbol{x}$, then $\boldsymbol{\omega}_N = 2\,\boldsymbol{\Omega}$. In this paper, we follow the relativistic definition of vorticity.

The geometry of the Gödel-type model has been studied by a number of authors [27–29]. The Weyl curvature of Gödel-type spacetime is of type D in the Petrov classification. The Gödel-type universe admits five Killing vector fields, namely, ∂_t, ∂_y, ∂_z, $\partial_x - \mu y\,\partial_y$ and [24,30]:

$$K = \frac{2\sqrt{\sigma}\,e^{-\,\mu x}}{\sigma+\kappa}\,\partial_t - 2\mu y\,\partial_x + \left(\mu^2 y^2 - \frac{e^{-2\mu x}}{\sigma+\kappa}\right)\partial_y\,. \tag{6}$$

We are interested in the measurements of an observer that is free and spatially at rest in spacetime with a 4-velocity vector $u^\mu = dx^\mu/d\tau$ and proper time τ, where $\tau = t + \text{constant}$. The observer carries along its geodesic world line a natural tetrad frame $e^\mu{}_{\hat\alpha}$ that is orthonormal, namely,

$$g_{\mu\nu}\,e^\mu{}_{\hat\alpha}\,e^\nu{}_{\hat\beta} = \eta_{\hat\alpha\hat\beta}\,, \tag{7}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ is the Minkowski metric tensor. Indeed,

$$e_{\hat 0} = \partial_t\,, \qquad e_{\hat 1} = \partial_x\,, \qquad e_{\hat 2} = -\sqrt{\frac{\sigma}{\sigma+\kappa}}\,\partial_t + \frac{e^{-\,\mu x}}{\sqrt{\sigma+\kappa}}\,\partial_y\,, \qquad e_{\hat 3} = \partial_z\,, \tag{8}$$

where the spatial axes of the observer's frame are primarily along the background coordinate axes. By introducing the dual coframe $\vartheta^{\hat\alpha}$,

$$\vartheta^{\hat 0} = dt + \sqrt{\sigma}\,e^{\mu x}dy, \qquad \vartheta^{\hat 1} = dx, \qquad \vartheta^{\hat 2} = \sqrt{\sigma+\kappa}\,e^{\mu x}dy, \qquad \vartheta^{\hat 3} = dz\,, \tag{9}$$

such that $e_{\hat\alpha}\rfloor\vartheta^{\hat\beta} = \delta^\beta_\alpha$, the line element (1) is recast into

$$ds^2 = -\,\big(dt + \sqrt{\sigma}\,e^{\mu x}\,dy\big)^2 + dx^2 + (\sigma+\kappa)\,e^{2\mu x}\,dy^2 + dz^2\,. \tag{10}$$

Let $\lambda^\mu{}_{\hat\alpha}$ be the orthonormal tetrad frame that is parallel transported along the observer's geodesic world line such that $D\lambda^\mu{}_{\hat\alpha}/d\tau = 0$. We find that

$$\lambda^\mu{}_{\hat 1} = e^\mu{}_{\hat 1}\cos\Omega\tau + e^\mu{}_{\hat 2}\sin\Omega\tau\,, \qquad \lambda^\mu{}_{\hat 2} = -e^\mu{}_{\hat 1}\sin\Omega\tau + e^\mu{}_{\hat 2}\cos\Omega\tau\,, \tag{11}$$

while $\lambda^{\mu}{}_{\hat{3}} = e^{\mu}{}_{\hat{3}}$, and naturally $\lambda^{\mu}{}_{\hat{0}} = e^{\mu}{}_{\hat{0}} = u^{\mu}$. It is simple to check these results using the Christoffel symbols:

$$\Gamma^0_{10} = \sqrt{\frac{\sigma}{\sigma+\kappa}}\,\Omega\,, \qquad \Gamma^1_{20} = \sqrt{\sigma+\kappa}\,e^{\mu x}\,\Omega\,, \qquad \Gamma^2_{10} = -\frac{e^{-\mu x}}{\sqrt{\sigma+\kappa}}\,\Omega\,, \tag{12}$$

which are the only nonzero components of $\Gamma^{\mu}_{\nu 0}$. Therefore, the observer's natural frame rotates with respect to the parallel-transported frame about their common z axis with frequency $-\Omega$, which is consistent with vorticity (5).

Let us now consider the special case of metric (1) with parameters

$$\mu = \sqrt{2}\,\Omega, \qquad \sigma = 2, \qquad \kappa = -1\,. \tag{13}$$

With these parameters, metric (1) reduces to the Gödel line element

$$ds^2 = -\,dt^2 - 2\sqrt{2}\,e^{\sqrt{2}\Omega x}\,dt\,dy + dx^2 - e^{2\sqrt{2}\Omega x}\,dy^2 + dz^2\,. \tag{14}$$

For the Gödel universe, Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G\,T_{\mu\nu} \tag{15}$$

have a perfect fluid source of

$$T_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} + p g_{\mu\nu}\,, \tag{16}$$

where ρ is the energy density, p is the pressure, and $u^{\mu} = \delta^{\mu}_0$ is the 4-velocity vector of the perfect fluid. In this special case, $R_{\mu\nu} = 2\Omega^2 u_{\mu}u_{\nu}$, and

$$2\,\Omega^2 = 8\pi G(\rho + p)\,, \qquad \Lambda + \Omega^2 = 8\pi G p\,. \tag{17}$$

In the absence of the cosmological constant Λ, we have as the source of the Gödel universe a perfect fluid with a stiff equation of state $\rho = p = \Omega^2/(8\pi G)$. Another possibility is dust ($p = 0$) with $4\pi G\rho = -\Lambda = \Omega^2$. It follows from Equation (17) that $-\Lambda = 4\pi G(\rho - p)$; therefore, in any realistic situation, the cosmological constant of the Gödel universe must be negative or zero ($\Lambda \le 0$).

The spinning test particle in the Gödel universe is immersed in the perfect fluid source, and its intrinsic spin couples to the vorticity of the fluid. The nature of the spin–gravity coupling and its connection with Mathisson's classical spin-curvature force provided the original motivation for the present work.

After this brief digression regarding the Gödel universe, we return to the Gödel-type metric with explicit components:

$$(g_{\mu\nu}) = \begin{bmatrix} -1 & 0 & -\sqrt{\sigma}\,W & 0 \\ 0 & 1 & 0 & 0 \\ -\sqrt{\sigma}\,W & 0 & \kappa\,W^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (g^{\mu\nu}) = \begin{bmatrix} -\frac{\kappa}{\sigma+\kappa} & 0 & -\frac{\sqrt{\sigma}}{\sigma+\kappa}W^{-1} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{\sqrt{\sigma}}{\sigma+\kappa}W^{-1} & 0 & \frac{1}{\sigma+\kappa}W^{-2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{18}$$

where $W(x) = e^{\mu x}$, and $\sqrt{-g} = \sqrt{\sigma+\kappa}\,W(x)$.

3. Mathisson's Spin-Curvature Force

To connect Mathisson's classical spin-curvature force in the correspondence limit with intrinsic spin that is purely of quantum origin, it proves useful to introduce a classical model of intrinsic spin. To simplify matters, we permanently attach a free spin vector $\boldsymbol{S}$ to a Newtonian point particle resulting in a "pole-dipole" particle. The particle thus carries the spin vector along its world line, and the corresponding equations of motion in a gravitational field are the Mathisson–Papapetrou pole–dipole equations [21,31]:

$$\frac{DP^\mu}{d\varsigma} = -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}\,U^\nu S^{\alpha\beta}\,, \tag{19}$$

$$\frac{DS^{\mu\nu}}{d\varsigma} = P^\mu U^\nu - P^\nu U^\mu\,, \tag{20}$$

where $U^\mu = dx^\mu/d\varsigma$ is the 4-velocity of the pole–dipole particle, $U^\mu U_\mu = -1$, and ς is its proper time. The particle's 4-momentum is P^μ and its spin tensor is $S^{\mu\nu}$, which satisfies the Frenkel–Pirani supplementary condition [32,33]

$$S^{\mu\nu}\,U_\nu = 0\,. \tag{21}$$

In this system, the inertial mass of the particle m, $m := -P^\mu U_\mu$, and the magnitude of its spin s, $s^2 := \frac{1}{2}S^{\mu\nu}S_{\mu\nu}$, are constants of the motion. Moreover, Pirani has shown that the spin vector S^μ, $S^\mu\,U_\mu = 0$,

$$S^\mu = -\frac{1}{2}\,\eta^{\mu\nu\rho\sigma}\,U_\nu S_{\rho\sigma}\,, \qquad S^{\alpha\beta} = \eta^{\alpha\beta\gamma\delta}\,U_\gamma S_\delta\,. \tag{22}$$

is Fermi–Walker-transported along the particle's world line [33]. That is, the Mathisson–Papapetrou equations for a spinning test particle, together with the Frenkel–Pirani supplementary condition, imply that the spin vector of a test pole–dipole particle is nonrotating in this classical model, which is consistent with the inertia of the intrinsic spin. Furthermore, the Mathisson–Papapetrou equations, together with the Frenkel–Pirani supplementary condition, imply that, in the massless limit, the spinning massless test particle follows a null geodesic with the spin vector parallel or is antiparallel to its direction of motion [34]. Hence, our classical model is consistent with physical expectations.

What is the influence of the inertia of the intrinsic spin on the motion of the spinning particle? From Equation (20), we find

$$P^\mu = m\,U^\mu + S^{\mu\nu}\,\frac{DU_\nu}{d\varsigma}\,; \tag{23}$$

thus, in the absence of spin, $P^\mu = m\,U^\mu$, and the particle simply follows a timelike geodesic of the background gravitational field. In the presence of spin, on the other hand, the Mathisson spin-curvature force $\mathcal{F}^\mu$, $\mathcal{F}^\mu U_\mu = 0$,

$$\mathcal{F}^\mu = -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}\,U^\nu S^{\alpha\beta} = {}^*R^\mu{}_{\nu\rho\sigma}\,U^\nu\,S^\rho\,U^\sigma\,, \qquad {}^*R_{\mu\nu\rho\sigma} = \frac{1}{2}\,\eta_{\mu\nu\alpha\beta}\,R^{\alpha\beta}{}_{\rho\sigma}\,, \tag{24}$$

must be taken into account [22]. It follows from Equation (23) that $P^\mu - m\,U^\mu$ is of the second order in spin; hence, the Mathisson–Papapetrou equations of motion to first order in spin become [35]

$$\frac{DS^{\mu\nu}}{d\varsigma} \approx 0\,, \tag{25}$$

and

$$m\,\frac{DU^\mu}{d\varsigma} \approx \mathcal{F}^\mu = -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}\,U^\nu S^{\alpha\beta}\,. \tag{26}$$

4. Spin-Vorticity–Gravity Coupling

We now turn to the behavior of spinning test particles in the Gödel-type spacetime. Within the framework of linearized general relativity, it can be shown in general that, in source-free Ricci-flat regions of the gravitational field, the Mathisson force corresponds to the Stern–Gerlach force associated with the spin–gravitomagnetic field coupling [20]. In the

Gödel-type universe, on the other hand, the spinning particle is immersed in the source of the gravitational field. Is $\mathcal{F}_\mu = -\partial_\mu(\mathcal{H}_{sg})$ still valid for the Gödel-type spacetime?

