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Solutions of Fractional Differential Inclusions and Stationary Points of Intuitionistic Fuzzy-Set-Valued Maps

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Abstract: One of the tools for building new fixed-point results is the use of symmetry in the distance functions. The symmetric property of metrics is particularly useful in constructing contractive inequalities for analyzing different models of practical consequences. A lot of important invariant point results of crisp mappings have been improved by using the symmetry of metrics. However, more than a handful of fixed-point theorems in symmetric spaces are yet to be investigated in fuzzy versions. In accordance with the aforementioned orientation, the idea of Presic-type intuitionistic fuzzy stationary point results is introduced in this study within a space endowed with a symmetrical structure. The stability of intuitionistic fuzzy fixed-point problems and the associated new concepts are proposed herein to complement their corresponding concepts related to multi-valued and single-valued mappings. In the instance where the intuitionistic fuzzy-set-valued map is reduced to its crisp counterparts, our results complement and generalize a few well-known fixed-point theorems with symmetric structure, including the main results of Banach, Ciric, Presic, Rhoades, and some others in the comparable literature. A significant number of consequences of our results in the set-up of fuzzy-set- and crisp-set-valued as well as point-to-point-valued mappings are emphasized and discussed. One of our findings is utilized to assess situations from the perspective of an application for the existence of solutions to non-convex fractional differential inclusions involving Caputo fractional derivatives with nonlocal boundary conditions. Some nontrivial examples are constructed to support the assertions and usability of our main ideas.

Keywords: intuitionistic fuzzy set; intuitionistic fuzzy fixed point; metric space; multi-valued mapping; stability; Caputo fractional derivative; differential inclusion; symmetry

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1. Introduction

Fixed point (FP) results occur in almost all areas of mathematical disciplines, including fuzzy set theory and mathematical logic. In relevant contexts concerning spaces with metric space structure, an appropriate kind of completeness is usually assumed. This requirement is, with regards to FP theorems, common and fundamental. A typical FP statement is of the form $\vartheta(a) = a$, where ϑ is a self-mapping on a non-empty set W. This issue may be rephrased as $\rho(a) = 0$, where $\rho(a) = a - \vartheta(a)$. Even though this problem statement is straightforward, finding a solution to it can be highly challenging and occasionally impossible. The first announcement of a solution to this problem came from Banach [1] under certain appropriate circumstances, i.e., when ϑ is a contraction and W is complete. As a matter of fact, the contraction principle is a rephrasing of the sequential approximation techniques adopted by Cauchy, Liouville, Picard, Lipschitz, and other early mathematicians. Presic [2] gave one of the well-known generalizations of the Banach FP theorem, which has deep applications in the study of the equilibrium of nonlinear difference equations that arise in dynamic systems.

One of the difficulties in the mathematical simulation of real-world problems is the uncertainty that results from our inability to accurately classify events. It has been observed that classical mathematics struggles to deal with errors that are due to uncertainty. In an effort to overcome the aforementioned challenges, Zadeh [3] introduced the idea of fuzzy sets in 1965 as one of the uncertainty approaches and tools to build mathematical models that are compatible with real-world issues. In the meantime, other frameworks have updated the basic concepts of fuzzy sets. In 1981, Heilpern [4] established fuzzy versions of Nadler's [5] FP theorem for fuzzy contraction mappings by introducing a class of fuzzy-set-valued maps. Following that, other authors investigated various criteria under which fuzzy-set-valued maps possess invariant points. For some related results in this lane, we refer to [6–9].

Following Zadeh [3], and as a further expansion of fuzzy set theory, Atanassov introduced the idea of intuitionistic fuzzy sets (IFS). IFS offers a practical mathematical framework to deal with inaccuracy and hesitation resulting from incomplete knowledge. Because IFS measures membership and non-membership degrees, as well as the degree of non-determinacy, it is more applicable than fuzzy sets. As a result, it has found widespread use in fields such as decision-making issues, image processing, drug selection, and medical diagnostics. In the meantime, research in IFS has been expanding at a geometric rate, and various outcomes have been attained in a variety of areas. Azam et al. [10,11] recently unveiled novel methods for analyzing common and coincidence points of intuitionistic fuzzy-set-valued maps.

In accordance with our surveyed literature, invariant point results of Presic-type involving intuitionistic fuzzy mappings (IFMs) are yet to be examined, not minding the usefulness of the latter notion. Following the various progressions described above, we begin this work by introducing the notion of Presic-type intuitionistic fuzzy FP results in the context of metric spaces (MSs). To highlight the connection of the ideas herein with multi-valued and single-valued mappings, the stability of intuitionistic fuzzy FP problems is proposed. The former concepts have hitherto been considered only for crisp mappings. As a result, our findings enhance some crucial metric FP theorems when the intuitionistic fuzzy-set-valued map is reduced to its crisp equivalents, some of which are the results of Abbas et al. [12], Banach [1], Ciric and Presic [13], Heilpern [4], Presic [2], Rhoades [14], and some others in the related literature. As an application, one of our findings is used to explore the requirements for the existence of solutions to non-convex fractional differential inclusions using Caputo fractional derivatives of arbitrary order with nonlocal boundary conditions.

2. Preliminaries

Hereafter, the sets $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{N}$ represent the set of reals, positive reals, and the natural numbers, respectively.

Definition 1 ([15])**.** *Let (W, σ) be an MS. A mapping $\vartheta : W \longrightarrow W$ is weakly contractive if, for all $a, b \in W$,*

$$\sigma(\vartheta(a), \vartheta(b)) \leq \sigma(a, b) - \varphi(\sigma(a, b)),$$

where $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous and non-decreasing function with $\varphi(0) = 0$ and $\varphi(t) \longrightarrow \infty$ as $t \longrightarrow \infty$.

Alber and Guerre-Delabriere [15] showed that every weakly contractive mapping on a Hilbert space corresponds to a Picard operator. Rhoades [14] demonstrated, on a complete MS, the validity of the ideas of [15]. Dutta et al. [16] enlarged the ideas of weak contractive results and generated an FP result that improved the key discoveries in [14,15] and some results therefrom.

Definition 2 ([2])**.** *Let $l \geq 1$ be a non-negative number. A point $u \in W$ is called an FP of $\vartheta : W^l \longrightarrow W$ if $\vartheta(u, \cdots, u) = u$.*

Consider the equation:

$$a_{\check{p}+l} = \vartheta(a_{\check{p}}, \cdots, a_{\check{p}+l-1}),\ \check{p} \in \mathbb{N}, \tag{1}$$

containing starting values $a_1, \cdots, a_l \in W$. Equation (1) develops into an FP problem in the sense that $u \in W$ is a solution of (1) only if and when u is an FP of $\rho : W \longrightarrow W$ specified as

$$\rho(u) = \vartheta(u, \cdots, u), \text{ for all } u \in W.$$

Presic [2] established the following highly noteworthy result in view of (1):

Theorem 1 ([2])**.** *Let (W, σ) be a complete MS, l a non-negative number, and $\vartheta : W^l \longrightarrow W$ a mapping meeting the criterion*

$$\sigma(\vartheta(a_1, \cdots, a_l), \vartheta(a_2, \cdots, a_{l+1})) \leq \check{\lambda} \max\{\sigma(a_1, a_2), \cdots, \sigma(a_l, a_{l+1})\},$$

for all $a_1, \cdots, a_l \in W$, where $\check{\lambda} \in (0,1)$. Then, we can find $u \in W$ with $\vartheta(u, \cdots, u) = u$. For any random points, in addition, $a_1, \cdots, a_l \in W$, the sequence specified by (1) converges to u and

$$\lim_{\check{p} \longrightarrow \infty} a_{\check{p}} = \vartheta(\lim_{\check{p} \longrightarrow \infty} a_{\check{p}}, \cdots, \lim_{\check{p} \longrightarrow \infty} a_{\check{p}}).$$

We see that by setting $l = 1$, Theorem 1 has the Banach FP theorem as its special result. Theorem 1 has drawn a lot of attention due to its importance in the study of the global asymptotic stability of equilibrium for the FP problem (1).

Just recently, Abbas et al. [12] studied a certain family of operators satisfying Presic-type contractive criteria, the convergence of a generalized weak Presic-type l-step technique was investigated as follows:

Theorem 2 ([12])**.** *Consider a complete MS (W, σ). If a mapping $\vartheta : W^l \longrightarrow W$, for a positive l satisfies*

$$\begin{aligned} \sigma(\vartheta(a_1, \cdots, a_l), \vartheta(a_2, \cdots, a_{l+1})) \leq\ & \max\{d(a_i, a_{i+1}) : 1 \leq i \leq l\} \\ & - \varphi(\max\{\sigma(a_i, a_{i+1}) : 1 \leq i \leq l\}), \end{aligned}$$

for all $(a_1, \cdots, a_{l+1}) \in W^{l+1}$, where $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a lower semi-continuous function with $\varphi(t) = 0$ if and only if $t = 0$, then, for random points $a_1, \cdots, a_l \in W$, the sequence specified by (1) converges to $u \in W$ and $\vartheta(u, \cdots, u) = u$. Moreover, if

$$\sigma(\vartheta(a, \cdots, a), \vartheta(b, \cdots, b)) \leq \sigma(a, b) - \varphi(\sigma(a, b))$$

holds for all $a, b \in W$ with $a \neq b$, then ϑ has a unique FP in W.

For a few publications involving Presic-type ideas, we refer to [13,17,18] and some citations therein.

Let the set of all nonempty compact subsets of W be denoted by $\mathcal{K}(W)$, where (W, σ) is an MS. For $\widetilde{A}, \widetilde{B} \in \mathcal{K}(W)$, the function $\underline{H} : \mathcal{K}(W) \times \mathcal{K}(W) \longrightarrow \mathbb{R}_+$ specified by

$$\underline{H}(\widetilde{A}, \widetilde{B}) = \max\left\{\sup_{a \in \widetilde{A}} \sigma(a, \widetilde{B}), \sup_{a \in \widetilde{B}} \sigma(a, \widetilde{A})\right\},$$

known as the Hausdorff–Pompeiu metric $\mathcal{K}(W)$ induced by the metric σ, where

$$\sigma(a, \widetilde{A}) = \inf_{b \in \widetilde{A}} \sigma(a, b).$$

The following lemma due to Nadler [5] is useful in establishing our results.

Lemma 1. *Let (W,σ) be an MS and $\widetilde{A},\widetilde{B}\in\mathcal{K}(W)$. Then, for each $a\in\widetilde{A}$, we can find $b\in\widetilde{B}$ with*

$$\sigma(a,b)\leq\underline{H}(\widetilde{A},\widetilde{B}).$$

In the paragraphs that follow, we review particular fuzzy set notions and intuitionistic fuzzy set concepts that are necessary below. These ideas follow from [3,19,20]. Let the universal set W be given. A function with the domain W and values in the range $[0,1]=I$ is known as a fuzzy set in W. The function value $\widetilde{A}(a)$ is referred to as the grade of membership of a in $\widetilde{A}$. If $\widetilde{A}$ is a fuzzy set in W, then $[\widetilde{A}]_\alpha$ represents the α-level set and is specified as

$$[\widetilde{A}]_\alpha=\begin{cases}\overline{\{a\in W:\widetilde{A}(a)>0\}}, & \text{if } \alpha=0\\ \{a\in W:\widetilde{A}(a)\geq\alpha\}, & \text{if } \alpha\in(0,1].\end{cases}$$

where, by $\overline{M}$, we mean the closure of the crisp set M. We denote the family of fuzzy sets in W by I^W.

A fuzzy set $\widetilde{A}$ in an MS W is considered an approximate quantity provided $[\widetilde{A}]_\alpha$ is compact and convex in W and $\sup_{a\in W}\widetilde{A}(a)=1$. We indicate the totality of all approximations in W by $W(W)$. If we can find an $\alpha\in[0,1]$ with $[\widetilde{A}]_\alpha,[\widetilde{B}]_\alpha\in\mathcal{K}(W)$, then we define

$$D_\alpha(\widetilde{A},\widetilde{B})=\underline{H}([\widetilde{A}]_\alpha,[\widetilde{B}]_\alpha).$$

$$\sigma_\infty(\widetilde{A},\widetilde{B})=\sup_\alpha D_\alpha(\widetilde{A},\widetilde{B}).$$

Definition 3. *Let W be a nonempty set. Then, an IFS Ψ in W is an ordered triple set that is provided by*

$$\Psi=\{\langle\jmath,\mu_\Psi(\jmath),\nu_\Psi(\jmath)\rangle:\jmath\in W\},$$

where $\mu_\Psi:W\longrightarrow[0,1]$ and $\nu_\Psi:W\longrightarrow[0,1]$ define the degrees of membership and non-membership, respectively, of $\jmath$ in W and satisfy $0\leq\mu_\Psi+\nu_\Psi\leq1$, for each $\jmath\in W$. Moreover, the degree of non-determinacy (or hesitancy) of $\jmath\in W$ is specified as

$$h_\Psi(\jmath)=1-\mu_\Psi(\jmath)-\nu_\Psi(\jmath).$$

In particular, if $h_\Psi(\jmath)=0$ for all $\jmath\in W$, then an IFS reduces to an ordinary fuzzy set.

