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Conditional Uncertainty Distribution of Two Uncertain Variables and Conditional Inverse Uncertainty Distribution

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Abstract: It is noted that some uncertain variables are independent while others are not. In general, there is a symmetrical relationship between independence and dependence among uncertain variables. The utilization of conditional uncertain measures as well as conditional uncertainty distributions proves highly efficacious in resolving uncertainties pertaining to an event subsequent to the acquisition of knowledge about other events. In this paper, the theorem about the conditional uncertainty distribution of two uncertain variables is proposed. It is demonstrated that the theorem holds regardless of whether the two variables are independent or not. In addition, it is also found that uncertainty distribution possesses an inherent inverse function when it is a regular uncertainty distribution within the framework of Uncertainty Theory; therefore, this paper delves into investigating the conditional inverse uncertainty distribution, including specific cases of the conditional inverse uncertainty distributions. Meanwhile, illustrative examples are applied to clarify the findings.

Keywords: Uncertainty Theory; independent; conditional inverse uncertainty distribution; conditional uncertainty distribution

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1. Introduction

Some uncertain variables are independent while others are not, and the relationship between independence and dependence is symmetrical. Liu's Uncertainty Theory provides a useful approach to studying indeterminacy since it is a normal characteristic of real life. Liu's work on Uncertainty Theory [1] expounds on concepts encompassing uncertain measures, uncertain variables, uncertainty distributions and so on, providing an encompassing grasp of Uncertainty Theory.

Currently, Liu's Uncertainty Theory offers a valuable approach for studying indeterminacy given its alignment with the inherent nature of real-world scenarios. The utility of uncertain programming holds immense importance in the project scheduling problem (Liu [1], Ning et al. [2]), the vehicle routing problem, and the machine scheduling problem. The uncertain stock model (Liu [3], Gao et al. [4]), the uncertain interest rate model (Zhang et al. [5]), and the uncertain currency model (Liu et al. [6]) all belong to uncertain finance. One of the tools for quantifying risks is uncertain risk analysis (Liu [7], Peng [8], Liu [9]), which is at the bottom of Uncertainty Theory. Furthermore, uncertain reliability analysis (Liu [7], Gao and Yao [10]) proves to be a highly efficacious implementation in system reliability. Hybrid logic and uncertain logic were defined by Li and Liu [11]. In order to calculate the true value of uncertain propositions, Chen and Relescu [12] provided the Chen–Ralescu theorem. Uncertain entailment (Liu [13]) was given in 2009, which calculates the truth value of an uncertain formula via the maximum uncertainty principle; meanwhile, an entailment model was given in order to find the truth value of additional formulas.

The conditional uncertain set and the conditional membership function were proposed by Yao [14]. In addition, the uncertain differential equation (Chen and Liu [15], Yao [16]), uncertain regression analysis (Lio and Liu [17], Liu and Yang [18], Wang et al. [19]), uncertain time-series analysis (Yang and Liu [20]), the uncertain process (Liu [21,22], Chen et al. [23]), and so on, have been applied to many fields, and satisfactory conclusions have been obtained. There are a number of theorems about independent uncertain variables that can be derived from Uncertainty Theory, such as the Extreme Value Theorem (Liu [24]), the Order Statistic (Liu [25]), the Minkowski Inequality (Liu [1]) and so on. In both practical scenarios and production settings, numerous quantities are interdependent, consequently leading to the interdependence of uncertain variables that describe these real-world quantities. As a result of a large number of uncertain variables not being independent, the conditional uncertain measure (Liu [1]), as well as the conditional uncertainty distribution (Liu [25]), can be highly effective in solving the uncertain problem of an event subsequent to the acquisition of knowledge about other events. Nevertheless, Liu [25] has exclusively presented a theorem concerning the conditional uncertainty distribution of a single uncertain variable, outlining the uncertainty distribution of said variable across distinct intervals. As we know from Uncertainty Theory, uncertainty distribution possesses an inverse function if it is a regular uncertainty distribution, but conditional inverse uncertainty distribution has never been proposed.

To summarize the above, uncertain variables are often not independent but related. In order to accurately describe this relationship, it is necessary to analyze the conditional inverse uncertainty distribution and the conditional uncertainty distribution, which are of great significance in describing the relationship between uncertain variables, inference and prediction, data modeling, classification, risk assessment, and decision analysis. In this paper, a list of fundamental definitions and theorems are presented in Section 2. Following this, the theorem about conditional uncertainty distribution of two uncertain variables has been proved in Section 3. This theorem applies no matter whether two uncertain variables are independent or not. In addition, Section 4 presents the study of the conditional inverse uncertainty distribution that has never been proposed, and the conditional inverse uncertainty distribution of the special conditional uncertainty distribution in some cases is obtained. Meanwhile, illustrative examples are applied to clarify the findings of the conditional inverse uncertainty distribution. Finally, a conclusion to this thesis is given in Section 5.

2. Preliminaries

This section introduces a lot of fundamental definitions and theorems that need to be used in this article. For more details, readers should refer to Uncertainty Theory (Liu [26]).

Assume $(\Gamma, \mathcal{L})$ is a measurable space, $\mathcal{L}$ is a σ-algebra over Γ, each element Λ in $\mathcal{L}$ is a measurable set, and Λ^c is a complementary set of Λ. Liu [1] provided three axioms (i.e., normality, duality, and subadditivity).