Let us consider a spinning test particle held at rest in space at fixed (x, y, z) coordinates in the Gödel-type spacetime. According to the free reference observer with adapted tetrad frames of $e^\mu{}_{\hat{\alpha}}$ and $\lambda^\mu{}_{\hat{\alpha}}$ at the same location, the spin vector to linear order stays fixed with respect to the parallel-propagated frame as a consequence of Equation (25); that is, $S_{\hat{i}}$, $i = 1, 2, 3$, are constants of the motion, where $S_{\hat{\alpha}} = S_\mu \lambda^\mu{}_{\hat{\alpha}}$; hence,

$$S_{\hat{0}} = 0\,, \qquad S_{\hat{i}} = S_\mu \lambda^\mu{}_{\hat{i}}\,. \tag{27}$$

The motion of the comoving observer has vorticity in accordance with Equation (4), and we therefore expect that the spin should couple to the vorticity resulting in the spin-vorticity Hamiltonian given by

$$\mathcal{H}_{sv} = -\boldsymbol{S} \cdot \boldsymbol{\omega} = \Omega\, S^{\hat{3}}\,. \tag{28}$$

Furthermore, the spin vector precesses with frequency $\Omega\,\partial_z$ with respect to the observer's natural frame $e^\mu{}_{\hat{i}}$ based on the spatial coordinate axes. The Hamiltonian associated with this motion is the spin–gravity Hamiltonian given by

$$\mathcal{H}_{sg} = \boldsymbol{S} \cdot \boldsymbol{B}\,, \tag{29}$$

where $\boldsymbol{B} = \boldsymbol{\Omega} = \Omega\,\partial_z$ is the gravitomagnetic field in this case. The result is

$$\mathcal{H}_{sg} = \Omega\, S^{\hat{3}}\,. \tag{30}$$

The spin–gravity coupling is indeed the same as the spin–vorticity coupling in this case, since the spinning particle, while engulfed by the source of the gravitational field, is fixed in space and comoving with the observer. It is clear that in this case $\partial_\mu(\mathcal{H}_{sg}) = 0$, so that the Stern–Gerlach force vanishes. To calculate the Mathisson force in this case, we need to find the Riemann curvature tensor for the Gödel-type universe, since the Mathisson force is directly proportional to the spacetime curvature.

In metric (1), the nonzero components of the Riemann tensor can be obtained from

$$R_{0101} = \Omega^2,\ R_{0202} = (\kappa + \sigma)e^{2\mu x}\Omega^2,\ R_{0112} = -\sqrt{\sigma}e^{\mu x}\Omega^2,\ R_{1212} = -\kappa\Big(\frac{4\kappa}{\sigma} + 5\Big)e^{2\mu x}\Omega^2\,. \tag{31}$$

We are interested in the components of the curvature tensor projected onto the orthonormal tetrad frame $\lambda^\mu{}_{\hat{\alpha}}$ adapted to our fiducial observer, namely,

$$R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} = R_{\mu\nu\rho\sigma}\, \lambda^\mu{}_{\hat{\alpha}}\, \lambda^\nu{}_{\hat{\beta}}\, \lambda^\rho{}_{\hat{\gamma}}\, \lambda^\sigma{}_{\hat{\delta}}\,. \tag{32}$$

The measured components of the Riemann tensor can be expressed via its symmetries as a 6×6 matrix in the standard manner with indices that range over the set $\{01, 02, 03, 23, 31, 12\}$. The end result is of the general form:

$$\begin{bmatrix} \mathbb{E} & \mathbb{H} \\ \mathbb{H}^T & \mathbb{S} \end{bmatrix}, \tag{33}$$

where $\mathbb{E}$, $\mathbb{H}$, and $\mathbb{S}$ represent the measured gravitoelectric, gravitomagnetic, and spatial components of the Riemann curvature tensor, respectively, and $\mathbb{E}$ and $\mathbb{S}$ are symmetric matrices, while $\mathbb{H}$ is traceless. In the case of Gödel-type spacetime (1), we find that $\mathbb{H} = 0$, and

$$(\mathbb{E}_{\hat{i}\hat{j}}) = \begin{bmatrix} \Omega^2 & 0 & 0 \\ 0 & \Omega^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad (\mathbb{S}_{\hat{i}\hat{j}}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\Big(1 + \frac{4\kappa}{\sigma}\Big)\Omega^2 \end{bmatrix}. \tag{34}$$

These results are equally valid if the curvature tensor is projected onto the natural frame $e^{\mu}{}_{\hat{\alpha}}$ of the reference observer.

We find that the Mathisson force, given by Equation (24), can be expressed as

$$\mathcal{F}^{\mu} = \lambda^{\mu}{}_{\hat{\alpha}}\, \mathcal{F}^{\hat{\alpha}}\,, \qquad \mathcal{F}^{\hat{0}} = 0\,, \qquad \mathcal{F}^{\hat{i}} = \mathbb{H}^{\hat{j}\hat{i}} S_{\hat{j}}\,. \tag{35}$$

However, $\mathbb{H} = 0$; therefore, the measured components of the Mathisson force vanish as well. That is,

$$\mathcal{F}_{\mu} = -\,\partial_{\mu}(\mathcal{H}_{sg}) = 0\,. \tag{36}$$

It is important to verify this result in a quasi-inertial Fermi normal coordinate system established about the world line of an arbitrary reference observer that is spatially at rest.

5. Fermi Coordinates in Gödel-Type Spacetimes

To explore spin–gravity coupling in Fermi coordinates, it is convenient to set up a quasi-inertial system of coordinates based on the nonrotating spatial frame adapted to a fiducial geodesic observer that is at rest in space with fixed (x, y, z) coordinates and a 4-velocity vector $u^{\mu} = \delta^{\mu}_{0}$ in Gödel-type spacetime (1). The reference observer establishes in the neighborhood of its world line a Fermi normal coordinate system based on the parallel-propagated spatial frame $\lambda^{\mu}{}_{\hat{i}}$, $i = 1, 2, 3$, given by Equation (11). That is, at each event $\bar{x}^{\mu}(\tau)$ on its world line, there is a local hypersurface formed by all spacelike geodesic curves that are orthogonal to the observer's world line at $\bar{x}^{\mu}(\tau)$. Consider an event with coordinates x^{μ} on this hypersurface that can be connected to $\bar{x}^{\mu}(\tau)$ by a unique spacelike geodesic of proper length ℓ. Then, the reference observer can assign Fermi coordinates $X^{\mu} = (T, X^{i})$ to x^{μ} such that

$$T := \tau\,, \qquad X^{i} := \ell\, \xi^{\mu} \lambda_{\mu}{}^{\hat{i}}(\tau)\,. \tag{37}$$

Here, ξ^{μ}, $\xi^{\mu}\, u_{\mu} = 0$, is a unit spacelike vector tangent to the unique spacelike geodesic at $\bar{x}^{\mu}(\tau)$.

For the case of Gödel's universe, one can find the exact Fermi metric coefficients [30]. The previous results are generalized here for Gödel-type spacetime (1). For the spacelike geodesics $x^{\mu}(\ell)$, we use Killing vector fields ∂_t, ∂_y, and ∂_z to derive the equations of motion:

$$t' + \sqrt{\sigma}\, e^{\mu x}\, y' = E\,, \quad \sqrt{\sigma}\, e^{\mu x}\, t' - \kappa\, e^{2\mu x}\, y' = k\,. \tag{38}$$

$$z' = h\,, \quad -\,t'^{2} - 2\sqrt{\sigma}\, e^{\mu x}\, t'\, y' + x'^{2} + \kappa e^{2\mu x}\, y'^{2} + z'^{2} = 1\,. \tag{39}$$

Here, E, k, and h are integration constants; moreover, a prime denotes the derivative of a spacetime coordinate with respect to proper length ℓ, e.g., $t' = dt/d\ell$. The condition $\xi_{\mu}\lambda^{\mu}{}_{\hat{0}} = 0$, where $\xi^{\mu} = dx^{\mu}/d\ell$, implies that $E = 0$. Then, with $z = h\,\ell$ and $E = 0$, we find

$$t' = \frac{\sqrt{\sigma}\, e^{-\,\mu x}\, k}{\sigma + \kappa}\,, \tag{40}$$

$$y' = -\,\frac{e^{-\,2\mu x}\, k}{\sigma + \kappa}\,, \tag{41}$$

$$x'^{2} + \frac{e^{-\,2\mu x}\, k^{2}}{\sigma + \kappa} = 1 - h^{2}\,. \tag{42}$$

The ordinary differential Equation (42) has the general solution for $x(\ell)$ given by

$$e^{\mu x} = \alpha_0\, \cosh(a\ell + b), \tag{43}$$

where the constant parameters are fixed as

$$\alpha_0\, a = \frac{|k|\, \mu}{\sqrt{\sigma + \kappa}}\,, \qquad a = \mu\sqrt{1 - h^{2}}\,, \tag{44}$$

and the condition

$$\alpha_0 \cosh b = 1 \tag{45}$$

is imposed to satisfy $x(0) = 0$.

Substituting Equation (43) into Equations (40) and (41), we find the solutions for $t(\ell)$ and $y(\ell)$:

$$t - \tau = \frac{2}{\mu}\sqrt{\frac{\sigma}{\sigma+\kappa}}\,\frac{k}{|k|}\left[\arctan e^{a\ell+b} - \arctan e^{b}\right]. \tag{46}$$

$$y = -\frac{1}{\mu\,\alpha_0}\,\frac{1}{\sqrt{\sigma+\kappa}}\,\frac{k}{|k|}\left[\tanh\left(a\ell+b\right) - \tanh b\right]. \tag{47}$$

Then, making use of Equations (8)–(11) and (37), we derive for the Fermi coordinates

$$T = \tau\,, \qquad Z = \ell\, h\,. \tag{48}$$

$$X\cos(\Omega T) - Y\sin(\Omega T) = \frac{\ell\,|k|}{\sqrt{\sigma+\kappa}}\,\sinh b\,. \tag{49}$$

$$X\sin(\Omega T) + Y\cos(\Omega T) = -\frac{\ell\, k}{\sqrt{\sigma+\kappa}}\,. \tag{50}$$

As in [30], we introduce the cylindrical coordinates

$$X = \rho\cos\theta\,, \qquad Y = \rho\sin\theta\,, \tag{51}$$

and recast Equations (49) and (50) into

$$\cos(\theta+\Omega T) = \tanh b\,, \tag{52}$$

$$\sin(\theta+\Omega T) = -\frac{|k|}{k\,\cosh b}\,, \tag{53}$$

$$\mu\rho = \ell\, a\,. \tag{54}$$

As a result, we rewrite the solutions (43), (47), and (46) as

$$e^{\mu x} = \cosh(\mu\rho) + \sinh(\mu\rho)\,\cos(\theta+\Omega T)\,, \tag{55}$$

$$\sqrt{\sigma+\kappa}\,\mu\, y = \frac{\tanh(\mu\rho)\,\sin(\theta+\Omega T)}{1+\tanh(\mu\rho)\,\cos(\theta+\Omega T)}\,, \tag{56}$$

$$\tan\left[\frac{\mu}{2}\sqrt{\frac{\sigma+\kappa}{\sigma}}(T-t)\right] = \frac{(e^{\mu\rho}-1)\sin(\theta+\Omega T)}{1-\cos(\theta+\Omega T)+[1+\cos(\theta+\Omega T)]\,e^{\mu\rho}}\,. \tag{57}$$

Finally, the transformation from (t,x,y,z) to Fermi coordinates (T,X,Y,Z) can be conveniently written in terms of the new variables:

$$\mathfrak{R} = \mu\rho\,, \qquad \mathfrak{F} = \theta + \Omega T\,, \tag{58}$$

as follows:

$$e^{\mu x} = \cosh\mathfrak{R} + \sinh\mathfrak{R}\,\cos\mathfrak{F}\,, \tag{59}$$

$$\sqrt{\sigma+\kappa}\,\mu\, y = \frac{\sinh\mathfrak{R}\,\sin\mathfrak{F}}{\cosh\mathfrak{R}+\sinh\mathfrak{R}\,\cos\mathfrak{F}}\,, \tag{60}$$

$$\tan\left[\frac{\mu}{2}\sqrt{\frac{\sigma+\kappa}{\sigma}}(T-t)\right] = \frac{(e^{\mathfrak{R}}-1)\,\sin\mathfrak{F}}{1-\cos\mathfrak{F}+(1+\cos\mathfrak{F})\,e^{\mathfrak{R}}}\,. \tag{61}$$