The symbol $(IFS)^W$ represents the collection of all intuitionistic fuzzy sets.

Definition 4 ([21])**.** *Let Ψ be an intuitionistic fuzzy set in W. Then, the α-level set of Ψ is a crisp subset of W denoted by $[\Psi]_\alpha$ and is specified as*

$$[\Psi]_\alpha=\{\jmath\in W:\mu_\Psi(\jmath)\geq\alpha \text{ and } \nu_\Psi(\jmath)\leq1-\alpha\},\ \text{if } \alpha\in[0,1].$$

Definition 5 ([22])**.** *Let $L=\{(\alpha,\beta):\alpha+\beta\leq1,\ (\alpha,\beta)\in(0,1]\times[0,1)\}$ and Ψ be an intuitionistic fuzzy set in W. Then the (α,β)-level set of Ψ is specified as*

$$[\Psi]_{(\alpha,\beta)}=\{\jmath\in W:\mu_\Psi(\jmath)\geq\alpha \text{ and } \nu_\Psi(\jmath)\leq\beta\}.$$

Example 1. *Let $W=\{\jmath_1,\jmath_2,\jmath_3,\jmath_4,\jmath_5\}$ and Ψ be an IFS in W specified by*

$$\Psi=\{(\jmath_1,0.6,0.2),(\jmath_2,0.5,0.4),(\jmath_3,0.1,0.7),(\jmath_4,0.3,0.5),(\jmath_5,0.4,0.3)\}.$$

Then, the (α,β)-level sets of Ψ are given by

$[\Psi]_{(0.4,0.3)}=\{\jmath_1,\jmath_5\}$.
$[\Psi]_{(0.1,0.7)}=\{\jmath_1,\jmath_2,\jmath_3,\jmath_4,\jmath_5\}$.
$[\Psi]_{(0.3,0.5)}=\{\jmath_1,\jmath_2,\jmath_4,\jmath_5\}$.

Definition 6 ([23]). *Let W be a nonempty set. A mapping $\Theta = \langle \mu_\Theta, \nu_\Theta \rangle : W \longrightarrow (IFS)^W$ is called an intuitionistic fuzzy-set-valued map. A point $u \in W$ is called an intuitionistic fuzzy FP of Θ if there exists $(\alpha, \beta) \in (0,1] \times [0,1)$ with $u \in [\Theta u]_{(\alpha,\beta)}$.*

Definition 7 ([11]). *An IFS Ψ in an MS Ψ is called an approximate quantity if and only if $[\Psi]_{(\alpha,\beta)}$ is compact and convex in Ψ for each $(\alpha, \beta) \in (0,1] \times [0,1)$ with*

$$\sup_{\jmath \in W} \mu_\Psi(\jmath) = 1 \text{ and } \inf_{\jmath \in W} \nu_\Psi(\jmath) = 0.$$

Remark 1. *Any crisp set M can be represented as an intuitionistic fuzzy set based on its intuitionistic characteristic function $\mathfrak{M} = \langle \digamma_M, \urcorner_M \rangle$ specified as*

$$\digamma_M(\jmath) = \begin{cases} 1, & \text{if } \jmath \in M \\ 0, & \text{if } \jmath \notin M, \end{cases} \quad \urcorner_M(\jmath) = \begin{cases} 0, & \text{if } \jmath \in M \\ 1, & \text{if } \jmath \notin M. \end{cases}$$

Consistent with Azam and Tabassum [11,24], for each $(\alpha, \beta) \in (0,1] \times [0,1)$ with $[\Psi_1]_{(\alpha,\beta)}, [\Psi_2]_{(\alpha,\beta)} \in \mathcal{K}(W)$, we define the following distance functions:

$$D_{(\alpha,\beta)}(\Psi_1, \Psi_2) = \underline{H}([\Psi_1]_{(\alpha,\beta)}, [\Psi_2]_{(\alpha,\beta)}).$$

$$\sigma_{(\infty,\infty)}(\Psi_1, \Psi_2) = \sup_{(\alpha,\beta)} D_{(\alpha,\beta)}(\Psi_1, \Psi_2).$$

3. Main Results

This section's introduction presents the concept of stationary points (also known as endpoints) for intuitionistic fuzzy-set-valued maps, which is informed by the uniqueness of the FP of single-valued mappings. The reader may refer to analogous articles on endpoint notions by Amini-Harandi [25] and Choudhury [26].

Definition 8. *Let W be a nonempty set. An element $u \in W$ is called a stationary point of an intuitionistic fuzzy-set-valued map $\omega : W \longrightarrow (IFS)^W$ if we can find an $(\alpha, \beta) \in (0,1] \times [0,1)$ with $[\omega u]_{(\alpha,\beta)} = \{u\}$. Similarly, for each $l \in \mathbb{N}$, u is called a stationary point of $\omega(a_1, \cdots, a_l) = \langle \mu_\omega(a_1, \cdots, a_l), \nu_\omega(a_1, \cdots, a_l) \rangle : W^l \longrightarrow [0,1]$ if we can find an $(\alpha, \beta) \in (0,1] \times [0,1)$ with $[\omega(u, \cdots, u)]_{(\alpha,\beta)} = \{u\}$.*

The following example recognizes the existence of a stationary point of intuitionistic fuzzy-set-valued maps.

Example 2. *Let $W = [0,3)$, $\check{\lambda}, \zeta \in (0,1]$, $\sigma(a,b) = |a-b|$ for all $a, b \in W$, and define an intuitionistic fuzzy-set-valued map $\mathcal{W} : W \longrightarrow (IFS)^W$ as*

$$\mu_{\mathcal{W}(W)}(t) = \begin{cases} \frac{\check{\lambda}}{30}, & \text{if } 0 \le t < 3 - \frac{a}{40} \\ \frac{\check{\lambda}}{9}, & \text{if } 3 - \frac{a}{40} \le t \le 3 - \frac{a}{5^2} \\ \frac{\check{\lambda}}{2}, & \text{if } 3 - \frac{a}{5^2} < t \le 3 - \frac{a}{4^2} \\ 0, & \text{if } 3 - \frac{a}{4^2} < t < 3, \end{cases} \quad \nu_{\mathcal{W}(W)}(t) = \begin{cases} \frac{\zeta}{19}, & \text{if } 0 \le t < 3 - \frac{a}{40} \\ \frac{\zeta}{28}, & \text{if } 3 - \frac{a}{40} \le t \le 3 - \frac{a}{5^2} \\ \frac{\zeta}{7^2}, & \text{if } 3 - \frac{a}{5^2} < t \le 3 - \frac{a}{4^2} \\ \zeta, & \text{if } 3 - \frac{a}{4^2} < t < 3. \end{cases}$$

Now, we define $\omega : W \longrightarrow (IFS)^W$ as follows:

$$\mu_{\omega(a)}(t) = \begin{cases} \digamma_{\{0\}}, & \text{if } a = 0 \\ \mu_{\mathcal{W}(a)}, & \text{if } a > 0, \end{cases} \quad \nu_{\omega(a)}(t) = \begin{cases} \urcorner_{\{0\}}, & \text{if } a = 0 \\ \nu_{\mathcal{W}(a)}, & \text{if } a > 0, \end{cases}$$

If $(\alpha,\beta)=\left(\frac{\check{\lambda}}{9},\frac{\zeta}{28}\right)$, *then*

$$[\omega a]_{(\alpha,\beta)}=\begin{cases}\{0\}, & \text{if } a=0\\ \left[3-\frac{a}{40},3-\frac{a}{4^2}\right], & \text{if } a>0.\end{cases}$$

We see that $\{0\}=[\omega 0]_{\left(\frac{\check{\lambda}}{9},\frac{\zeta}{28}\right)}$; *that is,* $a=0\in W$ *is the stationary point of* ω.

Theorem 3. *Let* (W,σ) *be a complete MS,* l *a non-negative number, and* $\omega(a_1,\cdots,a_l):W^l\longrightarrow[0,1]$ *be an intuitionistic fuzzy-set-valued map. Assume that the following conditions hold:*

(i) *We can find an* $(\alpha,\beta)\in(0,1]\times[0,1)$ *where* $[\omega(a_1,\cdots,a_l)]_{(\alpha,\beta)}$ *is a nonempty compact subset of* W;

(ii) *We can find a lower semi-continuous function* $\varphi:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ *satisfying* $\varphi(t)=0$ *if and only if* $t=0$ *with*

$$\begin{aligned}\underline{H}([\omega(a_1,\cdots,a_l)]_{(\alpha,\beta)},[\omega(a_2,\cdots,a_{l+1})]_{(\alpha,\beta)})\leq\ &\max\{\sigma(a_i,a_{i+1}):1\leq i\leq l\}\\ &-\varphi(\max\{\sigma(a_i,a_{i+1}):1\leq i\leq l\}),\end{aligned}\tag{2}$$

for all $(a_1,\cdots,a_{l+1})\in W^{l+1}$. *Then, for any random points* $a_1,\cdots,a_l\in W$, *the sequence* $\{a_{\check{p}+l}\}_{\check{p}\geq1}$ *specified by*

$$a_{\check{p}+l}\in[\omega(a_{\check{p}},\cdots,a_{\check{p}+l-1})]_{(\alpha,\beta)},\ \check{p}\in\mathbb{N}\tag{3}$$

converges to $u\in W$ *and* $u\in[\omega(u,\cdots,u)]_{(\alpha,\beta)}$. *Moreover, if*

$$\underline{H}([\omega(a_1,\cdots,a)]_{(\alpha,\beta)},[\omega(b,\cdots,b)]_{(\alpha,\beta)})\leq\sigma(a,b)-\varphi(\sigma(a,b))\tag{4}$$

holds for all $a,b\in W$ *with* $a\neq b$, *then* ω *has a stationary point in* W.

Proof. Let $a_1,\cdots,a_l$ be random l elements in W. Consider the sequence specified by (3). If we can find an $(\alpha,\beta)\in(0,1]\times[0,1)$ with $a_i=a_{i+1}$ for all $i=\check{p},\check{p}+1,\cdots,\check{p}+l-1$, then $a_i\in[\omega(a_i,\cdots,a_i)]_{(\alpha,\beta)}$, that is, a_i is an intutionistic fuzzy FP of ω, and the proof is finished. Hence, we assume that $a_i\neq a_{i+1}$ for all $i=\check{p},\check{p}+1,\cdots,\check{p}+l-1$. For $l\geq\check{p}$, from (2) and Lemma 1, we have the following inequations:

$$\begin{aligned}\sigma(a_{l+\check{p}},a_{l+\check{p}-1})&\leq\underline{H}([\omega(a_{\check{p}},\cdots,a_{l+\check{p}-1})]_{(\alpha,\beta)},[\omega(a_{\check{p}+1},\cdots,a_{\check{p}+l})]_{(\alpha,\beta)})\\&\leq\max\{\sigma(a_i,a_{i+1}):\check{p}\leq i\leq l+\check{p}-1\}\\&\quad-\varphi(\max\{\sigma(a_i,a_{i+1}):\check{p}\leq i\leq l+\check{p}-1\})\\&<\max\{\sigma(a_i,a_{i+1}):\check{p}\leq i\leq l+\check{p}-1\}.\end{aligned}$$

$$\begin{aligned}\sigma(a_{l+1},a_{l+2})&\leq\underline{H}([\omega(a_1,\cdots,a_l)]_{(\alpha,\beta)},[\omega(a_2,\cdots,a_{l+1})]_{(\alpha,\beta)})\\&\leq\max\{\sigma(a_i,a_{i+1}):1\leq i\leq l\}\\&\quad-\varphi(\max\{\sigma(a_i,a_{i+1}):1\leq i\leq l\})\\&<\max\{\sigma(a_i,a_{i+1}):1\leq i\leq l\}.\end{aligned}$$

$$\begin{aligned}\sigma(a_l,a_{l+1})&\leq\underline{H}([\omega(a_1,\cdots,a_{l-1})]_{(\alpha,\beta)},[\omega(a_2,\cdots,a_l)]_{(\alpha,\beta)})\\&\leq\max\{\sigma(a_i,a_{i+1}):1\leq i\leq l-1\}\\&\quad-\varphi(\max\{\sigma(a_i,a_{i+1}):1\leq i\leq l-1\})\\&<\max\{\sigma(a_i,a_{i+1}):1\leq i\leq l-1\}.\end{aligned}$$

$$\begin{aligned}\sigma(a_{l-\check{p}}, a_{l-\check{p}+1}) &\leq \underline{H}([\omega(a_1,\cdots,a_{l-\check{p}-1})]_{(\alpha,\beta)}, [\omega(a_2,\cdots,a_{l-\check{p}})]_{(\alpha,\beta)}) \\ &\leq \max\{\sigma(a_i,a_{i+1}) : 1 \leq i \leq l-\check{p}-1\} \\ &\quad -\varphi(\max\{\sigma(a_i,a_{i+1}) : 1 \leq i \leq l-\check{p}-1\}) \\ &< \max\{\sigma(a_i,a_{i+1}) : 1 \leq i \leq l-\check{p}-1\}.\end{aligned}$$