Normality Axiom: For the universal set Γ, $\mathcal{M}\{\Gamma\} = 1$.

Duality Axiom: For every event Λ in $\mathcal{L}$, $\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Lambda^c\}=1$.

Subadditivity Axiom:

$$\mathcal{M}\{\bigcup_{i=1}^{\infty} \Lambda_i\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$$

for any countable sequence of events $\Lambda_1, \Lambda_2, \ldots$.

Definition 1 (Liu [1])**.** *An uncertain measure $\mathcal{M}$ is a set function that satisfies the above three axioms.*

Definition 2 (Liu [1])**.** *The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space if $(\Gamma, \mathcal{L})$ is a measurable space, Γ is a nonempty set, $\mathcal{L}$ is a σ-algebra over Γ, and $\mathcal{M}$ is an uncertain measure.*

Definition 3 (Liu [1]). *An uncertain variable is a function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\xi \in B$ is an event for any Borel set B of real numbers.*

Definition 4 (Liu [1]). *The uncertainty distribution Ψ of an uncertain variable ξ is defined by*

$$\Psi(x) = \mathcal{M}\{\xi \leq x\}$$

for any real number x.

Theorem 1. *The uncertain measure is a set function characterized by monotonically increasing. In other words, for any event Λ_1 and Λ_2 with $\Lambda_1 \subseteq \Lambda_2$, the following inequality holds.*

$$\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}.$$

Definition 5 (Liu [25]). *A regular uncertainty distribution $\Psi(x)$ is a continuous and strictly increasing uncertainty distribution, and*

$$\lim_{x \to -\infty} \Psi(x) = 0, \lim_{x \to +\infty} \Psi(x) = 1.$$

with $0 < \Psi(x) < 1$.

Definition 6 (Liu [25]). *Assume $\Psi(x)$ is a regular uncertainty distribution for an uncertain variable ξ. Then, the inverse uncertainty distribution $\Psi^{-1}(\alpha)$ is defined by*

$$\Psi^{-1}(\alpha) = \mathcal{M}^{-1}\{\xi \leq x\}$$

Definition 7 (Liu [1]). *The conditional uncertainty distribution Ψ of an uncertain variable ξ given A is defined by*

$$\Psi(x|A) = \mathcal{M}\{\xi \leq x|A\}$$

provided that $\mathcal{M}\{A\} > 0$.

Definition 8 (Liu [1]). *Assume $(\Gamma, \mathcal{L})$ is a measurable space, $\mathcal{M}$ is an uncertain measure, Λ and A are measurable set in $\mathcal{L}$. For Λ given A, the conditional uncertain measure is following definition*

$$\mathcal{M}\{\Lambda|A\} = \begin{cases} \frac{\mathcal{M}\{\Lambda \cap A\}}{\mathcal{M}\{A\}}, & if \quad \frac{\mathcal{M}\{\Lambda \cap A\}}{\mathcal{M}\{A\}} < 0.5 \\ 1 - \frac{\mathcal{M}\{\Lambda^c \cap A\}}{\mathcal{M}\{A\}}, & if \quad \frac{\mathcal{M}\{\Lambda^c \cap A\}}{\mathcal{M}\{A\}} < 0.5 \\ 0.5, & otherwise, \end{cases}$$

provided that $\mathcal{M}\{A\} > 0$.

Theorem 2. *Assume ξ and η be uncertain sets. Then, the following equality holds.*

$$(\xi \cup \eta)^c = \xi^c \cap \eta^c, (\xi \cap \eta)^c = \xi^c \cup \eta^c.$$

Theorem 3 (Liu [25]). *Assume $\Psi(x)$ is a regular uncertainty distribution for an uncertain variable ξ, and $\Psi(y) < 1$ for a real number y. Then, for $\xi > y$, the conditional uncertainty distribution is*

$$\Psi(x|(y, +\infty)) = \mathcal{M}\{\xi \leq x|\xi > y\} = \begin{cases} 0, & if \quad \Psi(x) \leq \Psi(y) \\ \frac{\Psi(x)}{1-\Psi(y)} \wedge 0.5, & if \quad \Psi(y) < \Psi(x) \leq \frac{1+\Psi(y)}{2} \\ \frac{\Psi(x)-\Psi(y)}{1-\Psi(y),} & if \quad \frac{1+\Psi(y)}{2} < \Psi(x). \end{cases}$$

where "$\wedge$" is a logical operation symbol that means minimum.

Theorem 4 (Liu [25])**.** *Assume $\Psi(x)$ is a regular uncertainty distribution of an uncertain variable ξ, and $\Psi(y) > 0$ for a real number y. Then, for $\xi \leq y$, the conditional uncertainty distribution is*

$$\Psi(x|(-\infty, y]) = \mathcal{M}\{\xi \leq x|\xi \leq y\} = \begin{cases} \frac{\Psi(x)}{\Psi(y)}, & if \quad \Psi(x) < \frac{\Psi(y)}{2} \\ \frac{\Psi(x)+\Psi(y)-1}{\Psi(y)} \vee 0.5, & if \quad \frac{\Psi(y)}{2} \leq \Psi(x) < \Psi(y) \\ 1, & if \quad \Psi(y) \leq \Psi(x). \end{cases}$$

where "$\vee$" is a logical operation symbol that means maximum.