By differentiation, we obtain

$$dt+\sqrt{\sigma}\,e^{\mu x}dy=dT+\frac{1}{\mu}\sqrt{\frac{\sigma}{\sigma+\kappa}}\,(\cosh\mathfrak{R}-1)\,d\mathfrak{F}\,, \tag{62}$$

$$dx^2+(\kappa+\sigma)\,e^{2\mu x}\,dy^2=\frac{1}{\mu^2}\Big(d\mathfrak{R}^2+\sinh^2\!\mathfrak{R}\,d\mathfrak{F}^2\Big)\,. \tag{63}$$

It remains to substitute these results into Equation (10) to derive the line element of the Gödel-type universe in terms of the Fermi coordinates. We find

$$\begin{aligned} ds^2 = &-\,(1+\mathbb{L})\,dT^2-2\Omega\,\mathbb{K}\,dT\,(XdY-YdX)\\ &+dX^2+dY^2+dZ^2+\frac{\mathbb{F}}{X^2+Y^2}(XdY-YdX)^2\,, \end{aligned} \tag{64}$$

where

$$\mathbb{L}=\frac{\sigma}{4(\sigma+\kappa)}\left[\sinh^2\mathfrak{R}-\frac{\sigma+2\kappa}{\sigma+\kappa}(\cosh\mathfrak{R}-1)^2\right], \tag{65}$$

$$\mathbb{K}=-\,\frac{\kappa}{\sigma+\kappa}\,\frac{(\cosh\mathfrak{R}-1)^2}{\mathfrak{R}^2}\,, \tag{66}$$

$$\mathbb{F}=\frac{\sinh^2\mathfrak{R}}{\mathfrak{R}^2}-1-\frac{\sigma}{\sigma+\kappa}\,\frac{(\cosh\mathfrak{R}-1)^2}{\mathfrak{R}^2} \tag{67}$$

are functions of the variable

$$\mathfrak{R}=2\Omega\,\sqrt{\frac{\sigma+\kappa}{\sigma}}\,(X^2+Y^2)^{1/2}\,. \tag{68}$$

6. Spin–Gravity Coupling in Fermi Coordinates

In general, the spacetime metric in the Fermi system is given by

$$ds^2=\hat{g}_{\mu\nu}\,dX^\mu dX^\nu\,, \tag{69}$$

where

$$\hat{g}_{00}=-1-R_{\hat{0}\hat{i}\hat{0}\hat{j}}(T)\,X^iX^j+\cdots\,,\quad \hat{g}_{0i}=-\frac{2}{3}\,R_{\hat{0}\hat{j}\hat{i}\hat{k}}(T)\,X^jX^k+\cdots\,, \tag{70}$$

and

$$\hat{g}_{ij}=\delta_{ij}-\frac{1}{3}\,R_{\hat{i}\hat{k}\hat{j}\hat{l}}(T)\,X^kX^l+\cdots\,. \tag{71}$$

In these expansions in powers of spatial Fermi coordinates, the coefficients are, in general, functions of T and consist of components of the Riemann curvature tensor and its covariant derivatives as measured by the reference observer that permanently occupies the spatial origin of the Fermi coordinate system. That is, the metric of the Fermi normal coordinate system established on the basis of a parallel-propagated spatial frame along the world line of a geodesic observer is the Minkowski metric plus perturbations caused by the curvature of spacetime. Fermi coordinates are admissible within a cylindrical spacetime region around the world line of the fiducial observer, and the radius of this cylinder is given by an appropriate radius of the curvature of spacetime [30].

As defined in Equation (32), $R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}$ are evaluated at the origin of spatial Fermi coordinates via the projection of the Riemann tensor on the tetrad frame $\lambda^\mu{}_{\hat{\alpha}}$ of the fiducial observer; indeed, for the stationary Gödel-type spacetime, the nonzero components of $R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}$ are constants and can be obtained from

$$R_{\hat{0}\hat{1}\hat{0}\hat{1}}=R_{\hat{0}\hat{2}\hat{0}\hat{2}}=\Omega^2,\qquad R_{\hat{1}\hat{2}\hat{1}\hat{2}}=-\left(1+\frac{4\kappa}{\sigma}\right)\Omega^2 \tag{72}$$

via the symmetries of the Riemann curvature tensor.

We define the curvature-based gravitoelectric potential $\hat{\Phi}$ and gravitomagnetic vector potential $\hat{\boldsymbol{A}}$ via $\hat{g}_{00} = -1 + 2\hat{\Phi}$ and $\hat{g}_{0i} = -2\hat{A}_i$ [36,37]. Indeed,

$$\hat{\Phi} = -\frac{1}{2} R_{\hat{0}\hat{i}\hat{0}\hat{j}} X^i X^j + \cdots , \qquad \hat{A}_i = \frac{1}{3} R_{\hat{0}\hat{j}\hat{i}\hat{k}} X^j X^k + \cdots . \tag{73}$$

The corresponding fields are given by

$$\hat{\boldsymbol{E}} = -\boldsymbol{\nabla}\hat{\Phi} , \qquad \hat{\boldsymbol{B}} = \boldsymbol{\nabla} \times \hat{\boldsymbol{A}} , \tag{74}$$

as expected; more explicitly,

$$\hat{E}_i = R_{\hat{0}\hat{i}\hat{0}\hat{j}} X^j + \cdots , \qquad \hat{B}_i = -\frac{1}{2} \epsilon_{ijk} R^{\hat{j}\hat{k}}{}_{\hat{0}\hat{l}} X^l + \cdots . \tag{75}$$

To the lowest order, the gravitomagnetic field vanishes in the Gödel-type spacetime; therefore, we need to compute higher-order terms.

In Section 5, we derived the exact Fermi metric coefficients for the Gödel-type universe. They are given explicitly by

$$\hat{g}_{00} = -1 - \frac{\sigma}{4(\sigma+\kappa)} \left[\sinh^2 \mathfrak{R} - \frac{\sigma + 2\kappa}{\sigma+\kappa} (\cosh \mathfrak{R} - 1)^2 \right], \tag{76}$$

$$\hat{g}_{0i} = -\frac{\kappa}{\sigma+\kappa}\, \Omega\, \frac{(\cosh \mathfrak{R} - 1)^2}{\mathfrak{R}^2} (Y, -X, 0) , \tag{77}$$

and

$$(\hat{g}_{ij}) = \begin{bmatrix} 1+\mathbb{A} & -\mathbb{C} & 0 \\ -\mathbb{C} & 1+\mathbb{B} & 0 \\ 0 & 0 & 1 \end{bmatrix} , \tag{78}$$

where

$$\mathbb{A} = \mathbb{F}\, \frac{Y^2}{X^2+Y^2} , \qquad \mathbb{B} = \mathbb{F}\, \frac{X^2}{X^2+Y^2} , \qquad \mathbb{C} = \mathbb{F}\, \frac{XY}{X^2+Y^2} . \tag{79}$$

The exact Fermi coordinate system has been established around the fiducial observer fixed at $X = Y = Z = 0$.

For $\kappa \geq 0$, there are no closed timelike curves. In the special case of the Gödel universe with parameters (13), there are no closed timelike curves within a cylindrical region about the Z axis with

$$\mathfrak{R} = \sqrt{2}\, \Omega\, (X^2 + Y^2)^{1/2} \leq \mathfrak{R}_{\max} , \qquad \mathfrak{R}_{\max} = 2 \ln(1 + \sqrt{2}) . \tag{80}$$

Indeed, a circle in the (X, Y) plane inside this domain is spacelike; however, it becomes null for $\mathfrak{R} = \mathfrak{R}_{\max}$ and timelike for $\mathfrak{R} > \mathfrak{R}_{\max}$.

The stationary and divergence-free gravitomagnetic vector field of the Gödel-type universe is given by $\hat{B}_1 = \hat{B}_2 = 0$ and

$$\hat{B}_3 = -\frac{\kappa}{\sigma+\kappa}\, \Omega (\cosh \mathfrak{R} - 1) \frac{\sinh \mathfrak{R}}{\mathfrak{R}} . \tag{81}$$

It is interesting to note that $\hat{B}_3$ and its first derivative with respect to $\mathfrak{R}$ vanish at $\mathfrak{R} = 0$; then, $\hat{B}_3$ monotonically increases with increasing $\mathfrak{R}$ and diverges as $\mathfrak{R} \to \infty$. More explicitly,

$$\hat{B}_3 = -\frac{2\kappa}{\sigma} \Omega^3 (X^2+Y^2) \left[1 + \Omega^2 \left(\frac{\sigma+\kappa}{\sigma} \right) (X^2+Y^2) + \frac{2\Omega^4}{5} \left(\frac{\sigma+\kappa}{\sigma} \right)^2 (X^2+Y^2)^2 + \cdots \right] , \tag{82}$$

so that the fiducial observer measures a null gravitomagnetic field at its location $(X = Y = Z = 0)$. Furthermore, the gravitomagnetic field away from the Z axis points along Z and is cylindrically symmetric; indeed, it vanishes all along Z but increases monotonically away from the Z axis and eventually diverges as the radius of the cylinder about the Z axis approaches infinity.

Within the Fermi coordinate system, it is useful to define the class of fundamental observers that remain at rest in space, each with fixed (X, Y, Z) coordinates. For our present purposes, we concentrate on the set of fundamental observers that occupy a cylindrical region in the neighborhood of the Z axis. Specifically, in this region, we can express the metric tensor in Fermi coordinates as

$$\hat{g}_{\mu\nu} = \eta_{\mu\nu} + \hat{h}_{\mu\nu}\,, \tag{83}$$

where the nonzero components of the gravitational potentials are given by

$$\hat{h}_{00} = -\,\Omega^2(X^2+Y^2)\,, \quad \hat{h}_{01} = -\,\frac{\kappa}{\sigma}\Omega^3(X^2+Y^2)Y\,, \quad \hat{h}_{02} = \frac{\kappa}{\sigma}\Omega^3(X^2+Y^2)X\,, \tag{84}$$

and

$$\hat{h}_{11} = \frac{1}{3}\Big(1+\frac{4\kappa}{\sigma}\Big)\Omega^2 Y^2\,, \quad \hat{h}_{12} = -\,\frac{1}{3}\Big(1+\frac{4\kappa}{\sigma}\Big)\Omega^2 XY\,, \quad \hat{h}_{22} = \frac{1}{3}\Big(1+\frac{4\kappa}{\sigma}\Big)\Omega^2 X^2\,. \tag{85}$$

That is, for the sake of simplicity, we confine our considerations to a cylindrical region about the Z axis such that $\Omega\,|X| = \Omega\,|Y| \lesssim \epsilon$, where $0 < \epsilon \ll 1$ and all terms of order ϵ^4 and higher are neglected in our analysis.