Hence, we conclude that the sequence $\{\sigma(a_{\check{p}+l-1}, a_{\check{p}+l})\}_{\check{p}\geq 1}$ is monotone, non-increasing, and bounded below. We can therefore find $\tau \geq 0$ with

$$\lim_{\check{p}\longrightarrow\infty} \sigma(a_{\check{p}+l-1}, a_{\check{p}+l}) = \lim_{\check{p}\longrightarrow\infty} \max\{\sigma(a_{\check{p}+i}, a_{\check{p}+i+1}) : 1 \leq i \leq l-1\} = \tau. \tag{5}$$

We claim that $\tau = 0$. To see this, consider the following inequalities:

$$\begin{aligned}\sigma(a_{l+\check{p}}, a_{l+\check{p}+1}) \leq\ & \underline{H}([\omega(a_{\check{p}},\cdots,a_{l+\check{p}-1})]_{(\alpha,\beta)}, [\omega(a_{\check{p}+1},\cdots,a_{l+\check{p}})]_{(\alpha,\beta)}) \\ \leq\ & \max\{\sigma(a_i,a_{i+1}) : \check{p} \leq i \leq l+\check{p}-1\} \\ & - \varphi(\max\{\sigma(a_i,a_{i+1}) : \check{p} \leq i \leq l+\check{p}-1\}).\end{aligned} \tag{6}$$

Taking the upper limit in (6) as $\check{p} \longrightarrow \infty$, we have $\tau \leq \tau - \varphi(\tau)$, which implies that $\varphi(\tau) \leq 0$, and hence $\varphi(\tau) = 0$, from which it follows that $\tau = 0$. Therefore, $\lim_{\check{p}\longrightarrow\infty} \sigma(a_{l+\check{p}}, a_{l+\check{p}+1}) = 0$.

Next, we show that $\{a_{\check{p}}\}_{\check{p}\geq 1}$ is a Cauchy sequence in W. Let $m, \check{p} \in \mathbb{N}$ with $\check{p} \geq m$. Then, from (2) and Lemma 1, we obtain

$$\begin{aligned}\sigma(a_{l+\check{p}}, a_{l+m}) \leq\ & \underline{H}([\omega(a_{\check{p}},\cdots,a_{l+\check{p}-1})]_{(\alpha,\beta)}, [\omega(a_m,\cdots,a_{l+m-1})]_{(\alpha,\beta)}) \\ \leq\ & \underline{H}([\omega(a_{\check{p}},\cdots,a_{l+\check{p}-1})]_{(\alpha,\beta)}, [\omega(a_{\check{p}+1},\cdots,a_{l+\check{p}})]_{(\alpha,\beta)}) \\ & + \underline{H}([\omega(a_{\check{p}+1},\cdots,a_{l+\check{p}})]_{(\alpha,\beta)}, [\omega(a_{\check{p}+2},\cdots,a_{l+\check{p}+1})]_{(\alpha,\beta)}) \\ & + \cdots + \underline{H}([\omega(a_{m-1},\cdots,a_{l+m-2})]_{(\alpha,\beta)}, [\omega(a_m,\cdots,a_{l+m-1})]_{(\alpha,\beta)}) \\ \leq\ & \max\{\sigma(a_i,a_{i+1}) : \check{p} \leq i \leq l+\check{p}-1\} \\ & - \varphi(\max\{\sigma(a_i,a_{i+1}) : \check{p} \leq i \leq l+\check{p}-1\}) \\ & + \max\{\sigma(a_i,a_{i+1}) : \check{p}+1 \leq i \leq l+\check{p}\} \\ & - \varphi(\max\{\sigma(a_i,a_{i+1}) : \check{p}+1 \leq i \leq l+\check{p}\}) \\ & + \cdots + \max\{\sigma(a_i,a_{i+1}) : m-1 \leq i \leq l+m-2\} \\ & - \varphi(\max\{\sigma(a_i,a_{i+1}) : m-1 \leq i \leq l+m-2\}).\end{aligned} \tag{7}$$

Taking the upper limit in (7) as $\check{p} \longrightarrow \infty$ gives $\lim_{\check{p}\longrightarrow\infty} \sigma(a_{l+\check{p}}, a_{l+m}) = 0$. This shows that $\{a_{\check{p}}\}_{\check{p}\geq 1}$ is a Cauchy sequence in W. Hence, the completeness of this space guarantees the existence of $u \in W$ with

$$\lim_{\check{p}\longrightarrow\infty} \sigma(a_{\check{p}}, u) = 0. \tag{8}$$

Now, to show that u is an intuitionistic fuzzy FP of ω, let $\check{p} \in \mathbb{N}$, and then consider

$$\begin{aligned}
\sigma(u,[\omega(u,\cdots,u)]_{(\alpha,\beta)}) \leq\ & \sigma(u,a_{\check{p}+l})+\sigma(a_{\check{p}+l},[\omega(u,\cdots,u)]_{(\alpha,\beta)}) \\
\leq\ & \sigma(u,a_{\check{p}+l})+\underline{H}([\omega(a_{\check{p}},\cdots,a_{\check{p}+l-1})]_{(\alpha,\beta)},[\omega(u,\cdots,u)]_{(\alpha,\beta)}) \\
\leq\ & \sigma(u,a_{\check{p}+l})+\underline{H}([\omega(u,\cdots,u)]_{(\alpha,\beta)}),[\omega(u,\cdots,a_{\check{p}})]_{(\alpha,\beta)} \\
& +\underline{H}([\omega(u,\cdots,a_{\check{p}})]_{(\alpha,\beta)},[\omega(u,\cdots,a_{\check{p}},a_{\check{p}+1})]_{(\alpha,\beta)}) \\
& +\cdots+\underline{H}([\omega(u,a_{\check{p}},\cdots,a_{\check{p}+l-2})]_{(\alpha,\beta)},[\omega(a_{\check{p}},\cdots,a_{\check{p}+l-1})]_{(\alpha,\beta)}). \\
\leq\ & \sigma(u,a_{\check{p}+l})+\max\{\sigma(u,a_i):1\leq i\leq\check{p}\} \\
& -\varphi(\max\{\sigma(u,a_i):1\leq i\leq\check{p}\}) \\
& +\max\{\sigma(u,a_{\check{p}}),\sigma(a_{\check{p}},a_{\check{p}+1})\} \\
& -\varphi(\max\{\sigma(u,a_{\check{p}}),\sigma(a_{\check{p}},a_{\check{p}+1})\}) \\
& +\cdots+\max\{\sigma(u,a_{\check{p}}),\sigma(a_{\check{p}},a_{\check{p}+1}),\cdots,\sigma(a_{\check{p}+l-2},a_{\check{p}+l-1})\} \\
& -\varphi(\max\{\sigma(u,a_{\check{p}}),\sigma(a_{\check{p}},a_{\check{p}+1}),\cdots,\sigma(a_{\check{p}+l-2},a_{\check{p}+l-1})\}).
\end{aligned} \tag{9}$$

Taking the upper limit in (9) gives $\sigma(u,[\omega(u,\cdots,u)]_{(\alpha,\beta)}) \leq 0$, which implies that $u \in [\omega(u,\cdots,u)]_{(\alpha,\beta)}$; that is, u is an intuitionistic fuzzy FP of ω. Now, we prove that under condition (4), ω has a stationary point in W. For this, assume that we can find $u^* \in [\omega(u^*,\cdots,u^*)]_{(\alpha,\beta)}$ with $u \neq u^*$ and $[\omega(u,\cdots,u)]_{(\alpha,\beta)} \neq \{u\}$. Then, via Lemma 1, we have

$$\begin{aligned}
\sigma(u,u^*) &\leq \underline{H}([\omega(u,\cdots,u)]_{(\alpha,\beta)},[\omega(u^*,\cdots,u^*)]_{(\alpha,\beta)}) \\
&\leq \sigma(u,u^*)-\varphi(\sigma(u,u^*)) \\
&< \sigma(u,u^*),
\end{aligned} \tag{10}$$

a contradiction. Hence, ω has a stationary point in W. □

The following Theorem is a Presic-type generalization of the main result of Heilpern [4] using the concept of $\sigma_{(\infty,\infty)}$-distance function for intuitionistic fuzzy sets.

Theorem 4. *Let (W,σ) be a complete MS, l a non-negative number, and $\omega : W^l \longrightarrow \mathcal{K}(W)$ an intuitionistic fuzzy-set-valued map. Assume that we can find a lower semi-continuous function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with $\varphi(t)=0$ if and only if $t=0$ and*

$$\begin{aligned}
\sigma_{(\infty,\infty)}(\omega(a_1,\cdots,a_l),\omega(a_2,\cdots,a_{l+1})) \leq\ & \max\{\sigma(a_i,a_{i+1}):1\leq i\leq l\} \\
& -\varphi(\max\{\sigma(a_i,a_{i+1}):1\leq i\leq l\}),
\end{aligned} \tag{11}$$

for all $(a_1,\cdots,a_{l+1}) \in W^{l+1}$. Then, for each random point $a_1,\cdots,a_l \in W$, the sequence $\{a_{\check{p}+l}\}_{\check{p}\geq 1}$ specified by

$$a_{\check{p}+l} \in \omega(a_{\check{p}},\cdots,a_{\check{p}+l-1}),\ \check{p}\in\mathbb{N}$$

converges to $u \in W$ and $\{u\} \subset \omega(u,\cdots,u)$.

Proof. Let $a_1,\cdots,a_l \in W$ and $(\alpha,\beta) \in (0,1]\times[0,1)$. Then, by hypothesis, $\omega(a_1,\cdots,a_l)]_{(\alpha,\beta)} \in \mathcal{K}(W)$. Now, by definitions of $D_{(\alpha,\beta)}$ and $\sigma_{(\infty,\infty)}$-metric for intuitionistic fuzzy sets, for all $a_1,\cdots,a_{l+1} \in W^{l+1}$, we have

$$\begin{aligned}
D_{(\alpha,\beta)}(\omega(a_1,\cdots,a_l),\omega(a_2,\cdots,a_{l+1})) &= \underline{H}([\omega(a_1,\cdots,a_l)]_{(\alpha,\beta)},[\omega(a_2,\cdots,a_{l+1})]_{(\alpha,\beta)}) \\
&\leq \sup_{(\alpha,\beta)} \underline{H}([\omega(a_1,\cdots,a_l)]_{(\alpha,\beta)},[\omega(a_2,\cdots,a_{l+1})]_{(\alpha,\beta)}) \\
&= \sigma_{(\infty,\infty)}(\omega(a_1,\cdots,a_l),\omega(a_2,\cdots,a_{l+1})) \\
&\leq \max\{\sigma(a_i,a_{i+1}) : 1 \leq i \leq l\} \\
&\quad -\varphi(\max\{\sigma(a_i,a_{i+1}) : 1 \leq i \leq l\}).
\end{aligned}$$

Hence, Theorem 3 can be applied to find $u \in W$ with $\{u\} \subset \omega(u,\cdots,u)$. □

We provide the next example to verify the hypotheses of Theorems 3 and 4.