Definition 9 (Liu [26])**.** *It is called linear uncertain variable ξ if it has the following linear uncertainty distribution*

$$\Psi(x) = \begin{cases} 0, & if \quad x \leq a \\ \frac{x-a}{b-a}, & if \quad a < x \leq b \\ 1, & if \quad b < x \end{cases}$$

named $\mathcal{L}(a, b)$, where $a < b$ for real numbers a and b.

Definition 10 (Liu [26])**.** *It is called the normal uncertain variable ξ if it has the following normal uncertainty distribution*

$$\Psi(x) = (1 + exp(\frac{\pi(e-x)}{\sqrt{3}\sigma}))^{(-1)}, x \in \mathcal{R}$$

named $\mathcal{N}(e, \sigma)$, where e, σ are real numbers with $\sigma > 0$.

3. Conditional Uncertainty Distribution of Two Uncertain Variables

There are a number of theorems about independent uncertain variables that can be derived from Uncertainty Theory (Liu [26]). Given the prevalence of such interdependencies among a substantial number of uncertain variables, the utilization of conditional uncertain measures (Liu [1]) and conditional uncertainty distributions (Liu [25]) proves highly efficacious in resolving uncertainties pertaining to an event subsequent to the acquisition of knowledge about other events. Liu ([25]) has exclusively presented Theorem 3 and Theorem 4 concerning the conditional uncertainty distribution of a single uncertain variable, outlining the uncertainty distribution of said variable across distinct intervals.

In this section, from Definition 8, Theorem 3, and Theorem 4, we come up with the following theorems about the conditional uncertainty distribution for two uncertain variables, and it is proven that the theorem remains valid regardless of the independence status of the uncertain variables.

Theorem 5. *Assume Ψ_1, Ψ_2 are regular uncertainty distributions for uncertain variables η_1, η_2, respectively, and $\Psi_2(y) < 1$ for a real number y. Then, for $\eta_2 > y$, the conditional uncertain measure is*

$$\mathcal{M}\{\eta_1 \leq x|\eta_2 > y\} = \begin{cases} \frac{\Psi_1(x)}{1-\Psi_2(y)}, & if \quad \Psi_2(y) < \Psi_1(x) < \frac{1-\Psi_2(y)}{2} \\ \frac{\Psi_1(x)-\Psi_2(y)}{1-\Psi_2(y)}, & if \quad \frac{1+\Psi_2(y)}{2} \leq \Psi_1(x) \\ 0.5, & otherwise. \end{cases}$$

Proof. From Definition 8, defined as $\mathcal{M}\{\Lambda|A\}$, we have

$$\mathcal{M}\{\eta_1 \leq x|\eta_2 > y\} = \begin{cases} \frac{\mathcal{M}\{(\eta_1 \leq x)\cap(\eta_2 > y)\}}{\mathcal{M}\{\eta_2 > y\}}, & if \quad \frac{\mathcal{M}\{(\eta_1 \leq x)\cap(\eta_2 > y)\}}{\mathcal{M}\{\eta_2 > y\}} < 0.5 \\ 1 - \frac{\mathcal{M}\{(\eta_1 > x)\cap(\eta_2 > y)\}}{\mathcal{M}\{\eta_2 > y\}}, & if \quad \frac{\mathcal{M}\{(\eta_1 > x)\cap(\eta_2 > y)\}}{\mathcal{M}\{\eta_2 > y\}} < 0.5 \\ 0.5, & otherwise. \end{cases}$$

From the duality axiom, defined by Definition 1, we obtain

$$\frac{1-\mathcal{M}\{((\eta_1 \leq x)\cap(\eta_2 > y))^c\}}{\mathcal{M}\{\eta_2 > y\}} = \frac{\mathcal{M}\{(\eta_1 \leq x)\cap(\eta_2 > y)\}}{\mathcal{M}\{\eta_2 > y\}},$$

and from De Morgan's Law (Theorem 2) and the subadditivity axioms, we have

$$\mathcal{M}\{((\eta_1 \leq x)\cap(\eta_2 > y))^c\} = \mathcal{M}\{(\eta_1 > x)\cup(\eta_2 \leq y)\} \leq \mathcal{M}\{\eta_1 > x\} + \mathcal{M}\{\eta_2 \leq y\}.$$

Thus

$$\frac{1-\mathcal{M}\{\eta_1 > x\} - \mathcal{M}\{\eta_2 \leq y\}}{\mathcal{M}\{\eta_2 > y\}} \leq \frac{1-\mathcal{M}\{(\eta_1 > x)\cup(\eta_2 \leq y)\}}{\mathcal{M}\{\eta_2 > y\}}$$

and

$$\frac{1-\mathcal{M}\{(\eta_1 > x)\cup(\eta_2 \leq y)\}}{\mathcal{M}\{\eta_2 > y\}} = \frac{\mathcal{M}\{(\eta_1 \leq x)\cap(\eta_2 > y)\}}{\mathcal{M}\{\eta_2 > y\}} \leq \frac{\mathcal{M}\{\eta_1 \leq x\}}{\mathcal{M}\{\eta_2 > y\}}.$$

