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Novel q-Differentiable Inequalities

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Abstract: The striking goal of this study is to introduce a q-identity for a parameterized integral operator via differentiable function. First, we discover the parameterized lemma for the q-integral. After that, we provide several q-differentiable inequalities. By taking suitable choices, some interesting results are obtained. With all of these, we displayed the findings from the traditional analysis utilizing $q \to 1^-$.

Keywords: quantum derivative; quantum integral; quantum inequalities; Hölder inequality; power mean inequality

MSC: 26D07; 26D10; 26D15; 26A33

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1. Introduction

The theory of inequality has a unique and important place for the function class known as convex functions, which has a very helpful definition and feature-based structure. We really encounter convexity frequently and in a variety of ways. The most common example is while we are standing up, which is safe as long as our center of gravity's vertical projection is contained inside the concave area of our feet. Convexity also significantly influences our daily lives through its myriad uses in business, health, the arts, and other fields. This function class has enhanced its relevance by being used in studies of inequality theory and in several application domains by recognizing novel types of inequality. It has been found that there is a very strong relationship between inequalities and convex functions. Both theoretical and practical domains greatly benefit from the convex function. Through its diverse uses in commerce, industry, and medicine, convexity also has a profound influence on our daily life. It is one of the most sophisticated disciplines of mathematical modeling because of the variety of implementations available. The definition of convex functions is

Definition 1 ([1]). *If $\Omega : [\mu, \nu] \subseteq \Re \to \Re$ is convex, then the inequality*

$$\Omega(\mathrm{t}\,\phi_2 + (1-\mathrm{t})\phi_1) \leq \mathrm{t}\,\Omega(\phi_2) + (1-\mathrm{t})\,\Omega(\phi_1),$$

holds for every $\phi_1,\ \phi_2 \in [\mu, \nu]$ and every $\mathrm{t} \in [0,1]$.

The Hermite–Hadamard inequality is the utmost important and extensively used result involving convex functions [2]. The most familiar inequality related to the integral mean of a convex function is stated as

$$\Omega\left(\frac{\phi_1 + \phi_2}{2}\right) \leq \frac{1}{\phi_2 - \phi_1}\int_{\phi_1}^{\phi_2} \Omega(\mathrm{t})dt \leq \frac{\Omega(\phi_1) + \Omega(\phi_2)}{2} \tag{1}$$

where $\Omega : \Im \subseteq \Re \to \Re$ is a convex function and $\phi_1, \phi_2 \in \Im$ with $\phi_1 < \phi_2$.

Mathematicians have puzzled about how to give estimates for some midpoint and trapezoid differences, where the concept of classical derivatives has been insufficient for years. This curiosity has also spurred mathematicians to embark on a new search for the practical uses for their theories that classical analysis lacks. This quest has led to the discovery of fractional derivative and integral operators, which has sped up research on fractional analysis.

U.S. Kirmaci proved the following midpoint inequality for differentiable convex functions in [3].

Theorem 1. *Let* $\Omega : [\mu, \nu] \to \Re$ *be differentiable on* (μ, ν)*. If* $|\Omega'(\mathrm{t})|$ *is convex on* $[\mu, \nu]$*, then*

$$\left|\Omega\left(\frac{\mu+\nu}{2}\right) - \frac{1}{\nu-\mu}\int_{\mu}^{\nu} \Omega(\mathrm{t})\, d\mathrm{t}\right| \le \frac{(\nu-\mu)(|\Omega'(\mu)| + |\Omega'(\nu)|)}{8}. \tag{2}$$

In [4], F. Qi and B. Y. Xi presented a new inequality for differentiable convex functions called the Bullen inequality, which can be described as

Theorem 2. *Let* $\Omega : [\mu, \nu] \to \Re$ *be a differentiable function on* (μ, ν)*. If* $|\Omega'(\mathrm{t})|$ *is integrable and convex on* $[\mu, \nu]$*, we attain the following identity:*

$$\left|\frac{1}{2}\left[\Omega\left(\frac{\mu+\nu}{2}\right) + \frac{\Omega(\mu)+\Omega(\nu)}{2}\right] - \frac{1}{(\nu-\mu)}\int_{\mu}^{\nu} \Omega(\mathrm{t})d\mathrm{t}\right| \le \frac{\nu-\mu}{16}\left[|\Omega'(\mu)| + |\Omega'(\nu)|\right]. \tag{3}$$

Due to their extensive examination in the literature, researchers have concentrated on inequalities and convex functions [5–9]. The concept of "calculus without limits", sometimes known as "quantum calculus", is an infinitesimal one without constraints. The study of quantum theory is crucial to mathematics and related fields. Mathematicians turned their attention to q-calculus, which had previously been used in physics, philosophy, cryptology, computer science, and mechanics, to study the theory of inequalities, numerical theory, fundamental hyper-geometric functions, and orthogonal polynomials (see [10–13]); a lot of research has been done in this area recently. The credit for the creation of this field goes to Euler, who deployed the q-parameter to Newton's investigation of infinite series. Jackson was the one to introduce the q-calculus, according to [10]. As the first phase of his symmetrical study in the nineteenth century, Jackson created q-definite integrals. Tariboon first introduced the ${}_{\mu}D_{\mathrm{q}}$-difference operator in 2013 [14].

Firstly, in this research article, we will establish general q-parameterized identity. Then, by employing this identity against convex functions, we attain new quantum variants of Mid-point, Simpson, Trapezoid, and Bullen-type inequalities. With the particular selection of $\mathrm{q} \to 1^-$, we can recapture the above inequalities in limiting calculus.

2. Some Basics of q-Calculus

In the present section, we recall concepts of q-derivatives, q-integrals, and related results:

Definition 2 ([14])**.** *If* $\Omega : [\mu, \nu] \to \Re$*, the* ${}_{\mu}$q*-derivative of* Ω *at* $\phi_1 \in [\mu, \nu]$ *is defined as*

$${}_{\mu}D_{\mathrm{q}}\Omega(\phi_1) = \frac{\Omega(\phi_1) - \Omega(\mathrm{q}\phi_1 + (1-\mathrm{q})\mu)}{(1-\mathrm{q})(\phi_1-\mu)}, \quad \phi_1 \neq \mu. \tag{4}$$

If $\phi_1 = \mu$*, we define* ${}_{\mu}D_{\mathrm{q}}\phi(\mu) = \lim_{\phi_1 \to \mu} \; {}_{\mu}D_{\mathrm{q}}\phi(\phi_1)$ *if it exists and is finite.*

In [15], the notion of q-Riemann integral was demonstrated in terms of Jackson q-integral on $[\mu,\nu]$:

$$\int_{\mu}^{\phi_1} \Omega(\mathrm{t}) \ {}_{\mu}d_{\mathrm{q}}\mathrm{t} \ = (1-\mathrm{q})(\phi_1-\mu)\sum_{\aleph=0}^{\infty}\mathrm{q}^{\aleph}\Omega\Big(\mathrm{q}^{\aleph}\phi_1+\Big(1-\mathrm{q}^{\aleph}\Big)\mu\Big), \ \ \phi_1\in[\mu,\nu]. \tag{5}$$

Definition 3. *If $\mu=0$ in (5), then $\int_0^{\phi_1}\Omega(\mathrm{t}) \ {}_0d_{\mathrm{q}}\mathrm{t} \ = \int_0^{\phi_1}\Omega(\mathrm{t}) \ d_{\mathrm{q}}\mathrm{t}$,*

where: $\int_0^{\phi_1}\Omega(\mathrm{t}) \ d_{\mathrm{q}}\mathrm{t}$ is q-definite integral on $[0,\phi_1]$ and defined as [13]

$$\int_0^{\phi_1}\Omega(\mathrm{t}) \ {}_0d_{\mathrm{q}}\mathrm{t} \ = \int_0^{\phi_1}\Omega(\mathrm{t}) \ d_{\mathrm{q}}\mathrm{t} \ = (1-\mathrm{q})\phi_1\sum_{\aleph=0}^{\infty}\mathrm{q}^{\aleph}\Omega\Big(\mathrm{q}^{\aleph}\phi_1\Big). \tag{6}$$

If $c\in(\mu,\phi_1)$, then the q-definite integral on $[c,\phi_1]$ is expressed as:

$$\int_c^{\phi_1}\Omega(\mathrm{t}) \ {}_{\mu}d_{\mathrm{q}}\mathrm{t} \ = \int_{\mu}^{\phi_1}\Omega(\mathrm{t}) \ {}_{\mu}d_{\mathrm{q}}\mathrm{t} \ - \int_{\mu}^{c}\Omega(\mathrm{t}) \ {}_{\mu}d_{\mathrm{q}}\mathrm{t} \ . \tag{7}$$

