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Review

Field-Theoretic Renormalization Group in Models of Growth Processes, Surface Roughening and Non-Linear Diffusion in Random Environment: Mobilis in Mobili

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Abstract: This paper is concerned with intriguing possibilities for non-conventional critical behavior that arise when a nearly critical strongly non-equilibrium system is subjected to chaotic or turbulent motion of the environment. We briefly explain the connection between the critical behavior theory and the quantum field theory that allows the application of the powerful methods of the latter to the study of stochastic systems. Then, we use the results of our recent research to illustrate several interesting effects of turbulent environment on the non-equilibrium critical behavior. Specifically, we couple the Kazantsev–Kraichnan "rapid-change" velocity ensemble that describes the environment to the three different stochastic models: the Kardar–Parisi–Zhang equation with time-independent random noise for randomly growing surface, the Hwa–Kardar model of a "running sandpile" and the generalized Pavlik model of non-linear diffusion with infinite number of coupling constants. Using field-theoretic renormalization group analysis, we show that the effect can be quite significant leading to the emergence of induced non-linearity or making the original anisotropic scaling appear only through certain "dimensional transmutation".

Keywords: cooperative systems; critical behavior; renormalizaton group; universality; scaling; kinetic roughening; random growth; self-organized criticality

Citation: Antonov, N.V.; Gulitskiy, N.M.; Kakin, P.I.; Lebedev, N.M.; Tumakova, M.M. Field-Theoretic Renormalization Group in Models of Growth Processes, Surface Roughening and Non-Linear Diffusion in Random Environment: Mobilis in Mobili. *Symmetry* **2023**, *15*, 1556. https://doi.org/10.3390/sym15081556

Academic Editors: Michal Hnatič and Juha Honkonen

Received: 10 June 2023
Revised: 24 July 2023
Accepted: 3 August 2023
Published: 8 August 2023

1. Introduction

To see a World in a Grain of Sand
And a Heaven in a Wild Flower
Hold Infinity in the palm of your hand
And Eternity in an hour
Auguries of Innocence by William Blake

Numerous physical systems of different physical natures demonstrate very interesting singular behaviors in the vicinity of their critical points. Critical behavior of strongly non-equilibrium systems has attracted constant attention over the decades; see, e.g., [1–12] and the literature cited therein. Examples are provided by kinetic roughening of surfaces or interfaces [6,7,9], propagation of flame, smoke and solidification fronts, percolation processes [5], random walks and diffusion in random (e.g., porous or turbulent) media [12], models with self-organized criticality [11], transitions between fluctuating and absorbing states [1,2], reaction–diffusion problems [3] and many others. In comparison with more conventional equilibrium cases, their critical behavior is much richer but much less understood.

Any system of this kind necessarily involves many strongly interacting degrees of freedom of different scales. The very formulation of corresponding models is an extremely demanding task, and the same is true for their effort- and time-consuming quantitative and qualitative analysis.

However, experience with equilibrium nearly-critical states shows that principal universal properties of many-body cooperative systems can be adequately and nearly exhaustively described by phenomenological or simple microscopic models, designed on the basis of considerations of simplicity, dimensions and symmetries. The guiding examples are provided by various mean-field theories and by the Ising and Heisenberg models of second-order phase transitions in liquid–vapor or magnetic systems.

Such systems share a characteristic feature which is that the correlation length diverges near the critical point despite the microscopic interactions being short range. It means that the precise nature of the systems (whether they correspond to liquids or solid states like magnets, superconductors, etc.) becomes unimportant as their behavior is governed by highly universal power laws for correlation functions. Universality here refers to sole dependence of essential characteristics (critical exponents, normalized scaling functions and amplitude ratios) on the "global" features such as the systems' symmetries and dimensions. This is expressed in the idea of "universality classes". All of that makes it possible to develop the critical state theory for all those different systems instead of creating a new particular theory for every example of critical phenomena (which is, of course, a very important undertaking in itself).

The modern critical state theory relies on the field-theoretic renormalization group (RG); see [13–16] for the details and the references.

The RG analysis allows us to formulate so-called RG equations for renormalizable field theories and to calculate the fixed points coordinates of those equations. Infrared (IR) attractive fixed points define critical behavior, i.e., universality classes. The field-theoretic ϕ^4 model is written for a scalar order parameter with $O(n)$-symmetry, and it describes the universality class of the most common equilibrium phase transitions. The number of scalar components n and the space dimension d are universal parameters, so universal features of the critical behavior depend only on them. The latter are calculated via perturbation theory where the role of the small parameter is usually played by $\varepsilon = 4 - d$, i.e., the deviation of the space dimension from its "logarithmic" value $d = 4$. (In general, the choice of small parameter depends on the model. In particular, for the ϕ^m interaction, the deviation from the logarithmic dimension $d = 2m/(m-2)$ plays that role. For many other models (turbulence, advection), the logarithmicity is not achieved by changing d, and the analog of ε has a different physical nature and is not related to d (see below). For some models, there can be several such parameters, which leads to generalized "multiple ε expansions"; see, e.g., [17–28] and references therein).

By the early 1980s, the RG approach was largely complete and became the common language of physicists when discussing phase transitions and critical phenomena. In subsequent years, the realm of applicability of the RG rapidly grew; it now includes critical dynamics [29,30] and Chapter 5 in [16], strongly non-equilibrium phase transitions, diffusion-limited chemical reactions [1–4,31], driven diffusive systems [32,33], percolation [5], roughening of fluctuating surfaces, growth processes and propagation of fronts [6–10,34–36], turbulence and turbulent transport [37,38], systems with self-organized criticality [11,39–46] and random walks and anomalous diffusion in random media [12,47–52] (not an exhaustive list).

In many semi-phenomenological formulations, those phenomena are described by stochastic differential equations for smoothed (coarse-grained) fields with additive random forces (noises). The key point for the applicability of the field-theoretic RG approach is the general statement that such equations can be reformulated as certain field-theoretic models for extended sets of fields; see the original references [53–59] and Chapter 5 in [16] for the review, proof and discussion.

Thus, problems of non-equilibrium dynamics can be studied with the well-developed techniques of quantum field theory (QFT): functional methods (including various Schwinger–Dyson equations, Ward identities, Legendre transformations of generating functionals), various methods of calculation of higher-order Feynman diagrams, renormalization theory (including composite operators), short-distance operator–product expansions and instanton calculus and resummations of asymptotic series; see [13–16] for the details and the references.

Another important direction to study is focused on more realistic ("non-ideal" or "dirty") systems and the effects of various external disturbances, perturbations and that of internal irregularities, heterogeneities and disorder.

The available experience with real nearly-critical systems has revealed their dramatic susceptibility to all kinds of perturbations, including gravity, experimental setup topology, presence of impurities, environment movement, etc.; see, e.g., [60] for the liquid–vapor transition. "Pure" systems in a state of thermodynamic equilibrium (that ordinarily would display ideal critical behavior in infinite volume) may be still troubled by finite size effects, non-equilibrium impurities, finite evolution (aging) time, etc.

The effect can be as considerable asa phase-transition-type change; new universality classes with unforeseen features can also appear [61]. In particular, the effects of disorder on original "ideal" models were covered in a vast number of papers [62–68].

As another prominent example, the critical behavior of a system can also be highly affected by the motion of the surrounding medium: see [24,69–74]. In practice, real systems can hardly be isolated from the influence of surrounding mediums, like turbulent motion in the atmosphere, in the ocean, or, especially, in forest fires.

In this paper, we briefly review our recent research concerning the effects of the turbulent environment on non-equilibrium critical behavior. In Section 2, we introduce the basic stochastic models of growth phenomena, surface roughening, landscape erosion and self-organized criticality. In Section 3, we introduce two basic models of turbulent fluid: the Kazantsev–Kraichnan ensemble and the stirred Navier–Stokes equation. Then we couple the Kazantsev–Kraichnan ensemble to the three different stochastic models: to the Kardar–Parisi–Zhang equation with time-independent random noise (Section 4) of surface roughening, to the Hwa–Kardar model of self-organized criticality (Section 5) and to the extended Pavlik model of non-linear diffusion (Section 6). We apply the field-theoretic approach to the resulting problems and then discuss the findings and perspectives of this kind of research in Section 7.

We show that the effect of environment can be as strong as leading to emergence of induced non-linearity practically changing the original model itself (the case of the Kardar–Parisi–Zhang equation) or making the original anisotropic scaling to appear only through certain "dimensional transmutation" in a specific regime of critical behavior (the case of the Hwa–Kardar model). In those instances, the competition between intrinsic dynamics and external disturbance resulted in a symmetry breaking: Galilean symmetry of the Kardar–Parisi–Zhang equation and spatial anisotropy of the Hwa–Kardar equation. In the case of the extended Pavlik model, the symmetry of the original model held resulting in two-dimensional infinite surfaces of fixed points that govern its critical behavior.

2. Basic Stochastic Models

2.1. Kardar–Parisi–Zhang Equation

Over decades, constant interest has been attracted to the random growth phenomena and surface roughening. The vast number of examples include bacterial colony spread, growth of tumors, fronts propagation (roughening of a phase boundary between, e.g., a burning part of a rye field and an unaffected part), molecular beam epitaxy and many others; see [6–10,34–36] and references therein.

The experience shows that various correlation and response (Green's) functions of these problems demonstrate scaling (self-similar) behavior in the IR range (long times, large distances). In particular, for the structure functions, one has

$$S_n(t,r) = \langle [h(t,\mathbf{x}) - h(0,\mathbf{0})]^n \rangle \simeq r^{n\chi} F_n(t/r^z), \quad r = |\mathbf{x}|, \tag{1}$$

while for the linear response function one usually has

$$G(t,r) = \langle h(t,\mathbf{x}) h'(0,\mathbf{0}) \rangle \simeq r^{-d} F(t/r^z), \tag{2}$$

where the brackets $\langle \cdot \rangle$ denote averaging over the statistical ensemble, χ and z are the traditional notation for the two main critical exponents, the roughness exponent and the dynamical exponent, respectively, and $F_n(\cdot)$, $F(\cdot)$ are certain scaling functions.

Here and below, throughout the paper, we denote $x = \{t, \mathbf{x}\}$, where $\mathbf{x} = \{x_i\}$, $i = 1, \ldots, d$ is the spatial coordinate of the d-dimensional substrate, $h = h(x) = h(t,\mathbf{x})$ is a certain basic scalar field, e.g., fluctuating part of the surface height), $h' = h'(x)$ is an auxiliary response field (see below).

The goal of the theory is to establish scaling relations of the type (1) and (2) on the base of certain dynamical models, to calculate the critical exponents like χ and z and scaling functions like $F(\cdot)$ in a systematic way and to investigate their universality (that is, the dependence on the dimension of space d and other parameters of the model).

The most celebrated model of the surface dynamics is the Kardar–Parisi–Zhang (KPZ) equation [75], sometimes referred to as "the Ising model for non-equilibrium phenomena". It is described by the stochastic equation of the form:

$$\partial_t h = \nu_0 \partial^2 h + V(h) + f. \tag{3}$$

Here and below, $\nu_0 > 0$ is the surface tension coefficient (in other applications, the viscosity or diffusivity coefficient), $\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial x_i$, $\partial^2 = \partial_i \partial_i$ is the Laplace operator and f is the random noise to be specified below; summation over repeated vector indices is always implied.