On the other hand, from the duality axiom, defined by Definition 1, we have

$$\frac{\mathcal{M}\{(\eta_1 \leq x)^c\cap(\eta_2 > y)\}}{\mathcal{M}\{\eta_2 > y\}} = \frac{1-\mathcal{M}\{((\eta_1 \leq x)^c\cap(\eta_2 > y))^c\}}{\mathcal{M}\{\eta_2 > y\}},$$

and from De Morgan's Law (Theorem 2) and the subadditivity axioms, we obtain

$$\mathcal{M}\{((\eta_1 \leq x)^c\cap(\eta_2 > y))^c\} = \mathcal{M}\{(\eta_1 \leq x)\cup(\eta_2 \leq y)\} \leq \mathcal{M}\{\eta_1 \leq x\} + \mathcal{M}\{\eta_2 \leq y\}.$$

Thus

$$\frac{1-\mathcal{M}\{\eta_1 \leq x\} - \mathcal{M}\{\eta_2 \leq y\}}{\mathcal{M}\{\eta_2 > y\}} \leq \frac{1-\mathcal{M}\{(\eta_1 \leq x)\cup(\eta_2 \leq y)\}}{\mathcal{M}\{\eta_2 > y\}}$$

and

$$\frac{1-\mathcal{M}\{(\eta_1 \leq x)\cup(\eta_2 \leq y)\}}{\mathcal{M}\{\eta_2 > y\}} = \frac{\mathcal{M}\{(\eta_1 \leq x)^c\cap(\eta_2 > y)\}}{\mathcal{M}\{\eta_2 > y\}} \leq \frac{\mathcal{M}\{(\eta_1 \leq x)^c\}}{\mathcal{M}\{\eta_2 > y\}}.$$

Next, the following discussion can be divided into three parts.

Part 1: If

$$\frac{\mathcal{M}\{\eta_1 \leq x\}}{\mathcal{M}\{\eta_2 > y\}} < 0.5$$

and

$$0.5 \leq \frac{1-\mathcal{M}\{(\eta_1 \leq x) - \mathcal{M}(\{\eta_2 \leq y)\}}{\mathcal{M}\{\eta_2 > y\}},$$

we have

$$\Psi_2(y) < \Psi_1(x) < \frac{1}{2}(1-\Psi_2(y)).$$

Then, applying the maximum uncertainty principle (Liu [1])

$$\frac{\mathcal{M}\{(\eta_1 \leq x)\cap(\eta_2 > y)\}}{\mathcal{M}\{\eta_2 > y\}} = \frac{1-\mathcal{M}\{\{((\eta_1 \leq x)\cap(\eta_2 > y))^c\}}{\mathcal{M}\{\eta_2 > y\}} = \frac{\Psi_1(x)}{1-\Psi_2(y)}.$$

Part 2: If

$$0.5 \leq \frac{1-\mathcal{M}\{\eta_1 > x\} - \mathcal{M}\{\eta_2 \leq y\}}{\mathcal{M}\{\eta_2 > y\}}$$

and

$$\frac{\mathcal{M}\{(\eta_1 \le x)^c\}}{\mathcal{M}\{\eta_2 > y\}} < 0.5,$$

we have

$$\Psi_1(x) \ge \frac{1}{2}(1 + \Psi_2(y)).$$

Then, applying the maximum uncertainty principle (Liu [1])

$$\frac{\mathcal{M}\{(\eta_1 \le x)^c \cap (\eta_2 > y)\}}{\mathcal{M}\{\eta_2 > y\}} = \frac{1 - \mathcal{M}\{((\eta_1 \le x)^c \cap (\eta_2 > y))^c\}}{\mathcal{M}\{\eta_2 > y\}} = \frac{1 - \Psi_1(x)}{1 - \Psi_2(y)}.$$

Part 3: If part 1 and part 2 are not met, then

$$\mathcal{M}\{\eta_1 \le x | \eta_2 > y\} = 0.5.$$

Thus

$$\mathcal{M}\{\eta_1 \le x | \eta_2 > y\} = \begin{cases} \frac{\Psi_1(x)}{1-\Psi_2(y)}, & if \quad \Psi_2(y) < \Psi_1(x) < \frac{1-\Psi_2(y)}{2} \\ \frac{\Psi_1(x)-\Psi_2(y)}{1-\Psi_2(y)}, & if \quad \frac{1+\Psi_2(y)}{2} \le \Psi_1(x) \\ 0.5, & otherwise. \end{cases}$$

The theorem is proved. □

Theorem 6. *Assume Ψ_1, Ψ_2 are regular uncertainty distributions for uncertain variables η_1, η_2, respectively, and $\Psi_2(y) > 0$ for a real number y. Then, for $\eta_2 \le y$, the conditional uncertain measure is*

$$\mathcal{M}\{\eta_1 \le x | \eta_2 \le y\} = \begin{cases} \frac{\Psi_1(x)}{\Psi_2(y)}, & if \quad \Psi_1(x) < \frac{\Psi_2(y)}{2} \\ \frac{\Psi_1(x)+\Psi_2(y)-1}{\Psi_2(y)}, & if \quad 1 - \frac{\Psi_2(y)}{2} < \Psi_1(x) < \Psi_2(y) \\ 0.5, & otherwise. \end{cases}$$