In order to solve differential equations, integral inequalities are quite useful to estimate bounds. Numerous researchers have looked into how integral inequality can be used in both classical and quantum calculus to explore new useful possibilities. Since mathematical inequality's significance has long been understood, inequalities like Hermite–Hadamard, Jensen, Ostrowski, and Hölder were frequently utilized in quantum calculus. Tariboon and Ntouyas described the q-derivative and q-integral of ongoing work at intervals and confirmed some of its features in 2013 [14]. Numerous well-known inequalities, including those based on Hermite–Hadamard, trapezoid, Ostrowski, Cauchy–Bunyakovsky–Schwarz, Gruss, and Gruss–Cebyvsev, have been examined for q-calculus in [15]. In 2020, Bermudo et al. [16] explained new q-derivative and q-integral for continuous work on a regular basis cost ${}^{\nu}$q-calculus, while the previous one is called ${}_{\mu}$q-calculus. In [17], Alp et al. proved some Midpoint-type inequalities for ${}_{\mu}$q-integrals. Noor et al. established some inequalities of trapezoid type for ${}_{\mu}$q-integrals in [18]. On the other hand, Budak et al. present several midpoint and trapezoid type inequalities for ${}^{\nu}$q-integrals in [19,20]. In [21], Budak et al. proved some Simpson–Newton type inequalities by using the concept of quantum integrals. Many mathematicians have conducted research in the area of quantum calculus; the interested reader can check [17,22–24]. Recently in [25], q-calculus has been used to define positive operators involving Bézier bases. In [17], several variants of q-Hermite–Hadamard inequalities were established by utilizing the idea of support line for convex functions. They also provided the corrected version of q-Hermite–Hadamard inequality given as follows:

Let $\Omega:[\mu,\nu]\to\Re$ be a convex function on $[\mu,\nu]$; we have

$$\Omega\left(\frac{\mathrm{q}\mu+\nu}{1+\mathrm{q}}\right)\leq\frac{1}{\nu-\mu}\int_{\mu}^{\nu}\Omega(\mathrm{t}) \ {}_{\mu}d_{\mathrm{q}}\mathrm{t} \ \leq\frac{\mathrm{q}\Omega(\mu)+\Omega(\nu)}{1+\mathrm{q}}.$$

Also, another useful variant was given in [17] as follows:

$$\Omega\left(\frac{\mu+\nu}{2}\right)-\frac{(1-\mathrm{q})(\nu-\mu)}{2(1+\mathrm{q})}\Omega'\left(\frac{\mu+\nu}{2}\right)\leq\frac{1}{\nu-\mu}\int_{\mu}^{\nu}\Omega(\mathrm{t}) \ {}_{\mu}d_{\mathrm{q}}\mathrm{t} \ \leq\frac{\mathrm{q}\Omega(\mu)+\Omega(\nu)}{1+\mathrm{q}}.$$

Later, Bermudo et al. in [16] presented another picture of the q-Hermite–Hadamard inequality considering ${}^{\nu}$q-integral as follows:

Theorem 3. *For a convex function $\Omega : [\mu, \nu] \to \Re$, the following inequalities hold for* $q \in (0,1)$*:*

$$\Omega\left(\frac{\mu + q\nu}{1+q}\right) \le \frac{1}{\nu - \mu}\int_{\mu}^{\nu} \Omega(t)\, {}^{\nu}d_q t \le \frac{\Omega(\mu) + q\Omega(\nu)}{1+q} \tag{8}$$

and

$$\Omega\left(\frac{\mu + \nu}{2}\right) \le \frac{1}{2(\nu - \mu)}\left[\int_{\mu}^{\nu} \Omega(t)\, {}_{\mu}d_q t + \int_{\mu}^{\nu} \Omega(t)\, {}^{\nu}d_q t\right] \le \frac{\Omega(\mu) + \Omega(\nu)}{2}. \tag{9}$$

In [26], M. A. Ali et al. proved the following new version of quantum Hermite–Hadamard inequality involving ${}_{\mu}$q and ${}^{\nu}$q-integrals.

Theorem 4. *If $\Omega : [\mu, \nu] \to \Re$ is a convex function, then we have*

$$\Omega\left(\frac{\mu + \nu}{2}\right) \le \frac{1}{\nu - \mu}\left[\int_{\mu}^{\frac{\mu+\nu}{2}} \Omega(t)\, {}_{\mu}d_q t + \int_{\frac{\mu+\nu}{2}}^{\nu} \Omega(t)\, {}^{\nu}d_q t\right] \le \frac{\Omega(\mu) + \Omega(\nu)}{2}.$$

In [17], the authors established the following midpoint inequalities for q-differentiable convex functions.

Theorem 5. *Let $\Omega : [\mu, \nu] \to \Re$ be a* q*-differentiable on (μ, ν). If $|{}_{\mu}D_q\Omega(x)|$ is convex on $[\mu, \nu]$, then the following inequality holds for* $q \in (0,1)$*:*

$$\begin{aligned} &\left|\Omega\left(\frac{q\mu + \nu}{1+q}\right) - \frac{1}{\nu - \mu}\int_{\mu}^{\nu} \Omega(t)\, {}_{\mu}d_q t\right| \\ \le\ & \frac{q(\nu - \mu)}{(1+q)^3(1+q+q^2)}\left[3|{}_{\mu}D_q\Omega(\nu)| + \left(2q^2 + 2q - 1\right)|{}_{\mu}D_q\Omega(\mu)|\right]. \end{aligned} \tag{10}$$

After that, M. A. Noor et al. [18] proved some new trapezoidal inequalities for q-differentiable convex functions.

Theorem 6. *Let $\Omega : [\mu, \nu] \to \Re$ be* q*-differentiable on (μ, ν). If $|{}_{\mu}D_q\Omega(x)|$ is convex on $[\mu, \nu]$, then following inequality holds for* $q \in (0,1)$*:*

$$\begin{aligned} &\left|\frac{q\Omega(\mu) + \Omega(\nu)}{1+q} - \frac{1}{\nu - \mu}\int_{\mu}^{\nu} \Omega(t)\, {}_{\mu}d_q t\right| \\ \le\ & \frac{q^2(\nu - \mu)}{(1+q)^4(1+q+q^2)}\left[\left(1 + 4q + q^2\right)|{}_{\mu}D_q\Omega(\nu)| + \left(1 + 3q^2 + 2q^3\right)|{}_{\mu}D_q\Omega(\mu)|\right]. \end{aligned} \tag{11}$$

Very recently, H. Budak established midpoint and trapezoidal-type inequalities for q-differentiable convex functions.

Theorem 7 ([19])**.** *Let $\Omega : [\mu, \nu] \to \Re$ be* q*-differentiable on (μ, ν). If $|{}^{\nu}D_q\Omega(x)|$ is convex on $[\mu, \nu]$, then the following inequality holds for* $q \in (0,1)$*:*

$$\begin{aligned} &\left|\Omega\left(\frac{\mu + q\nu}{1+q}\right) - \frac{1}{\nu - \mu}\int_{\mu}^{\nu} \Omega(t)\, {}^{\nu}d_q t\right| \\ \le\ & \frac{q(\nu - \mu)}{(1+q)^3(1+q+q^2)}\left[3|{}^{\nu}D_q\Omega(\mu)| + \left(2q^2 + 2q - 1\right)|{}^{\nu}D_q\Omega(\nu)|\right] \end{aligned} \tag{12}$$

and

$$\left|\frac{\Omega(\mu)+q\Omega(\nu)}{1+q}-\frac{1}{\nu-\mu}\int_{\mu}^{\nu}\Omega(t)\ {}^{\nu}d_qt\right| \tag{13}$$
$$\leq \frac{q^2(\nu-\mu)}{(1+q)^4(1+q+q^2)}\left[\left(1+4q+q^2\right)\left|{}^{\nu}D_q\Omega(\mu)\right|+\left(1+3q^2+2q^3\right)\left|{}^{\nu}D_q\Omega(\nu)\right|\right].$$

The following notations will be frequently used:

$$[\aleph]_q=\sum_{i=0}^{\aleph-1}q^i=1+q+q^2+\ldots+q^{\aleph-1}$$

and

$$(1-t)_q^{\aleph}=(t,q)_{\aleph}=\prod_{i=0}^{\aleph-1}\left(1-q^it\right), \tag{14}$$

Lemma 1 ([17])**.** *The following equality holds:*

$$\int_{\mu}^{\phi_1}(t-\mu)^{\alpha}\ {}_{\mu}d_qt\ =\frac{(\phi_1-\mu)^{\alpha+1}}{[\alpha+1]_q}, \tag{15}$$

for $\alpha\in\Re\backslash\{-1\}$.