Example 3. *Let $W = [0,\infty)$ and define $\sigma : W \times W \longrightarrow \mathbb{R}$ with $\sigma(a,b) = |a-b|$ for all $a,b \in W$. Clearly, (W,σ) is a complete MS. Let $\check{\lambda},\zeta \in (0,1]$. For all $a_1,\cdots,a_l \in W$, consider an intuitionistic fuzzy-set-valued map $\mathcal{W}(a_1,\cdots,a_l) = \langle \mu_{\mathcal{W}}(a_1,\cdots,a_l),\nu_{\mathcal{W}}(a_1,\cdots,a_l)\rangle : W^l \longrightarrow [0,1]$ specified as follows:*

$$\mu_{\mathcal{W}}(a_1,\cdots,a_l)(t_1,\cdots,t_l) = \begin{cases} \check{\lambda}, & \text{if } (t_1,\cdots,t_l) \in \left[0,\frac{a_1+\cdots+a_l}{7l^2}\right] \\ 1-\frac{\check{\lambda}^2}{3}, & \text{if } (t_1,\cdots,t_l) \in \left(\frac{a_1+\cdots+a_l}{7l^2},\frac{a_1+\cdots+a_l}{6l^2}\right] \\ \frac{\sqrt{\check{\lambda}}}{4} & \text{if } (t_1,\cdots,t_l) \in \left(\frac{a_1+\cdots+a_l}{6l^2},\frac{a_1+\cdots+a_l}{5l^2}\right] \\ 0, & \text{if } (t_1,\cdots,t_l) \in \left(\frac{a_1+\cdots+a_l}{5l^2},\infty\right), \end{cases}$$

$$\nu_{\mathcal{W}}(a_1,\cdots,a_l)(t_1,\cdots,t_l) = \begin{cases} 0, & \text{if } (t_1,\cdots,t_l) \in \left[0,\frac{a_1+\cdots+a_l}{7l^2}\right] \\ \frac{\zeta}{8}, & \text{if } (t_1,\cdots,t_l) \in \left(\frac{a_1+\cdots+a_l}{7l^2},\frac{a_1+\cdots+a_l}{6l^2}\right] \\ \frac{\zeta}{5} & \text{if } (t_1,\cdots,t_l) \in \left(\frac{a_1+\cdots+a_l}{6l^2},\frac{a_1+\cdots+a_l}{5l^2}\right] \\ \zeta, & \text{if } (t_1,\cdots,t_l) \in \left(\frac{a_1+\cdots+a_l}{5l^2},\infty\right). \end{cases}$$

Now, define $\omega(a_1,\cdots,a_l) = \langle \mu_\omega(a_1,\cdots,a_l),\nu_\omega(a_1,\cdots,a_l)\rangle : W^l \longrightarrow [0,1]$ as follows:

$$\mu_\omega(a_1,\cdots,a_l)(t_1,\cdots,t_l) = \begin{cases} \digamma_{\{0\}}, & \text{if } (a_1,\cdots,a_l) = (0,\cdots,0) \\ \mu_{\mathcal{W}(a_1,\cdots,a_l)}, & \text{if } (a_1,\cdots,a_l) > (0,\cdots,0), \end{cases}$$

$$\nu_\omega(a_1,\cdots,a_l)(t_1,\cdots,t_l) = \begin{cases} \urcorner_{\{0\}}, & \text{if } (a_1,\cdots,a_l) = (0,\cdots,0) \\ \mu_{\mathcal{W}(a_1,\cdots,a_l)}, & \text{if } (a_1,\cdots,a_l) > (0,\cdots,0). \end{cases}$$

Assume that $(\alpha,\beta) = (\check{\lambda},0)$; then

$$[\omega(a_1\cdots,a_l)]_{\left(\check{\lambda},\frac{\zeta}{20}\right)} = \begin{cases} \{0\}, & \text{if } (a_1,\cdots,a_l) = (0,\cdots,0) \\ \left[0,\frac{a_1+\cdots+a_l}{7l^2}\right], & \text{if } (a_1,\cdots,a_l) > (0,\cdots,0). \end{cases}$$

Define the function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with

$$\varphi(t) = \begin{cases} \frac{t}{8}, & \text{if } t \in \left[0,\frac{11}{3}\right) \\ \frac{3^{\check{p}}(3^{\check{p}+1}t)}{3^{2\check{p}+1}-1}, & \text{if } t \in \left[\frac{3^{2\check{p}}+1}{3^{\check{p}}},\frac{3^{2(\check{p}+1)}+1}{3^{\check{p}+1}}\right], \check{p} \in \mathbb{N}. \end{cases}$$

Direct computation demonstrates that φ is lower semi-continuous on $\mathbb{R}_+$ and $\varphi(t) = 0$ if and only if $t = 0$. Now, for all $(a_1,\cdots,a_{l+1}) \in W$ (note that there is nothing to show if $(a_1,\cdots,a_{l+1}) = (0,\cdots,0)$), we have

$$
\begin{aligned}
&D_{(\alpha,\beta)}(\omega(a_1,\cdots,a_l),\omega(a_2,\cdots,a_{l+1}))\\
&=\underline{H}([\omega(a_1,\cdots,a_l)]_{(\alpha,\beta)},[\omega(a_2,\cdots,a_{l+1})]_{(\alpha,\beta)})\\
&=\underline{H}\left(\left[0,\frac{a_1+\cdots+a_l}{7l^2}\right],\left[0,\frac{a_2+\cdots+a_{l+1}}{7l^2}\right]\right)\\
&\leq \frac{1}{7l}|a_1-a_{l+1}|\leq \frac{1}{7}\max\{|a_i-a_{i+1}|:1\leq i\leq l\}\\
&\leq \frac{7}{8}\max\{\sigma(a_i,a_{i+1}):1\leq i\leq l\}\\
&=\max\{\sigma(a_i,a_{i+1}:1\leq i\leq l)\}\\
&\ -\varphi(\max\{\sigma(a_i,a_{i+1}):1\leq i\leq l\}).
\end{aligned}
$$

Moreover, for all $a,b\in W$, we have

$$
\begin{aligned}
\underline{H}([\omega(a,\cdots,a)]_{(\alpha,\beta)},[\omega(b,\cdots,b)]_{(\alpha,\beta)}) &\leq \frac{1}{30}|a-b|\\
&\leq \frac{7}{8}\sigma(a,b)\\
&= \sigma(a,b)-\varphi(\sigma(a,b)).
\end{aligned}
$$

Therefore, all the conditions of Theorem 3 are satisfied. From it, we can find $u=0\in W$ with $[\omega(0,\cdots,0)]_{(\alpha,\beta)}=\{0\}$, that is, 0 is both a stationary point and intuitionistic fuzzy FP of ω.

Corollary 1. *Let (W,σ) be a complete MS, l a non-negative number and $\omega(a_1,\cdots,a_l):W^l\longrightarrow[0,1]$ be an intuitionistic fuzzy-set-valued map. Assume that the following conditions hold:*

(i) We can find $(\alpha,\beta)\in(0,1]\times[0,1)$ such that $[\omega(a_1,\cdots,a_l)]_{(\alpha,\beta)}$ is a nonempty compact subset of W;

(ii) We can find $\check{\lambda}\in(0,1)$ with

$$
\begin{aligned}
&\underline{H}([\omega(a_1,\cdots,a_l)]_{(\alpha,\beta)},[\omega(a_2,\cdots,a_{l+1})]_{(\alpha,\beta)})\\
&\leq \check{\lambda}\max\{\sigma(a_i,a_{i+1}):1\leq i\leq l\}
\end{aligned}
\tag{12}
$$

for all $(a_1,\cdots,a_{l+1})\in W^{l+1}$. Then, for any random points $a_1,\cdots,a_l\in W$, the sequence $\{a_{\check{p}+l}\}_{\check{p}\geq 1}$ specified by $a_{\check{p}+l}\in[\omega(a_{\check{p}},\cdots,a_{\check{p}+l-1})]_{(\alpha,\beta)}$, $\check{p}\in\mathbb{N}$ converges to $u\in W$ and $u\in[\omega(u,\cdots,u)]_{(\alpha,\beta)}$. Moreover, if for all $a,b\in W$ with $a\neq b$,

$$
\underline{H}([\omega(a,\cdots,a)]_{(\alpha,\beta)}[\omega(b,\cdots,b)]_{(\alpha,\beta)})\leq\check{\lambda}\sigma(a,b),
$$

then ω has a stationary point in W.

Proof. Put $\varphi(t)=(1-\check{\lambda})t$, where $\check{\lambda}\in(0,1)$ and $t\geq 0$ in Theorem 3. □

Corollary 2. *Let (W,σ) be a complete MS, l a non-negative number, and $\omega(a_1,\cdots,a_l):W^l\longrightarrow[0,1]$ an intuitionistic fuzzy-set-valued map. Assume that the following conditions are satisfied:*

(i) We can find $(\alpha,\beta)\in(0,1]\times[0,1)$ such that $[\omega(a_1,\cdots,a_l)]_{(\alpha,\beta)}$ is a nonempty compact subset of W;

(ii) There exist non-negative constants $\check{\lambda}_1,\cdots,\check{\lambda}_l$ with $\sum_{i=1}^{l}\check{\lambda}_i<1$ and

$$
\begin{aligned}
\underline{H}([\omega(a_1,\cdots,a_l)]_{(\alpha,\beta)},[\omega(a_2,\cdots,a_{l+1})]_{(\alpha,\beta)})\leq\ &\check{\lambda}_1\sigma(a_1,a_2)+\check{\lambda}_2\sigma(a_2,a_3)\\
&+\cdots+\check{\lambda}_l\sigma(a_l,a_{l+1}),
\end{aligned}
\tag{13}
$$

for all $(a_1, \cdots, a_{l+1}) \in W^{l+1}$. *Then, for any random points* $a_1, \cdots, a_l \in W$, *the sequence* $\{a_{\check{p}+l}\}_{\check{p}\geq 1}$ *specified by* $a_{\check{p}+l} \in [\omega(a_{\check{p}}, \cdots, a_{\check{p}+l-1})]_{(\alpha,\beta)}$ *converges to* $u \in W$ *and* $u \in [\omega(u, \cdots, u)]_{(\alpha,\beta)}$. *Moreover, if*

$$\begin{aligned} &\underline{H}([\omega(a, \cdots, a)]_{(\alpha,\beta)}, [\omega(b, \cdots, b)]_{(\alpha,\beta)}) \\ &\leq \sum_{i=1}^{l} \check{\lambda}_i \sigma(a, b) \end{aligned} \tag{14}$$

holds for all $a, b \in W$ *with* $a \neq b$, *then* ω *has a stationary point in* W.

Proof. Obviously, condition (12) follows from condition (13) by taking $\check{\lambda} = \sum_{i=1}^{l} \check{\lambda}_i$. Furthermore, let $a, b \in W$ with $a \neq b$. Then, from (14), we have

$$\begin{aligned} \underline{H}([\omega(a, \cdots, a)]_{(\alpha,\beta)}, [\omega(b, \cdots, b)]_{(\alpha,\beta)}) &\leq \underline{H}([\omega(a, \cdots, a)]_{(\alpha,\beta)}, [\omega(b, \cdots, a, b)]_{(\alpha,\beta)}) \\ &\quad + \underline{H}([\omega(a, \cdots, a, b)]_{(\alpha,\beta)}, [\omega(a, \cdots, a, b, b)]_{(\alpha,\beta)}) \\ &\quad + \cdots + \underline{H}([\omega(a, b, \cdots, b)]_{(\alpha,\beta)}, [\omega(b, \cdots, b)]_{(\alpha,\beta)}) \\ &\leq \sum_{i=1}^{l} \check{\lambda}_i \sigma(a, b). \end{aligned}$$

Thus, all the assertions of Corollary 1 are satisfied with $\check{\lambda} = \sum_{i=1}^{l} \check{\lambda}_i$, and that completes the proof. □

Definition 9 ([4])**.** *Let* (W, σ) *be an MS. A fuzzy-set-valued map* $\omega : W \longrightarrow \mathcal{K}(W)$ *is called a fuzzy* $\check{\lambda}$*-contraction if we can find a constant* $\check{\lambda} \in (0, 1)$ *with* $a, b \in W$,

$$\sigma_\infty(\omega(a), \omega(b)) \leq \check{\lambda} \sigma(a, b).$$

In ([4], Theorem 3.1), it has been shown that every fuzzy $\check{\lambda}$-contraction on a complete MS has a fuzzy FP. Following this idea, we present the next definition to enable us to establish a significant consequence of Theorem 4.