Proof. From Definition 8, defined $\mathcal{M}\{\Lambda | A\}$, we have

$$\mathcal{M}\{\eta_1 \le x | \eta_2 \le y\} = \begin{cases} \frac{\mathcal{M}\{(\eta_1 \le x) \cap (\eta_2 \le y)\}}{\mathcal{M}\{\eta_2 \le y\}}, & if \quad \frac{\mathcal{M}\{(\eta_1 \le x) \cap (\eta_2 \le y)\}}{\mathcal{M}\{\eta_2 \le y\}} < 0.5 \\ \frac{1-\mathcal{M}\{(\eta_1 > x) \cap (\eta_2 \le y)\}}{\mathcal{M}\{\eta_2 \le y\}}, & if \quad \frac{\mathcal{M}\{(\eta_1 > x) \cap (\eta_2 \le y)\}}{\mathcal{M}\{\eta_2 \le y\}} < 0.5 \\ 0.5, & otherwise. \end{cases}$$

From the duality axiom, defined by Definition 1, we have

$$\frac{\mathcal{M}\{(\eta_1 \le x) \cap (\eta_2 \le y)\}}{\mathcal{M}\{\eta_2 \le y\}} = \frac{1 - \mathcal{M}\{((\eta_1 \le x) \cap (\eta_2 \le y))^c\}}{\mathcal{M}\{\eta_2 \le y\}},$$

and from De Morgan's Law (Theorem 2) and the subadditivity axioms, we obtain

$$\mathcal{M}\{((\eta_1 \le x) \cap (\eta_2 \le y))^c\} = \mathcal{M}\{(\eta_1 > x) \cup (\eta_2 > y)\} \le \mathcal{M}\{(\eta_1 > x)\} + \mathcal{M}\{(\eta_2 > y)\}.$$

Thus

$$\frac{1 - \mathcal{M}\{\eta_1 > x\} - \mathcal{M}\{\eta_2 > y\}}{\mathcal{M}\{\eta_2 \le y\}} \le \frac{1 - \mathcal{M}\{(\eta_1 > x) \cup (\eta_2 > y)\}}{\mathcal{M}\{\eta_2 \le y\}} = \frac{\mathcal{M}\{(\eta_1 \le x) \cap (\eta_2 \le y)\}}{\mathcal{M}\{\eta_2 \le y\}}$$

and

$$\frac{1 - \mathcal{M}\{(\eta_1 > x) \cup (\eta_2 > y)\}}{\mathcal{M}\{\eta_2 \le y\}} = \frac{\mathcal{M}\{(\eta_1 \le x) \cap (\eta_2 \le y)\}}{\mathcal{M}\{\eta_2 \le y\}} \le \frac{\mathcal{M}\{\eta_1 \le x\}}{\mathcal{M}\{\eta_2 \le y\}}.$$

On the other hand, from the duality axiom, defined by Definition 1, we have

$$\frac{\mathcal{M}\{(\eta_1 \le x)^c \cap (\eta_2 \le y)\}}{\mathcal{M}\{\eta_2 \le y\}} = \frac{1 - \mathcal{M}\{((\eta_1 \le x)^c \cap (\eta_2 \le y))^c\}}{\mathcal{M}\{\eta_2 \le y\}},$$

and from De Morgan's Law (Theorem 2) and the subadditivity axioms, we obtain

$$\mathcal{M}\{((\eta_1 \le x)^c \cap (\eta_2 \le y))^c\} = \mathcal{M}\{(\eta_1 \le x) \cup (\eta_2 > y)\} \le \mathcal{M}\{\eta_1 \le x\} + \mathcal{M}\{\eta_2 > y\}.$$

Thus

$$\frac{1 - \mathcal{M}\{\eta_1 \le x\} - \mathcal{M}\{\eta_2 > y\}}{\mathcal{M}\{\eta_2 \le y\}} \le \frac{1 - \mathcal{M}\{(\eta_1 \le x) \cup (\eta_2 > y)\}}{\mathcal{M}\{\eta_2 \le y\}} = \frac{\mathcal{M}\{(\eta_1 \le x)^c \cap (\eta_2 \le y)\}}{\mathcal{M}\{\eta_2 \le y\}}$$

and

$$\frac{1 - \mathcal{M}\{(\eta_1 \le x) \cup (\eta_2 > y)\}}{\mathcal{M}\{\eta_2 \le y\}} = \frac{\mathcal{M}\{(\eta_1 \le x)^c \cap (\eta_2 \le y)\}}{\mathcal{M}\{\eta_2 \le y\}} \le \frac{\mathcal{M}\{(\eta_1 \le x)^c\}}{\mathcal{M}\{\eta_2 \le y\}}.$$

Next, the following discussion can be divided into three parts.
Part 1: If

$$\frac{\mathcal{M}\{\eta_1 \le x\}}{\mathcal{M}\{\eta_2 \le y\}} < 0.5$$

and

$$0.5 \le \frac{1 - \mathcal{M}\{\eta_1 \le x\} - \mathcal{M}\{\eta_2 > y\}}{\mathcal{M}\{\eta_2 \le y\}},$$

then

$$\Psi_1(x) < \frac{1}{2}\Psi_2(y).$$

Then, applying the maximum uncertainty principle (Liu [1])

$$\frac{\mathcal{M}\{(\eta_1 \le x) \cap (\eta_2 \le y)\}}{\mathcal{M}\{\eta_2 \le y\}} = \frac{1 - \mathcal{M}\{((\eta_1 \le x) \cap (\eta_2 \le y))^c\}}{\mathcal{M}\{\eta_2 \le y\}} = \frac{\Psi_1(x)}{\Psi_2(y)}.$$