In the cylindrical neighborhood of the fiducial observer under consideration, fundamental observers have access to adapted orthonormal tetrad frames $\varphi^\mu{}_{\hat{\alpha}}$, given in the Fermi coordinate system (T, X, Y, Z) by

$$\varphi^\mu{}_{\hat{0}} = (1+\tfrac{1}{2}\hat{h}_{00}, 0, 0, 0)\,, \qquad \varphi^\mu{}_{\hat{1}} = (\hat{h}_{01}, 1-\tfrac{1}{2}\hat{h}_{11}, 0, 0)\,, \tag{86}$$

$$\varphi^\mu{}_{\hat{2}} = (\hat{h}_{02}, -\hat{h}_{12}, 1-\tfrac{1}{2}\hat{h}_{22}, 0)\,, \qquad \varphi^\mu{}_{\hat{3}} = (0, 0, 0, 1)\,. \tag{87}$$

These tetrad axes are primarily along the Fermi coordinate directions; indeed, for $X = Y = 0$, $\varphi^\mu{}_{\hat{\alpha}} \to \lambda^\mu{}_{\hat{\alpha}}$. According to these fundamental observers, a spinning particle held at rest in space has a 4-velocity vector in the Fermi system given by $\hat{U}^\mu = \varphi^\mu{}_{\hat{0}}$; moreover, its spin vector has the following measured components:

$$\hat{S}_{\hat{0}} = 0\,, \qquad \hat{S}_{\hat{i}} = \hat{S}_\mu\, \varphi^\mu{}_{\hat{i}}\,, \tag{88}$$

since $\hat{S}^\mu\,\hat{U}_\mu = 0$. Furthermore, the gravitomagnetic field at the location of the spin is given by

$$\hat{B}_1 = 0\,, \qquad \hat{B}_2 = 0\,, \qquad \hat{B}_3 = -\,\frac{1}{2}(\partial_X\,\hat{h}_{02} - \partial_Y\,\hat{h}_{01}) = -\,\frac{2\kappa}{\sigma}\Omega^3(X^2+Y^2)\,, \tag{89}$$

in agreement with Equation (82) within our approximation scheme. The Hamiltonian for spin-gravity coupling in the Fermi frame is thus given by

$$\hat{\mathcal{H}}_{sg} = \hat{\boldsymbol{S}}\cdot\hat{\boldsymbol{B}} = -\,\frac{2\kappa}{\sigma}\Omega^3(X^2+Y^2)\hat{S}^{\hat{3}}\,, \tag{90}$$

which reduces in our approximation to $-\,\frac{2\kappa}{\sigma}\Omega^3(X^2+Y^2)S^{\hat{3}}$, where $S^{\hat{3}}$ is a constant. The corresponding Stern–Gerlach force is then

$$-\,\partial_\mu\,\hat{\mathcal{H}}_{sg} = \frac{4\kappa}{\sigma}\,\Omega^3\,S^{\hat{3}}(0, X, Y, 0)\,. \tag{91}$$

Next, we need to compute the Mathisson force in the Fermi frame, namely,

$$\hat{\mathcal{F}}_\mu = -\frac{1}{2}\,\hat{R}_{\mu\nu\alpha\beta}\hat{U}^\nu\,\hat{S}^{\alpha\beta}\,. \tag{92}$$

For metric (83), the curvature tensor to the first order in the perturbation is given by

$$\hat{R}_{\mu\nu\alpha\beta} = \frac{1}{2}\,(\hat{h}_{\mu\beta,\,\nu\alpha} + \hat{h}_{\nu\alpha,\,\mu\beta} - \hat{h}_{\nu\beta,\,\mu\alpha} - \hat{h}_{\mu\alpha,\,\nu\beta})\,. \tag{93}$$

We are interested in the gravitomagnetic components of this curvature tensor as measured by the fundamental observers. Projection of this tensor on the tetrad frame $\varphi^\mu{}_{\hat{\alpha}}$ does not affect its components in our approximation scheme. We find in this case

$$(\hat{\mathbb{H}}_{\hat{i}\hat{j}}) = \begin{bmatrix} 0 & 0 & \kappa_1 \\ 0 & 0 & \kappa_2 \\ 0 & 0 & 0 \end{bmatrix}, \tag{94}$$

where

$$\kappa_1 = \frac{1}{2}\partial_X\,(\partial_X\,\hat{h}_{02} - \partial_Y\,\hat{h}_{01})\,, \qquad \kappa_2 = \frac{1}{2}\partial_Y\,(\partial_X\,\hat{h}_{02} - \partial_Y\,\hat{h}_{01})\,. \tag{95}$$

Hence, $\hat{\mathcal{F}}_{\hat{0}} = 0$, and $\hat{\mathcal{F}}_{\hat{i}} = \hat{\mathbb{H}}_{\hat{i}\hat{j}}\hat{S}^{\hat{j}} = (\kappa_1, \kappa_2, 0)S^{\hat{3}}$ at the level of approximation under consideration here. Moreover, Equation (89) implies

$$\kappa_1 = \frac{4\kappa}{\sigma}\,\Omega^3 X\,, \qquad \kappa_2 = \frac{4\kappa}{\sigma}\,\Omega^3 Y\,. \tag{96}$$

Therefore, $\hat{\mathcal{F}}_\mu = -\,\partial_\mu\,\hat{\mathcal{H}}_{sg}$ as measured by the fundamental observers within the cylindrical domain in the Fermi frame.

We have thus far relied on the classical pole–dipole model for the evaluation of spin–gravity coupling. It is important to demonstrate that our considerations are consistent with the solutions of the Dirac equation in the Gödel-type universe.

7. Dirac Equation in the Gödel-Type Universe

Let us start with the Dirac equation in the form [38,39]

$$(i\gamma^\alpha\nabla_\alpha - m)\,\Psi = 0\,, \qquad \nabla_\mu = \partial_\mu + \Gamma_\mu\,, \tag{97}$$

where the fermion wave function Ψ is a 4-component spacetime scalar variable composed of the pair of 2-spinors φ and χ:

$$\Psi = \begin{bmatrix} \varphi \\ \chi \end{bmatrix}, \qquad \varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}, \qquad \chi = \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix}. \tag{98}$$

As before, we assume that the observer in the gravitational field has a natural adapted orthonormal tetrad field and

$$\gamma^\alpha = e^\alpha{}_{\hat{\beta}}\,\gamma^{\hat{\beta}}\,, \qquad \{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}(x)I_4\,, \tag{99}$$

where I_n is the n-dimensional identity matrix, and

$$\gamma^{\hat{0}} = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}, \qquad \gamma^{\hat{i}} = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}. \tag{100}$$

Here, σ_i are Pauli matrices, namely,

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{101}$$

The spin connection Γ_μ (also known as Fock–Ivanenko coefficients) is given by

$$\Gamma_\mu = -\frac{i}{4}\, e^{\nu}{}_{\hat{\alpha}}\, e_{\nu\,\hat{\beta};\mu}\, \sigma^{\hat{\alpha}\hat{\beta}}\,, \qquad \sigma^{\hat{\alpha}\hat{\beta}} := \frac{i}{2}[\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}]\,. \tag{102}$$

Making use of tetrad frame (8), we find, after some algebra, the explicit form of the Dirac Equation (97) in Gödel-type spacetime (1):

$$\Big[\Big(\gamma^{\hat{0}} - \sqrt{\frac{\sigma}{\sigma+\kappa}}\,\gamma^{\hat{2}}\Big) i\partial_t - \gamma^{\hat{1}}\, p_x - \frac{e^{-\mu x}}{\sqrt{\kappa+\sigma}}\,\gamma^{\hat{2}}\, p_y - \gamma^{\hat{3}}\, p_z \\ + \frac{i\mu}{2}\gamma^{\hat{1}} + \frac{\mu}{4}\sqrt{\frac{\sigma}{\sigma+\kappa}}\,\gamma^{\hat{0}}\,\Sigma^{\hat{3}} - m\Big]\Psi = 0\,. \tag{103}$$

Here, as usual, the momentum operator is $\boldsymbol{p} = -\,i\boldsymbol{\nabla}$ and the spin operator $\boldsymbol{\Sigma}$ is given by the matrix

$$\Sigma^{\hat{i}} = \begin{bmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{bmatrix}. \tag{104}$$

Next, due to the symmetries of Gödel-type spacetime, we assume a solution of the form

$$\Psi = \psi(x)\exp(-i\,\omega\, t + i\, k_2\, y + i\, k_3\, z)\,, \tag{105}$$

where the four components of $\psi(x)$ satisfy ordinary differential equations, namely,

$$\frac{d\psi}{dx} = \mathcal{M}\psi\,, \tag{106}$$

where $\mathcal{M}$ is the 4×4 matrix

$$\mathcal{M} = \begin{bmatrix} \mathcal{A}_+ & ik_3 & 0 & i\mathcal{B}_+ \\ -ik_3 & -\mathcal{A}_- & i\mathcal{B}_- & 0 \\ 0 & i\mathcal{B}_+ & \mathcal{A}_+ & ik_3 \\ i\mathcal{B}_- & 0 & -ik_3 & -\mathcal{A}_- \end{bmatrix} + im \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}. \tag{107}$$

Here, $\mathcal{A}_\pm$ and $\mathcal{B}_\pm$ are given by

$$\mathcal{A}_\pm = \omega\sqrt{\frac{\sigma}{\sigma+\kappa}} \pm \Omega\,\sqrt{\frac{\sigma+\kappa}{\sigma}} + k_2\, e^{-\,\mu x}\,, \qquad \mathcal{B}_\pm = \omega \pm \frac{\Omega}{2}\,. \tag{108}$$

The spin–vorticity–gravity coupling is evident in the way the frequency of the radiation is changed by $\pm\Omega/2$, in agreement with previous results [37,40,41]. If $k_2 = 0$, the waves can only travel parallel or antiparallel to the rotation axis. In this case, matrix $\mathcal{M}$ has constant elements, and the general solution of Equation (106) can be expressed in terms of the eigenvalues and eigenfunctions of $\mathcal{M}$. It turns out that no propagation can occur in this case due to the requirement that the wave amplitude be finite at all times [41]. These general results for the Dirac equation are consistent with the propagation of the scalar and electromagnetic waves in the Gödel-type universe; for brief accounts of these latter topics, see the appendices at the end of this paper.

To deal with the general case, we henceforth assume $k_2 \neq 0$ and change to $\xi = e^{-\,\mu x}$ instead of x as the independent variable. Let us recall here that $\mu > 0$, since we have explicitly assumed $\Omega > 0$. For $\infty > x > -\infty$, we find that ξ goes from zero to $+\infty$; hence, ξ is a radial coordinate. In terms of ξ, Equation (106) takes the form

$$\xi\,\frac{d\psi}{d\xi} = \mathbb{M}\psi\,, \tag{109}$$

where matrix $\mathbb{M}$ is simply related to $\mathcal{M}$, namely,

$$\mathbb{M} = \begin{bmatrix} -\bar{\mathcal{A}}_+ & -i\gamma & 0 & -i\bar{\mathcal{B}}_+ \\ i\gamma & \bar{\mathcal{A}}_- & -i\bar{\mathcal{B}}_- & 0 \\ 0 & -i\bar{\mathcal{B}}_+ & -\bar{\mathcal{A}}_+ & -i\gamma \\ -i\bar{\mathcal{B}}_- & 0 & i\gamma & \bar{\mathcal{A}}_- \end{bmatrix} - \frac{im}{2\Omega}\sqrt{\frac{\sigma}{\sigma+\kappa}} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}. \tag{110}$$

Here, $\bar{\mathcal{A}}_\pm$ and $\bar{\mathcal{B}}_\pm$ are given by

$$\bar{\mathcal{A}}_\pm = \frac{\omega}{2\Omega}\left(\frac{\sigma}{\sigma+\kappa}\right) \pm \frac{1}{2} + \beta\,\xi\,, \qquad \bar{\mathcal{B}}_\pm = \frac{1}{2}\sqrt{\frac{\sigma}{\sigma+\kappa}}\left(\frac{\omega}{\Omega} \pm \frac{1}{2}\right), \tag{111}$$

and we have introduced dimensionless parameters

$$\beta = \frac{k_2}{2\Omega}\sqrt{\frac{\sigma}{\sigma+\kappa}}\,, \qquad \gamma = \frac{k_3}{2\Omega}\sqrt{\frac{\sigma}{\sigma+\kappa}}\,. \tag{112}$$

To clarify the structure of the resulting system (109)–(110), we note that the 4-spinor (98) can be decomposed into the sum of the left and right spinors,

$$\psi = \psi^L + \psi^R\,, \quad \psi^L = \frac{1}{2}(1-\gamma_5)\psi\,, \quad \psi^R = \frac{1}{2}(1+\gamma_5)\psi\,, \tag{113}$$

where $\gamma_5 := i\,\gamma^{\hat{0}}\gamma^{\hat{1}}\gamma^{\hat{2}}\gamma^{\hat{3}}$. By definition, the left and right spinors are eigenstates of the γ_5 matrix: $\gamma_5\psi^L = -\psi^L$ and $\gamma_5\psi^R = \psi^R$. Furthermore, we decompose the left and right spinors into the eigenstates of the $\Sigma^{\hat{3}}$ spin matrix (i.e., the "spin-up" and "spin-down" states):

$$\psi^L = \psi^L_+ + \psi^L_-\,, \quad \psi^R = \psi^R_+ + \psi^R_-\,, \quad \Sigma^{\hat{3}}\psi^L_\pm = \pm\psi^L_\pm\,, \quad \Sigma^{\hat{3}}\psi^R_\pm = \pm\psi^R_\pm\,. \tag{114}$$