Definition 10. *Let* (W, σ) *be an MS. An intuitionistic fuzzy-set-valued map* $\omega : W \longrightarrow \mathcal{K}(W)$ *is called an intuitionistic fuzzy weakcontraction if we can find a lower semi-continuous function* $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *with* $\varphi(t) = 0$ *if and only if* $t = 0$ *and for all* $a, b \in W$, $a \neq b$,

$$\sigma_{(\infty,\infty)}(\omega(a), \omega(b)) \leq \sigma(a, b) - \varphi(\sigma(a, b)).$$

Corollary 3. *Let* (W, σ) *be a complete MS and* $\omega : W \longrightarrow \mathcal{K}(W)$ *be an intuitionistic fuzzy weak contraction on* W. *Then,* ω *has at least one intuitionistic fuzzy FP in* W.

Proof. It is enough to take $l = 1$ in Theorem 4. □

Remark 2. *If we take* $l = 1$, $\varphi(t) = (1 - \check{\lambda})t$ *for all* $t \in \mathbb{R}_+$ *and* $\check{\lambda} \in (0, 1)$, *and* ω *is reduced to a fuzzy-set-valued map, then Corollary 3 becomes the main result of Heilpern ([4], Theorem 3.1).*

4. Applications in Fuzzy, Multivalued, and Single-Valued Mappings

Here, we describe some new FP results of fuzzy, multi-valued, and single-valued mappings using the findings from Section 3. To this goal, keep in mind that if a point is $u \in Au$ ($u = Au$), it is referred to as an FP of a multi-valued (single-valued) mapping $\widetilde{A}$ on W. If $Au = u$, a point $u \in W$ is said to be a stationary point of a multi-valued mapping $\widetilde{A}$.

Theorem 5. *Let* (W, σ) *be a complete MS,* l *a non-negative number, and* $\omega(a_1, \cdots, a_l) : W^l \longrightarrow [0, 1]$ *a fuzzy-set-valued map. Assume that the following conditions hold:*

(i) *We can find an* $\alpha \in (0,1]$ *such that* $[\omega(a_1, \cdots, a_l)]_\alpha$ *is a nonempty compact subset of* W*;*
(ii) *We can find a lower semi-continuous function* $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *satisfying* $\varphi(t) = 0$ *if and only if* $t = 0$ *with*

$$\underline{H}([\omega(a_1, \cdots, a_l)]_\alpha, [\omega(a_2, \cdots, a_{l+1})]_\alpha) \leq \max\{\sigma(a_i, a_{i+1}) : 1 \leq i \leq l\} - \varphi(\max\{\sigma(a_i, a_{i+1}) : 1 \leq i \leq l\}),$$

for all $(a_1, \cdots, a_{l+1}) \in W^{l+1}$*. Then, for any random points* $a_1, \cdots, a_l \in W$*, the sequence* $\{a_{\check{p}+l}\}_{\check{p}\geq 1}$ *specified by*

$$a_{\check{p}+l} \in [\omega(a_{\check{p}}, \cdots, a_{\check{p}+l-1})]_\alpha, \ \check{p} \in \mathbb{N}$$

converges to $u \in W$ *and* $u \in [\omega(u, \cdots, u)]_\alpha$*. Moreover, if*

$$\underline{H}([\omega(a_1, \cdots, a)]_\alpha, [\omega(b, \cdots, b)]_\alpha) \leq \sigma(a, b) - \varphi(\sigma(a, b))$$

holds for all $a, b \in W$ *with* $a \neq b$*, then* ω *has a stationary point in* W*.*

Proof. It suffices to take the degree of hesitancy $h_{\omega(a)}(t) = 1 - \mu_{\omega(a)}(t) - \nu_{\omega(a)}(t) = 0$ in Theorem 3. □

Theorem 6. *Let* (W, σ) *be a complete MS,* l *be a non-negative number, and* $\widetilde{A} : W^l \longrightarrow \mathcal{K}(W)$ *be a multi-valued mapping. Assume that*

$$\begin{aligned} \underline{H}(\widetilde{A}(a_1, \cdots, a_l), \widetilde{A}(a_2, \cdots, a_{l+1})) &\leq \max\{\sigma(a_i, a_{i+1}) : 1 \leq i \leq l\} \\ &= -\varphi(\max\{\sigma(a_i, a_{i+1}) : 1 \leq i \leq l\}) \end{aligned}$$

holds for all $(a_1, \cdots, a_{l+1}) \in W^{l+1}$*, where* $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *is a lower semi-continuous function with* $\varphi(t) = 0$ *if and only if* $t = 0$*. Then, for any random points* $a_1, \cdots, a_l \in W$*, the sequence* $\{a_{\check{p}+l}\}_{\check{p}\geq 1}$ *specified by* $a_{\check{p}+l} \in \widetilde{A}(a_{\check{p}}, \cdots, a_{\check{p}+l-1})$*,* $\check{p} \in \mathbb{N}$ *converges to* $u \in W$ *and* $u \in \widetilde{A}(u, \cdots, u)$*. Moreover, if*

$$\underline{H}(\widetilde{A}(a, \cdots, a), \widetilde{A}(b, \cdots, b)) \leq \sigma(a, b) - \varphi(\sigma(a, b))$$

holds for all $a, b \in W$ *with* $a \neq b$*, then* $\widetilde{A}$ *has a stationary point in* W*.*

Proof. Consider an intuitionistic fuzzy-set-valued map $\Theta = \langle \mu_\Theta, \nu_\Theta \rangle : W \longrightarrow (IFS)^W$ specified by

$$\mu_{\Theta(\jmath_1, \cdots, \jmath_l)}(t_1, \cdots, t_l) = \begin{cases} 1, & \text{if } (t_1, \cdots, t_l) \in \widetilde{A}(\jmath_1, \cdots, \jmath_l) \\ 0, & \text{if } (t_1, \cdots, t_l) \notin \widetilde{A}(\jmath_1, \cdots, \jmath_l), \end{cases}$$

$$\nu_{\Theta(\jmath_1, \cdots, \jmath_l))}(t_1, \cdots, t_l) = \begin{cases} 0, & \text{if } (t_1, \cdots, t_l) \in \widetilde{A}(\jmath_1, \cdots, \jmath_l) \\ 1, & \text{if } (t_1, \cdots, t_l) \notin \widetilde{A}(\jmath_1, \cdots, \jmath_l). \end{cases}$$

Then, we can find $(\alpha, \beta) = (1, 0) \in (0, 1] \times [0, 1)$ with $[\Theta(\jmath_1, \cdots, \jmath_l)]_{(1,0)} = \widetilde{A}(\jmath_1, \cdots, \jmath_l)$. Hence, Theorem 3 can be applied to find $u \in W$ with $u \in [\Theta(u, \cdots, u)]_{(1,0)} = \widetilde{A}(u, \cdots, u)$. □

Theorem 7 (see [12], Theorem 2.1)**.** *Let* (W, σ) *be a complete MS,* l *a non-negative number, and* $\vartheta : W^l \longrightarrow W$ *a single-valued mapping. Assume that we can find a lower semi-continuous function* $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *with* $\varphi(t) = 0$ *if and only if* $t = 0$ *and*

$$\sigma(\vartheta(a_1, \cdots, a_l), \vartheta(a_2, \cdots, a_{l+1})) \leq \max\{\sigma(a_i, a_{i+1}) : 1 \leq i \leq l\} - \varphi(\max\{\sigma(a_i, a_{i+1}) : 1 \leq i \leq l\})$$

holds for all $(a_1, \cdots, a_{l+1}) \in W^{l+1}$*. Then, for any random points* $a_1, \cdots, a_l \in W$*, the sequence* $\{a_{\check{p}+l}\}_{\check{p}\in\mathbb{N}}$ *specified by* $a_{\check{p}+l} = \vartheta(a_{\check{p}}, \cdots, a_{\check{p}+l-1})$*,* $\check{p} \in \mathbb{N}$ *converges to* $u \in W$ *and* $u = \vartheta(u, \cdots, u)$*. Moreover, if*

$$\sigma(\vartheta(a, \cdots, a), \vartheta(b, \cdots, b)) \leq \sigma(a, b) - \varphi(\sigma(a, b))$$

holds for all $a, b \in W$ *with* $a \neq b$*, then* $u \in W$ *is the unique FP of* ϑ*.*

Proof. Consider an intuitionistic fuzzy-set-valued map $\Theta = \langle \mu_\Theta, \nu_\Theta \rangle : W \longrightarrow (IFS)^W$ specified by

$$\mu_{\Theta(\jmath_1,\cdots,\jmath_l)}(t_1, \cdots, t_l) = \begin{cases} 1, & \text{if } (t_1, \cdots, t_l) = \vartheta(\jmath_1, \cdots, \jmath_l) \\ 0, & \text{if } (t_1, \cdots, t_l) \neq \vartheta(\jmath_1, \cdots, \jmath_l), \end{cases}$$

$$\nu_{\Theta(\jmath_1,\cdots,\jmath_l))}(t_1, \cdots, t_l) = \begin{cases} 0, & \text{if } (t_1, \cdots, t_l) = \vartheta(\jmath_1, \cdots, \jmath_l) \\ 1, & \text{if } (t_1, \cdots, t_l) \neq \vartheta(\jmath_1, \cdots, \jmath_l). \end{cases}$$

Then, we can find $(\alpha, \beta) = (1, 0) \in (0, 1] \times [0, 1)$ with $[\Theta(\jmath_1, \cdots, \jmath_l)]_{(1,0)} = \{\vartheta(\jmath_1, \cdots, \jmath_l)\}$. Hence, Theorem 3 can be applied to find $u \in W$ with $u \in [\Theta(u, \cdots, u)]_{(1,0)} = \{\vartheta(\jmath_1, \cdots, \jmath_l)\}$, which implies that $\vartheta(u, \cdots, u) = u$. □

Remark 3.

(i) *Theorems 3 and 6 are intuitionistic fuzzy-set-valued and multi-valued extensions of the result of Abbas et al. ([12], Theorem 2.1).*
(ii) *Theorem 3 is an intuitionistic fuzzy generalization of the results of Ciric [13] and Presic [2].*
(iii) *If* $l = 1$*, Theorem 3 is an intuitionistic fuzzy improvement to the result of Rhoades [14].*
(iv) *By setting* $\varphi(t) = (1 - \check{\lambda})t$*, where* $\check{\lambda} \in (0, 1)$ *and* $t \geq 0$*, we can derive the Banach contraction theorem from Theorem 3 by employing the method of proving Theorem 7.*

5. Stability of Intuitionistic Fuzzy FP Inclusions

The idea of stability is connected to a system's limiting behaviors. Both discrete and continuous dynamical systems have been used to study it (see [27], for example). In-depth research has also been performed in many frameworks on the topic of the convergence of a series of mappings and their FPs (see, for instance, [28]). A set-valued mapping frequently has a higher frame rate than a single-valued mapping. For the study of stability, the set of FPs of set-valued mappings thus becomes more intriguing. This part launches the investigation of the stability of Presic-type intuitionistic fuzzy FP problems.