Part 2: If

$$0.5 \le \frac{1 - \mathcal{M}\{(\eta_1 > x)\} - \mathcal{M}\{(\eta_2 > y)\}}{\mathcal{M}\{\eta_2 \le y\}}$$

and

$$\frac{\mathcal{M}\{(\eta_1 \le x)^c\}}{\mathcal{M}\{\eta_2 \le y\}} < 0.5,$$

then

$$1 - \frac{1}{2}\Psi_2(y) < \Psi_1(x) < \Psi_2(y).$$

Then, applying the maximum uncertainty principle (Liu [1])

$$\frac{1 - \mathcal{M}\{((\eta_1 \le x)^c \cap (\eta_2 \le y))^c\}}{\mathcal{M}\{\eta_2 \le y\}} = \frac{\mathcal{M}\{(\eta_1 \le x)^c \cap (\eta_2 \le y)\}}{\mathcal{M}\{\eta_2 \le y\}} = \frac{\Psi_1(x) + \Psi_2(y) - 1}{\Psi_2(y)}.$$

Part 3: If Part 1 and Part 2 are not met, then

$$\mathcal{M}\{\eta_1 \le x | \eta_2 \le y\} = 0.5.$$

Thus

$$\mathcal{M}\{\eta_1 \le x | \eta_2 \le y\} = \begin{cases} \frac{\Psi_1(x)}{\Psi_2(y)}, & if \quad \Psi_1(x) < \frac{\Psi_2(y)}{2} \\ \frac{\Psi_1(x)+\Psi_2(y)-1}{\Psi_2(y)}, & if \quad 1 - \frac{\Psi_2(y)}{2} < \Psi_1(x) < \Psi_2(y) \\ 0.5, & otherwise. \end{cases}$$

The theorem is proved. □

4. Conditional Inverse Uncertainty Distribution

From Theorem 5 and 6, we know the conditional uncertainty distribution is a strictly increasing function in some ranges. Then, $\mathcal{M}\{\eta_1 \le x|\eta_2 > y\}$ have an inverse function due to $\mathcal{M}\{\eta_1 \le x|\eta_2 > y\}$ being a regular function in the range of $\Psi_2(y) < \Psi_1(x) < \frac{1-\Psi_2(y)}{2}$ or $\frac{1+\Psi_2(y)}{2} \le \Psi_1(x)$. Similarly, $\mathcal{M}\{\eta_1 \le x|\eta_2 \le y\}$ have an inverse function in the range of $\Psi_1(x) < \frac{\Psi_2(y)}{2}$ or $1 - \frac{\Psi_2(y)}{2} < \Psi_1(x) < \Psi_2(y)$.

In Uncertainty Theory, the role of inverse uncertainty distribution is crucial for solving many uncertainty problems, for example, calculating the expected value and so on. In this part, we give the definition of the conditional inverse uncertainty distribution, and two instances about the conditional inverse uncertainty distribution of some special uncertainty distributions are provided.

Definition 11. *Assume Ψ_1, Ψ_2 are uncertainty distributions for uncertain variables η_1, η_2, respectively. Then, the conditional inverse uncertainty distribution $\Psi^{-1}_{x|y}(\alpha)$ is defined by*

$$\Psi^{-1}_{x|y}(\alpha) = \mathcal{M}^{-1}\{\eta_1 \le x|\eta_2 > y\}$$

for $\Psi_2(y) < 1$ and $\Psi_2(y) < \Psi_1(x) < \frac{1-\Psi_2(y)}{2}$ or $\frac{1+\Psi_2(y)}{2} \le \Psi_1(x)$ with real numbers x, y.

Similarly,

$$\Psi^{-1}_{x|y}(\alpha) = \mathcal{M}^{-1}\{\eta_1 \le x|\eta_2 \le y\}$$

for $\Psi_2(y) > 0$ and $\Psi_1(x) < \frac{\Psi_2(y)}{2}$ or $1 - \frac{\Psi_2(y)}{2} < \Psi_1(x) < \Psi_2(y)$ with real numbers x, y.

Next, we give the conditional inverse uncertainty distribution of some special uncertainty distributions.

4.1. Conditional Inverse Uncertainty Distribution Of Linear Uncertainty Distribution

In this subsection, the conditional inverse uncertainty distribution of linear uncertainty distributions is given, where $\eta_1 \sim \mathcal{L}(a,b)$ and $\eta_2 \sim \mathcal{L}(a,b)$ are defined by Definition 9. Thus, η_1 and η_2 have the same uncertainty distributions Ψ.