After these steps, we thus have

$$\psi^L_+ = \mathcal{L}_+\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \quad \psi^L_- = \mathcal{L}_-\begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \quad \psi^R_+ = \mathcal{R}_+\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, \quad \psi^R_- = \mathcal{R}_-\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}, \tag{115}$$

where explicitly

$$\mathcal{L}_+ = \frac{1}{2}(\varphi_1 + \chi_1)\,, \qquad \mathcal{L}_- = \frac{1}{2}(\varphi_2 + \chi_2)\,, \tag{116}$$

$$\mathcal{R}_+ = \frac{1}{2}(\varphi_1 - \chi_1)\,, \qquad \mathcal{R}_- = \frac{1}{2}(\varphi_2 - \chi_2)\,. \tag{117}$$

Taking these definitions into account, we can straightforwardly recast system (109)–(110) into an equivalent but more transparent form:

$$\left(\xi\frac{d}{d\xi} + \bar{\mathcal{A}}_+\right)\mathcal{L}_+ = -i(\bar{\mathcal{B}}_+ + \gamma)\mathcal{L}_- + i\frac{m}{2\Omega}\sqrt{\frac{\sigma}{\sigma+\kappa}}\,\mathcal{R}_-\,, \tag{118}$$

$$\left(\xi\frac{d}{d\xi} - \bar{\mathcal{A}}_-\right)\mathcal{L}_- = -i(\bar{\mathcal{B}}_- - \gamma)\mathcal{L}_+ + i\frac{m}{2\Omega}\sqrt{\frac{\sigma}{\sigma+\kappa}}\,\mathcal{R}_+\,, \tag{119}$$

$$\left(\xi\frac{d}{d\xi} + \bar{\mathcal{A}}_+\right)\mathcal{R}_+ = \quad i(\bar{\mathcal{B}}_+ - \gamma)\mathcal{R}_- - i\frac{m}{2\Omega}\sqrt{\frac{\sigma}{\sigma+\kappa}}\,\mathcal{L}_-\,, \tag{120}$$

$$\left(\xi\frac{d}{d\xi} - \bar{\mathcal{A}}_-\right)\mathcal{R}_- = \quad i(\bar{\mathcal{B}}_- + \gamma)\mathcal{R}_+ - i\frac{m}{2\Omega}\sqrt{\frac{\sigma}{\sigma+\kappa}}\,\mathcal{L}_+\,. \tag{121}$$

The nontrivial mass mixes the left and right modes. However, for the massless ($m = 0$) case or in the high-energy approximation ($\frac{mc^2}{\hbar\Omega} \ll 1$), we can neglect the last terms on the right-hand sides. As a result, the left modes $\mathcal{L}_\pm$ decouple from the right modes $\mathcal{R}_\pm$ and the system reduces to

$$\Big(\xi\frac{d}{d\xi} + \bar{\mathcal{A}}_+\Big)\mathcal{L}_+ = -i(\bar{\mathcal{B}}_+ + \gamma)\mathcal{L}_-\,, \tag{122}$$

$$\Big(\xi\frac{d}{d\xi} - \bar{\mathcal{A}}_-\Big)\mathcal{L}_- = -i(\bar{\mathcal{B}}_- - \gamma)\mathcal{L}_+\,, \tag{123}$$

$$\Big(\xi\frac{d}{d\xi} + \bar{\mathcal{A}}_+\Big)\mathcal{R}_+ = \quad i(\bar{\mathcal{B}}_+ - \gamma)\mathcal{R}_-\,, \tag{124}$$

$$\Big(\xi\frac{d}{d\xi} - \bar{\mathcal{A}}_-\Big)\mathcal{R}_- = \quad i(\bar{\mathcal{B}}_- + \gamma)\mathcal{R}_+\,. \tag{125}$$

It is interesting to mention that in this approximation scheme Equation (109) can also be solved by a different approach that is briefly described in Appendix A.

7.1. Explicit Solutions

Multiplying Equation (122) by $-i(\bar{\mathcal{B}}_- - \gamma)$ and Equation (123) by $-i(\bar{\mathcal{B}}_+ + \gamma)$, we derive the second-order equations for the *left modes*:

$$\Big(\xi\frac{d}{d\xi} + \bar{\mathcal{A}}_+\Big)\Big(\xi\frac{d}{d\xi} - \bar{\mathcal{A}}_-\Big)\mathcal{L}_- = \Big[-\bar{\mathcal{B}}_+\bar{\mathcal{B}}_- + \gamma(\bar{\mathcal{B}}_+ - \bar{\mathcal{B}}_-) + \gamma^2\Big]\mathcal{L}_-\,, \tag{126}$$

$$\Big(\xi\frac{d}{d\xi} - \bar{\mathcal{A}}_+\Big)\Big(\xi\frac{d}{d\xi} + \bar{\mathcal{A}}_-\Big)\mathcal{L}_+ = \Big[-\bar{\mathcal{B}}_+\bar{\mathcal{B}}_- + \gamma(\bar{\mathcal{B}}_+ - \bar{\mathcal{B}}_-) + \gamma^2\Big]\mathcal{L}_+\,. \tag{127}$$

In Equation (111), it is useful to introduce a dimensionless parameter α,

$$\alpha := \frac{\omega}{2\Omega}\Big(\frac{\sigma}{\sigma+\kappa}\Big)\,, \qquad \bar{\mathcal{A}}_\pm = \alpha + \beta\,\xi \pm \frac{1}{2}\,; \tag{128}$$

then,

$$\bar{\mathcal{A}}_+\bar{\mathcal{A}}_- = (\alpha+\beta\xi)^2 - \frac{1}{4}\,, \qquad \bar{\mathcal{A}}_+ - \bar{\mathcal{A}}_- = 1\,, \tag{129}$$

$$\bar{\mathcal{B}}_+\bar{\mathcal{B}}_- = \frac{\sigma}{\sigma+\kappa}\Big[\frac{\omega^2}{(2\Omega)^2} - \frac{1}{16}\Big]\,, \qquad \bar{\mathcal{B}}_+ - \bar{\mathcal{B}}_- = \frac{1}{2}\sqrt{\frac{\sigma}{\sigma+\kappa}}\,. \tag{130}$$

Employing the ansatz

$$\mathcal{L}_\pm = \xi^{-1}\,u_{\mp\frac{1}{2}}\,, \tag{131}$$

we can recast Equations (126) and (127) into the form

$$\xi^2\frac{d^2}{d\xi^2}u_s + \Big[\frac{1}{4} - \tilde{\mu}_f^2 - \beta^2\xi^2 - 2\beta\xi\,(\alpha+s)\Big]\,u_s = 0\,, \tag{132}$$

where $s = \pm\frac{1}{2}$ and

$$\begin{aligned}\tilde{\mu}_f^2 &= \alpha^2 + \gamma^2 - \bar{\mathcal{B}}_+\bar{\mathcal{B}}_- + \gamma(\bar{\mathcal{B}}_+ - \bar{\mathcal{B}}_-)\\ &= \frac{1}{\mu^2}\Big[-\omega^2\,\frac{\kappa}{\sigma+\kappa} + (k_3 - \Omega/2)^2\Big]\,.\end{aligned} \tag{133}$$

With a new independent variable $\tilde{\xi} = 2|\beta|\xi$, Equation (132) can be reduced to Whittaker's equation [42]:

$$\frac{d^2u_s}{d\tilde{\xi}^2} + \Bigg[-\frac{1}{4} + \frac{\tilde{\kappa}_f}{\tilde{\xi}} + \frac{\frac{1}{4} - \tilde{\mu}_f^2}{\tilde{\xi}^2}\Bigg]u_s = 0\,, \tag{134}$$

where

$$\tilde{\kappa}_f = -\frac{\beta}{|\beta|}(\alpha + s)\,. \tag{135}$$

The Dirac field is a linear perturbation on the Gödel-type spacetime; therefore, $\psi(x)$ should be bounded. Demanding that $\psi(x)$ be finite everywhere, the acceptable solution of Whittaker's equation is given via the confluent hypergeometric functions by

$$u_s = u_s^0 \exp(-\tfrac{1}{2}\tilde{\xi})\, \tilde{\xi}^{\frac{1}{2}+\tilde{\mu}_f}\, {}_1F_1(\tfrac{1}{2} + \tilde{\mu}_f - \tilde{\kappa}_f, 1 + 2\tilde{\mu}_f; \tilde{\xi})\,, \tag{136}$$

where

$$\frac{1}{2} + \tilde{\mu}_f - \tilde{\kappa}_f = -n\,, \quad n = 0, 1, 2, \dots\,. \tag{137}$$

In this case, the confluent hypergeometric function can be expressed in terms of the associated Laguerre polynomial.

For $k_2 < 0$, β is negative and $\tilde{\kappa}_f = \alpha + s = \frac{\omega}{2\Omega}\big(\frac{\sigma}{\sigma+\kappa}\big) + s$, with $s = \pm\frac{1}{2}$. Then, combining Equations (137) and (133), we derive the dispersion relation

$$\omega = (2n + 1 + 2s)\Omega \pm \Big[(k_3 - \Omega/2)^2 - \frac{\kappa}{\sigma}(2n + 1 + 2s)^2\Omega^2\Big]^{1/2}. \tag{138}$$

Note that solutions with both signs of energy are admissible.

Similarly, multiplying Equation (124) by $i(\bar{\mathcal{B}}_- + \gamma)$ and Equation (125) by $i(\bar{\mathcal{B}}_+ - \gamma)$, we derive the second-order equations for the *right modes*:

$$\Big(\xi\frac{d}{d\xi} + \bar{\mathcal{A}}_+\Big)\Big(\xi\frac{d}{d\xi} - \bar{\mathcal{A}}_-\Big)\mathcal{R}_- = \Big[-\bar{\mathcal{B}}_+\bar{\mathcal{B}}_- - \gamma(\bar{\mathcal{B}}_+ - \bar{\mathcal{B}}_-) + \gamma^2\Big]\mathcal{R}_-\,, \tag{139}$$

$$\Big(\xi\frac{d}{d\xi} - \bar{\mathcal{A}}_+\Big)\Big(\xi\frac{d}{d\xi} + \bar{\mathcal{A}}_-\Big)\mathcal{R}_+ = \Big[-\bar{\mathcal{B}}_+\bar{\mathcal{B}}_- - \gamma(\bar{\mathcal{B}}_+ - \bar{\mathcal{B}}_-) + \gamma^2\Big]\mathcal{R}_+\,. \tag{140}$$

Using the ansatz

$$\mathcal{R}_\pm = \xi^{-1}\, v_{\mp\frac{1}{2}}\,, \tag{141}$$

we recast Equations (139) and (140) into

$$\xi^2\frac{d^2}{d\xi^2}v_s + \Big[\frac{1}{4} - \bar{\mu}_f^2 - \beta^2\xi^2 - 2\beta\xi\,(\alpha + s)\Big]\, v_s = 0\,, \tag{142}$$

where $s = \pm\frac{1}{2}$, as before, but now we have

$$\begin{aligned}\bar{\mu}_f^2 &= \alpha^2 + \gamma^2 - \bar{\mathcal{B}}_+\bar{\mathcal{B}}_- - \gamma(\bar{\mathcal{B}}_+ - \bar{\mathcal{B}}_-)\\ &= \frac{1}{\mu^2}\Big[-\omega^2\frac{\kappa}{\sigma+\kappa} + (k_3 + \Omega/2)^2\Big]\,.\end{aligned} \tag{143}$$

With the independent variable $\tilde{\xi} = 2|\beta|\xi$, Equation (142) can again be reduced to Whittaker's equation:

$$\frac{d^2 v_s}{d\tilde{\xi}^2} + \Bigg[-\frac{1}{4} + \frac{\tilde{\kappa}_f}{\tilde{\xi}} + \frac{\frac{1}{4} - \bar{\mu}_f^2}{\tilde{\xi}^2}\Bigg] v_s = 0\,, \tag{144}$$

where $\tilde{\kappa}_f$ is given by Equation (135). The regular solution of Equation (144) is given by

$$v_s = v_s^0 \exp(-\tfrac{1}{2}\tilde{\xi})\, \tilde{\xi}^{\frac{1}{2}+\bar{\mu}_f}\, {}_1F_1(\tfrac{1}{2} + \bar{\mu}_f - \tilde{\kappa}_f, 1 + 2\bar{\mu}_f; \tilde{\xi})\,, \tag{145}$$

where

$$\frac{1}{2} + \bar{\mu}_f - \tilde{\kappa}_f = -n\,, \quad n = 0, 1, 2, \dots\,. \tag{146}$$

As before, we can combine Equations (146) and (143) to derive the dispersion relation

$$\omega = (2n+1+2s)\Omega \pm \left[(k_3 + \Omega/2)^2 - \frac{\kappa}{\sigma}(2n+1+2s)^2\Omega^2 \right]^{1/2}. \tag{147}$$

The motion of Dirac waves in the Gödel-type universe is in general agreement with the corresponding results for the scalar and electromagnetic wave propagation described in Appendices B and C.