Theorem 8. *Let* (W, σ) *be a complete MS,* l *a non-negative number, and* $\omega_i(a_1, \cdots, a_l) : W^l \longrightarrow [0, 1]$ *a sequence of intuitionisic fuzzy-set-valued maps for* $i = 1, 2$*. Assume that the following assertions hold:*

(i) *We can find* $(\alpha, \beta) \in (0, 1] \times [0, 1)$ *such that* $[\omega_i(a_1, \cdots, a_l)]_{(\alpha,\beta)}$ *is a nonempty compact subset of* W*;*
(ii) *We can find a lower semi-continuous function* $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *satisfying* $\varphi(t) = 0$ *if and only if* $t = 0$ *with*

$$\underline{H}([\omega_i(a_1, \cdots, a_l)]_{(\alpha,\beta)}, [\omega_i(a_2, \cdots, a_{l+1})]_{(\alpha,\beta)}) \leq \max\{\sigma(a_i, a_{i+1}) : 1 \leq i \leq l\} - \varphi(\max\{\sigma(a_i, a_{i+1}) : 1 \leq i \leq l\}) \quad (15)$$

for all $(a_1, \cdots, a_{l+1}) \in W^{l+1}$ *and*

$$\underline{H}([\omega_i(a, \cdots, a)]_{(\alpha,\beta)}, [\omega_i(b, \cdots, b)]_{(\alpha,\beta)}) \leq \sigma(a, b) - \varphi(\sigma(a, b)) \quad (16)$$

holds for all $a, b \in W$. Then,

$$\varphi(\underline{H}(\mathcal{F}_{ix}(\omega_1), \mathcal{F}_{ix}(\omega_2))) \leq \delta,$$

where

$$\delta = \sup_{a \in W} \underline{H}([\omega_1(a, \cdots, a)]_{(\alpha,\beta)}, [\omega_2(a, \cdots, a)]_{(\alpha,\beta)}).$$

Proof. Following Theorem 3, we have the fact that $\mathcal{F}_{ix}(\omega_i)$ is nonempty. Let $\theta_0 \in [\omega_1(\theta_0, \cdots, \theta_0)]_{(\alpha,\beta)}$. Then, by Lemma 1, we can find $\theta_1 \in [\omega_2(\theta_0, \cdots, \theta_0)]_{(\alpha,\beta)}$ with

$$\sigma(\theta_0, \theta_1) \leq \underline{H}([\omega_1(\theta_0, \cdots, \theta_0)]_{(\alpha,\beta)}, [\omega_2(\theta_0, \cdots, \theta_0)]_{(\alpha,\beta)}).$$

Since $[\omega_1(\theta_1, \cdots, \theta_1)]_{(\alpha,\beta)}, [\omega_2(\theta_0, \cdots, \theta_0)]_{(\alpha,\beta)} \in \mathcal{K}(W)$, and $\theta_1 \in [\omega_2(\theta_0, \cdots, \theta_0)]_{(\alpha,\beta)}$, then, via Lemma 1, we can find $\theta_2 \in [\omega_1(\theta_1, \cdots, \theta_1)]_{(\alpha,\beta)}$ with

$$\sigma(\theta_1, \theta_2) \leq \underline{H}([\omega_2(\theta_0, \cdots, \theta_0)]_{(\alpha,\beta)}, [\omega_1(\theta_1, \cdots, \theta_1)]_{(\alpha,\beta)}).$$

Continuing in this way, we generate a sequence $\{\theta_{\check{p}}\}_{\check{p} \geq 1}$ in W with $\theta_{\check{p}} \in [\omega_2(\theta_{\check{p}-1}, \cdots, \theta_{\check{p}-1})]_{(\alpha,\beta)}, \theta_{\check{p}+1} \in [\omega_1(\theta_{\check{p}}, \cdots, \theta_{\check{p}})]_{(\alpha,\beta)}$ and

$$\begin{aligned}\sigma(\theta_{l+1}, \theta_{l+1}) &\leq \underline{H}([\omega_1(\theta_l, \cdots, \theta_l)]_{(\alpha,\beta)}, [\omega_2(\theta_{l+1}, \cdots, \theta_{l+1})]_{(\alpha,\beta)}) \\ &\leq \sigma(\theta_l, \theta_{l+1}) - \varphi(\sigma(\theta_l, \theta_{l+1})).\end{aligned}$$

Similarly,

$$\begin{aligned}\sigma(\theta_l, \theta_{l+1}) &\leq \underline{H}([\omega_1(\theta_{l-1}, \cdots, \theta_{l-1})]_{(\alpha,\beta)}, [\omega_2(\theta_l, \cdots, \theta_l)]_{(\alpha,\beta)}) \\ &\leq \sigma(\theta_{l-1}, \theta_l) - \varphi(\sigma(\theta_{l-1}, \theta_l)).\end{aligned}$$

Therefore, for all $l \geq \check{p}$, we have

$$\begin{aligned}\sigma(\theta_{l-\check{p}}, \theta_{l-\check{p}+1}) &\leq \underline{H}([\omega_1(\theta_{l-\check{p}-1}, \cdots, \theta_{l-\check{p}-1})]_{(\alpha,\beta)}, [\omega_2(\theta_{l-\check{p}}, \cdots, \theta_{l-\check{p}})]_{(\alpha,\beta)}) \quad (17) \\ &\leq \sigma(\theta_{l-\check{p}-1}, \theta_{l-\check{p}}) - \varphi(\sigma(\theta_{l-\check{p}-1}, \theta_{l-\check{p}})). \quad (18)\end{aligned}$$

Continuing as in Theorem 3, it follows that $\{\theta_{\check{p}}\}_{\check{p} \geq 1}$ is a Cauchy sequence in W, and the completeness of this space implies that we can find $u \in W$ with $\theta_{\check{p}} \longrightarrow u$ as $\check{p} \longrightarrow \infty$. Now, let $u \in [\omega_2(u, \cdots, u)]_{(\alpha,\beta)}$. Then, by assumption, we have

$$\begin{aligned}\sigma(\theta_0, \theta_1) &\leq \underline{H}([\omega_1(\theta_0, \cdots, \theta_0)]_{(\alpha,\beta)}, [\omega_2(\theta_0, \cdots, \theta_0)]_{(\alpha,\beta)}) \\ &\leq \sup_{a \in W} \underline{H}([\omega_1(a, \cdots, a)]_{(\alpha,\beta)}, [\omega_2(a, \cdots, a)]_{(\alpha,\beta)}) = \delta.\end{aligned}$$

Thus, via triangle inequality, we obtain

$$\begin{aligned}\sigma(\theta_0, u) &\leq \sigma(\theta_0, \theta_1) + \sigma(\theta_1, u) \\ &\leq \sigma(\theta_0, \theta_1) + \underline{H}([\omega_1(\theta_0, \cdots, \theta_0)]_{(\alpha,\beta)}, [\omega_2(u, \cdots, u)]_{(\alpha,\beta)}) \\ &\leq \delta + \sigma(\theta_0, u) - \varphi(\sigma(\theta_0, u)),\end{aligned}$$

which implies that $\varphi(\sigma(\theta_0, u)) \leq \delta$. It follows that for any random point $\theta_0 \in \mathcal{F}_{ix}(\omega_1)$, we can find $u \in \mathcal{F}_{ix}(\omega_2)$ with $\varphi(\sigma(\theta_0, u)) \leq \delta$. Based on similar steps, for any point $\xi_0 \in \mathcal{F}_{ix}(\omega_2)$, we can find an element $\gamma \in \mathcal{F}_{ix}(\omega_1)$ with $\varphi(\sigma(\xi_0, \gamma)) \leq \delta$. Consequently, it follows that $\underline{H}(\mathcal{F}_{ix}(\omega_1), \mathcal{F}_{ix}(\omega_2)) \leq \delta$. □

In order to achieve the following conclusions, we present the idea of uniform convergence of a series of intuitionistic fuzzy-set-valued maps.

Definition 11. *Let (W, σ) be an MS. A sequence of intuitionistic fuzzy-set-valued maps $\{\omega_{\check{p}}(a) : W \longrightarrow [0,1],\ \check{p} \in \mathbb{N}\}$ is said to converge uniformly to an intuitionistic fuzzy-set-valued map $\omega(a) : W \longrightarrow [0,1]$ if. for every $\epsilon > 0$ and for all $a \in W$, there exist $\check{p}_\epsilon \in \mathbb{N}$ and $(\alpha, \beta) \in (0,1] \times [0,1)$ with, for all $\check{p} \geq \check{p}_\epsilon$,*

$$\underline{H}([\omega_{\check{p}} a]_{(\alpha,\beta)}, [\omega a]_{(\alpha,\beta)}) < \epsilon. \tag{19}$$

If (19) holds, then we write

$$\lim_{\check{p} \longrightarrow \infty} \underline{H}([\omega_{\check{p}} a]_{(\alpha,\beta)}, [\omega a]_{(\alpha,\beta)}) = 0,$$

where $[\omega a]_{(\alpha,\beta)}$ is called the limiting cut set, and is given by

$$[\omega a]_{(\alpha,\beta)} = \lim_{\check{p} \longrightarrow \infty} [\omega_{\check{p}} a]_{(\alpha,\beta)}.$$

Example 4. *Take $W = [0, 10]$ and define $\sigma : W \times W \longrightarrow \mathbb{R}$ as $\sigma(a, b) = |a - b|$ for all $a, b \in W$. Consider two mappings $\alpha, \beta : W \longrightarrow (0,1]$ and a sequence of intuitionistic fuzzy-set-valued maps $\{\omega_{\check{p}}\}_{\check{p} \geq 1} = \{\langle \mu_{\omega_{\check{p}}}, \nu_{\omega_{\check{p}}} \rangle\}_{\check{p} \geq 1}$ specified by*

$$\mu_{\omega_{\check{p}}(a)}(t) = \begin{cases} \alpha(a), & \text{if } 0 \leq t \leq \frac{1}{(4\check{p}^2 + a)} \\ 0, & \text{if } \frac{1}{(4\check{p}^2 + a)} < t \leq 10, \end{cases} \quad \nu_{\omega_{\check{p}}(a)}(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq \frac{1}{(4\check{p}^2 + a)} \\ \beta(a), & \text{if } \frac{1}{(4\check{p}^2 + a)} < t \leq 10. \end{cases}$$

Suppose that $(\alpha, \beta) := (\alpha(a), 0)$ for all $a \in W$; then,

$$[\omega_{\check{p}} a]_{(\alpha,\beta)} = \left[0, \frac{1}{(4\check{p}^2 + a)}\right].$$

Given $\epsilon > 0$, we have

$$\underline{H}([\omega_{\check{p}} a]_{(\alpha,\beta)}, [\omega a]_{(\alpha,\beta)}) = \frac{1}{(4\check{p}^2 + a)} < \epsilon.$$

Observe that $\check{p} \geq \frac{1}{2}\sqrt{\left(\frac{1}{\epsilon} - a\right)}$ decreases with a and the maximum value is $\frac{1}{2\sqrt{\epsilon}}$. Hence, choose $\check{p}_\epsilon \geq \frac{1}{2\sqrt{\epsilon}}$ so that for $\epsilon > 0$, we can find $\check{p}_\epsilon \in \mathbb{N}$ with, for all $\check{p} \geq \check{p}_\epsilon$, $\underline{H}([\omega_n a]_{(\alpha,\beta)}, [\omega a]_{(\alpha,\beta)}) < \epsilon$. It follows that $\{\omega_{\check{p}}\}_{\check{p} \geq 1}$ converges uniformly to ω on W.

We recall that the FP sets $\mathcal{F}_{ix}(\omega_{\check{p}})$ of a sequence of multi-valued mappings $\omega_{\check{p}} : W \longrightarrow \mathcal{K}(W)$ are stable if $\underline{H}(\mathcal{F}_{ix}(\omega_{\check{p}}), \mathcal{F}_{ix}(\omega)) \longrightarrow 0$ as $\check{p} \longrightarrow \infty$, where $\omega = \lim_{\check{p} \longrightarrow \infty} \omega_{\check{p}}$. Similar to the concept of stability of FPs in [29,30], we propose the following definition of stability of FP sets of a sequence of intuitionistic fuzzy-set-valued maps.