Due to

$$\mathcal{M}\{\eta_1 \le x|\eta_2 > y\} = \begin{cases} \frac{\Psi(x)}{1-\Psi(y)}, & if \quad \Psi(y) < \Psi(x) < \frac{1-\Psi(y)}{2} \\ \frac{\Psi(x)-\Psi(y)}{1-\Psi(y)}, & if \quad \frac{1+\Psi(y)}{2} \le \Psi(x) \\ 0.5, & otherwise. \end{cases}$$

$$\mathcal{M}\{\eta_1 \le x|\eta_2 \le y\} = \begin{cases} \frac{\Psi(x)}{\Psi(y)}, & if \quad \Psi(x) < \frac{\Psi(y)}{2} \\ \frac{\Psi(x)+\Psi(y)-1}{\Psi(y)}, & if \quad 1 - \frac{\Psi(y)}{2} < \Psi(x) < \Psi(y) \\ 0.5, & otherwise. \end{cases}$$

So, the argument for the conditional inverse uncertainty distribution can be divided into four cases.

Case 1: If $\Psi(y) < \Psi(x) < \frac{1-\Psi(y)}{2}$, we have

$$\mathcal{M}\{\eta_1 \le x|\eta_2 > y\} = \frac{\Psi(x)}{1-\Psi(y)}.$$

Then,

$$\alpha = \frac{\frac{x-a}{b-a}}{1 - \frac{y-a}{b-a}} = \frac{x-a}{b-y}.$$

Thus

$$\Psi^{-1}_{x|y}(\alpha) = x = (b-y)\alpha + a.$$

Case 2: If $\frac{1+\Psi(y)}{2} \le \Psi(x)$, we easily obtain

$$\mathcal{M}\{\eta_1 \le x | \eta_2 > y\} = \frac{\Psi(x) - \Psi(y)}{1-\Psi(y)}.$$

Then,

$$\alpha = \frac{\frac{x-a}{b-a} - \frac{y-a}{b-a}}{1 - \frac{y-a}{b-a}} = \frac{x-y}{b-y}.$$

Thus

$$\Psi^{-1}_{x|y}(\alpha) = x = (b-y)\alpha + y.$$

Case 3: If $\Psi(x) < \frac{\Psi(y)}{2}$, we also easily obtain

$$\mathcal{M}\{\eta_1 \le x | \eta_2 \le y\} = \frac{\Psi(x)}{\Psi(y)}.$$

Then

$$\alpha = \frac{x-a}{y-a}.$$

Thus

$$\Psi^{-1}_{x|y}(\alpha) = x = a + (y-a)\alpha.$$

Case 4: If $1 - \frac{\Psi(y)}{2} < \Psi(x) < \Psi(y)$, we also easily obtain

$$\mathcal{M}\{\eta_1 \le x | \eta_2 \le y\} = \frac{\Psi(x) + \Psi(y) - 1}{\Psi(y)}.$$

Then

$$\alpha = \frac{x-a+y-a-b+a}{y-a}.$$

Thus

$$\Psi^{-1}_{x|y}(\alpha) = x = a + b - y + (y-a)\alpha.$$

Remark 1. *Assume Ψ is the linear uncertainty distribution for uncertain variables η_1, η_2, that $\eta_1 \sim \mathcal{L}(a,b)$, $\eta_2 \sim \mathcal{L}(a,b)$, and $\Psi(y) < 1$ for a real number y. Then, the conditional inverse uncertainty distribution of $\mathcal{M}\{\eta_1 \le x | \eta_2 > y\}$ defined by Definition 11 is*

$$\Psi^{-1}_{x|y}(\alpha) = \begin{cases} (b-y)\alpha + a, & if \quad \Psi(y) < \Psi(x) < \frac{1-\Psi(y)}{2} \\ (b-y)\alpha + y, & if \quad \frac{1+\Psi(y)}{2} \le \Psi(x). \end{cases}$$

Example 1. *Assume $\eta_1 \sim \mathcal{L}(1,6)$, $\eta_2 \sim \mathcal{L}(1,6)$, and Ψ is the linear uncertainty distribution of η_1,η_2, and $\Psi(y) < 1$ for a real number y. Then, the conditional inverse uncertainty distribution of $\mathcal{M}\{\eta_1 \le x | \eta_2 > y\}$ defined by Definition 11 is*

$$\Psi^{-1}_{x|y}(\alpha) = \begin{cases} (6-y)\alpha + 1, & if \quad y < x < \frac{4-y}{2} \\ (6-y)\alpha + y, & if \quad \frac{3+y}{2} \le x. \end{cases}$$

for $0 \le \alpha \le 1$.

Remark 2. *Assume Ψ is the linear uncertainty distribution for uncertain variables η_1, η_2, that $\eta_1 \sim \mathcal{L}(a,b)$, $\eta_2 \sim \mathcal{L}(a,b)$, and $\Psi(y) > 0$ for a real number y. Then, the conditional inverse uncertainty distribution of $\mathcal{M}\{\eta_1 \leq x | \eta_2 \leq y\}$ defined by Definition 11 is*

$$\Psi^{-1}_{x|y}(\alpha) = \begin{cases} a + (y-a)\alpha, & if \quad \Psi(x) < \frac{\Psi(y)}{2} \\ a + b - y + (y-a)\alpha, & if \quad 1 - \frac{\Psi(y)}{2} < \Psi(x) < \Psi(y). \end{cases}$$

Example 2. *Assume $\eta_1 \sim \mathcal{L}(2,10)$, $\eta_2 \sim \mathcal{L}(2,10)$ and Ψ is the linear uncertainty distribution of η_1,η_2, and $\Psi(y) > 0$ for a real number y. Then, the conditional inverse uncertainty distribution of $\mathcal{M}\{\eta_1 \leq x | \eta_2 \leq y\}$ defined by Definition 11 is*

$$\Psi^{-1}_{x|y}(\alpha) = \begin{cases} 2 + (y+2)\alpha, & if \quad x < \frac{y}{2+1} \\ 12 - y + (y-2)\alpha, & if \quad 11 - \frac{y}{2} < x < y\,. \end{cases}$$

for $0 \leq \alpha \leq 1$.