7.2. Dealing with Subtle Points of Dirac Theory on Curved Spacetimes

In order to have a correct quantum-mechanical interpretation, Dirac Equation (97) should be recast into the form of the Schrödinger equation:

$$i\frac{\partial \Psi}{\partial t} = \mathcal{H}\Psi\,. \tag{148}$$

In flat spacetime with the Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1,1,1,1)$, the trivial frame $e^{\mu}{}_{\hat{\alpha}} = \delta^{\mu}_{\alpha}$, and the spin connection $\Gamma_{\mu} = 0$, this is straightforward. Multiplying Equation (97) by $\gamma^{\hat{0}}$, we derive Schrödinger Equation (148) with the Hermitian Hamiltonian

$$\mathcal{H} = \beta_{\rm D}\, m + \boldsymbol{\alpha}_{\rm D} \cdot \boldsymbol{p}\,. \tag{149}$$

Here we denote, as usual, the matrices

$$\beta_{\rm D} := \gamma^{\hat{0}}\,, \qquad \alpha^{\hat{i}}_{\rm D} := \gamma^{\hat{0}}\gamma^{\hat{i}} = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix}, \qquad i = 1,2,3\,. \tag{150}$$

In addition, one also needs a quantum-probabilistic picture, which is related to the normalization of the wave function. As is well known, a direct consequence of the Dirac Equation (97) is the conservation of the vector current, which, in flat spacetime, can be expressed as

$$\partial_{\mu} J^{\mu} = 0\,, \qquad J^{\mu} = \overline{\Psi}\gamma^{\mu}\Psi\,. \tag{151}$$

Integration over 3-space yields a global conservation law

$$\int d^3x\, J^0 = \int d^3x \Psi^{\dagger}\Psi = \text{constant} = 1\,. \tag{152}$$

The physical interpretation of the Dirac fermion dynamics is based on Equations (148) and (152), especially when the fermionic particle interacts with external fields.

Dirac theory on curved manifolds, however, involves a number of subtleties. In particular, the differential conservation law (151) is replaced by its curved version:

$$\nabla_{\mu} J^{\mu} = \frac{1}{\sqrt{-g}}\,\partial_{\mu}\left(\sqrt{-g} J^{\mu}\right) = 0\,, \qquad J^{\mu} = e^{\mu}{}_{\hat{\alpha}}\,\overline{\Psi}\gamma^{\hat{\alpha}}\Psi\,, \tag{153}$$

which yields the global conservation law

$$\int d^3x\,\sqrt{-g}\,J^0 = \int d^3x \sqrt{-g}\, e^{0}{}_{\hat{\alpha}}\Psi^{\dagger}\gamma^{\hat{0}}\gamma^{\hat{\alpha}}\Psi = \text{constant} = 1\,. \tag{154}$$

For the natural Gödel-type tetrad frame (8), we have $e^{0}{}_{\hat{\alpha}}\gamma^{\hat{0}}\gamma^{\hat{\alpha}} = 1 - \sqrt{\frac{\sigma}{\sigma+\kappa}}\,\alpha^{\hat{2}}_{\rm D}$; therefore, the physical interpretation of the solutions is unclear. In addition, Dirac Equation (103) obviously cannot be directly recast into the form of the Schrödinger wave Equation (148).

Both issues are related to the choice of the tetrad frame, which is defined up to an arbitrary local Lorentz transformation. The choice (8) corresponds to the so-called Landau-Lifshitz gauge with ${e_0}^{\hat{i}} = 0$ and $e^{i}{}_{\hat{0}} = 0$. The situation is essentially improved when one chooses the Schwinger gauge for the frame, where $e_i{}^{\hat{0}} = 0$ and $e^0{}_{\hat{\imath}} = 0$. Then, Equation (154) reduces to an "almost flat" form:

$$\int d^3x \sqrt{-g}\, e^0{}_{\hat{0}}\, \Psi^\dagger \Psi = 1\,, \tag{155}$$

and the Dirac equation is straightforwardly recast into the Schrödinger form [38,39].

This suggests replacing the original tetrad frame (8) with a new one

$$\widetilde{e}_{\hat{0}} = \sqrt{\frac{\kappa}{\sigma+\kappa}}\left(\partial_t + \frac{\sqrt{\sigma}}{\kappa} e^{-\mu x}\partial_y\right), \quad \widetilde{e}_{\hat{1}} = \partial_x\,, \quad \widetilde{e}_{\hat{2}} = \frac{e^{-\mu x}}{\sqrt{\kappa}}\partial_y\,, \quad \widetilde{e}_{\hat{3}} = \partial_z, \tag{156}$$

where we assume $\kappa > 0$. Obviously, this choice corresponds to the Schwinger gauge $\widetilde{e}_i{}^{\hat{0}} = 0$ and $\widetilde{e}^0{}_{\hat{\imath}} = 0$ for $i = 1, 2, 3$.

For Gödel-type spacetimes, the two frames (8) and (156) are related by the Lorentz transformation

$$\widetilde{e}_{\hat{\alpha}} = \Lambda^{\hat{\beta}}{}_{\hat{\alpha}}\, e_{\hat{\beta}}, \tag{157}$$

where explicitly

$$\Lambda^{\hat{\alpha}}{}_{\hat{\beta}} = \begin{pmatrix} \sqrt{\frac{\sigma+\kappa}{\kappa}} & 0 & \sqrt{\frac{\sigma}{\kappa}} & 0 \\ 0 & 1 & 0 & 0 \\ \sqrt{\frac{\sigma}{\kappa}} & 0 & \sqrt{\frac{\sigma+\kappa}{\kappa}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{158}$$

Interestingly, the transformation with constant matrix elements is global, whereas, in general, only local Lorentz transformations are possible.

The change of a frame on the spacetime affects the fermionic wave function

$$\Psi \quad \longrightarrow \quad \widetilde{\Psi} = L^{-1}\,\Psi \tag{159}$$

via the spinor matrix L that satisfies

$$L^{-1}\gamma^{\hat{\alpha}} L = \Lambda^{\hat{\alpha}}{}_{\hat{\beta}}\,\gamma^{\hat{\beta}}. \tag{160}$$

Using a convenient parametrization with $\cosh\zeta = \sqrt{\frac{\sigma+\kappa}{\kappa}}$ and $\sinh\zeta = \sqrt{\frac{\sigma}{\kappa}}$, we easily derive

$$L = \cosh(\zeta/2)\, I_4 + \sinh(\zeta/2)\, \alpha_{\mathrm{D}}^{\hat{2}} = \begin{bmatrix} \cosh(\zeta/2)\, I_2 & \sinh(\zeta/2)\, \sigma_2 \\ \sinh(\zeta/2)\, \sigma_2 & \cosh(\zeta/2)\, I_2 \end{bmatrix}. \tag{161}$$

The spinor transformation (159) mixes the spin-up and spin-down states ($\mathcal{L}_\pm$) for the left modes (and similarly for the right modes), and an appropriate normalization of the solutions should be fixed for the squares $\widetilde{\Psi}^\dagger\widetilde{\Psi}$ of the transformed wave functions.

8. Dirac Equation in Fermi Frame

Let us next consider the Dirac equation in the quasi-inertial Fermi frame of Section 6. We are interested in the propagation of Dirac particles as described by fundamental observers that are all spatially at rest in the Fermi frame and occupy the limited cylindrical region about the Z axis such that $\Omega|X| = \Omega|Y| \lesssim \varepsilon$. As before, we ignore all terms of order ε^4 and higher. The preferred observers have adapted orthonormal tetrad frames $\varphi^\mu{}_{\hat{\alpha}}$ given in Equations (86)–(87). Let us note that $\varphi_{\mu\,\hat{\alpha}}$ can be written in the (T, X, Y, Z) coordinate system as

$$\varphi_{\mu\,\hat{0}} = (-1 + \tfrac{1}{2}\hat{h}_{00}, \hat{h}_{01}, \hat{h}_{02}, 0)\,, \qquad \varphi_{\mu\,\hat{1}} = (0, 1 + \tfrac{1}{2}\hat{h}_{11}, \hat{h}_{12}, 0)\,, \tag{162}$$

$$\varphi_{\mu\,\hat{2}} = (0, 0, 1 + \tfrac{1}{2}\hat{h}_{22}, 0)\,, \qquad \varphi_{\mu\,\hat{3}} = (0, 0, 0, 1)\,. \tag{163}$$

We employ perturbations beyond Minkowski spacetime in our Fermi frame; hence, in the absence of $\hat{h}_{\mu\nu}$, we have $\varphi^{\mu}{}_{\hat{\alpha}} \to \delta^{\mu}_{\alpha}$. To simplify matters even further, we assume henceforth that the deviation from Minkowski spacetime is only due to the gravitomagnetic potentials $\hat{h}_{01} = -\frac{\kappa}{\sigma}\,\Omega^3 Y(X^2+Y^2)$ and $\hat{h}_{02} = \frac{\kappa}{\sigma}\,\Omega^3 X(X^2+Y^2)$ that give rise to the gravitomagnetic field $\hat{\boldsymbol{B}} = (0,0,\hat{B}_3)$, where $\hat{B}_3 = -2\frac{\kappa}{\sigma}\,\Omega^3(X^2+Y^2)$.

With these assumptions, the spin connection in (102) can be computed using the tetrad system $\varphi_{\mu\,\hat{\alpha}}$ that is adapted to our reference observers and we find

$$\gamma^{\mu}\hat{\Gamma}_{\mu} = \frac{i}{2}\hat{B}_3\begin{bmatrix}\sigma_3 & 0\\ 0 & -\sigma_3\end{bmatrix}. \tag{164}$$

That is, the spin connection is proportional to the gravitomagnetic field of the Gödel-type universe in the Fermi frame under consideration here.