Lemma 2 ([27])**.** *The following equality holds:*

$$\int_{\frac{1}{[2]_q}}^{1}(1-qt)_q^{\aleph}d_qt=\frac{\left(1-\frac{1}{[2]_q}\right)_q^{\aleph+1}}{[\aleph+1]_q}.$$

In [28,29], the authors provide q-integration by parts as follows:

Lemma 3. *For continuous functions* $h,\Omega:[\mu,\nu]\to\mathbb{R}$, *the following equality holds:*

$$\int_0^{c}h(t)_{\mu}D_q\Omega(t\nu+(1-t)\mu)_0d_qt$$
$$=\frac{h(t)\Omega(t\nu+(1-t)\mu)}{\nu-\mu}\bigg|_0^{c}-\frac{1}{\nu-\mu}\int_0^{c}\Omega(qt\nu+(1-qt)\mu)_0D_qh(t)_0d_qt. \tag{16}$$

Lemma 4. *For continuous functions* $h,\Omega:[\mu,\nu]\to\mathbb{R}$, *the following equality holds:*

$$\int_0^{c}h(t)^{\nu}D_q\Omega(t\mu+(1-t)\nu)\ {}_0d_qt$$
$$=\frac{1}{\nu-\mu}\int_0^{c}\Omega(qt\mu+(1-qt)\nu)_0D_qh(t)_0d_qt-\frac{h(t)\Omega(t\mu+(1-t)\nu)}{\nu-\mu}\bigg|_0^{c}. \tag{17}$$

Finding novel parameterized quantum inequalities with convex derivatives is the main objective of this study. The findings were then calculated using Hölder's inequality and Power Mean inequality. Several exceptional cases have been proven to justify disparities in the literature.

3. Parameterized Quantum Inequalities

We firstly prove the following lemma which is required for our main results.

Lemma 5. *Let* $\Omega : [\mu, \nu] \to \Re$ *be a* q*-differentiable function on* (μ, ν)*. If* ${}_{\mu}D_{\mathrm{q}}\Omega$ *and* ${}^{\nu}D_{\mathrm{q}}\Omega$ *are continuous and integrable on* $[\mu, \nu]$*, we attain the following identity:*

$$\frac{(\nu-\mu)}{4}\left[\int_0^1 (\lambda - \mathrm{qt})\, {}^{\nu}D_{\mathrm{q}}\Omega\left(\frac{\mathrm{t}}{2}\mu + \frac{2-\mathrm{t}}{2}\nu\right)d_{\mathrm{q}}\mathrm{t} + \int_0^1 (\mathrm{qt} - \lambda)\, {}_{\mu}D_{\mathrm{q}}\Omega\left(\frac{\mathrm{t}}{2}\nu + \frac{2-\mathrm{t}}{2}\mu\right)d_{\mathrm{q}}\mathrm{t}\right]$$
$$= (1-\lambda)\Omega\left(\frac{\mu+\nu}{2}\right) + \lambda\frac{\Omega(\mu)+\Omega(\nu)}{2} - \frac{1}{(\nu-\mu)}\left[\int_{\mu}^{\frac{\mu+\nu}{2}} \Omega(\mathrm{t})\, {}_{\mu}d_{\mathrm{q}}\mathrm{t} + \int_{\frac{\mu+\nu}{2}}^{\nu} \Omega(\mathrm{t})\, {}^{\nu}d_{\mathrm{q}}\mathrm{t}\right]$$

for $\mathrm{q} \in (0,1)$ *and* $\lambda \in \mathbb{R}$.

Proof.

$$\frac{(\nu-\mu)}{4}\left[\int_0^1 (\lambda - \mathrm{qt})\, {}^{\nu}D_{\mathrm{q}}\Omega\left(\frac{\mathrm{t}}{2}\mu + \frac{2-\mathrm{t}}{2}\nu\right)d_{\mathrm{q}}\mathrm{t} + \int_0^1 (\mathrm{qt} - \lambda)\, {}_{\mu}D_{\mathrm{q}}\Omega\left(\frac{\mathrm{t}}{2}\nu + \frac{2-\mathrm{t}}{2}\mu\right)d_{\mathrm{q}}\mathrm{t}\right]$$
$$= \frac{(\nu-\mu)}{4}[E_1 + E_2] \tag{18}$$

Calculate the value of E_1 by using Lemma 4; we have

$$E_1 = \int_0^1 (\lambda - \mathrm{qt})\, {}^{\nu}D_{\mathrm{q}}\Omega\left(\frac{\mathrm{t}}{2}\mu + \frac{2-\mathrm{t}}{2}\nu\right)d_{\mathrm{q}}\mathrm{t} = \int_0^1 (\lambda - \mathrm{qt})\, {}^{\nu}D_{\mathrm{q}}\Omega\left(\frac{\mu+\nu}{2}\mathrm{t} + (1-\mathrm{t})\nu\right)d_{\mathrm{q}}\mathrm{t}$$
$$= \frac{(\lambda - \mathrm{q}t)\Omega\left(\frac{\mu+\nu}{2}\mathrm{t} + (1-\mathrm{t})\nu\right)}{\nu - \frac{\mu+\nu}{2}}\Bigg|_1^0 + \frac{1}{\nu - \frac{\mu+\nu}{2}}\int_0^1 (-\mathrm{q})\Omega\left(\frac{\mu+\nu}{2}\mathrm{qt} + (1-\mathrm{qt})\nu\right)d_{\mathrm{q}}\mathrm{t}$$
$$= \frac{2}{\nu-\mu}(\mathrm{q}-\lambda)\Omega\left(\frac{\mu+\nu}{2}\right) + \frac{2\lambda}{\nu-\mu}\Omega(\nu) + \frac{-2\mathrm{q}}{\nu-\mu}\int_0^1 \Omega\left(\frac{\mu+\nu}{2}\mathrm{qt} + (1-\mathrm{qt})\nu\right)d_{\mathrm{q}}\mathrm{t}$$
$$= \frac{2}{\nu-\mu}(\mathrm{q}-\lambda)\Omega\left(\frac{\mu+\nu}{2}\right) + \frac{2\lambda}{\nu-\mu}\Omega(\nu) - \frac{2\mathrm{q}}{\nu-\mu}(1-\mathrm{q})\sum_{n=0}^{\infty}\mathrm{q}^n\Omega\left(\frac{\mu+\nu}{2}\mathrm{q}^{n+1} + (1-\mathrm{q}^{n+1})\nu\right)$$
$$= \frac{2}{\nu-\mu}(\mathrm{q}-\lambda)\Omega\left(\frac{\mu+\nu}{2}\right) + \frac{2\lambda}{\nu-\mu}\Omega(\nu) - \frac{2}{\nu-\mu}(1-\mathrm{q})\left[\sum_{m=0}^{\infty}\mathrm{q}^m\Omega\left(\frac{\mu+\nu}{2}\mathrm{q}^m + (1-\mathrm{q}^m)\nu\right) - \Omega\left(\frac{\mu+\nu}{2}\right)\right]$$
$$= \frac{2(1-\lambda)}{\nu-\mu}\Omega\left(\frac{\mu+\nu}{2}\right) + \frac{2\lambda}{\nu-\mu}\Omega(\nu) - \frac{2}{\nu-\mu}(1-\mathrm{q})\sum_{m=0}^{\infty}\mathrm{q}^m\Omega\left(\frac{\mu+\nu}{2}\mathrm{q}^m + (1-\mathrm{q}^m)\nu\right)$$
$$= \frac{2(1-\lambda)}{\nu-\mu}\Omega\left(\frac{\mu+\nu}{2}\right) + \frac{2\lambda}{\nu-\mu}\Omega(\nu) - \frac{4}{(\nu-\mu)^2}\int_{\frac{\mu+\nu}{2}}^{\nu}\Omega(\mathrm{t})\, {}^{\nu}d_{\mathrm{q}}\mathrm{t}.$$

Similarly, by Lemma 3, we obtain

$$E_2 = \int_0^1 (\mathrm{qt}-\lambda)\ {}_{\mu}D_{\mathrm{q}}\Omega\left(\frac{\mathrm{t}}{2}\nu + \frac{2-\mathrm{t}}{2}\mu\right)d_{\mathrm{q}}\mathrm{t} = \int_0^1 (\mathrm{qt}-\lambda)\ {}_{\mu}D_{\mathrm{q}}\Omega\left(\frac{\mu+\nu}{2}\mathrm{t} + (1-\mathrm{t})\mu\right)d_{\mathrm{q}}\mathrm{t}$$

$$= \frac{(\mathrm{qt}-\lambda)\Omega\left(\frac{\mu+\nu}{2}\mathrm{t} + (1-\mathrm{t})\mu\right)}{\frac{\mu+\nu}{2}-\mu}\Bigg|_0^1 - \frac{1}{\frac{\mu+\nu}{2}-\mu}\int_0^1 \mathrm{q}\Omega\left(\frac{\mu+\nu}{2}\mathrm{qt} + (1-\mathrm{qt})\mu\right)d_{\mathrm{q}}\mathrm{t}$$