In the original KPZ model, the non-linearity $V(h)$ is taken in the form

$$V(h) = \frac{\lambda_0}{2} (\partial h)^2 = \frac{\lambda_0}{2} (\partial_i h)(\partial_i h), \tag{4}$$

where the coupling constant λ_0 can be of either sign.

To be precise, the model pioneered in [76], where it was written for a potential vector field $v_i = \partial_i h$ and included as a stochastic d-dimensional generalization of the Burgers equation. Then d has the meaning of the full coordinate space.

In the majority of studies, the noise $f = f(x)$ is taken to be Gaussian and white in time and space. Then, it is defined by the pair correlation function. (The noise f is supposed to have a certain constant component $\langle f \rangle$ that guarantees that $\langle h(x) \rangle = 0$, which follows from the meaning of h as a fluctuating part, but in practical calculations both of them can be simultaneously ignored).

$$\langle f(x) f(x') \rangle = D_0 \, \delta(x - x') = D_0 \, \delta(t - t') \, \delta^{(d)}(\mathbf{x} - \mathbf{x}'), \quad D_0 > 0. \tag{5}$$

Other relevant types of the noise statistics will be introduced later in Section 2.4.

Equation (3) is studied on the entire t axis; the retardation condition is assumed; the asymptotic condition for the field h at $t \to -\infty$ is irrelevant due to the presence of the random noise.

The RG was applied to the stochastic problem (3)–(5) in the very first papers [75,76] in the form of Wilson's recursion relations. Later, the more advanced field-theoretic RG, suitable for higher-order calculations, was employed; see [77] for the references and critical discussion.

The original stochastic problem can be reformulated as a certain multiplicatively renormalizable field-theoretic model which is logarithmic at $d = 2$, so that the part of a formal small expansion parameter is played by the deviation $\varepsilon = 2 - d$.

The corresponding β function (coefficient in the differential RG equations) for the coupling constant u, the renormalized analog of the combination $u_0 = \lambda_0^2 D_0 / 4\pi\nu_0^3$ (the actual expansion parameter in ordinary perturbation theory) has the form (in the most convenient minimal subtraction renormalization scheme):

$$\beta(u) = -u\,(\varepsilon + u). \tag{6}$$

This result is exact to all orders of perturbation theory; see [78,79].

Possible asymptotic scaling regimes are governed by the fixed points u^*, solutions of the equation $\beta(u^*) = 0$. The point is IR attractive for $\beta'(u^*) > 0$ and IR repulsive (ultraviolet (UV) attractive) for $\beta'(u^*) < 0$. From (6), it follows that the trivial (free or Gaussian) fixed point $u^* = 0$ is IR repulsive for $d < 2$ and attractive for $d > 2$. For this fixed point, the exponents in (1) are found exactly: $\chi = (d-2)/2$, $z = 2$ and $\chi = 0$, $z = 2$, respectively.

The nontrivial point $u^* = -\varepsilon$ is IR attractive for $d < 2$ but lies in the unphysical region $u^* < 0$. For $d > 2$, this fixed point lies in the physical region $u^* > 0$ and is IR repulsive. It is interpreted as the boundary between the interval $0 < u < u^*$, where the IR behavior is governed by the Gaussian point $u^* = 0$, and the region $u > u^*$. There, the IR behavior is supposed to be described by a certain strong-coupling regime; see, e.g., the discussion in [80] and references therein.

Within the RG framework, it is natural to associate this strong-coupling regime with a certain nonperturbative IR attractive fixed point, not "visible" in the perturbative β function (6). By continuity, this hypothetical point should also exist for $d \leq 2$, because the scaling behavior is observed there, too. Existence of the strong-coupling fixed point is supported by the functional RG [81–84].

Thus, according to the common opinion, nontrivial scaling behavior exists for all $1 \leq d < d_c$, where d_c is the upper critical dimension. In early works [77,80], it was argued that $d_c \leq 4$, while more recent studies suggest that $d_c = \infty$; see [85,86].

It is interesting to note that this pattern of fixed points of the KPZ equation is nicely modeled by the simple modification of (6) of the form $\beta(u) = -u\,(\varepsilon + u - au^2)$ with $a > 0$. In particular, it "predicts" the finite value $d_c = 2 - 1/4a\varepsilon > 2$. At $d = d_c$ the analog of the strong-coupling fixed point coalesces with the perturbative UV fixed point, so that for $d > d_c$, they both disappear. Then, for all $u > 0$, the IR behavior is governed by the Gaussian point $u^* = 0$. For $d = 2$, the β function of this form was obtained in [87], where a non-minimal renormalization scheme was employed.

For general d, the symmetry of the model (which takes on the form of Galilean symmetry in terms of the potential vector field $v_i = \partial_i h$) gives the exact relation $\chi + z = 2$; see, e.g., [80] for the references. For $d = 1$, the fluctuation–dissipation relation makes it possible to find the exact values $\chi = 1/2$, $z = 3/2$ [76].

Some attempts were undertaken to derive exact results for $d > 1$. In [88], an infinite sequence of possible "quantized" exact values of χ were obtained using certain nonperturbative requirements imposed onto the operator product expansion; two of them were identified as the values for $d = 2$ and 3. In [89], the fractal nature of a rough surface plays the central role and the exact values of the exponents are expressed in terms of its fractal dimension. In both cases, the results are in reasonable agreement with numerical simulations.

2.2. Generalized Pavlik's Model

Numerous modifications of the original KPZ equation have been proposed: "colored" noise f with finite correlation time [90,91], "quenched" h-dependent and time-independent noise [92,93], vector or matrix field h [94–96], random coupling constant [66], inclusion of superdiffusion [97], long-range temporal correlations [98], inclusion of anisotropy [92,99–101], conservation law for the field h [102–104], modified non-linearity $V(h)$ [105] and so on.

Probably the most interesting modification was introduced by Pavlik [105], where the non-linear term in (3) is taken in the form

$$V(h) \sim \partial^2 h^2/2 = (\partial h)^2 + h\partial^2 h. \tag{7}$$

The first term is the original KPZ non-linearity (4), while the second can be viewed as an h-dependent contribution to the surface tension, which makes it essentially non-linear.

Note that in the absence of the noise, Equations (3) and (7) take on the form of the conservation law for the quantity h. In this respect, Pavlik's model has a close formal resemblance to the anisotropic models of self-organized criticality [106,107] and of landscape erosion [108,109], where the non-linearities also have the forms of total derivatives, and the model can be viewed as an isotropic analog of the latter ones. In fact, a variation of the Pavlik's model first appeared in [110] in connection to SOC where the need to extend the variation to include an infinite number of terms in the presence of time-independent non-conserved noise was established.

It should be noted that the idea that the non-linearity in diffusion equation might include arbitrary powers of the scalar field has naturally been around for a long time. For example, it was suggested as early as 1937 that a fluid flow through porous media may be described by a deterministic diffusion equation with non-linearity of the form $\partial^2 h^{(1+m)/m}$, see Equation (5) in [111]. Another examples are provided by the equations [112,113].

Indeed, the model [105] in its original formulation appears not to be self-sufficient: it is not closed concerning renormalization. The dimensional analysis and more sophisticated renormalization considerations show that an infinite number of non-linear terms $\partial^2 h^n$ with all $n \geq 2$ are equally relevant and should all be included in the model from the very beginning [114]. Thus, the generalized renormalizable model necessarily involves infinitely many coupling constants:

$$V(h) = \sum_{n=2}^{\infty} \lambda_{n0} \partial^2 h^n / n!. \tag{8}$$

Later, a similar infinite-charge situation was also encountered in stochastic models of strongly non-linear diffusion [115,116]. More examples are provided by stochastic models of landscape erosion: there, correct RG analysis also requires extending the original models [108,109] by adding infinite number of interaction terms and corresponding coupling constants [117–121]. These issues are discussed below in Section 2.4.

Recently, the extended Pavlik model (8) was used in the papers [122,123] to describe the critical activity of brain neurons and was introduced as the IR limit of the Wilson–Cowan equation, while before that it had been mostly discussed in connection to diffusion [124,125].

2.3. Hwa–Kardar Continuous Model of SOC

The concept of self-organized criticality (SOC) was famously introduced to explain the abundance of phenomena with the IR range power-law scaling of spatial, temporal or spatial–temporal correlations in Nature [11,39–46]. Unlike equilibrium systems that undergo phase transitions when a tuning parameter made to arrive at its critical value, non-equilibrium systems with SOC are believed to evolve to their critical state without any external tuning taking place. Rather, their intrinsic dynamics drive them towards the critical point.

The pursuit of verifying this exciting concept gathered scientists from very different fields such as biology [126,127], neurology [128–133], studies of social networks [134–139] and others. While many questions surrounding the concept and its applicability are still open despite the high maturity of the field [140–143], the rapid pace of technological advancement, especially in data analysis, makes one hopeful for swift resolution of some of these questions.

In the study of the equilibrium critical behavior of discrete systems, continuous models have been successfully used to investigate universal scaling. It was established, for example, that the discrete Ising and Heisenberg lattice models belong to the universality class of the continuous $O(n)$-symmetric φ^4 model; see [13–16]. A non-equilibrium example is provided by the KPZ universality class [6] that covers numerous discrete models of surface roughening; for the reaction-diffusion models, see, e.g., [144]. The conserved directed percolation was also considered from this point of view [145,146]; its relation to the Manna universality class of SOC makes that example particularly pertinent.

Thus, one can hope that basic universal IR properties of the discrete models of SOC can be described by certain continuous (coarse-grained) models, similar to models of critical dynamics, without "throwing the baby out with the water".

A continuous model of SOC was introduced by Hwa and Kardar in [106,107]; it is an anisotropic stochastic differential equation describing the dynamics of a "running" sandpile (a sand-dune or a barkhan), in particular, its surface. Sand entering the system from above causes avalanches that come down in a set direction. The surface grows rougher and rougher with time while remaining flat on average.

The equation for $h(x) = h(t, \mathbf{x})$, the deviation of the sand profile height from its average, reads

$$\partial_t h = \nu_{\perp 0}\, \boldsymbol{\partial}_\perp^2 h + \nu_{\|0}\, \partial_\|^2 h + V(h) + f \tag{9}$$

with

$$V(h) = \lambda_0 \partial_\| h^2/2. \tag{10}$$

Here, we used several new notations; a unit constant vector $\mathbf{n}$ denotes the set direction for the sand transport; decomposition $\mathbf{x} = \mathbf{x}_\perp + \mathbf{n}\, x_\|$ with $(\mathbf{x}_\perp \cdot \mathbf{n}) = 0$ allows us to introduce two spatial derivatives: the $(d-1)$-dimensional gradient $\boldsymbol{\partial}_\perp = \{\partial_i\}$, $i = 1, \ldots, (d-1)$, and the one-dimensional gradient $\partial_\| = (\mathbf{n} \cdot \boldsymbol{\partial})$. There are two different diffusivity coefficients $\nu_{\|0}$ and $\nu_{\perp 0}$ and a coupling λ_0. The random noise $f(x)$ is defined as in Equation (5).

The model is logarithmic at $d = 4$ and has a nontrivial IR attractive fixed point for $d < 4$ [106,107].