For the sake of simplicity, we assume a solution of the Dirac equation that propagates along the Z axis and is of the form

$$\hat{\Psi} = \hat{\psi}(X,Y)\exp(-i\,\omega\,T + i\,k_3\,Z)\,. \tag{165}$$

Moreover, it is convenient to define

$$\hat{\mathbb{X}} = \begin{pmatrix}\hat{\psi}_1\\ \hat{\psi}_3\end{pmatrix}, \qquad \hat{\mathbb{Y}} = \begin{pmatrix}\hat{\psi}_2\\ \hat{\psi}_4\end{pmatrix}. \tag{166}$$

In this case, Dirac's equation reduces to

$$\Big[\partial_X + i\partial_Y + \frac{\kappa}{\sigma}\,\omega\Omega^3(X^2+Y^2)(X+iY)\Big]\hat{\mathbb{X}} = -\frac{i\kappa}{\sigma}\,\Omega^3(X^2+Y^2)\sigma_1\hat{\mathbb{Y}} + i\begin{bmatrix}k_3 & \omega+m\\ \omega-m & k_3\end{bmatrix}\hat{\mathbb{Y}}, \tag{167}$$

and

$$\Big[\partial_X - i\partial_Y - \frac{\kappa}{\sigma}\,\omega\Omega^3(X^2+Y^2)(X-iY)\Big]\hat{\mathbb{Y}} = \frac{i\kappa}{\sigma}\,\Omega^3(X^2+Y^2)\sigma_1\hat{\mathbb{X}} + i\begin{bmatrix}-k_3 & \omega+m\\ \omega-m & -k_3\end{bmatrix}\hat{\mathbb{X}}. \tag{168}$$

Here, $\partial_X := \partial/\partial X$, etc.; furthermore, we note that

$$(\partial_X \pm i\partial_Y)(X^2+Y^2)^2 = 4(X^2+Y^2)(X \pm iY)\,, \tag{169}$$

$$(\partial_X \pm i\partial_Y)[(X^2+Y^2)(X \mp iY)] = 4(X^2+Y^2)\,. \tag{170}$$

In the absence of the gravitational perturbation, the positive-frequency plane wave solutions of the free Dirac equation propagating in the Z direction are given by

$$\hat{w}^{\pm}\,e^{-i\,\omega\,T + i\,k_3\,Z}, \tag{171}$$

where the spin of the Dirac particle is either parallel ($\hat{w}^{+}$) or antiparallel ($\hat{w}^{-}$) to the Z direction; that is,

$$\hat{w}^{+} = N^{\Uparrow}\begin{bmatrix}1\\0\\ \varrho\\0\end{bmatrix}, \qquad \hat{w}^{-} = N^{\Downarrow}\begin{bmatrix}0\\1\\0\\-\varrho\end{bmatrix}. \tag{172}$$

Here, $N^{\Uparrow}$ and $N^{\Downarrow}$ are positive normalization constants, $\omega = (m^2+k_3^2)^{1/2}$, and

$$\varrho := \frac{k_3}{\omega+m} = \frac{\omega-m}{k_3}\,. \tag{173}$$

With these background states, we solve Equations (167) and (168) to the linear order in the gravitomagnetic perturbation and obtain, after some algebra,

$$\hat{\Psi}^{+} = N^{\Uparrow} \begin{bmatrix} \exp[-\frac{3\kappa}{8\sigma}\omega\Omega^3(X^2+Y^2)^2] \\ \frac{i\kappa}{4\sigma}\Omega^3\varrho(X^2+Y^2)(X+iY) \\ \varrho\exp[-\frac{3\kappa}{8\sigma}\omega\Omega^3(X^2+Y^2)^2] \\ \frac{i\kappa}{4\sigma}\Omega^3(X^2+Y^2)(X+iY) \end{bmatrix} e^{-i\,\omega\,T+i\,k_3\,Z} \tag{174}$$

$$\hat{\Psi}^{-} = N^{\Downarrow} \begin{bmatrix} \frac{i\kappa}{4\sigma}\Omega^3\varrho(X^2+Y^2)(X-iY) \\ \exp[\frac{3\kappa}{8\sigma}\omega\Omega^3(X^2+Y^2)^2] \\ -\frac{i\kappa}{4\sigma}\Omega^3(X^2+Y^2)(X-iY) \\ -\varrho\exp[\frac{3\kappa}{8\sigma}\omega\Omega^3(X^2+Y^2)^2] \end{bmatrix} e^{-i\,\omega\,T+i\,k_3\,Z}\,. \tag{175}$$

These solutions of Dirac's equation exhibit the coupling of spin with the gravitomagnetic field of a Gödel-type universe and may be compared and contrasted with the results of Appendix C for the propagation of circularly polarized electromagnetic waves along the Z axis in the Fermi frame.

We should note that fermions in Gödel-type universes have been the subject of a number of previous studies; see, for instance, refs. [43–48] and the references cited therein.

9. Discussion

Spin–gravity coupling represents a physically important subject matter in view of the basic nature of the intrinsic spin of particles and the universality of the gravitational interaction. We have investigated in detail the coupling of intrinsic spin with the gravitomagnetic fields of a three-parameter class of Gödel-type spacetimes. These stationary and homogeneous rotating universes are characterized by the set of constant parameters (κ, σ, μ); for $\kappa < 0$; there are closed timelike curves (CTCs) in spacetime, while, for $\kappa \geq 0$, the CTCs are absent. For $(\kappa, \sigma, \mu) \to (-1, 2, \sqrt{2}\,\Omega)$, we recover Gödel's rotating universe model, where $\Omega > 0$ is the frequency of rotation. Regarding the background Gödel-type spacetimes, we have studied Dirac's equation and worked out its solutions; furthermore, we have extended our results to exact Fermi normal coordinate systems in these universes. We have shown that the Stern–Gerlach force due to the coupling of intrinsic spin with the gravitomagnetic field of a Gödel-type spacetime is in agreement with the correspondence limit with the classical Mathisson spin-curvature force. This is a nonlinear generalization of previous work that focused on linearized general relativity [20]. Our main results turn out to be independent of the possible causality difficulties of the Gödel-type spacetimes.

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Appendix A. Alternative Solution of Equation (109)

The purpose of this appendix is to present a different approach to the solution of Equation (109).

We can write Equation (109) in the form

$$\xi\,\frac{d(\mathcal{U}\psi)}{d\xi} = \mathcal{U}\,\mathbb{M}\,\mathcal{U}^{-1}(\mathcal{U}\psi)\,, \tag{A1}$$

where $\mathcal{U}$ is a constant unitary matrix given by

$$\mathcal{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} I_2 & -I_2 \\ I_2 & I_2 \end{bmatrix}. \tag{A2}$$

Under this similarity transformation, we have

$$\mathcal{U}\,\gamma^{\hat{0}}\,\mathcal{U}^{\dagger} = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix} = \gamma_5\,, \qquad \mathcal{U}\,\gamma^{\hat{i}}\,\mathcal{U}^{\dagger} = \gamma^{\hat{i}}\,. \tag{A3}$$

That is, the standard representation of the Dirac matrices is thus transformed to the chiral (Weyl) representation. Employing this representation, we find

$$\begin{aligned} \mathcal{U}\,\mathbb{M}\,\mathcal{U}^{-1} = &\begin{bmatrix} -\bar{\mathcal{A}}_+ & i\bar{\mathcal{B}}_+ - i\gamma & 0 & 0 \\ i\bar{\mathcal{B}}_- + i\gamma & \bar{\mathcal{A}}_- & 0 & 0 \\ 0 & 0 & -\bar{\mathcal{A}}_+ & -i\bar{\mathcal{B}}_+ - i\gamma \\ 0 & 0 & -i\bar{\mathcal{B}}_- + i\gamma & \bar{\mathcal{A}}_- \end{bmatrix} \\ &- \frac{im}{2\Omega}\sqrt{\frac{\sigma}{\sigma+\kappa}}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \end{aligned} \tag{A4}$$

where $\bar{\mathcal{A}}_\pm$ and $\bar{\mathcal{B}}_\pm$ are given by Equation (111). By expressing $\mathcal{U}\psi$ in the form

$$\mathcal{U}\psi = \sqrt{2}\begin{bmatrix} \mathcal{R} \\ \mathcal{L} \end{bmatrix}, \qquad \mathcal{R} = \begin{bmatrix} \mathcal{R}_+ \\ \mathcal{R}_- \end{bmatrix}, \qquad \mathcal{L} = \begin{bmatrix} \mathcal{L}_+ \\ \mathcal{L}_- \end{bmatrix}, \tag{A5}$$

where $\mathcal{R}$ and $\mathcal{L}$ are now right-handed and left-handed two-component Weyl spinors, we recover the system of Equations (118)–(121). The rest of the analysis would follow the treatment presented in Section 7.

Appendix B. Scalar Waves in the Gödel-Type Universe

Consider first a scalar field ϕ of inertial mass m propagating on the background Gödel-type spacetime (1). The wave equation is

$$g^{\mu\nu}\phi_{;\mu\nu} - \frac{m^2c^2}{\hbar^2}\phi = 0\,, \tag{A6}$$

where $\hbar/(mc)$ is the Compton wavelength of the particle. The back reaction is of the second order in the perturbation and can be neglected. The scalar wave equation can be written as

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\,g^{\mu\nu}\frac{\partial\phi}{\partial x^\nu}\right) - \frac{m^2c^2}{\hbar^2}\phi = 0\,, \tag{A7}$$

where, for metric (1), $\sqrt{-g} = e^{\mu x}\sqrt{\sigma+\kappa}$. Moreover, ∂_t, ∂_y, and ∂_z are Killing vector fields; therefore, we assume that

$$\phi(x) = e^{-i\omega t + ik_2 y + ik_3 z}\,\bar{\phi}(\xi)\,, \qquad \xi := e^{-\mu x}\,, \tag{A8}$$

where ξ increases from 0 to ∞ as the x coordinate decreases from $+\infty$ to $-\infty$. In terms of the new radial variable ξ, the equation for $\bar{\phi}$ reduces to

$$\frac{d^2\bar{\phi}}{d\xi^2} - \left[\alpha_s^2 + \frac{\beta_s}{\xi} + \frac{\zeta_s(\zeta_s+1)}{\xi^2}\right]\bar{\phi} = 0\,, \tag{A9}$$

where

$$\alpha_s = \frac{k_2}{\mu\sqrt{\sigma+\kappa}}, \quad \beta_s = \frac{2\omega\, k_2}{c\,\mu^2}\,\frac{\sqrt{\sigma}}{\sigma+\kappa}, \quad \zeta_s(\zeta_s+1) = \frac{1}{\mu^2}\Big(-\frac{\omega^2}{c^2}\,\frac{\kappa}{\sigma+\kappa} + k_3^2 + \frac{m^2c^2}{\hbar^2}\Big). \tag{A10}$$

Let us assume $\zeta_s > 0$ and note that for $k_2 = 0$, Equation (A9) for $\bar{\phi}$ has solutions of the form $\xi^{-\zeta_s}$ and ξ^{ζ_s+1} that diverge at $\xi = 0$ and $\xi = \infty$, respectively. However, the scalar perturbation must be finite everywhere; therefore, waves cannot freely propagate parallel or antiparallel to the axis of rotation of the Gödel-type spacetime. Next, for $k_2 \neq 0$, we introduce a new variable, $\bar{\xi} := \frac{|k_2|\,\sqrt{\sigma}}{\Omega(\sigma+\kappa)}\,\xi$, in terms of which Equation (A9) takes the form of Whittaker's equation [42]:

$$\frac{d^2\bar{\phi}}{d\bar{\xi}^2} + \left[-\frac{1}{4} + \frac{\bar{\kappa}_s}{\bar{\xi}} + \frac{\frac{1}{4}-\bar{\mu}_s^2}{\bar{\xi}^2}\right]\bar{\phi} = 0\,, \tag{A11}$$

where

$$\bar{\kappa}_s = -\frac{\omega}{2\Omega}\,\frac{k_2}{|k_2|}\,\frac{\sigma}{\sigma+\kappa}\,, \qquad \bar{\mu}_s = \pm(\zeta_s + \tfrac{1}{2})\,. \tag{A12}$$

In terms of the confluent hypergeometric functions, bounded solutions of this equation can be expressed up to proportionality constants by the following:

$$\exp(-\tfrac{1}{2}\bar{\xi})\,\bar{\xi}^{\zeta_s+1}\,{}_1F_1(-n, 2\zeta_s+2; \bar{\xi})\,, \qquad n = 0, 1, 2, \dots\,. \tag{A13}$$

Here, $\zeta_s > 0$, $\bar{\mu}_s = \zeta_s + 1/2$, and

$$\zeta_s + 1 + \frac{\omega}{2\Omega}\,\frac{k_2}{|k_2|}\,\frac{\sigma}{\sigma+\kappa} = -\,n\,, \qquad \omega = \pm\,2\Omega\,(n+\zeta_s+1)\,\frac{\sigma+\kappa}{\sigma}\,, \tag{A14}$$

for $k_2 < 0$ (upper plus sign) or $k_2 > 0$ (lower minus sign), respectively. Negative frequency in the case of $k_2 > 0$ indicates that waves traveling forward in time move backward along the y direction. Finally, we note that only certain frequencies are allowed for the scalar waves; for instance, for $k_2 < 0$, we have $\omega_n = 2\Omega\,(n+\zeta_s+1)(\sigma+\kappa)/\sigma$. That is,

$$\omega_n^{\pm} = (2n+1)\Omega \pm \left[\Big(-4n(n+1)\,\frac{\kappa}{\sigma} + 1\Big)\Omega^2 + k_3^2 + \frac{m^2c^2}{\hbar^2}\right]^{1/2}, \tag{A15}$$

where $\omega_n^+ > 0$ for all n by definition, while $\omega_n^- > 0$ for $n = 1, 2, 3, \dots$ only if

$$\omega_n^+\,\omega_n^- = 4n(n+1)\,\frac{\sigma+\kappa}{\sigma}\,\Omega^2 - k_3^2 - \frac{m^2c^2}{\hbar^2} > 0\,. \tag{A16}$$

For further work on the scalar perturbations of the Gödel-type universe and its extensions, see [49–51].