$$= \frac{2}{\nu-\mu}(\mathrm{q}-\lambda)\Omega\left(\frac{\mu+\nu}{2}\right) + \frac{2\lambda}{\nu-\mu}\Omega(\nu) - \frac{2}{\nu-\mu}(1-\mathrm{q})\sum_{n=0}^{\infty}\mathrm{q}^{n+1}\Omega\left(\frac{\mu+\nu}{2}\mathrm{q}^{n+1} + (1-\mathrm{q}^{n+1})\mu\right)$$

$$= \frac{2}{\nu-\mu}(\mathrm{q}-\lambda)\Omega\left(\frac{\mu+\nu}{2}\right) + \frac{2\lambda}{\nu-\mu}\Omega(\mu) - \frac{2}{\nu-\mu}(1-\mathrm{q})\left[\sum_{m=0}^{\infty}\mathrm{q}^{m}\Omega\left(\frac{\mu+\nu}{2}\mathrm{q}^{m} + (1-\mathrm{q}^{m})\mu\right) - \Omega\left(\frac{\mu+\nu}{2}\right)\right]$$

$$= \frac{2(1-\lambda)}{\nu-\mu}\Omega\left(\frac{\mu+\nu}{2}\right) + \frac{2\lambda}{\nu-\mu}\Omega(\mu) - \frac{2}{\nu-\mu}(1-\mathrm{q})\sum_{m=0}^{\infty}\mathrm{q}^{m}\Omega\left(\frac{\mu+\nu}{2}\mathrm{q}^{m} + (1-\mathrm{q}^{m})\nu\right)$$

$$= \frac{2(1-\lambda)}{\nu-\mu}\Omega\left(\frac{\mu+\nu}{2}\right) + \frac{2\lambda}{\nu-\mu}\Omega(\nu) - \frac{4}{(\nu-\mu)^2}\int_{\mu}^{\frac{\mu+\nu}{2}} \Omega(\mathrm{t})\ {}_{\mu}d_{\mathrm{q}}\mathrm{t}.$$

Putting the values of E_1 and E_2 in (18), we obtain the desire result. □

Corollary 1. *By setting* $\mathrm{q} \to 1^-$ *in Lemma 5, we have*

$$\frac{(\nu-\mu)}{4}\left[\int_0^1 (\lambda-\mathrm{t})\Omega'\left(\frac{\mathrm{t}}{2}\mu + \frac{2-\mathrm{t}}{2}\nu\right)d\mathrm{t} + \int_0^1 (\mathrm{t}-\lambda)\Omega'\left(\frac{\mathrm{t}}{2}\nu + \frac{2-\mathrm{t}}{2}\mu\right)d\mathrm{t}\right]$$

$$= (1-\lambda)\Omega\left(\frac{\mu+\nu}{2}\right) + \lambda\frac{\Omega(\mu)+\Omega(\nu)}{2} - \frac{1}{(\nu-\mu)}\int_{\mu}^{\nu}\Omega(\mathrm{t})d\mathrm{t}.$$

Remark 1. ***(i)*** *If we choose* $\lambda = 1$ *in Lemma 5, we obtain Lemma 4.1 of [26].*
(ii) *If we choose* $\lambda = \frac{1}{2}$ *in Lemma 5, we obtain Lemma 4 of [30].*

Theorem 8. *Let the assumptions of Lemma 5 hold. Then for* $\mathrm{q} \in (0,1)$ *and* $\lambda \in [0,1]$*, we have the following inequality:*

$$\left|(1-\lambda)\Omega\left(\frac{\mu+\nu}{2}\right) + \lambda\frac{\Omega(\mu)+\Omega(\nu)}{2} - \frac{1}{(\nu-\mu)}\left[\int_{\mu}^{\frac{\mu+\nu}{2}}\Omega(\mathrm{t})\ {}_{\mu}d_{\mathrm{q}}\mathrm{t} + \int_{\frac{\mu+\nu}{2}}^{\nu}\Omega(\mathrm{t})\ {}^{\nu}d_{\mathrm{q}}\mathrm{t}\right]\right| \tag{19}$$
$$\leq \frac{(\nu-\mu)}{4}\left[\Omega_1(\mathrm{q},\lambda)\left(\left|{}^{\nu}D_{\mathrm{q}}\Omega(\mu)\right| + \left|{}_{\mu}D_{\mathrm{q}}\Omega(\nu)\right|\right) + \Omega_2(\mathrm{q},\lambda)\left(\left|{}^{\nu}D_{\mathrm{q}}\Omega(\nu)\right| + \left|{}_{\mu}D_{\mathrm{q}}\Omega(\mu)\right|\right)\right],$$

where

$$\Omega_1(\mathrm{q},\lambda) = \frac{1}{2}\int_0^1 |\lambda-\mathrm{q}t|t\, d_{\mathrm{q}}t$$
$$= \begin{cases} \dfrac{2\lambda^3 + \mathrm{q}[2]_{\mathrm{q}} - \lambda[3]_{\mathrm{q}}}{2[2]_{\mathrm{q}}[3]_{\mathrm{q}}}, & 0 \leq \lambda \leq \mathrm{q}; \\[2ex] \dfrac{\lambda[3]_{\mathrm{q}} - \mathrm{q}[2]_{\mathrm{q}}}{2[2]_{\mathrm{q}}[3]_{\mathrm{q}}}, & \lambda > \mathrm{q}, \end{cases}$$

and

$$\Omega_2(\mathrm{q},\lambda) = \int_0^1 |\mathrm{q}t-\lambda|\left(\frac{2-t}{2}\right)d_\mathrm{q}t = \begin{cases} \frac{2\lambda^2+\mathrm{q}-\lambda[2]_\mathrm{q}}{[2]_\mathrm{q}} - \frac{2\lambda^3+\mathrm{q}[2]_\mathrm{q}-\lambda[3]_\mathrm{q}}{2[2]_\mathrm{q}[3]_\mathrm{q}}, & 0\le\lambda\le\mathrm{q}; \\ \frac{2\mathrm{q}+1}{2[2]_\mathrm{q}}\lambda - \frac{2\mathrm{q}^3+\mathrm{q}^2+\mathrm{q}}{2[2]_\mathrm{q}[3]_\mathrm{q}}, & \lambda>\mathrm{q}. \end{cases}$$

Proof. Taking modulus on Lemma 5, we obtain

$$\left|(1-\lambda)\Omega\left(\frac{\mu+\nu}{2}\right)+\lambda\frac{\Omega(\mu)+\Omega(\nu)}{2}-\frac{1}{(\nu-\mu)}\left[\int_\mu^{\frac{\mu+\nu}{2}}\Omega(\mathrm{t})\,{}_\mu d_\mathrm{q}\mathrm{t}+\int_{\frac{\mu+\nu}{2}}^{\nu}\Omega(\mathrm{t})\,{}^\nu d_\mathrm{q}\mathrm{t}\right]\right|$$
$$\le\frac{(\nu-\mu)}{4}\left[\int_0^1|\lambda-\mathrm{q}t|\left|{}^\nu D_\mathrm{q}\Omega\left(\frac{\mathrm{t}}{2}\mu+\frac{2-\mathrm{t}}{2}\nu\right)d_\mathrm{q}\mathrm{t}\right|+\int_0^1|\mathrm{q}t-\lambda|\left|{}_\mu D_\mathrm{q}\Omega\left(\frac{\mathrm{t}}{2}\nu+\frac{2-\mathrm{t}}{2}\mu\right)d_\mathrm{q}\mathrm{t}\right|\right]$$
$$\le\frac{(\nu-\mu)}{4}\left[\int_0^1|\lambda-\mathrm{q}t|\left(\frac{\mathrm{t}}{2}|{}^\nu D_\mathrm{q}\Omega(\mu)|+\frac{2-\mathrm{t}}{2}|{}^\nu D_\mathrm{q}\Omega(\nu)|\right)d_\mathrm{q}\mathrm{t}\right.$$
$$\left.+\int_0^1|\mathrm{q}t-\lambda|\left(\frac{\mathrm{t}}{2}|{}_\mu D_\mathrm{q}\Omega(\nu)|+\frac{2-\mathrm{t}}{2}|{}_\mu D_\mathrm{q}\Omega(\mu)|\right)d_\mathrm{q}\mathrm{t}\right],$$

by using simple calculations, we obtain the required result. □

Remark 2. *If we choose $\lambda=0$ in Theorem 8, then we have the following midpoint-type inequality:*

$$\left|\Omega\left(\frac{\mu+\nu}{2}\right)-\frac{1}{(\nu-\mu)}\left[\int_\mu^{\frac{\mu+\nu}{2}}\Omega(\mathrm{t})\,{}_\mu d_\mathrm{q}\mathrm{t}+\int_{\frac{\mu+\nu}{2}}^{\nu}\Omega(\mathrm{t})\,{}^\nu d_\mathrm{q}\mathrm{t}\right]\right| \quad (20)$$
$$\le\frac{\mathrm{q}(\nu-\mu)}{8[3]_\mathrm{q}[2]_\mathrm{q}}\left[[2]_\mathrm{q}(|{}^\nu D_\mathrm{q}\Omega(\mu)|+|{}_\mu D_\mathrm{q}\Omega(\nu)|)+\left([3]_\mathrm{q}+\mathrm{q}^2\right)(|{}^\nu D_\mathrm{q}\Omega(\nu)|+|{}_\mu D_\mathrm{q}\Omega(\mu)|)\right]$$

which is proved by Ali et al. in [26].