2.4. Pastor–Satorras–Rothman Model of Landscape Erosion

The Pastor-Satorras–Rothman anisotropic model [108,109] of landscape erosion was introduced in order to explain discrepancies in the results of experimental measurements of the roughness exponent χ. The reported results clustered at the opposite ends of a relatively wide range of values, prompting the authors of [108,109] to consider the role of anisotropy of the landscape at the small length scales.

The model is a stochastic differential Equation (9), but with

$$V(h) = \lambda_0 \partial_\|^2 h^3/2. \tag{11}$$

The function $V(h)$ is odd in h so as to preserve the symmetry $h, f \to -h, -f$. There is also symmetry $x_\| \to -x_\|$.

The authors of [108,109] considered two types of random noise f. The first type was ordinary white noise (5); the resulting model was logarithmic at $d = 2$. The second type was the time-independent (static or columnar) noise with the correlation function

$$\langle f(x) f(x') \rangle = D_0\, \delta^{(d)}(\mathbf{x} - \mathbf{x}'), \quad D_0 > 0. \tag{12}$$

The noise (12) corresponds to a case where the main cause of scaling in the process of erosion is heterogeneity of the soil rather that random external disturbances (such as rainfall and so on). It was introduced in [147] and was prompted by the features of erosion observed in experiments [148] (also cf. [63–65] regarding connection to non-universality in directed percolation).

The noise (12) is a special case of a quenched noise (the authors of the present paper cannot comment on the obvious oddity that the deterministic correlation function (13) depends on the random field h):

$$\langle f(x)f(x')\rangle = D_0\,\Delta(h-h')\,\delta^{(d)}(\mathbf{x}-\mathbf{x}'), \tag{13}$$

where h and h' are the values of the scalar field at the corresponding points $\mathbf{x}$ and $\mathbf{x}'$. Although this noise appears in many different problems [1,62,92,100], it resists analytical approaches due to the factor $\Delta(h-h')$.

It turns out that the models (9) and (11) are renormalizable only in their extended version with infinitely many coupling constants; this holds both for the white noise (5) and for the time-independent noise (12); see [117–121]. For the extended model,

$$V(h) = \sum_n \lambda_{n0}\partial_{\|}^2 h^n/2. \tag{14}$$

Note that it is possible to consider a narrowed version of $V(h)$ odd in h; that is, $n = 2k+1$ in (14). Then, the symmetry $h, f \to -h, -f$ guarantees its renormalizability. The original model [108,109] belongs to this subclass. This "odd" version of the model (14) with the time-independent noise (12) was investigated in [119] within the functional RG. A line of IR attractive fixed points in $d = 2$ was found confirming the hypothesis of non-universality (that would explain a wide range of experimental results).

3. Models of Turbulent Velocity Fields

The interaction with the surrounding fluid is usually introduced with the "minimal" substitution in Equations (3) and (9):

$$\partial_t \to \nabla_t = \partial_t + (\mathbf{v}\cdot\boldsymbol{\partial}) = \partial_t + v_l\partial_l. \tag{15}$$

Here ∇_t is the Galilean covariant (Lagrangian) derivative, while $\mathbf{v}(x) = \{v_i\}$ with $i = 1,\dots,d$ is the velocity field. In the following, we assume that the fluid is incompressible so that the velocity field is transverse: $(\partial\mathbf{v}) = 0$, i.e., $(\mathbf{k}\mathbf{v}) = 0$ in the momentum representation.

The Navier–Stokes stochastic differential equation for an isotropic incompressible viscous fluid with an external random stirring force has the form (see, e.g., original papers [76,149–151], monographs [16,37] and the review [38]):

$$\nabla_t v_i = \nu_0\partial^2 v_i - \partial_i\wp + f_i, \tag{16}$$

where ν_0 is the kinematic viscosity coefficient and $\wp$ is the pressure. The latter can be expressed in terms of $\mathbf{v}$ as $\wp = -\partial^{-2}\partial_i\partial_k v_i v_k$. To be precise,

$$\wp(t,\mathbf{x}) = -\int d\mathbf{x}'\,\Delta(\mathbf{x}-\mathbf{x}')\partial_i\partial_k\, v_i(t,\mathbf{x}')v_k(t,\mathbf{x}'), \tag{17}$$

where $\Delta(\mathbf{x}-\mathbf{x}')$ defined as the Green function for the Laplace equation, $\partial^2\Delta(\mathbf{x}-\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')$.

As the velocity field is transverse, so is the external random force per unit mass f_i. We employ for it often used Gaussian probability distribution with a zero mean and a correlation function of the form:

$$\langle f_i(t,\mathbf{x})f_j(t',\mathbf{x}')\rangle = \delta(t-t')\,D_{ij}(\mathbf{x}-\mathbf{x}'), \tag{18}$$

where

$$D_{ij}(\mathbf{x}-\mathbf{x}') = \int_{k>m}\frac{d\mathbf{k}}{(2\pi)^d}P_{ij}(\mathbf{k})\,D(k)\,\exp\{i\,\mathbf{k}(\mathbf{x}-\mathbf{x}')\} \tag{19}$$

with the transverse projector $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$ and an amplitude $D(k) > 0$ that depends on the wave number $k \equiv |\mathbf{k}|$. The correlation function (18) and (19) obviously satisfies the transversality condition $\partial_i D_{ij} = 0$.

The Dirac function $\delta(t - t')$ in (18) is chosen to preserve the Galilean symmetry. The cutoff at $k = m$ (related to the typical largest scale $L = 1/m$), provides an IR regularization. The sharp cutoff is chosen for convenience.

The simplest choice $D(k) = B_0$ with a constant $B_0 > 0$ describes uniform (in relation to the momentum and the frequency) random mixing of the fluid that takes place at all scales $k > m$ [149]. The presence of the projector causes entanglement between different $\mathbf{k}$-modes while also introducing non-locality. Indeed, in the coordinate representation, one obtains:

$$D_{ij}(\mathbf{x} - \mathbf{x}') = P_{ij}(\partial)\, \delta(\mathbf{x} - \mathbf{x}') = \delta_{ij}\delta(\mathbf{x} - \mathbf{x}') - \partial_i \partial_j\, \Delta(\mathbf{x} - \mathbf{x}'), \tag{20}$$

with $\Delta(\mathbf{x} - \mathbf{x}')$ from (17).

For $m = 0$, the case $D(k) = B_0$ involves a finite mode at $\mathbf{k} = 0$; it can be interpreted as an overall macroscopic random "shaking" of the fluid container as a whole; see footnote[15] in [149]. This model is logarithmic at $d = 4$.

The choice $D(k) = B_0 k^2$ describes a fluid in thermal equilibrium [37,76,149]; it is logarithmic at $d = 2$.

The choice

$$D(k) = B_0\, k^{4-d-y}, \quad B_0 > 0 \tag{21}$$

is often used when investigating fully developed turbulence with the RG approach. It was introduced in [150,151]; for a more detailed discussion, see, e.g., the monographs [16,37] and the review paper [38]. The model (21) is logarithmic for $y = 0$ and arbitrary d. The marginal value $y \to 4$ has the following physical interpretation: at that limit, $D(k)$ in (21) resembles representation of the function $\delta(\mathbf{k})$ in power terms. It is this function that models the energy pumping by the largest-scale vortices. Various extensions of the model (16) and (18), where the random force f_i has a finite correlation time, were studied, e.g., in [152–154].

Another widely used statistical ensemble for the velocity field is the Kazantsev–Kraichnan model, described by a random Gaussian field, white in time and self-similar in space. At the end of the 1990s, Kraichnan's model of passive scalar advection received much attention from the turbulence community because it elucidates the origin of intermittency and anomalous scaling in fluid turbulence; see [155] and references therein. For the first time, the anomalous multiscaling was established on the base of a dynamical model within controlled approximations and regular perturbative expansions. That was possible due to the relative simplicity of the model, which allows for some exact results and accurate simulations.

In the model, the velocity field has zero mean and the correlation function of the form similar to (18) and (19):

$$\begin{gathered}\langle v_i(t,\mathbf{x}) v_j(t',\mathbf{x}')\rangle = \delta(t - t') D_{ij}(\mathbf{x} - \mathbf{x}'), \\ D_{ij}(\mathbf{r}) = B_0 \int_{k>m} \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{k^{d+\xi}} P_{ij}(\mathbf{k}) \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}), \quad B_0 > 0.\end{gathered} \tag{22}$$

The quantity $0 < \xi < 2$ is similar to the Hölder exponent related to the "roughness" of the velocity field. As such, the marginal value $\xi \to 2$ corresponds to a smooth surface and is referred to as Batchelor's limit while the "Kolmogorov" value that models turbulent motion is $\xi = 4/3$.

Various extensions of the ensemble (22) were also studied: inclusion of anisotropy, compressibility, finite correlation time, and so on. Of special interest is the strongly anisotropic ensemble in $d = 1 + 1$ dimensions due to Avellaneda and Majda [156,157], where some rigorous exact results are available. The d-dimensional generalizations of this ensemble were studied in connection to the turbulent advection of the vector (magnetic) fields [158–161].

4. Surface Roughening in a Random Environment: Induced Non-Linearity

In this section, we consider the problem of roughening dynamics of a surface in the external turbulent medium. The surface is described by the profile field $h(x)$ governed by the KPZ Equation (3) with the time-independent noise (12) and subjected to a turbulent environment, represented by the Kazantsev–Kraichnan ensemble (22). The effect of the advection is initially introduced by the the "minimal" replacement (15). It turns out that this necessarily leads to the emergence of non-linear interaction, quadratic in the velocity field [162].

Before the proper exposition of the significant features related to this theory, let us discuss several convenient substitutions. Firstly, in the Kazantsev–Kraichnan ensemble, we put $B_0 = 1$ in (22) by dilating the velocity field. We also substitute $D_0 = 1$ in the expression (5). Both substitutions can be made without a loss of generality. Naturally, after that dilatation, the minimal subtraction (15) should be replaced by

$$\nabla_t = \partial_t + w_0 \nu_0^{1/2} (\mathbf{v} \cdot \boldsymbol{\partial}) = \partial_t + w_0 \nu_0^{1/2} v_l \partial_l. \tag{23}$$

Here, w_0 is a coupling constant that originally entered the amplitude B_0 with the relation $B_0 = w_0 \nu_0$ following from dimensional considerations (see below).

According to the general statement (see, e.g., Chapter 5 in [16]), the full stochastic problems (3), (12) and (22) can be reformulated as the field theory for an extended set of fields $\Phi = \{h, h', v_i\}$ with the action functional $S(\Phi)$:

$$S(\Phi) = \frac{1}{2} h' h' + h' \{ -\partial_t h - w_0 \nu_0^{\frac{1}{2}} (v_i \partial_i) h + \nu_0 \partial^2 h + \frac{1}{2} g_0 \nu_0^2 (\partial h)^2 \} + S_v(\Phi), \tag{24}$$

where h' is the Martin–Siggia–Rose response field and

$$S_v(\Phi) = -\frac{1}{2} \int dt \int d\mathbf{x} \int d\mathbf{x}' v_i(t, \mathbf{x}) D^{-1}_{ij} (\mathbf{x} - \mathbf{x}') v_j(t, \mathbf{x}') \tag{25}$$

with $D_{ij}(\mathbf{r})$ from (22). Here, we omitted the necessary integrations in the first line for convenience, but they are assumed. By employing the field-theoretic formulation, one can acquire a multitude of correlation and response functions, namely Green's functions, pertaining to the primary stochastic problem. These functions are derived via functional averaging, utilizing $S(\Phi)$ in (24) as the weight function. Note that λ_0 from (4) was replaced by $g_0 \nu_0^2$ in (24) as $\lambda_0 = g_0 \nu_0^2$ from dimensional considerations; g_0 here is another coupling constant.