Definition 12. *Let $\{\omega_{\check{p}}(a) = \langle \mu_{\omega_{\check{p}}}(a), \nu_{\omega_{\check{p}}}(a) \rangle : W \longrightarrow [0,1],\ a \in W, \check{p} \in \mathbb{N}\}$ be a sequence of intuitionistic fuzzy-set-valued maps that converges uniformly to an intuitionistic fuzzy-set-valued map $\omega(a) : W \longrightarrow [0,1]$. Suppose that $\{\mathcal{F}_{ix}(\omega_{\check{p}})\}_{\check{p} \geq 1}$ is the sequence of FP sets of the sequence $\{\omega_{\check{p}}\}_{\check{p} \geq 1}$ and $\{\mathcal{F}_{ix}(\omega)\}$ is the FP set of ω. Then, we say that the intuitionistic fuzzy FP sets of $\{\omega_{\check{p}}\}_{\check{p} \geq 1}$ are stable if*

$$\lim_{\check{p} \longrightarrow \infty} \underline{H}(\mathcal{F}_{ix}(\omega_{\check{p}}), \mathcal{F}_{ix}(\omega)) = 0.$$

Lemma 2. *Let (W, σ) be a complete MS, l a non-negative number, and $\{\omega_{\check{p}}(a_1, \cdots, a_l) : W^l \longrightarrow [0,1],\ \check{p} \in \mathbb{N}\}$ a sequence of intuitionistic fuzzy-set-valued maps uniformly convergent to $\omega(a_1, \cdots, a_l) : W^l \longrightarrow [0,1]$. If $\{\omega_{\check{p}}(a_1, \cdots, a_l)\}_{\check{p} \geq 1}$ satisfies (15) and (16) for each $\check{p} \in \mathbb{N}$, then ω also satisfies (15) and (16).*

Proof. Since $\omega_{\check{p}}$ satisfies (15) and (16) for each $\check{p} \in \mathbb{N}$, then for all $(a_1, \cdots, a_{l+1}) \in W$, we have

$$\begin{aligned} \underline{H}([\omega_{\check{p}}(a_1, \cdots, a_l)]_{(\alpha,\beta)}, [\omega_{\check{p}}(a_2, \cdots, a_{l+1})]_{(\alpha,\beta)}) \leq\ & \max\{\sigma(a_i, a_{i+1}) : 1 \leq i \leq l\} \\ & - \varphi(\max\{\sigma(a_i, a_{i+1}) : 1 \leq i \leq l\}), \end{aligned} \tag{20}$$

and

$$\underline{H}([\omega_{\check{p}}(a, \cdots, a)]_{(\alpha,\beta)}, [\omega_{\check{p}}(b, \cdots, b)]_{(\alpha,\beta)}) \leq \sigma(a, b) - \varphi(\sigma(a, b)). \tag{21}$$

As $\omega_{\check{p}}$ uniformly converges to ω and φ is lower semi-continuous, taking the upper limit in (20) and (21) yields

$$\begin{aligned} \underline{H}([\omega(a_1, \cdots, a_l)]_{(\alpha,\beta)}, [\omega(a_2, \cdots, a_{l+1})]_{(\alpha,\beta)}) \leq\ & \max\{\sigma(a_i, a_{i+1}) : 1 \leq i \leq l\} \\ & - \varphi(\max\{\sigma(a_i, a_{i+1}) : 1 \leq i \leq l\}), \end{aligned}$$

and

$$\underline{H}([\omega(a, \cdots, a)]_{(\alpha,\beta)}, [\omega(b, \cdots, b)]_{(\alpha,\beta)}) \leq \sigma(a, b) - \varphi(\sigma(a, b)).$$

□

In what follows, we apply Theorem 8 and Lemma 2 to establish a stability result for a sequence of intuitionistic fuzzy-set-valued maps.

Theorem 9. *Let (W, σ) be a complete MS and $\{\omega_{\check{p}}(a_1, \cdots, a_l) : W^l \longrightarrow [0,1],\ \check{p} \in \mathbb{N}\}$ be a sequence of intuitionistic fuzzy-set-valued maps, uniformly convergent to $\omega(a_1, \cdots, a_l) : W^l \longrightarrow [0,1]$. Assume that the following conditions are satisfied:*

(i) We can find $(\alpha, \beta) \in (0,1] \times [0,1)$ such that $[\omega_{\check{p}}(a_1, \cdots, a_l)]_{(\alpha,\beta)}$ and $[\omega(a_1, \cdots, a_l)]_{(\alpha,\beta)}$ are nonempty compact subsets of W;

(ii) $\omega_{\check{p}}$ satisfies (15) and (16) for each $\check{p} \in \mathbb{N}$.

Then,

$$\lim_{\check{p} \longrightarrow \infty} \underline{H}(\mathcal{F}_{ix}(\omega_{\check{p}}), \mathcal{F}_{ix}(\omega)) = 0,$$

that is, the set of all intuitionistic fuzzy FPs of $\omega_{\check{p}}$ is stable.

Proof. Via Lemma 2, ω satisfies (15) and (16). Let $\delta_{\check{p}} = \sup_{a \in W} \underline{H}([\omega_{\check{p}}(a)]_{(\alpha,\beta)}, [\omega a]_{(\alpha,\beta)})$. Since $\omega_{\check{p}}$ converges to ω uniformly on W, we have

$$\lim_{\check{p} \longrightarrow \infty} \delta_{\check{p}} = \lim_{\check{p} \longrightarrow \infty} \sup_{a \in W} \underline{H}([\omega_{\check{p}} a]_{(\alpha,\beta)}, [\omega a]_{(\alpha,\beta)}) = 0.$$

Applying Theorem 8 gives $\varphi(\underline{H}(\mathcal{F}_{ix}(\omega_{\check{p}}), \mathcal{F}_{ix}(\omega))) \leq \delta_{\check{p}}$ for all $\check{p} \in \mathbb{N}$. Given that φ is lower semi-continuous, we have

$$\lim_{\check{p} \longrightarrow \infty} \inf \varphi(\underline{H}(\mathcal{F}_{ix}(\omega_{\check{p}}), \mathcal{F}_{ix}(\omega))) \leq \lim_{\check{p} \longrightarrow \infty} \delta_{\check{p}} = 0,$$

from which it follows that

$$\lim_{\check{p} \longrightarrow \infty} \underline{H}(\mathcal{F}_{ix}(\omega_{\check{p}}), \mathcal{F}_{ix}(\omega)) = 0.$$

□

6. Applications to Non-Convex Fractional Differential Inclusions

One of the uses of invariant point theorems of contractive maps is in the analysis of differential equations. In this context, more than a handful of results have been presented. For example, Ali et al. [31] and Xu et al. [32] investigated some variants of predator–prey models using Caputo–Fabrizio operators. In the framework of complex-valued metric spaces, Hammad and De la Sen [33] applied a more general contractive criterion to discuss new existence conditions of solutions to Urysohn integral equations. Humairah et al. [34] presented new existence results for a coupled system of impulsive fractional differential equations in the context of complex-valued fuzzy metric spaces. Hammad and Zayed [35] put forward some types of boundary value problems for a system of coupled Atangana–Baleanu-type fractional differential equations and used Krasnoselskii's and Banach fixed-point techniques to examine their solvability criteria. For some related results involving the application of non-integer order differential operators, we refer to [36,37] and some citations therein.

Hereunder, the existence of solutions to non-convex fractional differential inclusions involving Caputo fractional derivatives of any order with nonlocal boundary conditions is next investigated, using one of the findings from Section 3. To this effect, consider

$$\begin{cases} {}^{C}D^{k}_{t_0} \in M(t, a(t)), & t \in \Omega = [0, \delta], \check{p} - 1 < k < \check{p}, \\ a^{(r)}(\phi) = a_r + \int_{t_0}^{t} f(s, a(s))ds, & r = 0, 1, \cdots, \check{p} - 1, \end{cases} \tag{22}$$

where $M : \Omega \times \mathbb{R} \longrightarrow P(\mathbb{R})$ is a multi-valued map, $P(\mathbb{R})$ is the power set of $\mathbb{R}$, $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given continuous function, and ${}^{C}D^{k}_{t_0}$ represents the Caputo fractional derivative of order k, $\check{p} = [k] + 1$, $[k]$ depicts the integer part of the real number k.

For convenience, we recall some necessary concepts of functional analysis and fractional calculus from [38,39] as follows. Let $C(\Omega, \mathbb{R})$ be the Banach space of all continuous real-valued functions defined on Ω with the norm specified by $\|a\| = \sup\{|a(t)| : t \in \Omega\}$. By $L^1(\Omega, \mathbb{R})$, we mean the Banach space of all measurable functions $a : \Omega \longrightarrow \mathbb{R}$ that are Lebesgue-integrable equipped with the norm $\|a\| = \int_{t_0}^{t} |a(t)|dt$.

Definition 13. *The fractional integral of order k with a lower limit zero of a function μ is given by*

$$I^{k}\mu(t) = \frac{1}{\Gamma(k)} \int_0^t \frac{\mu(s)}{(t-s)^{1-k}} ds, \ t > 0, k > 0,$$

provided the right-hand side is point-wise defined on $\mathbb{R}_+$, where $\Gamma(.)$ is the gamma function specified as

$$\Gamma(k) = \int_0^\infty t^{k-1} e^{-t} dt.$$

Definition 14. *The Riemann–Liouville fractional derivative of order $k > 0$, $\check{p} - 1 < k < \check{p}, \check{p} \in \mathbb{N}$ is specified as*

$$D^{k}_{0^+}\mu(t) = \frac{1}{\Gamma(\check{p} - k)} \left(\frac{d}{dt}\right)^{\check{p}} \int_0^t (t-s)^{\check{p}-k-1} \mu(s) ds,$$

where the function μ is absolutely continuous up to order $(\check{p} - 1)$.

Definition 15. *The Caputo fractional derivative of order k of a function $\mu : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is specified as*

$$\begin{aligned} {}^{C}D^{k}\mu(t) &= \frac{1}{\Gamma(\check{p} - k)} \int_0^t (t-s)^{\check{p}-k-1} \mu^{\check{p}}(s) d\, s \\ &= I^{\check{p}-k}\mu^{(\check{p})}(t), \ \check{p} - 1 < k < \check{p}, \check{p} = [k] + 1. \end{aligned}$$

Definition 16. *A multi-valued map $M : \Omega \longrightarrow P(\mathbb{R})$ with nonempty compact convex values is called measurable if for every $\varpi \in \mathbb{R}$, the function $t \longmapsto d(\varpi, M(t)) = \inf\{|\varpi - \varsigma| : \varsigma \in M(t)\}$ is measurable.*

Definition 17. *Let Ψ be a Banach space and S be a nonempty closed subset of Ψ. The multi-valued map $M : S \longrightarrow P(\Psi)$ is called lower semi-continuous if the set $\{\nu \in S : M(\nu) \cap G \neq \emptyset\}$ is open for every open set G in Ψ.*

Definition 18. *A subset W of $L^1(\Omega, \mathbb{R})$ is decomposable if, for all $u, v \in W$, and measurable set $\Delta \in \Omega$, the function $u\chi_\Delta + v\chi_{\Omega\setminus\Delta} \in W$, where χ_Δ denotes the characteristic function of Δ.*

Definition 19. *Let $M : \Omega \times \mathbb{R} \longrightarrow P(\mathbb{R})$ be a multi-valued map with nonempty compact values and $\varpi \in C(\Omega, \mathbb{R})$. Then, the set of selections of $M(.,.)$, denoted by $\mathcal{S}_{M,\varpi}$, is of lower semi-continuous type if*

$$\mathcal{S}_{M,\varpi} = \{\pi \in L^1(\Omega, \mathbb{R}) : \pi(t) \in M(t, \varpi(t))\},$$

for almost all $t \in \Omega$ is lower semi-continuous with nonempty closed and decomposable values.

To define the solution to Problem (22), we consider its linear variant given by

$$\begin{cases} {}^C D^k_{t_0} a(t) = g(t), & t \in \Omega \\ a^{(r)}(\phi) = a_r + \int_{t_0}^t f(s)ds, & r = 0, 1, \cdots, \check{p} - 1, \phi \in \Omega, \end{cases} \tag{23}$$

where $g \in C(\Omega, \mathbb{R})$.

Lemma 3 ([40])**.** *The fractional nonlocal boundary value problem (23) is equivalent to the integral equation:*

$$a(t) = I^k g(t) + \sum_{k=0}^{\check{p}-1} \frac{(t-\phi)^r}{r!} \left(a_r + \int_{t_0}^t f(s)ds - I^{k-r} g(\phi) \right), t \in \Omega. \tag{24}$$

Now, we apply Corollary 3 to investigate the existence of solutions to Problem (22) under the following assumptions.

Theorem 10. *Let $W = C(\Omega, \mathbb{R})$ and assume that the following conditions are satisfied:*

(C_1) : *the multi-valued map $M : \Omega \times \mathbb{R} \longrightarrow \mathcal{K}(\mathbb{R})$ is such that $M(., a) : \Omega \longrightarrow \mathcal{K}(\mathbb{R})$ is measurable for each $a \in \mathbb{R}$;*

(C_2) : *for almost all $t \in \Omega$ and $a, b \in \mathbb{R}$, we can find a function $\rho : \Omega \longrightarrow \mathbb{R}_+$ with*

$$\underline{H}(M(t, a(t)), M(t, b(t))) \leq \rho(t)|a - b| \text{ and } d(0, M(t, 0) \leq \rho(t));$$

(C_3) : *there exists a function $\tau : \Omega \longrightarrow \mathbb{R}_+$ such that for all $t \in \Omega$ and $a, b \in \mathbb{R}$,*

$$|f(t, a) - f(t, b)| \leq \tau(t)|a - b|;$$

(C_4) : *we can find $\check{\lambda} \in (0, 1)$ with $\xi_1\|\rho\| + \xi_2 \leq \check{\lambda}$, where*

$$\xi_1 = \left\{ \frac{2}{\Gamma(k+1)} + \sum_{r=1}^{\check{p}-1} \frac{1}{r!\Gamma(k-r+1)} \right\} (\delta - t_0)$$

$$\text{and } \xi_2 = \sum_{r=0}^{\check{p}-1} \frac{(\delta - t_0)^r}{r!} \|\tau\|.$$

Then, the fractional differential inclusion problem (22) has at least one solution in W.