Remark 3. *If we choose $\lambda=1$ in Theorem 8, then we have the following trapezoid-type inequality:*

$$\left|\frac{\Omega(\mu)+\Omega(\nu)}{2}-\frac{1}{(\nu-\mu)}\left[\int_\mu^{\frac{\mu+\nu}{2}}\Omega(\mathrm{t})\,{}_\mu d_\mathrm{q}\mathrm{t}+\int_{\frac{\mu+\nu}{2}}^{\nu}\Omega(\mathrm{t})\,{}^\nu d_\mathrm{q}\mathrm{t}\right]\right| \quad (21)$$
$$\le\frac{(\nu-\mu)}{8[2]_\mathrm{q}[3]_\mathrm{q}}\left[(|{}^\nu D_\mathrm{q}\Omega(\mu)|+|{}_\mu D_\mathrm{q}\Omega(\nu)|)+\left([3]_\mathrm{q}+\mathrm{q}+\mathrm{q}^2\right)(|{}^\nu D_\mathrm{q}\Omega(\nu)|+|{}_\mu D_\mathrm{q}\Omega(\mu)|)\right]$$

which is proved by Ali et al. in [26].

Remark 4. *If we choose $\lambda=\frac{1}{2}$ in Theorem 8, then we have the following Bullen-type inequality:*

$$\left|\frac{1}{2}\left[\Omega\left(\frac{\mu+\nu}{2}\right)+\frac{\Omega(\mu)+\Omega(\nu)}{2}\right]-\frac{1}{(\nu-\mu)}\left[\int_{\mu}^{\frac{\mu+\nu}{2}}\Omega(\mathrm{t})\,{}_{\mu}d_{\mathrm{q}}\mathrm{t}+\int_{\frac{\mu+\nu}{2}}^{\nu}\Omega(\mathrm{t})\,{}^{\nu}d_{\mathrm{q}}\mathrm{t}\right]\right| \tag{22}$$
$$\leq\frac{(\nu-\mu)}{4}\left[\Omega_1\left(\mathrm{q},\frac{1}{2}\right)\left(\left|{}^{\nu}D_{\mathrm{q}}\Omega(\mu)\right|+\left|{}_{\mu}D_{\mathrm{q}}\Omega(\nu)\right|\right)+\Omega_2\left(\mathrm{q},\frac{1}{2}\right)\left(\left|{}^{\nu}D_{\mathrm{q}}\Omega(\nu)\right|+\left|{}_{\mu}D_{\mathrm{q}}\Omega(\mu)\right|\right)\right],$$

where

$$\Omega_1\left(\mathrm{q},\frac{1}{2}\right)=\frac{1}{2}\int_0^1\left|\frac{1}{2}-\mathrm{q}t\right|t\,d_{\mathrm{q}}t=\begin{cases}\dfrac{2\mathrm{q}^2+2\mathrm{q}-1}{8[2]_{\mathrm{q}}[3]_{\mathrm{q}}}, & \mathrm{q}\geq\frac{1}{2};\\[2mm] \dfrac{1-\mathrm{q}-\mathrm{q}^2}{4[2]_{\mathrm{q}}[3]_{\mathrm{q}}}, & 0<\mathrm{q}<\frac{1}{2},\end{cases}$$

and

$$\Omega_2\left(\mathrm{q},\frac{1}{2}\right)=\int_0^1\left|\mathrm{q}t-\frac{1}{2}\right|\left(\frac{2-t}{2}\right)d_{\mathrm{q}}t=\begin{cases}\dfrac{4\mathrm{q}^3+2\mathrm{q}^2+2\mathrm{q}+1}{8[2]_{\mathrm{q}}[3]_{\mathrm{q}}}, & \mathrm{q}\geq\frac{1}{2};\\[2mm] \dfrac{1+\mathrm{q}+\mathrm{q}^2-2\mathrm{q}^3}{4[2]_{\mathrm{q}}[3]_{\mathrm{q}}}, & 0<\mathrm{q}<\frac{1}{2}\end{cases}$$

which is proved by Wannalookkhee et al. in [30].

Remark 5. *If we choose $\lambda=\frac{1}{2}$ and $\mathrm{q}\to 1^-$ in Theorem 8, then we have the inequality in (3).*

Remark 6. *If we choose $\lambda=\frac{1}{3}$ in Theorem 8, then we have the following Simpson-type inequality:*

$$\left|\frac{1}{6}\left[\Omega(\mu)+4\Omega\left(\frac{\mu+\nu}{2}\right)+\Omega(\nu)\right]-\frac{1}{(\nu-\mu)}\left[\int_{\mu}^{\frac{\mu+\nu}{2}}\Omega(\mathrm{t})\,{}_{\mu}d_{\mathrm{q}}\mathrm{t}+\int_{\frac{\mu+\nu}{2}}^{\nu}\Omega(\mathrm{t})\,{}^{\nu}d_{\mathrm{q}}\mathrm{t}\right]\right| \tag{23}$$
$$\leq\frac{(\nu-\mu)}{4}\left[\Omega_1\left(\mathrm{q},\frac{1}{3}\right)\left(\left|{}^{\nu}D_{\mathrm{q}}\Omega(\mu)\right|+\left|{}_{\mu}D_{\mathrm{q}}\Omega(\nu)\right|\right)+\Omega_2\left(\mathrm{q},\frac{1}{3}\right)\left(\left|{}^{\nu}D_{\mathrm{q}}\Omega(\nu)\right|+\left|{}_{\mu}D_{\mathrm{q}}\Omega(\mu)\right|\right)\right],$$

where

$$\Omega_1\left(\mathrm{q},\frac{1}{3}\right)=\frac{1}{2}\int_0^1\left|\frac{1}{3}-\mathrm{q}t\right|t\,d_{\mathrm{q}}t=\begin{cases}\dfrac{18\mathrm{q}^2+18\mathrm{q}-7}{54[2]_{\mathrm{q}}[3]_{\mathrm{q}}}, & \mathrm{q}\geq\frac{1}{3};\\[2mm] \dfrac{1-2\mathrm{q}-2\mathrm{q}^{\mathrm{q}}}{6[2]_{\mathrm{q}}[3]_{\mathrm{q}}}, & 0<\mathrm{q}<\frac{1}{3},\end{cases}$$

and

$$\Omega_2\left(\mathrm{q},\frac{1}{3}\right)=\int_0^1\left|\mathrm{q}t-\frac{1}{3}\right|\left(\frac{2-t}{2}\right)d_{\mathrm{q}}t=\begin{cases}\dfrac{36\mathrm{q}^3+12\mathrm{q}^2+12\mathrm{q}+1}{54[2]_{\mathrm{q}}[3]_{\mathrm{q}}}, & \mathrm{q}\geq\frac{1}{3};\\[2mm] \dfrac{1-4\mathrm{q}^3}{6[2]_{\mathrm{q}}[3]_{\mathrm{q}}}, & 0<\mathrm{q}<\frac{1}{3}.\end{cases}$$

Theorem 9. *Let the assumptions of Lemma 5 hold. Then for* $q \in (0,1)$, $\frac{1}{\mathfrak{r}_1} + \frac{1}{\mathfrak{r}_2} = 1$ *and* $\lambda \in [0,1]$, *we have the following inequality:*

$$\left|(1-\lambda)\Omega\left(\frac{\mu+\nu}{2}\right) + \lambda\frac{\Omega(\mu)+\Omega(\nu)}{2} - \frac{1}{(\nu-\mu)}\left[\int_{\mu}^{\frac{\mu+\nu}{2}} \Omega(t)\ {}_{\mu}d_q t + \int_{\frac{\mu+\nu}{2}}^{\nu} \Omega(t)\ {}^{\nu}d_q t\right]\right|$$
$$\leq \frac{\nu-\mu}{4}(\bar{}(q,\lambda))^{\frac{1}{\mathfrak{r}_1}}\left(\frac{1}{2[2]_q}\right)^{\frac{1}{\mathfrak{r}_2}}\left[\left(\left|{}^{\nu}D_q\Omega(\mu)\right|^{q} + (2q+1)\left|{}^{\nu}D_q\Omega(\nu)\right|^{\mathfrak{r}_2}\right)^{\frac{1}{\mathfrak{r}_2}}\right. \tag{24}$$
$$\left.+\left(\left|{}_{\mu}D_q\Omega(\nu)\right|^{q} + (2q+1)\left|{}_{\mu}D_q\Omega(\mu)\right|^{\mathfrak{r}_2}\right)^{\frac{1}{\mathfrak{r}_2}}\right],$$