The next step is to address UV divergences by implementing the renormalization procedure. The assessment of UV divergences relies on canonical dimensions, see, e.g., Sections 1.15 and 1.1 in [16]. Dynamic models have two independent scales: a time scale $[T]$ and a space scale $[L]$ (see Sections 1.17 and 5.14 in [16]); therefore, the determination of the canonical dimension for a given quantity F is contingent upon two numerical values, specifically the frequency dimension denoted as d_F^ω and the momentum dimension denoted as d_F^k.

$$[F] \sim [T]^{-d_F^\omega} [L]^{-d_F^k}. \tag{26}$$

As soon as all the terms in the action functional are required to be dimensionless with respect to both canonical dimensions, one can find the expressions for those canonical dimensions; the obvious normalization conditions are

$$d^k_{k_i} = -d^k_{x_i} = 1,\ d^\omega_{k_i} = d^\omega_{x_i} = 0,\ d^k_\omega = d^k_t = 0,\ d^\omega_\omega = -d^\omega_t = 1. \tag{27}$$

The full canonical dimension d_F is determined by the equation $d_F = d_F^k + 2 d_F^\omega$. The presence of the factor 2 in this equation arises from the relationship $\partial_t \propto \partial^2$ in the context of the free theory. All the canonical dimensions for the theory (24) and (25) are presented in the Table 1. The parameters x_0, x, μ will be defined later.

The model is logarithmic (all coupling constants become dimensionless) at $\varepsilon = 0$, i.e., at $d = 4$, and $\xi = 0$.

Table 1. Canonical dimensions of the fields and the parameters in the theory (25); $\varepsilon = 4 - d$.

F	h	h'	v_i	ν_0, ν	g_0	w_0	x_0	μ	g, w, x
d_F^ω	-1	1	$\frac{1}{2}$	1	0	0	0	0	0
d_F^k	$\frac{d}{2}$	$\frac{d}{2}$	$-\frac{\xi}{2}$	-2	$\frac{\varepsilon}{2}$	$\frac{\xi}{2}$	$\xi - \frac{\varepsilon}{2}$	1	0
d_F	$\frac{d}{2} - 2$	$\frac{d}{2} + 2$	$1 - \frac{\xi}{2}$	0	$\frac{\varepsilon}{2}$	$\frac{\xi}{2}$	$\xi - \frac{\varepsilon}{2}$	1	0

The determination of the full dimension d_Γ for a 1-irreducible Green's function Γ, which incorporates N_h fields h, $N_{h'}$ fields h', and N_v fields v_i, can be achieved through the utilization of the subsequent expression:

$$d_\Gamma = d + 2 - d_h N_h - d_{h'} N_h' - d_{v_i} N_v. \tag{28}$$

Within the framework of the logarithmic theory, the value of d_Γ serves as the formal index denoting the UV divergence δ of the corresponding Green's function. Consequently, the divergent component of the function Γ and the potential counterterms can be expressed as polynomials of degree $\delta = d_\Gamma|_{d=4}$.

For the model described by Equations (24) and (25), the actual divergence index δ' deviates from the formal index δ by $\delta' = \delta - N_h$. This deviation occurs because the field h is incorporated into the action functional solely through its spatial derivative form.

Taking into account the aforementioned condition, a list of potential counterterms can be compiled. These include $\partial^2 h'$ and $\partial_t h'$ (both responsible for renormalizing the mean value of h), $h'h'$, $h'\partial^2 h$, $h'\partial_t h$, $h'(\partial h)^2$, and $h'(v_i\partial_i)h$ (all five of which already exist in the action functional). Additionally, there is $h'(\partial_i v_i)$, which vanishes due to the incompressibility of the fluid. Finally, a new counterterm is introduced as $h'\mathbf{v}^2$.

In order for the model to be renormalizable, the additional term $\sim h'\mathbf{v}^2$ needs to be introduced into the action functional as given by Equation (25). Thus far, our analysis has revolved around the KPZ equation, which included turbulent advection through the Lagrangian derivative ∇_t. It is important to emphasize that the inclusion of this new term in the action functional effectively transforms the equation into a different form:

$$\partial_t h + w_0 \nu_0^{\frac{1}{2}} (v\partial) h = \nu_0 \partial^2 h + \frac{1}{2} g_0 \nu_0^2 (\partial h)^2 + \frac{1}{2} \frac{x_0}{\nu_0} \mathbf{v}^2 + f. \tag{29}$$

The full action functional therefore is as follows:

$$S(\Phi) = \frac{1}{2} h'h' + h' \left\{ -\partial_t h - w_0 \nu_0^{\frac{1}{2}} (v_i \partial_i) h + \nu_0 \partial^2 h + \frac{1}{2} g_0 \nu_0^2 (\partial h)^2 + \frac{1}{2} \frac{x_0}{\nu_0} \mathbf{v}^2 \right\} + S_v(\Phi). \tag{30}$$

The coupling constant corresponding to the new non-linearity is notated as x_0. The factor ν_0^{-1} appears from the dimensional considerations.

With the establishment of multiplicative renormalizability for the theory described by Equation (30), it becomes possible to express the bare fields and parameters in relation to their renormalized counterparts:

$$\begin{gathered} h_0 = Z_h h, \quad h_0' = Z_h' h', \quad \mathbf{v}_0 = Z_v \mathbf{v}, \\ g_0 = Z_g \mu^{\varepsilon/2} g, \quad w_0 = Z_w \mu^{\xi/2} w, \quad x_0 = Z_x \mu^{\xi - \varepsilon/2} x. \end{gathered} \tag{31}$$

In order to distinguish them from their renormalized counterparts, we have introduced the subscript 0 to the fields. Furthermore, the renormalized theory incorporates an additional parameter known as the renormalization mass μ, and it involves a set of renormalization constants denoted as Z.

The renormalized action has the form

$$S_R(\Phi) = \frac{1}{2}Z_1 h'h' + h'\left\{\partial_t h - Z_2 w \nu^2 (v_i \partial_i) h + Z_3 \nu \partial^2 h + \frac{1}{2} Z_4 g \nu^2 (\partial h)^2 + \frac{1}{2} Z_5 \frac{x}{\nu} \mathbf{v}^2 \right\} + S_v. \quad (32)$$

The determination of the constants Z is based on the requirement that the relevant Green's functions remain UV finite within the specified perturbation theory order. In our calculation, we employed the MS (minimal subtraction) scheme within the one-loop approximation. For a detailed illustration of the similar one-loop calculations, see [163].

Given the multiplicative renormalizability of the model, it is possible to derive the differential RG equation using conventional methods; see, e.g., [16] (Section 1.24). The RG equation in our case is as follows:

$$(D_\mu + \beta_g \partial_g + \beta_w \partial_w + \beta_x \partial_x - \gamma_\nu D_\nu - \gamma_{G_R}) G_R = 0. \quad (33)$$

Here the symbols $D_\mu = \mu \partial_\mu$ and $D_\nu = \nu \partial_\nu$ represent differential operators. Additionally, the quantities γ_ν, γ_{G_R} (anomalous dimensions) and β_g, β_w, and β_x (beta functions) are referred to as RG functions. These functions play a crucial role in the characterization of the model and are defined as follows:

$$\gamma_f = D_{RG} \ln Z_f, \quad \beta_f = D_{RG} f, \quad (34)$$

where $f = \{g, w, x\}$ and $D_{RG} = \mu \partial_\mu|_{e_0}$.

The asymptotic behavior of the Green's functions in a multiplicatively renormalizable model is determined by the attractors of the RG equations. These attractors (or IR attractive fixed points) are obtained by satisfying the condition that all β functions associated with the model vanish; see, e.g., [16] (Section 1.42). From (31) and (34), the following relations for β functions can be obtained:

$$\begin{aligned} \beta_g &= -g(\varepsilon/2 + \gamma_g), \\ \beta_w &= -w(\xi/2 + \gamma_w), \\ \beta_x &= -x(\xi - \varepsilon/2 + \gamma_x). \end{aligned} \quad (35)$$

Coordinates of the fixed points g^*, w^*, x^* could be found as the roots of the β functions. The fixed point is considered to be IR attractive if all the eigenvalues of the matrix $\Omega_{ij}(g, w, x) = \partial \beta_{f_j} / \partial f_i$ (where f again is a set of charges $\{g, w, x\}$) at the fixed point have strictly positive real parts; see, e.g., [16] (Section 1.42).

Including the solutions that account for the marginal values of the charges requires transitioning to a new set of charges to avoid nontrivial denominators in the β functions. The appropriate substitution is a set $\{g, y = xg, \alpha = w^2/y\}$. Omitting the calculation details of the RG procedure application (see Section 5), we present the following results for the fixed points in the Table 2.

Table 2. Fixed points for the model (35).

Name	Coordinates	Stability Region
FP1a	$g^* = w^* = x^* = 0$	$\varepsilon < 0, \xi < 0, \xi < \varepsilon/2$
FP2	$g^{*2} = -\varepsilon, x^* = w^* = 0$	$\varepsilon < -4\xi, \xi < 0, \varepsilon > 0$
FP3	$g^{*2} = -4\xi - 2\varepsilon, w^{*2} = -16\xi/3 - 4\varepsilon/3,$ $x^* = (-16\xi/3 - 4\varepsilon/3)/g$	$\varepsilon > -4\xi, \xi < 0$
FP4	$g^{*2} = 4\xi - 2\varepsilon, w^{*2} = 16\xi/3 - 4\varepsilon/3,$ $x = 2g/3$	$\varepsilon > 4\xi, \xi > 0$
FP5	$g^{*2} = -\varepsilon, w^* = 0, x^* = -8\xi/3g$	$\varepsilon < 4\xi, \xi > 0, \varepsilon > 0$
FP6	$g^* = x^* = 0, w^{*2} = 8\xi/3$	empty
FP1b	$g^{*2} = 0, y^* = 0,$ arbitrary α^*	$\varepsilon < 0, \xi < 0$
FP7	$g^{*2} = 0, y^* = -8\xi/3, \alpha^* = 0$	$\xi > 0, \varepsilon < 0$
FP8	$g^{*2} = 0, y^* = -8\xi/3, \alpha^* = 1$	empty

Stability regions on the $\varepsilon - \xi$ plane are shown in Figure 1. It is worth mentioning that in the regions depicted on the right-hand side of the graph in Figure 1, certain coordinates of the fixed points exhibit imaginary values, while others assume negative values.

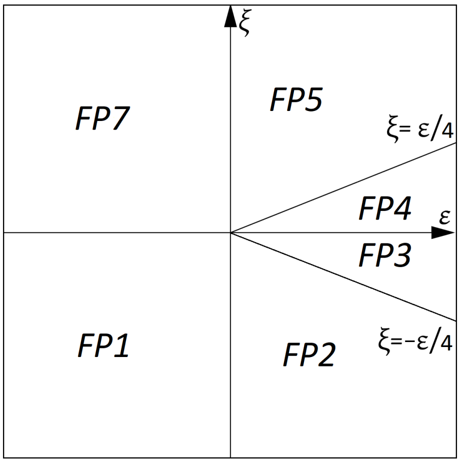

Figure 1. IR stability regions of the fixed points for the renormalized theory (32).