Proof. Under the assumptions (C_1)–(C_4), we shall show that all the conditions of Corollary 3 are satisfied. Let $G, \underline{H} : W \longrightarrow (0,1]$ be two mappings. For each $a \in W$, define a mapping $\Lambda_a : W \longrightarrow W$ by

$$\Lambda_a(t) = I^k g(t) + \sum_{k=0}^{\check{p}-1} \frac{(t-\phi)^r}{r!} \left(a_r + \int_{t_0}^{t} f(s)ds - I^{k-r} g(\phi) \right), t \in \Omega.$$

Then, consider an intuitionistic fuzzy-set-valued map $\Theta = \langle \mu_\Theta, \nu_\Theta \rangle : W \longrightarrow (IFS)^W$ specified by

$$\mu_{\Theta(a)}(\eta) = \begin{cases} G(a), & \text{if } \eta(t) = \Lambda_a(t),\ t \in \Omega \\ 0, & \text{otherwise,} \end{cases} \qquad \nu_{\Theta(a)}(\eta) = \begin{cases} 0, & \text{if } \eta(t) = \Lambda_a(t),\ t \in \Omega \\ \underline{H}(a), & \text{otherwise.} \end{cases}$$

If $(\alpha, \beta) := (G(a), 0)$ for all $a \in W$, then we obtain

$$\begin{aligned} [\Theta(a)]_{(\alpha,\beta)} = & \Bigg\{ \eta \in W : \mu_{\Theta(a)}(\eta) = G(a) \text{ and } \nu_{\Theta(a)}(\eta) = 0 \\ & = \int_{t_0}^{t} \frac{(t-s)^{k-1}}{\Gamma(k)} g(s)d\,s \\ & + \sum_{r=0}^{\check{p}-1} \frac{(t-\phi)^r}{r!} \left(a_r + \int_{t_0}^{t} \left(f(s, a(s)) - \frac{(\phi - s)^{k-r-1} g(s)}{\Gamma(k-r)} \right) ds \right),\ g \in \mathcal{S}_{M,a} \Bigg\}. \end{aligned}$$

Note that $\mathcal{S}_{M,a}$ is nonempty for each $a \in W$ due to Condition (C_1). Therefore, the intuitionistic fuzzy-set-valued map Θ has a measurable selection. Clearly, $[\Theta(a)]_{(\alpha,\beta)}$ is compact for each $a \in W$. Next, we show that Θ is an intuitionistic fuzzy weakly contraction. Let $a, b \in W$ and $\eta_1 \in [\Theta a]_{(\alpha,\beta)}$. Then, we can find $\theta_1(t) \in M(t, a(t))$ such that, for all $t \in \Omega$, we have

$$\begin{aligned} \eta_1(t) = & \int_{t_0}^{t} \frac{(t-s)^{k-1}}{\Gamma(k)} \theta_1(s)d\,s \\ & + \sum_{r=0}^{\check{p}-1} \frac{(t-\phi)^r}{r!} \left(a_r + \int_{t_0}^{\phi} \left(f(s, a(s)) - \frac{(\phi - s)^{k-r-1} \theta_1(s)}{\Gamma(k-r)} \right) ds \right). \end{aligned}$$

Based on Condition (C_2), we have

$$\underline{H}(M(t, a(t)), M(t, b(t))) \leq \rho(t)|a(t) - b(t)|.$$

Thus, we can find $l^* \in M(t, b(t))$ with

$$|\theta_1(t) - l^*| \leq \rho(t)|a(t) - b(t)|,\ t \in \Omega.$$

Consider an operator $\Xi : \Omega \longrightarrow P(\mathbb{R})$ specified by

$$\Xi(t) = \{l^* \in \mathbb{R} : |\theta_1(t) - l^*| \leq \rho(t)|a(t) - b(t)|\}.$$

Since $\Xi \cap M(t, b(t))$ is measurable (see Proposition 3.4), we can find a function θ_2 that is a measurable selection for Ξ. It follows that $\theta_2(t) \in M(t, b(t))$, and for each $t \in$, we obtain

$$|\theta_1(t) - \theta_2(t)| \leq \rho(t)|a(t) - b(t)|.$$

Now, let

$$\eta_2(t) = \int_{t_0}^{t} \frac{(t-s)^{k-1}}{\Gamma(k)} \theta_2(s) d\,s + \sum_{r=0}^{\check{p}-1} \frac{(t-\phi)^r}{r!} \left(a_r + \int_{t_0}^{\phi} \left(f(s, a(s)) - \frac{(\phi - s)^{k-r-1} \theta_2(s)}{\Gamma(k-r)} \right) ds \right).$$

Hence, for each $t \in \Omega$, we have

$$\begin{aligned} |\eta_1(t) - \eta_2(t)| \leq & \int_{t_0}^{t} \frac{(t-s)^{k-1}}{\Gamma(k)} |\theta_1(s) - \theta_2(s)| d\,s \\ & + \sum_{r=0}^{\check{p}-1} \frac{(t-\phi)^r}{r!} \int_{t_0}^{\phi} \frac{(\phi - s)^{k-r-1}}{\Gamma(k-r)} |\theta_1(s) - \theta_2(s)| d\,s \\ & + \sum_{r=0}^{\check{p}-1} \frac{(t-\phi)^r}{r!} \int_{t_0}^{t} |f(s, a(s)) - f(s, b(s))| d\,s \\ \leq & \left\{ \left\{ \frac{(\delta - t_0)^k}{\Gamma(k+1)} + \sum_{r=0}^{\check{p}-1} \frac{(\delta - t_0)^k}{r! \Gamma(k-r+1)} \right\} \|\rho\| \right. \\ & \left. + \sum_{r=0}^{\check{p}-1} \frac{(\delta - t_0)^r}{r!} \|\tau\| \right\} \|a - b\|. \end{aligned}$$

Therefore,

$$\|\eta_1 - \eta_2\| \leq (\xi_1 \|\rho\| + \xi_2) \|a - b\|. \tag{25}$$

Equivalently, interchanging the roles of a and b in (25) and applying Condition (C_4), we have

$$\begin{aligned} \underline{H}([\Theta a]_{(\alpha,\beta)}, [\Theta b]_{(\alpha,\beta)}) \leq & \ \check{\lambda} \|a - b\| \\ = & \ \|a - b\| - (1 - \check{\lambda}) \|a - b\|. \end{aligned} \tag{26}$$

By defining $\varphi(t) = (1 - \check{\lambda})t$ for all $t \geq 0$ and $\check{\lambda} \in (0, 1)$, (26) can be written as

$$\underline{H}([\Theta a]_{(\alpha,\beta)}, [\Theta b]_{(\alpha,\beta)}) \leq \|a - b\| - \varphi(\|a - b\|). \tag{27}$$

Now, taking the supremum over all of $(\alpha, \beta) \in (0, 1] \times [0, 1)$ in (27), we obtain

$$d_{(\infty,\infty)}(\Theta a, \Theta b)) \leq d(a, b) - \varphi(d(a, b))$$

for all $a, b \in W$. Note that φ is lower-semi-continuous and $\varphi(t) = 0$ if and only if $t = 0$. This proves that Θ is an intuitionistic fuzzy weak contraction on W. Therefore, by applying Corollary 3, Θ has at least one intuitionistic fuzzy FP in W, which corresponds to the solutions of Problem (22). □

Example 5. *Consider the fractional differential inclusion problem given by*

$$\begin{cases} {}^{C}D_0^{5.2} a(t) \in M(t, a(t)), & t \in [0, 1] \\ a^{(r)}(0.2) = 1 + \int_0^{0.2} \frac{s^r}{5(r+1)} e^{-\cos a(s)} ds, & r = 0, 1, \cdots, 4, \end{cases} \tag{28}$$

where $t_0 = 0$, $\delta = 1$, $k = 5.2$, $\phi = 0.2$, $a_r = 1$, $f(t, a(t)) = \frac{t^r}{5(r+1)} e^{-\cos a(t)}$ *and* $M : [0, 1] \times \mathbb{R} \longrightarrow P(\mathbb{R})$ *is a multi-valued map specified by*

$$M(t, a(t)) = \left[0, \frac{t^2 \cos^2 a(t)}{9(4 + \cos^2 a(t))} \right].$$

Obviously, the map $t \longmapsto \left[0, \frac{t^2\cos^2 a(t)}{9(4+\cos^2 a(t))}\right]$ *is measurable for each* $a \in \mathbb{R}$*. Moreover, for all* $a, b \in W = C([0,1], \mathbb{R})$*, we have*

$$\begin{aligned} |f(t,a(t)) - f(t,b(t))| &\leq \frac{t^r}{5(r+1)}\left|e^{-\cos a(t)} - e^{-\cos b(t)}\right| \\ &\leq \frac{t^r}{5(r+1)}|a(t) - b(t)|. \end{aligned}$$

In this case, $\tau(t) = \frac{t^r}{5(r+1)}$*, hence* $\|\tau\| = \frac{1}{5(r+1)}$*. In addition, note that for each* $(t,a) \in [0,1] \times \mathbb{R}$*,*

$$\begin{aligned} \sup\{\zeta : \zeta \in M(t,a)\} &\leq \frac{t^2\cos^2 a(t)}{9(4+\cos^2 a(t))} \\ &\leq \frac{1}{9}. \end{aligned}$$

Therefore,

$$\begin{aligned} \underline{H}(M(t,a(t)), M(t,b(t))) &= \left(\left[0, \frac{t^2\cos^2 a(t)}{9(4+\cos^2 a(t))}\right], \left[0, \frac{t^2\cos^2 b(t)}{9(4+\cos^2 b(t))}\right]\right) \\ &\leq \frac{1}{9}|a-b|. \end{aligned} \tag{29}$$

From (29), $\rho(t) = \frac{t}{9}$ *and* $\|\rho\| \approx 0.1111$*. Moreover, direct computation yields*

$$\xi_1 = \frac{2}{169.4061} + \frac{1}{32.5781} + \frac{1}{15.5134} + \frac{1}{14.544} \approx 0.1757.$$

and

$$\xi_2 = \frac{1}{5} + \frac{1}{10} + \frac{1}{30} + \frac{1}{120} \approx 0.3417.$$

Hence,

$$\xi_1\|\rho\| + \xi_2 = 0.3612 \leq \check{\lambda} = 0.5 \in (0,1).$$

Therefore, all the hypotheses of Theorem 10 are satisfied. Consequently, Example 5 has at least one solution in $C([0,1], \mathbb{R})$*.*

7. Concluding Remarks

Based on the notions of intuitionistic fuzzy sets and motivated by the fuzzy FP theorem due to Heilpern [4], a few ideas about Presic-type intuitionistic fuzzy FP results in the setting of MS are introduced in this article. The stability of intuitionistic fuzzy FP problems and some related concepts are suggested herein to add up their crisp analogs in the framework of set-valued and single-valued mappings in the existing literature. It is observed that by slimming down the intuitionistic fuzzy-set-valued maps in this work to their non-fuzzy counterparts, the ideas proposed herein complement a significant number of well-known results with metric structure. It is important to note that every crisp set is an intuitionistic fuzzy set with a zero degree of hesitancy. Moreover, as an application, one of our results is rendered to investigate novel conditions for the existence of solutions to non-convex fractional differential inclusions. The hypotheses of the latter result have also been supported with a relevant example (Example 5). As far as we know, the classical result of Presic [2] has never been examined in the framework of intuitionistic fuzzy maps. Consequently, the ideas of this paper are new.

It is worth noting that while the present results in this work are theoretical, a lot of recent studies dealing with existence problems of fractional calculus and its applications will likely encourage future investigations along the lines proposed in this article. Moreover, the underlying MS in this manuscript can be extended to some generalized metric and quasi MSs, such as b-MS, complex-valued MS, G-MS, metric-like space, C^*-algebra-valued MS, geodesic MS, fuzzy MS, and similar other ones.

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