Appendix C. Electromagnetic Waves in the Gödel-Type Universe

The propagation of electromagnetic radiation in the Gödel universe was originally investigated in the search for the coupling of photon helicity with the rotation of matter [52]. In Gödel-type spacetimes, Maxwell's equations can be reduced to an equation of the form of Equation (A9), where, instead of the quantities in Equation (A10), we find [53]

$$\alpha_s \to \alpha_{em} = \frac{K_2^{\pm}}{\mu\sqrt{\sigma+\kappa}}\,, \qquad \beta_s \to \beta_{em} = \frac{2\omega\,K_2^{\pm}}{c\,\mu^2}\,\frac{\sqrt{\sigma}}{\sigma+\kappa}\,, \tag{A17}$$

and $\zeta_s \to \zeta_{em}$, where

$$\zeta_{em}(\zeta_{em}+1) = \frac{1}{\mu^2}\Big(-\frac{\omega^2}{c^2}\,\frac{\kappa}{\sigma+\kappa} + (K_3^{\pm})^2 \mp 2\Omega\,K_3^{\pm}\Big)\,, \tag{A18}$$

since photon is massless ($m = 0$). The helicity coupling evident in Equation (A18) is consistent with the spin–vorticity–gravity coupling described in Section 4. That is, based on the results of Section 4, we would expect the corresponding Hamiltonian for a photon to be proportional to $\pm\hbar K_3^{\pm}\Omega/\omega$ so that, in terms of frequency, we would have $\pm K_3^{\pm}\Omega/\omega$. The effect should disappear in the case of a null geodesic consistent with the eikonal limit $\omega \to \infty$. For further extensions and generalizations to Gödel-type universes, see [53–59].

EM Waves in the Fermi Frame

We consider the propagation of electromagnetic radiation on the background quasi-inertial Fermi normal coordinate system. In terms of the Faraday tensor $F_{\mu\nu}$, the source-free Maxwell equations can be expressed as

$$F_{[\mu\nu,\rho]} = 0\,, \qquad (\sqrt{-g}\,F^{\mu\nu})_{,\nu} = 0\,. \tag{A19}$$

Using the same approach as in [52], we replace the gravitational field by a hypothetical optical medium that occupies Euclidean space with Cartesian Fermi coordinates (X, Y, Z). The electromagnetic field Equation (A19) reduces to the traditional form of Maxwell's equations in a medium with the following decompositions:

$$F_{\mu\nu} \to (\tilde{\boldsymbol{E}}, \tilde{\boldsymbol{B}})\,, \qquad \sqrt{-g}\,F^{\mu\nu} \to (-\tilde{\boldsymbol{D}}, \tilde{\boldsymbol{H}})\,. \tag{A20}$$

That is, $F_{0i} = -\tilde{E}_i$ and $F_{ij} = \epsilon_{ijk}\,\tilde{B}_k$; similarly, $\sqrt{-g}\,F^{0i} = \tilde{D}_i$ and $\sqrt{-g}\,F^{ij} = \epsilon_{ijk}\,\tilde{H}_k$. Here, ϵ_{ijk} is the totally antisymmetric symbol with $\epsilon_{123} = 1$. The corresponding optical medium turns out to be gyrotropic with constitutive relations [60–64]:

$$\tilde{D}_i = \hat{\epsilon}_{ij}\,\tilde{E}_j - (\hat{\boldsymbol{G}} \times \tilde{\boldsymbol{H}})_i\,, \qquad \tilde{B}_i = \hat{\mu}_{ij}\,\tilde{H}_j + (\hat{\boldsymbol{G}} \times \tilde{\boldsymbol{E}})_i\,, \tag{A21}$$

where the characteristics of the medium are conformally invariant and are given by

$$\hat{\epsilon}_{ij} = \hat{\mu}_{ij} = -\sqrt{-\hat{g}}\,\frac{\hat{g}^{ij}}{\hat{g}_{00}}\,, \qquad \hat{G}_i = -\frac{\hat{g}_{0i}}{\hat{g}_{00}}\,. \tag{A22}$$

Expressing electromagnetic fields in the standard complex form and introducing the Riemann–Silberstein vectors,

$$\tilde{\boldsymbol{F}}^{\pm} = \tilde{\boldsymbol{E}} \pm i\,\tilde{\boldsymbol{H}}\,, \qquad \tilde{\boldsymbol{S}}^{\pm} = \tilde{\boldsymbol{D}} \pm i\,\tilde{\boldsymbol{B}}\,, \tag{A23}$$

the wave propagation equation can be expressed as the Dirac equation for photons in the gravitational field. That is,

$$\boldsymbol{\nabla} \times \tilde{\boldsymbol{F}}^{\pm} = \pm\, i\,\frac{\partial \tilde{\boldsymbol{S}}^{\pm}}{\partial t}\,, \qquad \boldsymbol{\nabla} \cdot \tilde{\boldsymbol{S}}^{\pm} = 0\,, \tag{A24}$$

where

$$\tilde{S}_p^{\pm} = \hat{\epsilon}_{pq}\,\tilde{F}_q^{\pm} \pm i\,(\hat{\boldsymbol{G}} \times \tilde{\boldsymbol{F}}^{\pm})_p\,. \tag{A25}$$

The Dirac-type equation implies $\partial_t(\boldsymbol{\nabla} \cdot \tilde{\boldsymbol{S}}^{\pm}) = 0$; therefore, if $\boldsymbol{\nabla} \cdot \tilde{\boldsymbol{S}}^{\pm} = 0$ initially, then it is valid for all time.

To interpret the physical meaning of these results, it proves useful to consider plane electromagnetic waves of frequency ω propagating along the z axis in a global inertial frame with coordinates $x^{\mu} = (t, \boldsymbol{x})$ in Minkowski spacetime. Maxwell's equations are linear; therefore, we can use complex electric and magnetic fields and use the convention that only the real parts correspond to measurable quantities. The waves can have two independent orthogonal linear polarization states along the $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ directions, where $\hat{\boldsymbol{x}}$ is a unit vector along the x axis, etc. The circular polarization states are constructed from the linear polarization states via superposition; in this case, the electric ($\boldsymbol{e}$) and magnetic ($\boldsymbol{b}$) fields can be expressed as

$$\boldsymbol{e}_\pm = \frac{1}{2} a_\pm \, (\hat{\boldsymbol{x}} \pm i \, \hat{\boldsymbol{y}}) \, e^{-i\omega \, (t-z)} \,, \qquad \boldsymbol{b}_\pm = \mp \frac{i}{2} \, a_\pm \, (\hat{\boldsymbol{x}} \pm i \, \hat{\boldsymbol{y}}) \, e^{-i\omega \, (t-z)} \,, \tag{A26}$$

where a_+ and a_- are constant complex amplitudes. Here, the upper (lower) sign represents waves in which the orthogonal electric and magnetic fields rotate in the positive (negative) sense about the direction of the wave motion. In the case of a photon with positive (negative) circular polarization, the photon has positive (negative) helicity, namely, its spin is $+\hbar$ ($-\hbar$) along its direction of propagation. The Riemann–Silberstein vectors have interesting behaviors for helicity states of the photon; in fact, for *positive-helicity* radiation,

$$\boldsymbol{e}_+ + i \, \boldsymbol{b}_+ = a_+ \, (\hat{\boldsymbol{x}} + i \, \hat{\boldsymbol{y}}) \, e^{-i\omega \, (t-z)} \,, \qquad \boldsymbol{e}_+ - i \, \boldsymbol{b}_+ = 0 \,, \tag{A27}$$

while for radiation with *negative helicity*,

$$\boldsymbol{e}_- + i \, \boldsymbol{b}_- = 0 \,, \qquad \boldsymbol{e}_- - i \, \boldsymbol{b}_- = a_- \, (\hat{\boldsymbol{x}} - i \, \hat{\boldsymbol{y}}) \, e^{-i\omega \, (t-z)} \,. \tag{A28}$$

Hence, $\boldsymbol{e} + i \, \boldsymbol{b}$ ($\boldsymbol{e} - i \, \boldsymbol{b}$) represents in essence an electromagnetic wave with positive (negative) helicity. It is important to note that Equations (A24) and (A25) that represent the propagation of the electromagnetic test fields in a gravitational field completely decouple for different helicity states.

Imagine the propagation of electromagnetic waves with definite helicity along the Z axis in the Fermi normal coordinate system in the Gödel-type spacetime. The universe rotates in the negative sense about the Z axis. We confine our considerations to the cylindrical region near the rotation axis where the perturbation analysis contained in Equations (83)–(85) is valid. To simplify matters, we take into account only the gravitomagnetic potentials $\hat{h}_{01}$ and $\hat{h}_{02}$, and we ignore the other potentials; therefore, in Equation (A22), we have

$$\hat{\epsilon}_{ij} = \hat{\mu}_{ij} \approx 1 \,, \qquad \hat{\boldsymbol{G}} \approx -\frac{\kappa}{\sigma} \Omega^3 (X^2 + Y^2)(Y, -X, 0) \,. \tag{A29}$$

It is straightforward to show that, in this case, the field Equations (A24) and (A25) have the solution

$$\tilde{F}_1^{\pm} = \hat{a}_\pm \exp[-i\omega(T - Z) \mp \frac{\kappa}{4\sigma} \omega \Omega^3 \, (X^2 + Y^2)^2] \,, \tag{A30}$$

$$\tilde{F}_2^{\pm} = \pm i \hat{a}_\pm \exp[-i\omega(T - Z) \mp \frac{\kappa}{4\sigma} \omega \Omega^3 \, (X^2 + Y^2)^2] \,, \tag{A31}$$

and $\tilde{F}_3^{\pm} = 0$. Here, $\hat{a}_+$ and $\hat{a}_-$ are constant amplitudes for the positive and negative helicity waves in the Fermi frame, respectively. If the wave propagates along the axis of rotation (i.e., the $-Z$ direction), then, in Equations (A30) and (A31), we have $Z \to -Z$ and $\pm \to \mp$ in the exponents of these equations, as well as in the coefficient of the latter equation. For $\Omega = 0$, the Fermi frame reduces to a global inertial frame in Minkowski spacetime and we recover waves of the form given in Equations (A26)–(A28).

The helicity–gravitomagnetic field coupling is evident in these results and corresponds to Equations (89) and (90) of Section 6; indeed, the form of this coupling is reminiscent of the helicity–twist coupling studied in [65].

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