Let us discuss the stability regions in details. For different values of ε and ξ, different regimes take place. To begin with, let us consider the negative values first (quadrant III of $\varepsilon - \xi$ plane). The domain denoted as FP1 in Figure 1 includes the couple of Gaussian points FP1a and FP1b. In the regime characterized by negative values of ξ and ε, all the coupling constants converge to zero in the infrared (IR) limit. Consequently, this domain corresponds to a conventional diffusion behavior.

Regarding the domain referred to as FP2, it is noteworthy that $w^* = x^* = 0$, which implies the pure KPZ model (featuring spatially quenched noise and lacking turbulent mixing) being multiplicatively renormalizable without the inclusion of an additional term $h'\mathbf{v}^2$ in the action functional, as described in reference [121].

In a surprising revelation, the stability region associated with the point FP5 corresponds to a regime where the KPZ non-linearity holds significance, while turbulent mixing is considered irrelevant. Notably, in this regime, the coordinate x^* takes on a nontrivial value, indicating its significance within the system. This observation suggests that the newly introduced non-linearity $h'\mathbf{v}^2$ retains its relevance even in the absence of turbulent mixing, which initially gave rise to it. Furthermore, it is noteworthy that this regime exhibits a Kolmogorov exponent value of $\xi = 4/3$, irrespective of the system's dimensionality being $d = 2$ or $d = 3$.

The same situation arises within the regime associated with the point FP7: the presence of the new term $h'\mathbf{v}^2$ dictates the IR asymptotic behavior, while both the turbulent mixing and the KPZ non-linearity are considered irrelevant in this context. Finally, in the regimes characterized by the points FP3 and FP4, both non-linearities and turbulent mixing play significant roles, as evidenced by the nontrivial values of all coordinates associated with these fixed points.

5. Non-Conventional Scaling Behavior and Dimensional Transmutation

The goal of this paper is twofold: to illustrate how RG and QFT methods are applied to problems of non-equilibrium dynamics and to point out several interesting effects that occur specifically due to the added turbulent mixing. In the Section above, we discussed one of those effects, namely, the induced non-linearity. In this Section, we deal with another effect termed here as dimensional transmutation, a phenomenon that consists of a previously dimensionless parameter acquiring a nontrivial canonical dimension.

Let us formulate the problem to consider; we start with the Hwa–Kardar model (9) and (10) and put $\lambda = 1$ without the loss of generality. The noise is chosen in its simplest form (5) and the external velocity field is described by Equation (22). As a result, we combine in one model the anisotropic stochastic equation (9) and the isotropic velocity ensemble (22).

To describe the phenomenon of dimensional transmutation, we should first list the results of the RG analysis of the "pure" Hwa–Kardar model without external field v_i [106,107]. The model is anisotropic (i.e., it contains separate terms with $\partial_\perp^2 h$ and $\partial_\parallel^2 h$) and action functional for it reads (cf. Equation (24))

$$S(\Phi) = \frac{1}{2} h' D_0 h' + h' \left(-\partial_t h + \nu_{\parallel 0}\, \partial_\parallel^2 h + \nu_{\perp 0}\, \partial_\perp^2 h - \frac{1}{2}\, \partial_\parallel h^2 \right). \tag{36}$$

Since model (36) is anisotropic, it involves two independent scales $L_\parallel$ and $L_\perp$ instead of a single scale L (cf. Equation (26)), and an arbitrary quantity F is described by three canonical dimensions:

$$[F] \sim [T]^{-d_F^\omega} [L_\parallel]^{-d_F^\parallel} [L_\perp]^{-d_F^\perp}. \tag{37}$$

Table 3 contains the canonical dimensions of all fields and parameters for the theory (36), obtained from the dimensionless condition for each member of the action (36). We denoted a full canonical dimension d_F as $d_F = d_F^k + 2d_F^\omega$ and a full momentum dimension d_F^k as $d_F^k = d_F^\parallel + d_F^\perp$. The coupling constant g_0 is defined as $D_0 = g_0 \nu_{\perp 0}^{3/2} \nu_{\parallel 0}^{3/2}$; the parameter u_0 which we will need later is defined as $u_0 = \nu_{\parallel 0} / \nu_{\perp 0}$.

Table 3. Canonical dimensions for the theory (36); $\varepsilon = 4 - d$.

F	h'	h	D_0	$\nu_{\parallel 0}$	$\nu_{\perp 0}$	u_0	g_0	g	μ
d_F^ω	-1	1	3	1	1	0	0	0	0
$d_F^\parallel$	2	-1	-3	-2	0	-2	0	0	0
$d_F^\perp$	$d-1$	0	$1-d$	0	-2	2	ε	0	1
d_F^k	$d+1$	-1	$-d-2$	-2	-2	0	ε	0	1
d_F	$d-1$	1	$4-d$	0	0	0	ε	0	1

From Table 3 and canonical dimensions analysis, it follows that the logarithmic dimension of the theory is $d = 4$ and the model is multiplicatively renormalizable with the only nontrivial renormalization constant $Z_{\nu_\parallel}$. The only parameters that require renormalization are $\nu_{\parallel 0}$ and g_0:

$$\nu_{\parallel 0} = \nu_\parallel Z_{\nu_\parallel}, \quad g_0 = g \mu^\varepsilon Z_g, \tag{38}$$

where μ is the renormalization mass.

The canonical differential equations for a renormalized Green function $G^R = \langle \Phi \ldots \Phi \rangle$ read

$$\begin{gathered} \left(\sum_i d_i^\omega \mathcal{D}_i - d_G^\omega \right) G^R = 0, \\ \left(\sum_i d_i^\perp \mathcal{D}_i - d_G^\perp \right) G^R = 0, \quad \left(\sum_i d_i^\parallel \mathcal{D}_i - d_G^\parallel \right) G^R = 0. \end{gathered} \tag{39}$$

The index i enumerates all arguments of G^R (i.e., ω, $k_\perp$, $k_\parallel$, μ, $\nu_\perp$, and $\nu_\parallel$); the operator $\mathcal{D}_x$ is defined as $\mathcal{D}_x = x\partial_x$ for any x.

To calculate the critical exponents that stand in scaling power laws (i.e., dimensions), we need to use together above written Equation (39) and the differential RG equation

$$\left(\mathcal{D}_\mu + \beta_g \partial_g - \gamma_{\nu_\parallel} \mathcal{D}_{\nu_\parallel} - \gamma_{\nu_\perp} \mathcal{D}_{\nu_\perp} - \gamma_G \right) G^R = 0, \tag{40}$$

where β and γ denote β function for the coupling constant g and anomalous dimensions for the parameters of the system. To find universality classes of asymptotic (critical) behavior, we should find zeroes of the β functions, $\beta(g^*) = 0$, called also fixed points of the RG equations. To obtain IR (not UV) attractive point real parts of all the eigenvalues Ω_i of the matrix $\Omega_{ij} = \partial_{g_i}\beta_{g_j}$ must be positive; here $g = \{g_i\}$ is a full set of the charges.

The substitution $g \to g^*$ and, hence, $\gamma_F \to \gamma_F^*$ turns Equation (40) into an equation with constant coefficients, i.e., into an equation of the same type as Equation (39). This leads to the desired equation of critical scaling:

$$\left(\mathcal{D}_{k_\perp} + \mathcal{D}_{k_\parallel}\Delta_\parallel + \Delta_\omega\mathcal{D}_\omega - \Delta_G\right)G^R = 0 \tag{41}$$

with $\Delta_\parallel = 1 + \gamma^*_{\nu_\parallel}/2$ and $\Delta_\omega = 2 - \gamma^*_{\nu_\perp}$.

We want to study the critical scaling behavior where IR irrelevant parameters (namely, $\mu, \nu_\perp$ and $\nu_\parallel$) are kept fixed while frequencies and momenta are scaled. Thus, we combine all these equations to eliminate derivatives with respect to all non-IR parameters and arrive at the critical scaling equation for a given fixed point. As we will see below, such an exception is not always possible: it requires a certain balance between the number of IR and non-IR parameters and the number of independent scaling equations.

The model (36) has two fixed points. The first one is Gaussian (free) fixed point, which means that $g^* = 0$. It is IR attractive for $\varepsilon < 0$ and corresponding critical dimensions coincide with canonical ones. The second point has the coordinate $g^* = 32\varepsilon/9 + O(\varepsilon^2)$. It is IR attractive for $\varepsilon > 0$ and corresponding canonical dimensions read

$$\Delta_h = 1 - \varepsilon/3, \quad \Delta_{h'} = 3 - \varepsilon/3, \quad \Delta_\omega = 2, \quad \Delta_\parallel = 1 + \varepsilon/3. \tag{42}$$

This result reproduces a well known answer obtained firstly by Hwa and Kardar [106,107] for the (fully anisotropic) model (36).

To include turbulent advection in the Hwa–Kardar model we should replace ∂_t with ∇_t ("minimal" replacement) in the action (36) and add Kraichnan's velocity ensemble (22). This leads to the following action functional:

$$S(\Phi) = \frac{1}{2}h'D_0h' + h'\left(-\nabla_t h + \nu_{\parallel 0}\partial_\parallel^2 h + \nu_{\perp 0}\partial_\perp^2 h - \frac{1}{2}\partial_\parallel h^2\right) + S_{\mathbf{v}}, \tag{43}$$

$$S_{\mathbf{v}} = -\frac{1}{2}\int dt \int d\mathbf{x} \int d\mathbf{x}' v_i(t,\mathbf{x}) D_{ij}^{-1}(\mathbf{x}-\mathbf{x}') v_j(t,\mathbf{x}'). \tag{44}$$

Here, $D_{ij}^{-1}(\mathbf{x}-\mathbf{x}')$ is the kernel of the inverse operator D_{ij}^{-1} for the integral operator D_{ij} from (22) and $S_{\mathbf{v}}$ is responsible for Gaussian averaging over the field $\mathbf{v}$.

The main difference between this case and the previous one is that since that velocity ensemble is isotropic, it is no longer possible to define two independent spatial scales in this model. Thus,

$$[F] \sim [T]^{-d_F^\omega}[L]^{-d_F^k} \tag{45}$$

and $d_F = d_F^k + 2d_F^\omega$. Canonical dimensions of the fields and parameters of the model are presented in Table 4. From (45), it follows that the ratio $u_0 = \nu_{\parallel 0}/\nu_{\perp 0}$ is completely dimensionless, that is, dimensionless with respect to all possible dimensions (i.e., frequency and momentum dimensions) separately.

Table 4. Canonical dimensions for the theory (44); $\varepsilon = 4 - d$.

F	h'	h	D_0	$\nu_{\parallel 0}$	$\nu_{\perp 0}$	g_0	v	B_0	x_0	u_0, u	g, x	μ
d_F^ω	-1	1	3	1	1	0	1	0	0	0	0	0
d_F^k	$d+1$	-1	$-2-d$	-2	-2	ε	-1	$\xi-2$	ξ	0	0	1
d_F	$d-1$	1	$4-d$	0	0	ε	1	ξ	ξ	0	0	1

According to the general rules, in this model with mixing, the ratio u_0 should be treated as an additional charge. Another coupling constant x_0 is related to the amplitude B_0

of the correlation function (22) as $B_0 = x_0 \nu_{\perp 0}$. Thus, the theory is logarithmic when $\varepsilon = 0$ (i.e., $d = 4$) and $\xi = 0$.