where

$$\bar{}(q,\lambda) = \int_0^1 |\lambda - qt|^{\mathfrak{r}_1} d_q t$$

Proof. Using Hölder inequality on Lemma 5, we obtain

$$\left|(1-\lambda)\Omega\left(\frac{\mu+\nu}{2}\right) + \lambda\frac{\Omega(\mu)+\Omega(\nu)}{2} - \frac{1}{(\nu-\mu)}\left[\int_{\mu}^{\frac{\mu+\nu}{2}} \Omega(t)\ {}_{\mu}d_q t + \int_{\frac{\mu+\nu}{2}}^{\nu} \Omega(t)\ {}^{\nu}d_q t\right]\right|$$
$$\leq \frac{\nu-\mu}{4}\left(\int_0^1 |\lambda - qt|^{\mathfrak{r}_1} d_q t\right)^{\frac{1}{\mathfrak{r}_1}}\left(\left|{}^{\nu}D_q\Omega\left(\frac{t}{2}\mu + \frac{2-t}{2}\nu\right)d_q t\right|^{\mathfrak{r}_2}\right)^{\frac{1}{\mathfrak{r}_2}}$$
$$+\frac{\nu-\mu}{4}\left(\int_0^1 |qt-\lambda|^{\mathfrak{r}_1} d_q t\right)^{\frac{1}{\mathfrak{r}_1}}\left(\left|{}_{\mu}D_q\Omega\left(\frac{t}{2}\nu + \frac{2-t}{2}\mu\right)d_q t\right|^{\mathfrak{r}_2}\right)^{\frac{1}{\mathfrak{r}_2}}$$
$$\leq \frac{\nu-\mu}{4}\left(\int_0^1 |\lambda - qt|^{\mathfrak{r}_1} d_q t\right)^{\frac{1}{\mathfrak{r}_2}}\left(\int_0^1\left(\frac{t}{2}\left|{}^{\nu}D_q\Omega(\mu)\right|^{\mathfrak{r}_2} + \frac{2-t}{2}\left|{}^{\nu}D_q\Omega(\nu)\right|^{\mathfrak{r}_2}\right)d_q t\right)^{\frac{1}{\mathfrak{r}_2}}$$
$$+\frac{\nu-\mu}{4}\left(\int_0^1 |\lambda - qt|^{\mathfrak{r}_1} d_q t\right)^{\frac{1}{\mathfrak{r}_1}}\left(\int_0^1\left(\frac{t}{2}\left|{}_{\mu}D_q\Omega(\nu)\right|^{\mathfrak{r}_2} + \frac{2-t}{2}\left|{}_{\mu}D_q\Omega(\mu)\right|^{\mathfrak{r}_2}\right)d_q t\right)^{\frac{1}{\mathfrak{r}_2}}$$
$$\leq \frac{\nu-\mu}{4}(\bar{}(q,\lambda))^{\frac{1}{\mathfrak{r}_1}}\left(\frac{1}{2[2]_q}\right)^{\frac{1}{\mathfrak{r}_2}}\left[\left(\left|{}^{\nu}D_q\Omega(\mu)\right|^{q} + (2q+1)\left|{}^{\nu}D_q\Omega(\nu)\right|^{\mathfrak{r}_2}\right)^{\frac{1}{\mathfrak{r}_2}}\right.$$
$$\left.+\left(\left|{}_{\mu}D_q\Omega(\nu)\right|^{\mathfrak{r}_2} + (2q+1)\left|{}_{\mu}D_q\Omega(\mu)\right|^{\mathfrak{r}_2}\right)^{\frac{1}{\mathfrak{r}_2}}\right],$$

which completes the proof. □

Remark 7. *If we choose* $\lambda = 0$ *in Theorem 9, then we have the following midpoint-type inequality:*

$$\left|\Omega\left(\frac{\mu+\nu}{2}\right) - \frac{1}{(\nu-\mu)}\left[\int_{\mu}^{\frac{\mu+\nu}{2}} \Omega(t)\ {}_{\mu}d_q t + \int_{\frac{\mu+\nu}{2}}^{\nu} \Omega(t)\ {}^{\nu}d_q t\right]\right|$$
$$\leq \frac{q(\nu-\mu)}{4([\mathfrak{r}_1+1]_q)^{\frac{1}{\mathfrak{r}_1}}}\left(\frac{1}{2[2]_q}\right)^{\frac{1}{\mathfrak{r}_2}}\left[\left(\left|{}^{\nu}D_q\Omega(\mu)\right|^{\mathfrak{r}_2} + (2q+1)\left|{}^{\nu}D_q\Omega(\nu)\right|^{\mathfrak{r}_2}\right)^{\frac{1}{\mathfrak{r}_2}}\right.$$
$$\left.+\left(\left|{}_{\mu}D_q\Omega(\nu)\right|^{\mathfrak{r}_2} + (2q+1)\left|{}_{\mu}D_q\Omega(\mu)\right|^{\mathfrak{r}_2}\right)^{\frac{1}{\mathfrak{r}_2}}\right],$$

which is proved by Ali et al. in [26].

Remark 8. *If we choose $\lambda = 1$ in Theorem 9, then we have the following trapezoid-type inequality:*

$$\begin{aligned}
&\left|\frac{\Omega(\mu)+\Omega(\nu)}{2}-\frac{1}{(\nu-\mu)}\left[\int\limits_{\mu}^{\frac{\mu+\nu}{2}}\Omega(\mathrm{t})\,{}_{\mu}d_{\mathrm{q}}\mathrm{t}+\int\limits_{\frac{\mu+\nu}{2}}^{\nu}\Omega(\mathrm{t})\,{}^{\nu}d_{\mathrm{q}}\mathrm{t}\right]\right| \\
&\leq \frac{\nu-\mu}{4}\left({}^{-}(\mathrm{q},1)\right)^{\frac{1}{\mathfrak{r}_1}}\left(\frac{1}{2[2]_{\mathrm{q}}}\right)^{\frac{1}{\mathfrak{r}_2}}\left[\left(\left|{}^{\nu}D_{\mathrm{q}}\Omega(\mu)\right|^{\mathfrak{r}_2}+(2\mathrm{q}+1)\left|{}^{\nu}D_{\mathrm{q}}\Omega(\nu)\right|^{\mathfrak{r}_2}\right)^{\frac{1}{\mathfrak{r}_2}}\right. \\
&\left.+\left(\left|{}_{\mu}D_{\mathrm{q}}\Omega(\nu)\right|^{\mathfrak{r}_2}+(2\mathrm{q}+1)\left|{}_{\mu}D_{\mathrm{q}}\Omega(\mu)\right|^{\mathfrak{r}_2}\right)^{\frac{1}{\mathfrak{r}_2}}\right],
\end{aligned}$$

where

$${}^{-}(\mathrm{q},1)=\int_0^1|1-\mathrm{q}\mathrm{t}|^{\mathfrak{r}_1}d_{\mathrm{q}}\mathrm{t}$$

which is proved by Ali et al. in [26].

Remark 9. *If we choose $\lambda = \frac{1}{2}$ in Theorem 9, then we have the following Bullen-type inequality:*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\Omega\left(\frac{\mu+\nu}{2}\right)+\frac{\Omega(\mu)+\Omega(\nu)}{2}\right]-\frac{1}{(\nu-\mu)}\left[\int\limits_{\mu}^{\frac{\mu+\nu}{2}}\Omega(\mathrm{t})\,{}_{\mu}d_{\mathrm{q}}\mathrm{t}+\int\limits_{\frac{\mu+\nu}{2}}^{\nu}\Omega(\mathrm{t})\,{}^{\nu}d_{\mathrm{q}}\mathrm{t}\right]\right| \\
&\leq \frac{\nu-\mu}{4}\left({}^{-}\left(\mathrm{q},\frac{1}{2}\right)\right)^{\frac{1}{\mathfrak{r}_1}}\left(\frac{1}{2[2]_{\mathrm{q}}}\right)^{\frac{1}{\mathfrak{r}_2}}\left[\left(\left|{}^{\nu}D_{\mathrm{q}}\Omega(\mu)\right|^{\mathfrak{r}_2}+(2\mathrm{q}+1)\left|{}^{\nu}D_{\mathrm{q}}\Omega(\nu)\right|^{\mathfrak{r}_2}\right)^{\frac{1}{\mathfrak{r}_2}}\right. \\
&\left.+\left(\left|{}_{\mu}D_{\mathrm{q}}\Omega(\nu)\right|^{\mathfrak{r}_2}+(2\mathrm{q}+1)\left|{}_{\mu}D_{\mathrm{q}}\Omega(\mu)\right|^{\mathfrak{r}_2}\right)^{\frac{1}{\mathfrak{r}_2}}\right],
\end{aligned}$$

where

$${}^{-}\left(\mathrm{q},\frac{1}{2}\right)=\int_0^1\left|\frac{1}{2}-\mathrm{q}\mathrm{t}\right|^{\mathfrak{r}_1}d_{\mathrm{q}}\mathrm{t}$$

which is proved by Wannalookkhee et al. in [30].