The theory is renormalized by introducing five renormalization constants Z_i:

$$\nu_{\|0} = \nu_{\|} Z_{\nu_{\|}}, \quad \nu_{\perp 0} = \nu_{\perp} Z_{\nu_{\perp}}, \quad g_0 = Z_g g \mu^{\varepsilon}, \quad x_0 = \quad Z_x x \mu^{\xi}, \quad u_0 = Z_u u. \tag{46}$$

It follows from calculations that the one-loop expressions for β functions have the form

$$\begin{aligned} \beta_g &= g\left(-\varepsilon + \frac{9}{32}g + \frac{9}{16}\frac{x}{u} + \frac{9}{16}x\right), \quad \beta_x = x\left(-\xi + \frac{3}{8}x\right), \\ \beta_u &= u\left(-\frac{3}{16}g - \frac{3}{8}\frac{x}{u} + \frac{3}{8}x\right). \end{aligned} \tag{47}$$

From (47), it follows that the system has two possible IR attractive fixed points. The first one is Gaussian point FP1 with the coordinates $g^* = 0$, $x^* = 0$ and arbitrary u^*. The second point FP2 obtains coordinates $g^* = 0$, $x^* = 8\xi/3$, $u^* = 1$ and corresponds to the regime of simple turbulent advection (the non-linearity of the Hwa–Kardar equation is irrelevant in the sense of Wilson). Here, we denote a fixed point of the RG equation with "FP" as we did in the Section above.

The basins of attractions for points FP1 and FP2 are $\varepsilon < 0$, $\xi < 0$ (FP1) and $\xi > \varepsilon/3$, $\xi > 0$ (FP2). Since there are no fixed points with $g^* \neq 0$ in the set, the Hwa–Kardar universality class is not presented for such values of u.

However, other possibilities are possible solutions with $u^* = 0$ and $1/u^* = 0$. To study these cases, we should pass to new variables $w = x/u$ (for the case $u^* = 0$) and $\alpha = 1/u$ (for the case $u^* \to \infty$).

The first case yields no new IR attractive fixed points. But, the second case gives us two more fixed points FP3 and FP4 corresponding to coordinates $g^* = 32\varepsilon/9$, $x^* = 0$, $\alpha^* = 0$ (FP3) and $g^* = 32\varepsilon/9 - 16\xi/3$, $x^* = 8\xi/3$, $\alpha^* = 0$ (FP4). The point FP3 corresponds to the regime of critical behavior where only the non-linearity of the Hwa–Kardar equation is relevant. The point FP4 corresponds to the situation where both the non-linearity and the turbulent advection are relevant. The basins of attractions (IR) are $\varepsilon > 0$, $\xi < 0$ (FP3) and $\xi < \varepsilon/3$, $\xi > 0$ (FP4); see [164].

Since original Hwa–Kardar model (36) obtains two independent momentum dimensions, it involves three Equations (39) related to the canonical scale invariance. But, after we add isotropic turbulent advection (22) to the mix, only two such equations can be derived for the resulting problem (44). This is precisely what leads to dimensional transmutation.

By combining two scaling equations with differential RG equation taken at the fixed point, we arrive at the equation of critical scaling. For the Gaussian fixed point FP1 corresponding critical dimensions coincide with canonical ones. For the fixed points FP2, they are

$$\Delta_h = 1 - \xi, \quad \Delta_v = 1 - \xi, \quad \Delta_{h'} = 3 - \varepsilon + \xi, \quad \Delta_{\omega} = 2 - \xi. \tag{48}$$

Surprisingly, the scaling associated with fixed points FP3 and FP4 is significantly different from the scaling described by points FP1 and FP2. The difference arises when one sets $\beta_{\{g_i^*\}} = 0$ in the RG equation

$$\left(\mathcal{D}_{\mu} + \beta_g \partial_g + \beta_x \partial_x + \beta_u \partial_u - \gamma_{\nu_{\perp}} \mathcal{D}_{\nu_{\perp}} - \gamma_G\right) G^R = 0. \tag{49}$$

If Green functions have well-defined finite limits at $g \to 0$ and $x \to 0$ (it is the case of points FP1 and PF2) the β functions can be set to zero without any further analysis. But in the case of points FP3 and FP4 such a straightforward substitution cannot be performed for β_{α} and we should retain first nontrivial order of the expansion of β_{α} around $\alpha = 0$ in (49).

The main idea is the following. Let us consider the renormalized correlation function $G^R \sim R(k/\mu, g)$, where g is our coupling (let us suppose for brevity that we have only one charge), k is momenta and R is a function of dimensionless arguments. If the function

$R(1, g)$ is finite at $g = g^*$, we may put $g = g^*$ directly in the RG Equation (49) and obtain critical dimensions in ordinary way. But, if

$$R(k/\mu, g) = (g - g^*)^a F(k/\mu, g), \tag{50}$$

the naive substitution $\bar{g} \to g^*$ leads to the vanishing or divergence of the amplitude factor (here a is some exponent and we suppose that F has a finite limit for $g \to g^*$). To find the right IR behavior, one should take into account how the invariant constraint approaches its fixed point, namely $\bar{g} - g^* \simeq (k/\mu)^\Omega$, where $\Omega = \beta'(g^*) > 0$. This gives

$$G \simeq k^{\Delta_G + a\Omega} F(1, g^*). \tag{51}$$

This leads to the fact that the correct critical dimension of the function G appears to be $\Delta_G + a\Omega$ rather than Δ_G itself. This leads to modification of the RG equation at the fixed point:

$$\big(\mathcal{D}_\mu + \omega \mathcal{D}_{g-g^*} - \gamma_\nu^* \mathcal{D}_\nu - \gamma_G^* - a\Omega\big) F = 0, \tag{52}$$

which differs from the ordinary one by the replacement $\gamma_G^* \to \gamma_G^* + a\Omega$ (see Appendix A in [165]).

As a result, the critical scaling equation at the point FP3 has the form

$$\left(\mathcal{D}_{k_\parallel} + \mathcal{D}_{k_\perp} + 2\mathcal{D}_\omega - \frac{2\varepsilon}{3}\mathcal{D}_\alpha - d_G^k - 2d_G^\omega - \gamma_G^*\right) G^R = 0. \tag{53}$$

Now, we should find a general solution for the whole system which includes two canonical invariance equations, the RG Equation (49) taken at a fixed point, and homogeneous counterpart of equation (53). It is an arbitrary function of three independent variables which we may choose for definiteness as

$$z_1 = \frac{\omega}{\nu_\perp k_\perp^2}, \quad z_2 = \frac{k_\parallel}{k_\perp}, \quad \text{and} \quad z_3 = \alpha \left(\frac{k_\perp}{\mu}\right)^{2\varepsilon/3}. \tag{54}$$

However, it is the case where only the non-linearity is relevant, i.e., model (43) should coincide with the pure Hwa–Kardar model (36). This means that $d^\parallel$ and $d^\perp$ are two independent dimensions, see (39), and additional canonical symmetry arises. Additional symmetry requires the variables z_1, z_2 and z_3 to be dimensionless with respect to Table 3. The variables z_2 and z_3 do not satisfy this requirement, but it is possible to construct a new variable, $z_0 = z_2 z_3^{-1/2}$, which solves the problem. It obtains needed canonical dimensions and serves along with z_1 as the second solution of the homogeneous part of Equation (41):

$$z_0 = \frac{k_\parallel}{k_\perp^{\Delta_\parallel} \alpha^{\wp}} \mu^{\varepsilon/3} \quad \text{with} \quad \wp = \frac{3}{2\varepsilon}\left(\Delta_\parallel - 1\right). \tag{55}$$

Here, $\Delta_\parallel = 1 + \varepsilon/3$ and this is in agreement with expressions (42). The presence of variable z_0 with nontrivial $\Delta_\parallel \neq 1$ means that at fixed point FP3 the coordinates $x_\parallel$ and $\mathbf{x}_\perp$ are scaled independently, while all the IR irrelevant parameters (including α) are kept fixed. This reproduces the results derived for the pure Hwa–Kardar model (36).

The fixed point FP4 corresponds to the regime where both the turbulent advection and the non-linearity of the Hwa–Kardar equation are relevant. The critical scaling equation for it reads

$$\left(\mathcal{D}_{k_\parallel} + \mathcal{D}_{k_\perp} + \Delta_\omega \mathcal{D}_\omega - \left(\frac{2}{3}\varepsilon - 2\xi\right)\mathcal{D}_\alpha - d_G^k - \Delta_\omega d_G^\omega - \gamma_G^*\right) G^R = 0, \tag{56}$$

where $\Delta_\omega = 2 - \xi$. Three combinations $\hat{z}_1$, $\hat{z}_2$ and $\hat{z}_3$ are possible solutions of its homogeneous part:

$$\hat{z}_1 = \frac{\omega}{\nu_\perp k_\perp^2}\left(\frac{k_\perp}{\mu}\right)^{\xi}, \quad \hat{z}_2 = \frac{k_\parallel}{k_\perp}, \quad \hat{z}_3 = \alpha\left(\frac{k_\perp}{\mu}\right)^{2\varepsilon/3-2\xi}. \tag{57}$$

The variables (57) describe generalized critical scaling where α (i.e., the ratio of $\nu_\perp$ and $\nu_\parallel$) is also scaled. The self-similar behavior also involves some dilation of the ratio u, but to find ordinary critical behavior with fixed definite critical dimensions, we should know some special dependence on u.

This means that among four universality classes of critical behavior, two classes turned out to correspond to non-conventional scaling behavior. For point FP3, a kind of dimensional transmutation takes place, i.e., the ratio u of the two diffusivity coefficients $\nu_\parallel$ and $\nu_\perp$ acquires a nontrivial canonical dimension. Due to this, canonical dimensions $d^\parallel$ and $d^\perp$ scale independently from each other, which means that new canonical symmetry arises in the model.

For the point FP4, the IR behavior resembles various generalized self-similarity hypotheses, like the weak scaling of George Stell [166–168] or the parametric scaling in the spirit of Michael Fisher [169] for systems with different characteristic scales and modified scaling laws.

6. Non-Linear Diffusion in a Random Medium: An Infinite-Dimensional Model

In the current section, we investigate the infrared behavior of a scalar field undergoing anomalous diffusion in the turbulent media. Conservation of the total amount of the admixture during the diffusion and the advection processes can be described by the continuity equation of the most general form:

$$\nabla_t h = \partial_i J_i + f, \tag{58}$$

where ∇_t is given by (15) and introduces coupling with the media. The random force f models interaction with large-scale modes of the admixture distribution, which are out of inertial range and hence beyond the applicability of simplified model (22):

$$\langle f(x) f(x')\rangle = \delta(t-t')\int_{k>m}\frac{d\mathbf{k}}{(2\pi)^d}\, D_f(k)\, \exp\{\mathrm{i}\mathbf{k}(\mathbf{x}-\mathbf{x}')\}, \tag{59}$$

$$D_f(k) = D_0\, k^{2-d-y}, \quad D_0 > 0. \tag{60}$$

This correlation function is similar to the one written for the external random force in the Navier–Stokes Equations (18) and (21). The two differences are concerned with the scalar nature of the force f in (59) (compressibility condition vanishes and so does the projector P_{ij} in (21)) and with the power of momentum k in the kernel D_f (60). On the one hand, this form of the kernel may be viewed as a power law delta-like distribution, while, on the other hand, as we will see later, it appears to be of the same IR relevance as turbulent advection when $y = 0$. The latter means that the non-linear diffusion and the advection provide corrections to the possible IR scaling of the same magnitude and neither of them can be neglected.

Similarly, since we are interested in the asymptotic IR behavior, only the leading term of the gradient expansion should be kept in the expression for the current

$$J_i = \partial_i V(h) + O(\partial^3), \tag{61}$$

where $V(h)$ is from the extended Pavlik model (8). As we will see below, the decision to consider the extended model from the beginning is justified by the appearance of infinite number of counterterms needed for renormalization.

The action functional corresponding to the equivalent field-theoretic model is given by the sum:

$$S(\Phi) = S_h + S_v, \tag{62}$$

where

$$\mathcal{S}_h(\Phi) = \frac{1}{2} h' D_f h' + h' \left[-\nabla_t h + \partial^2 V(h) \right] \tag{63}$$

includes infinitely many coupling constants, and S_v was previously defined in (25).

Canonical dimension of all the fields and parameters of the model, including their renormalized counterparts to be introduced below, are calculated employing definitions (26) and (27) and presented in the Table 5. Note that in order to unambiguously define canonical dimensions, the amplitude factor D_0 is set to be equal to unity without the loss of generality.

Table 5. Canonical dimensions of the fields and the parameters in the model (62) and (63).

F	h	h'	$\mathbf{v}$	m, μ	B_0	λ_{n0}
d_F^k	$1 - \frac{y}{2}$	$d - 1 + \frac{y}{2}$	-1	1	$\xi - 2$	$\frac{1}{2} y(n-1) - (n+1)$
d_F^ω	$-\frac{1}{2}$	$\frac{1}{2}$	1	0	1	$\frac{1}{2}(n+1)$
d_F	$-\frac{y}{2}$	$d + \frac{y}{2}$	1	1	ξ	$\frac{1}{2} y(n-1)$
F	λ_n	ν_0, ν	w_0	g_{n0}	w	g_n
d_F^k	$-(n+1)$	-2	ξ	$\frac{1}{2} y(n-1)$	0	0
d_F^ω	$\frac{1}{2}(n+1)$	1	0	0	0	0
d_F	0	0	ξ	$\frac{1}{2} y(n-1)$	0	0

From Table 5, one can see that all the interactions $h' \partial^2 h^n$ become logarithmic simultaneously as couplings λ_{n0} turn dimensionless with exponent y approaching zero. This is the precise reason why their inclusion in the initial Equation (58) was a necessary prerequisite for the multiplicative renormalizability of the model. The full model (62) becomes logarithmic at $\xi = y = 0$.

Analysis of canonical dimensions shows that all the terms requiring the introduction of counterterms are already present in the action and, hence, the model is multiplicatively renormalizable. We also took into account that only the Green functions containing auxiliary field h' are non-vanishing due to causality reasons, that the Galilean symmetry forbids the counterterm $h'(v_i \partial_i) h$, and that due to incompressibility of the velocity, the field h' always enters Green functions only in the form of a spatial gradient.

The UV divergences manifest themselves as poles at $y \to 0$ and $\xi \to 0$ in the correlation functions. Their elimination requires an introduction of renormalization constants and of renormalized action functional, which in our case will take the form

$$\mathcal{S}_R(\Phi) = \frac{1}{2} h' D_f h' + h' \left\{ -\nabla_t h + \partial^2 V_R(h) \right\} + \mathcal{S}_{vR}(\mathbf{v}), \tag{64}$$

with the renormalized analog $V_R(h)$ of the series (8)

$$V_R(h) = \sum_{n=1}^{\infty} \frac{1}{n!} Z_n \lambda_n h^n, \tag{65}$$

and S_{vR} which is the functional S_v from (25) expressed in renormalized variables.

In what follows, it is convenient to introduce new notations and variables, namely

$$\lambda_{10} = \nu_0; \quad \lambda_{n0} = g_{n0} \, \nu_0^{(n+1)/2}, \ (n > 2); \quad B_0 = w_0 \nu_0. \tag{66}$$

Their renormalized counterparts are defined by multiplicative renormalization

$$\nu_0 = \nu Z_\nu, \quad w_0 = w \mu^{\xi} Z_w, \quad g_{n0} = g_n \mu^{(n-1)y/2} Z_{g_n} \ (n \geq 2). \tag{67}$$

The constants Z_n are calculated directly from the requirement of subtraction of the divergences in the loop contributions, and the others are found from the relations following from Equations (66) and (67):

$$Z_\nu = Z_w^{-1} = Z_1, \quad Z_{g_n} = Z_n\, Z_1^{-(n+1)/2}. \tag{68}$$

The first relation here follows from the fact the velocity distribution does not need particular renormalization and should only be expressed in terms of renormalized parameters, so that $B = B_0$.

In order to explicitly calculate the divergent part of the one-loop contribution, we employ the functional technique developed earlier in [114–118]. In particular, we consider the generating functional of the 1-irreducible Green's functions in renormalized theory. The expansion of the latter in the number of loops l has the form (see, e.g., Section 2.9 in [16]):

$$\Gamma_R(\Phi) = \sum_{l=0}^{\infty} \Gamma^{(l)}(\Phi), \quad \Gamma^{(0)}(\Phi) = \mathcal{S}_R(\Phi). \tag{69}$$

The loopless term is given by the renormalized action (64), while the one-loop contribution is given by the expression:

$$\Gamma^{(1)}(\Phi) = -(1/2)\,\mathrm{Tr}\,\ln(W/W_0), \tag{70}$$

where W is a linear operation with the kernel

$$W(x,x') = -\delta^2\, S_R(\Phi)\, /\delta\Phi(x)\delta\Phi(x'), \tag{71}$$

and W_0 is its analog in the free theory. The kernel W is a 3×3 matrix in a full set of the fields $\Phi = \{h, h', v\}$. Symbolically, this matrix can be represented as

$$W(\Phi) = \begin{pmatrix} -\partial^2 h'\cdot V'' & -\partial_t - V'(h)\partial^2 & -\partial h' \\ \partial_t - \partial^2 V'(h) & -D_f & \partial h \\ h'\partial & -\partial h & D_v \end{pmatrix} \tag{72}$$

where we omitted the vector indices and dependence on the arguments for the sake of brevity.

In order to calculate renormalization constants Z_n, we do not need to calculate the full expression (70) explicitly; rather, its divergent part is enough, which is already known to have the form

$$\int dx \partial^2 h'(x) R(h(x))$$

with some function $R(h)$ analogous to (8). This means that we have to calculate (70) only to the first order in its elements proportional to h', namely $W^{(hh)}$, $W^{(hv)}$ and $W^{(vh)}$. Moreover, within the accuracy needed for one-loop calculation, we can set $Z_n = 1$ and $\mathbf{v}(x) = 0$ in the one-loop contribution (70) and (71), while in the loopless contribution $\Gamma^{(0)}(\Phi)$, we keep only the first nontrivial terms in the renormalization constants Z_n with the additional convention that $g_n \sim g_2^{(n-1)}$. Employing the well-known formula $\delta(\mathrm{Tr}\ln K) = \mathrm{Tr}(K^{-1}\delta K)$ to vary (70) in θ', we finally obtain with required accuracy that $\mathrm{Tr}\,\ln(W/W_0) \simeq -I_1 + 2I_2$ where

$$I_1 = \int dx\, D^{(hh)}(x,x) V''(h) \partial^2 h'(x) \tag{73}$$

and

$$I_2 = \int dx \int dx' \;\partial_i h(x) D^{(h'h)}(x,x') D_{ij}^{(vv)}(x,x') \partial_j h'(x'). \tag{74}$$

In these expressions $D^{(\Phi\Phi)}$ are the corresponding elements of the matrix W^{-1} taken at the vanishing values of the field h'.

To proceed further, one should make use of the fact that once the two external derivatives are moved to the external fields $h(x)$ and $h'(x)$ in the expressions (73) and (74), the remaining integrals would contain only logarithmic divergences. According to the general statement, within the framework of dimensional regularization, logarithmic divergences do not depend on external frequencies or momenta. The latter means that as long as one concerned only with the calculation of the divergent parts of expressions (73) and (74), the inhomogeneity of the field $h(x)$ can be neglected and the integral can be evaluated treating field $h(x)$ as if it was merely a constant. Then $D^{(vv)}$ turns into the standard perturbative propagator of the velocity field, while elements $D^{(hh)}$ and $D^{(h'h)}$ become the corresponding propagators $\langle hh\rangle$ and $\langle h'h\rangle$ in the presence of homogeneous background field h, i.e., usual perturbative propagators but with substitution $\nu\partial^2 \to V'(h)\,\partial^2$ with $h = const$. The resulting integrals can be calculated in a standard fashion by going over to the Fourier (momentum-frequency) representation.

The result of those calculations yields the explicit expression for the divergent part of the one-loop contribution (70):

$$\Gamma^{(1)}(\Phi) = \frac{a_d}{4y}\left(\frac{\mu}{m}\right)^y \int dx\, h'(x)\,\partial^2 F(h(x)) + a_d\,\frac{(d-1)}{2d}\,\frac{w\nu}{\xi}\left(\frac{\mu}{m}\right)^\xi \int dx\, h'(x)\,\partial^2 h(x), \quad (75)$$

with

$$F(h) = \mu^{-y}\,\frac{V''(h)}{V'(h)} = \sum_{n=0}^{\infty} \frac{1}{n!}\,\mu^{y(n-1)/2}\,\nu^{(n+1)/2}\,r_n\,h^n. \quad (76)$$

Finally, the divergences obtained should be subtracted by the appropriate choice of the renormalization constants in the tree part of the expansion, which gives the desired one-loop expressions

$$Z_1 = 1 - \frac{a_d}{4}\frac{r_1}{y} - \frac{a_d(d-1)}{2d}\frac{w}{\xi}, \qquad Z_n = 1 - \frac{a_d}{4}\frac{r_n}{y}\frac{1}{g_n} \quad (n>1). \quad (77)$$

By adjusting standard formula (34) to the case of infinite number of charges,

$$\gamma_n = \widetilde{\mathcal{D}}_\mu \ln Z_n = \left[\beta_w\partial_w + \sum_{n=2}^{\infty}\beta_n\partial_{g_n}\right]\ln Z_n \quad (78)$$

and then making use of the relations (66) and (68), all the RG functions can be expressed in terms of the basic anomalous dimensions γ_n as follows:

$$\begin{aligned}
\gamma_{g_n} &= \gamma_n - (n+1)\gamma_1/2, \quad \gamma_\nu = -\gamma_w = \gamma_1, && (79)\\
\beta_w &= w[-\xi - \gamma_w] = w[-\xi + \gamma_1], && (80)\\
\beta_n &= g_n[-(n-1)y/2 - \gamma_{g_n}] = g_n[-(n-1)y/2 - \gamma_n + (n+1)\gamma_1/2]. && (81)
\end{aligned}$$