Remark 10. *If we choose $\lambda = \frac{1}{2}$ and $\mathrm{q} \to 1^{-}$ in Theorem 9, then we obtain Corollary 3 of [30].*

Remark 11. *If we choose $\lambda = \frac{1}{3}$ in Theorem 9, then we have the following Simpson-type inequality:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\Omega(\mu)+4\Omega\left(\frac{\mu+\nu}{2}\right)+\Omega(\nu)\right]-\frac{1}{(\nu-\mu)}\left[\int\limits_{\mu}^{\frac{\mu+\nu}{2}}\Omega(\mathrm{t})\,{}_{\mu}d_{\mathrm{q}}\mathrm{t}+\int\limits_{\frac{\mu+\nu}{2}}^{\nu}\Omega(\mathrm{t})\,{}^{\nu}d_{\mathrm{q}}\mathrm{t}\right]\right| \\
&\leq \frac{\nu-\mu}{4}\left({}^{-}\left(\mathrm{q},\frac{1}{3}\right)\right)^{\frac{1}{\mathfrak{r}_1}}\left(\frac{1}{2[2]_{\mathrm{q}}}\right)^{\frac{1}{\mathfrak{r}_2}}\left[\left(\left|{}^{\nu}D_{\mathrm{q}}\Omega(\mu)\right|^{\mathfrak{r}_2}+(2\mathrm{q}+1)\left|{}^{\nu}D_{\mathrm{q}}\Omega(\nu)\right|^{\mathfrak{r}_2}\right)^{\frac{1}{\mathfrak{r}_2}}\right. \\
&\left.+\left(\left|{}_{\mu}D_{\mathrm{q}}\Omega(\nu)\right|^{\mathfrak{r}_2}+(2\mathrm{q}+1)\left|{}_{\mu}D_{\mathrm{q}}\Omega(\mu)\right|^{\mathfrak{r}_2}\right)^{\frac{1}{\mathfrak{r}_2}}\right],
\end{aligned}$$

where

$${}^{-}\left(\mathrm{q},\frac{1}{3}\right)=\int_0^1\left|\frac{1}{3}-\mathrm{q}\mathrm{t}\right|^{\mathfrak{r}_1}d_{\mathrm{q}}\mathrm{t}$$

Theorem 10. *Let the assumptions of Lemma 5 hold. Then for* $q \in (0,1)$ *and* $\lambda \in [0,1]$, *we have the following inequality:*

$$\left|(1-\lambda)\Omega\left(\frac{\mu+\nu}{2}\right)+\lambda\frac{\Omega(\mu)+\Omega(\nu)}{2}-\frac{1}{(\nu-\mu)}\left[\int_{\mu}^{\frac{\mu+\nu}{2}}\Omega(t)\,{}_{\mu}d_q t+\int_{\frac{\mu+\nu}{2}}^{\nu}\Omega(t)\,{}^{\nu}d_q t\right]\right| \tag{25}$$
$$\leq \frac{\nu-\mu}{4}(\Omega_3(q,\lambda))^{1-\frac{1}{r_2}}\left[\left(\Omega_1(q,\lambda)\left|{}^{\nu}D_q\Omega(\mu)\right|^{r_2}+\Omega_2(q,\lambda)\left|{}_{\mu}D_q\Omega(\nu)\right|^{r_2}\right)^{\frac{1}{r_2}}\right.$$
$$\left.+\left(\Omega_1(q,\lambda)\left|{}^{\nu}D_q\Omega(\nu)\right|^{r_2}+\Omega_2(q,\lambda)\left|{}_{\mu}D_q\Omega(\mu)\right|^{r_2}\right)^{\frac{1}{r_2}}\right],$$

where

$$\Omega_3(q,\lambda)=\int_0^1|\lambda-qt|\,d_qt = \begin{cases}\frac{2\lambda^2+q}{[2]_q}-\lambda, & 0\leq\lambda\leq q;\\ \lambda-\frac{q}{[2]_q}, & \lambda>q,\end{cases}$$

Proof. Using Power Mean inequality on Lemma 5, we obtain

$$\left|(1-\lambda)\Omega\left(\frac{\mu+\nu}{2}\right)+\lambda\frac{\Omega(\mu)+\Omega(\nu)}{2}-\frac{1}{(\nu-\mu)}\left[\int_{\mu}^{\frac{\mu+\nu}{2}}\Omega(t)\,{}_{\mu}d_q t+\int_{\frac{\mu+\nu}{2}}^{\nu}\Omega(t)\,{}^{\nu}d_q t\right]\right|$$
$$\leq\frac{\nu-\mu}{4}\left(\int_0^1|\lambda-qt|d_qt\right)^{1-\frac{1}{r_2}}\left(\int_0^1|\lambda-qt|\left|{}^{\nu}D_q\Omega\left(\frac{t}{2}\mu+\frac{2-t}{2}\nu\right)\right|^{r_2}d_qt\right)^{\frac{1}{r_2}}$$
$$+\frac{\nu-\mu}{4}\left(\int_0^1|qt-\lambda|d_qt\right)^{1-\frac{1}{r_2}}\left(\int_0^1|qt-\lambda|\left|{}_{\mu}D_q\Omega\left(\frac{t}{2}\nu+\frac{2-t}{2}\mu\right)\right|^{r_2}d_qt\right)^{\frac{1}{r_2}}$$
$$\leq\frac{\nu-\mu}{4}\left(\int_0^1|\lambda-qt|d_qt\right)^{1-\frac{1}{r_2}}\left(\int_0^1|\lambda-qt|\left(\frac{t}{2}\left|{}^{\nu}D_q\Omega(\mu)\right|^{r_2}+\frac{2-t}{2}\left|{}^{\nu}D_q\Omega(\nu)\right|^{r_2}\right)d_qt\right)^{\frac{1}{r_2}}$$
$$+\frac{\nu-\mu}{4}\left(\int_0^1|\lambda-qt|d_qt\right)^{1-\frac{1}{r_2}}\left(\int_0^1|\lambda-qt|\left(\frac{t}{2}\left|{}_{\mu}D_q\Omega(\nu)\right|^{r_2}+\frac{2-t}{2}\left|{}_{\mu}D_q\Omega(\mu)\right|^{r_2}\right)d_qt\right)^{\frac{1}{r_2}}$$
$$\leq \frac{\nu-\mu}{4}(\Omega_3(q,\lambda))^{1-\frac{1}{r_2}}\left[\left(\Omega_1(q,\lambda)\left|{}^{\nu}D_q\Omega(\mu)\right|^{r_2}+\Omega_2(q,\lambda)\left|{}_{\mu}D_q\Omega(\nu)\right|^{r_2}\right)^{\frac{1}{r_2}}\right.$$
$$\left.+\left(\Omega_1(q,\lambda)\left|{}^{\nu}D_q\Omega(\nu)\right|^{r_2}+\Omega_2(q,\lambda)\left|{}_{\mu}D_q\Omega(\mu)\right|^{r_2}\right)^{\frac{1}{r_2}}\right],$$

which completes the proof. □

Remark 12. *If we choose* $\lambda = 0$ *in Theorem 10, then we have the following midpoint-type inequality:*

$$\left|\Omega\left(\frac{\mu+\nu}{2}\right)-\frac{1}{(\nu-\mu)}\left[\int\limits_{\mu}^{\frac{\mu+\nu}{2}}\Omega(\mathrm{t})\,{}_{\mu}d_{\mathrm{q}}\mathrm{t}+\int\limits_{\frac{\mu+\nu}{2}}^{\nu}\Omega(\mathrm{t})\,{}^{\nu}d_{\mathrm{q}}\mathrm{t}\right]\right|$$
$$\leq\frac{\nu-\mu}{4}\left(\frac{\mathrm{q}}{[2]_{\mathrm{q}}}\right)^{1-\frac{1}{\mathfrak{r}_2}}\left[\left(\frac{\mathrm{q}}{2[3]_{\mathrm{q}}}\left|{}^{\nu}D_{\mathrm{q}}\Omega(\mu)\right|^{\mathfrak{r}_2}+\frac{2\mathrm{q}^3+\mathrm{q}^2+\mathrm{q}}{2[2]_{\mathrm{q}}[3]_{\mathrm{q}}}\left|{}_{\mu}D_{\mathrm{q}}\Omega(\nu)\right|^{\mathfrak{r}_2}\right)^{\frac{1}{\mathfrak{r}_2}}\right.$$
$$\left.+\left(\frac{\mathrm{q}}{2[3]_{\mathrm{q}}}\left|{}^{\nu}D_{\mathrm{q}}\Omega(\nu)\right|^{\mathfrak{r}_2}+\frac{2\mathrm{q}^3+\mathrm{q}^2+\mathrm{q}}{2[2]_{\mathrm{q}}[3]_{\mathrm{q}}}\left|{}_{\mu}D_{\mathrm{q}}\Omega(\mu)\right|^{\mathfrak{r}_2}\right)^{\frac{1}{\mathfrak{r}_2}}\right]$$

which is proved by Ali et al. in [26].