For the further analysis, it is useful to present explicit expression for the beta-function of the coupling constant w and beta-functions of the first few couplings g_n:

$$\begin{aligned}
\beta_w &= -\xi\, w + (a_d/4)\left[2\,w(d-1)/d + g_3 - g_2^2\right],\\
\beta_2 &= -y\,g_2/2 + (a_d/8)\left[6\,w\,g_2(d-1)/d + 7g_2^3 + 9g_2\,g_3 - 2g_4\right],\\
\beta_3 &= -y\,g_3 + (a_d/4)\left[4\,w\,g_3(d-1)/d + 6g_2^4 - 14g_2^2\,g_3 + 5g_3^2 + 4g_2\,g_4 - g_5\right], \quad (82)\\
\beta_4 &= -3\,y\,g_4/2 + (a_d/8)\Big[10\,w\,g_4(d-1)/d - 48g_2^5 + 120g_2^3\,g_3 - 60g_2\,g_3^2\\
&\quad -45g_2^2\,g_4 - 25g_3\,g_4 + 10g_2\,g_5 - 2g_6\Big].
\end{aligned}$$

From (82), it follows that there are two intersecting two-dimensional surfaces of fixed points. Indeed, the equation $\beta_w(w^*, g_2^*, g_3^*) = 0$ has two solutions: trivial

$$w^* = 0, \tag{83}$$

corresponding to the IR asymptotic regime where the advection process appears to be irrelevant, and the solution

$$w^* = 2d\,[\xi - a_d(g_3^* - g_2^{*2})/4]/(a_d(d-1)), \tag{84}$$

depending on two parameters g_2^* and g_3^*. As for the rest of beta functions, one can see from their explicit representations that at either of the solutions (83) and (84), the function β_n depends on the first $(n+2)$ couplings g_n, which means that one can arbitrarily choose values of coordinates g_2^* and g_3^* and then recurrently determine all remaining coordinates g_n^* with $n > 3$ from the requirement of vanishing of the functions β_{n-2}.

According to the general rule, a fixed point is IR attractive if the real parts of all eigenvalues of the matrix $\Omega_{nm} = \partial\beta_n/\partial g_m|_{g^*}$ are non-negative. In the case under consideration, Ω is an infinite matrix and explicit calculation of its eigenvalues is a nontrivial task, which to date has not been solved. At best, it has been shown that there are regions on attractor surfaces at which all the diagonal elements $\Omega_{nn} > 0$ so that in these regions $tr[\Omega] > 0$. Nevertheless, the latter is the only necessary, but not sufficient, condition to ensure IR attractiveness.

However, if we assume that there are indeed IR attractive regions this will imply that correlation functions exhibit infrared scaling behavior described by the critical exponents

$$\Delta_h = (1 - y/2) - (1/2)\Delta_\omega, \quad \Delta_{h'} = (d - 1 + y/2) + (1/2)\Delta_\omega, \tag{85}$$

where Δ_ω differs for the solutions (83) and (84).

For the sheet with $w^* = 0$ in the one-loop approximation, we have a nonuniversal value of the frequency critical dimension

$$\Delta_\omega = 2 - a_d\,(g_3^* - g_2^{*2})/4 \tag{86}$$

which means that critical dimensions of the fields h, h' and hence scaling exponents of the correlation and response functions in the original model would also be nonuniversal in the sense that they depend on particular parameters of the admixture self-interaction. Meanwhile, for every fixed point on the sheet with $w^* \neq 0$

$$\Delta_\omega = 2 - \xi \tag{87}$$

is a universal quantity.

In both cases, critical dimensions of the scalar fields are subject to the exact relation

$$\Delta_{h'} + \Delta_h = d; \tag{88}$$

which is probably the main quantitative prediction of the present analysis that can be tested directly in the experiment even despite the possible non-universality of the IR scaling behavior.

As another possible application of the results obtained, one can consider a spread of a cloud of particles injected into the system at the origin. Then, the effective radius of the particle cloud at the moment t is given by [170]

$$R^2(t) = \int d\mathbf{x}\, x^2 \langle h(t,\mathbf{x}) h'(0,\mathbf{0})\rangle.$$

Substituting the scaling representation (2) and taking into account relation (88), one readily arrives at the spreading law

$$R^2(t) \propto t^{2/\Delta_\omega} \tag{89}$$

or, equivalently,

$$dR^2(t)/dt \propto R^{2-\Delta_\omega}(t). \tag{90}$$

By assuming that the system is at some fixed point that corresponds to the solution (83) and by substituting the most realistic value $\xi = 4/3$, we end up with nothing else but a well-known Richardson law [170]. Possible experimental observations of deviation from this law can be interpreted as evidence in favor of existence of attractive regions on the second sheet of fixed points (84).

Another interesting, albeit more theoretical, result of the RG analysis described in this Section is that coordinate w^* in (84) can be set to zero by appropriate choice of the difference $g_3^* - g_2^{*2}$; therefore, surfaces corresponding to the solutions (83) and (84) actually have a line of intersection. At this line of fixed points critical dimension of the frequency will coincide with that of the regime when advection is relevant. In this regime, the admixture behaves at large scales as if it was advected; however, this happens not because of interaction with the velocity field but due to admixture's own dynamic, which makes actual advection irrelevant. This implication refers us to the interesting question of whether admixture that locally undergoes regular diffusion in turbulent media can be described solely in terms of the anomalous diffusion equation with the suitable choice of the current which non-linearly depends on the admixture concentration.

7. Discussion and Conclusions

In this paper, we showcased different interesting effects that take place when a randomly moving environment interacts with a nearly-critical stochastic system. The effect can be as strong as leading to emergence of induced non-linearity that changes the original model itself; see how Equation (4) turns to Equation (29). Otherwise, it can make the original anisotropic scaling to appear only through certain "dimensional transmutation" in a specific regime of critical behavior; see solution (55) in Section 5. It also can be as relatively weak as just resulting in the emergence of new expected universality classes (see Section 6).

Specifically, we applied the field-theoretic RG approach to the three stochastic problems. In the first problem, a randomly growing surface described by the Kardar–Parisi–Zhang Equation (3) with the time-independent random noise (12) was studied in a moving environment modeled by the Kazantsev–Kraichnan ensemble (22). Through the RG analysis, the appearance of a new non-linearity was established. Surprisingly enough, the non-linearity turned out to be relevant in several regimes of critical behavior where the turbulent mixing that induced it is itself irrelevant. Thus, the advection plays the role of a trigger that turns on the non-linear interaction with the environment.

It is clear that the crucial reason for the strong effect of the moving environment in this problem is due to the time-independent noise (12) in Equation (3). Indeed, in [171], where the Hwa–Kardar model was considered with the Avellaneda–Majda ensemble [156,157] (anisotropic counterpart of the Kazantsev–Kraichnan ensemble) with finite correlation time, it was the case with the quenched noise that delivered the most interesting results (nonuniversal critical behavior and a complicated pattern of stability regions). The time-independent noise breaks the Galilean symmetry of the original stochastic model; it also raises logarithmic dimension of the model by the two units. Although the latter effect does not seem impressive at first sight, it plays a key role in the problem of landscape erosion (see Section 2.4), where only the model with time-independent noise is expected to predict nonuniversal nontrivial scaling [119].

The second problem involves the Hwa–Kardar model of self-organized criticality in (5), (9) and (10) that takes into account the environment motion described by the Kazantsev–Kraichnan ensemble (22). The effect that we named "dimensional transmutation" [164,165], is observed in the regime where these two systems with different symmetries coalesce and only the non-linearity of the Hwa–Kardar equation wins: the resulting critical dimensions are in agreement with those derived for the pure Hwa–Kardar model (5), (9) and (10). Even more complicatedly, non-conventional scaling was established for the regime where both the advection and the non-linearity are relevant; see Equation (57).

It seems clear that it is the interplay between anisotropic dynamics (9) and isotropic environment (22) that results in this versatility of critical behavior. Indeed, for the Hwa–Kardar model coupled to the environment described by the Navier–Stokes equation (see Equations (16) and (19), the case of $D(k) = B_0$), the curve of fixed points was established. Moreover, the points on the curve are IR-attractive, simultaneously implying nonuniversality [172,173].

In ordinary field theories like the φ^4 model, coupling constants ("charges") are the coefficients in the interaction terms and serve as expansion parameters in ordinary perturbation theory. In dynamical models like those considered in this review, the dimensionless ratios of various kinetic, diffusion and viscosity coefficients should be treated on the same footing as charges with their own β functions. The possible limits when the corresponding fixed points tend to zero or infinity give rise to a subtle pattern of IR attractors and, at the same time, lead to a non-conventional and nontrivial pattern of the critical behavior.

It has long been proposed that various interesting phenomena in complex strongly interacting many-body systems (e.g., formation of snow-flake structures described by universal self-similar laws, stochastic resonance etc.) may result from such a rivalry between intrinsic dynamics and various external disturbances that, in their turn, manifest themselves as a kind of effective external friction [174–176].

The creation of a framework that would explain all these effects, however, is a difficult undertaking. The concept of SOC, for example, has been insufficient for that task so far (see, e.g., discussion in [11]). Nevertheless, with the rapid progress in computational methods and data analysis, the scientific community is as close to the conclusive testing of SOC as it has ever been.

Another intriguing concept to explore was recently proposed and discussed in [177–180], where the interplay between complex topology and driving analyzed through the lens of geometry and competing higher-order interactions was suggested to be the key needed to understand critical dynamics.

In the third problem, we considered the non-linear diffusion (58) and (61) described by the extended Pavlik model (8) with a special choice of a random noise (59) and (60) in a moving medium modeled by the Kazantsev–Kraichnan ensemble (22). Inclusion of the velocity field in that case resulted in a pair of two-dimensional surfaces of fixed points in the infinite-dimensional parameter space (see Equations (83) and (84)) with the first surface corresponding to the regimes where turbulent advection is irrelevant. While the result (two sets of fixed points instead of one) is typical for this kind of stochastic problem, it is notable that the nontrivial regimes are represented with a whole surface of fixed points.

In the examples considered in our review, the velocity field was modeled by the Gaussian "rapid-change" Kazantsev–Kraichnan ensemble. An important direction of further study is to apply more realistic statistics or dynamics of the velocity. Specifically, it is interesting to include finite correlation time, non-Gaussianity, anisotropy, compressibility and back reaction of the system on the velocity dynamics. In particular, an anisotropic Gaussian model with finite correlation time was studied in [171], while the Navier–Stokes equation for incompressible fluid was studied in [73] for a fluid in thermal equilibrium and in [74] for a turbulent flow. This leads to appearance of new interesting universality classes and crossover phenomena.

It also seems especially promising to apply the functional RG to the models with an infinite number of counterterms. The Kazantsev–Kraichnan ensemble was analyzed with the functional RG in [181] while the landscape erosion was considered in [119]. It would be very interesting to see what the results of the analysis of the extended Pavlik model with turbulent motion included would be.

Author Contributions: All authors contributed to the paper equally. All authors have read and agreed to the published version of the manuscript.

Funding: The work of P.I.K. was supported by the Foundation for the Advancement of Theoretical Physics and Mathematics "BASIS", project 22-1-3-33-1.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflict of interest.

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