Remark 13. *If we choose $\lambda = 1$ in Theorem 10, then we have the following trapezoid-type inequality:*

$$\left|\frac{\Omega(\mu)+\Omega(\nu)}{2}-\frac{1}{(\nu-\mu)}\left[\int\limits_{\mu}^{\frac{\mu+\nu}{2}}\Omega(\mathrm{t})\,{}_{\mu}d_{\mathrm{q}}\mathrm{t}+\int\limits_{\frac{\mu+\nu}{2}}^{\nu}\Omega(\mathrm{t})\,{}^{\nu}d_{\mathrm{q}}\mathrm{t}\right]\right|$$
$$\leq\frac{\nu-\mu}{4}\left(\frac{1}{[2]_{\mathrm{q}}}\right)^{1-\frac{1}{\mathfrak{r}_2}}\left[\left(\frac{1}{2[2]_{\mathrm{q}}[3]_{\mathrm{q}}}\left|{}^{\nu}D_{\mathrm{q}}\Omega(\mu)\right|^{\mathfrak{r}_2}+\frac{\mathrm{q}}{[3]_{\mathrm{q}}}\left|{}_{\mu}D_{\mathrm{q}}\Omega(\nu)\right|^{\mathfrak{r}_2}\right)^{\frac{1}{\mathfrak{r}_2}}\right.$$
$$\left.+\left(\frac{1}{2[2]_{\mathrm{q}}[3]_{\mathrm{q}}}\left|{}^{\nu}D_{\mathrm{q}}\Omega(\nu)\right|^{\mathfrak{r}_2}+\frac{\mathrm{q}}{[3]_{\mathrm{q}}}\left|{}_{\mu}D_{\mathrm{q}}\Omega(\mu)\right|^{\mathfrak{r}_2}\right)^{\frac{1}{\mathfrak{r}_2}}\right]$$

which is proved by Ali et al. in [26].

Remark 14. *If we choose $\lambda = \frac{1}{2}$ in Theorem 10, then we have the following Bullen-type inequality:*

$$\left|\frac{1}{2}\left[\Omega\left(\frac{\mu+\nu}{2}\right)+\frac{\Omega(\mu)+\Omega(\nu)}{2}\right]-\frac{1}{(\nu-\mu)}\left[\int\limits_{\mu}^{\frac{\mu+\nu}{2}}\Omega(\mathrm{t})\,{}_{\mu}d_{\mathrm{q}}\mathrm{t}+\int\limits_{\frac{\mu+\nu}{2}}^{\nu}\Omega(\mathrm{t})\,{}^{\nu}d_{\mathrm{q}}\mathrm{t}\right]\right|$$
$$\leq\frac{\nu-\mu}{4}\left(\frac{\mathrm{q}}{2[2]_{\mathrm{q}}}\right)^{1-\frac{1}{\mathfrak{r}_2}}\left[\left(\Omega_1\left(\mathrm{q},\frac{1}{2}\right)\left|{}^{\nu}D_{\mathrm{q}}\Omega(\mu)\right|^{\mathfrak{r}_2}+\Omega_2\left(\mathrm{q},\frac{1}{2}\right)\left|{}_{\mu}D_{\mathrm{q}}\Omega(\nu)\right|^{\mathfrak{r}_2}\right)^{\frac{1}{\mathfrak{r}_2}}\right.$$
$$\left.+\left(\Omega_1\left(\mathrm{q},\frac{1}{2}\right)\left|{}^{\nu}D_{\mathrm{q}}\Omega(\nu)\right|^{\mathfrak{r}_2}+\Omega_2\left(\mathrm{q},\frac{1}{2}\right)\left|{}_{\mu}D_{\mathrm{q}}\Omega(\mu)\right|^{\mathfrak{r}_2}\right)^{\frac{1}{\mathfrak{r}_2}}\right]$$

where

$$\Omega_1\left(\mathrm{q},\frac{1}{2}\right)=\frac{1}{2}\int_0^1\left|\frac{1}{2}-\mathrm{q}t\right|t\,d_{\mathrm{q}}t$$
$$=\begin{cases}\dfrac{2\mathrm{q}^2+2\mathrm{q}-1}{8[2]_{\mathrm{q}}[3]_{\mathrm{q}}}, & \mathrm{q}\geq\frac{1}{2};\\ \dfrac{1-\mathrm{q}-\mathrm{q}^2}{4[2]_{\mathrm{q}}[3]_{\mathrm{q}}}, & 0<\mathrm{q}<\frac{1}{2},\end{cases}$$

and

$$\Omega_2\left(q, \frac{1}{2}\right) = \int_0^1 \left|qt - \frac{1}{2}\right| \left(\frac{2-t}{2}\right) d_q t$$
$$= \begin{cases} \dfrac{4q^3 + 2q^2 + 2q + 1}{8[2]_q[3]_q}, & q \geq \frac{1}{2}; \\ \dfrac{1 + q + q^2 - 2q^3}{4[2]_q[3]_q}, & 0 < q < \frac{1}{2} \end{cases}$$

which is proved by Wannalookkhee et al. in [30].

Remark 15. *If we choose $\lambda = \frac{1}{2}$ and $q \to 1^-$ in Theorem 10, then we obtain Corollary 4 of [30].*

Remark 16. *If we choose $\lambda = \frac{1}{3}$ in Theorem 10, then we have the following Simpson-type inequality:*

$$\left| \frac{1}{6}\left[\Omega(\mu) + 4\Omega\left(\frac{\mu+\nu}{2}\right) + \Omega(\nu)\right] - \frac{1}{(\nu-\mu)}\left[\int_{\mu}^{\frac{\mu+\nu}{2}} \Omega(t)\ {}_{\mu}d_q t + \int_{\frac{\mu+\nu}{2}}^{\nu} \Omega(t)\ {}^{\nu}d_q t\right]\right|$$
$$\leq \frac{\nu-\mu}{4}\left(\frac{6q-1}{9[2]_q}\right)^{1-\frac{1}{\mathfrak{r}_2}} \left[\left(\Omega_1\left(q, \frac{1}{3}\right)\left| {}^{\nu}D_q\Omega(\mu)\right|^{\mathfrak{r}_2} + \Omega_2\left(q, \frac{1}{3}\right)\left| {}_{\mu}D_q\Omega(\nu)\right|^{\mathfrak{r}_2}\right)^{\frac{1}{\mathfrak{r}_2}}\right.$$
$$\left. + \left(\Omega_1\left(q, \frac{1}{3}\right)\left| {}^{\nu}D_q\Omega(\nu)\right|^{\mathfrak{r}_2} + \Omega_2\left(q, \frac{1}{3}\right)\left| {}_{\mu}D_q\Omega(\mu)\right|^{\mathfrak{r}_2}\right)^{\frac{1}{\mathfrak{r}_2}}\right]$$

where

$$\Omega_1\left(q, \frac{1}{3}\right) = \frac{1}{2}\int_0^1 \left|\frac{1}{3} - qt\right| t\, d_q t = \begin{cases} \dfrac{18q^2 + 18q - 7}{54[2]_q[3]_q}, & q \geq \frac{1}{3}; \\ \dfrac{1 - 2q - 2q}{6[2]_q[3]_q}, & 0 < q < \frac{1}{3}, \end{cases}$$

and

$$\Omega_2\left(q, \frac{1}{3}\right) = \int_0^1 \left|qt - \frac{1}{3}\right| \left(\frac{2-t}{2}\right) d_q t$$
$$= \begin{cases} \dfrac{36q^3 + 12q^2 + 12q + 1}{54[2]_q[3]_q}, & q \geq \frac{1}{3}; \\ \dfrac{1 - 4q^3}{6[2]_q[3]_q}, & 0 < q < \frac{1}{3}. \end{cases}$$

4. Concluding Remarks

We will sum up our findings by saying that some novel estimates of Midpoint, Simpson, Trapezoid, and Bullen-type inequalities are obtained for convex functions. We also illustrate that the findings of this research represent a significant generalization of previously published related results. In future research, by using strong and uniform convexity, researchers can also find new bounds. By utilizing strong convexities, we may improve bounds obtained for our Quadrature quantum estimations. Moreover, we can extend the idea by taking into account the (p, q)-calculus, which will provide further insight to such studies to explore. It is necessary to state that by choosing $q \to 1^-$ in our primary results, our results turned into classical calculus. We feel it is a fascinating and novel topic for scholars who can achieve analogous inequalities by using different types of convexities.

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