4.2. Conditional Inverse Uncertainty Distribution of Normal Uncertainty Distribution

In this subsection, the conditional inverse uncertainty distribution of normal uncertainty distributions is given, that $\eta_1 \sim \mathcal{N}(\mathrm{e}, \sigma)$, $\eta_2 \sim \mathcal{N}(\mathrm{e}, \sigma)$ defined by Definition 10, Thus η_1 and η_2 have same uncertainty distributions Ψ.

Due to

$$\mathcal{M}\{\eta_1 \leq x | \eta_2 > y\} = \begin{cases} \frac{\Psi(x)}{1-\Psi(y)}, & if \quad \Psi(y) < \Psi(x) < \frac{1-\Psi(y)}{2} \\ \frac{\Psi(x)-\Psi(y)}{1-\Psi(y)}, & if \quad \frac{1+\Psi(y)}{2} \leq \Psi(x) \\ 0.5, & otherwise. \end{cases}$$

$$\mathcal{M}\{\eta_1 \leq x | \eta_2 \leq y\} = \begin{cases} \frac{\Psi(x)}{\Psi(y)}, & if \quad \Psi(x) < \frac{\Psi(y)}{2} \\ \frac{\Psi(x)+\Psi(y)-1}{\Psi(y)}, & if \quad 1 - \frac{\Psi(y)}{2} < \Psi(x) < \Psi(y) \\ 0.5, & otherwise. \end{cases}$$

Next, the argument for the conditional inverse uncertainty distribution can be divided into four cases.

Case 1: If $\Psi(y) < \Psi(x) < \frac{1-\Psi(y)}{2}$, we obtain

$$\mathcal{M}\{\eta_1 \leq x | \eta_2 > y\} = \frac{\Psi(x)}{1-\Psi(y)}$$

.

Then

$$\alpha = \frac{(1 + exp(\frac{\pi(e-x)}{\sqrt{3}\sigma}))^{(-1)}}{1 - (1 + exp(\frac{\pi(e-y)}{\sqrt{3}\sigma}))^{(-1)}}.$$

Then

$$\alpha = \frac{1 + exp(\frac{\pi(e-y)}{\sqrt{3}\sigma})}{(exp(\frac{\pi(e-x)}{\sqrt{3}\sigma}))(exp(\frac{\pi(e-y)}{\sqrt{3}\sigma})) + exp(\frac{\pi(e-y)}{\sqrt{3}\sigma})}.$$

Thus

$$\Psi^{-1}_{x|y}(\alpha) = x = 2e - y - \frac{\sqrt{3}\sigma}{\pi}(ln(1/\alpha + exp(\frac{\pi(e-y)}{\sqrt{3}\sigma})(1/\alpha - 1))).$$

Case 2: If $\frac{1+\Psi(y)}{2} \leq \Psi(x)$, we easily obtain

$$\mathcal{M}\{\eta_1 \leq x | \eta_2 > y\} = \frac{\Psi(x) - \Psi(y)}{1 - \Psi(y)}$$

Then

$$\alpha = \frac{(1+exp(\frac{\pi(e-x)}{\sqrt{3}\sigma}))^{(-1)} - (1+exp(\frac{\pi(e-y)}{\sqrt{3}\sigma}))^{(-1)}}{1-(1+exp(\frac{\pi(e-y)}{\sqrt{3}\sigma}))^{(-1)}}.$$

Then

$$\alpha = \frac{exp(\frac{\pi(e-y)}{\sqrt{3}\sigma}) - exp(\frac{\pi(e-x)}{\sqrt{3}\sigma})}{(exp(\frac{\pi(e-x)}{\sqrt{3}\sigma}))(exp(\frac{\pi(e-y)}{\sqrt{3}\sigma})) + exp(\frac{\pi(e-y)}{\sqrt{3}\sigma})}.$$

Thus

$$\Psi^{-1}_{x|y}(\alpha) = x = y - \frac{\sqrt{3}\sigma}{\pi}(ln(1-\alpha) - ln(\alpha exp(\frac{\pi(e-y)}{\sqrt{3}\sigma}) + 1)).$$

Case 3: If $\Psi(x) < \frac{\Psi(y)}{2}$, we also easily obtain

$$\mathcal{M}\{\eta_1 \le x|\eta_2 \le y\} = \frac{\Psi(x)}{\Psi(y)}.$$

Then,

$$\alpha = \frac{(1+exp(\frac{\pi(e-x)}{\sqrt{3}\sigma}))^{(-1)}}{(1+exp(\frac{\pi(e-y)}{\sqrt{3}\sigma}))^{(-1)}}.$$

Thus

$$\Psi^{-1}_{x|y}(\alpha) = x = e - \frac{\sqrt{3}\sigma}{\pi} ln(1-\alpha + exp(\frac{\pi(e-y)}{\sqrt{3}\sigma})).$$

Case 4: If $1 - \frac{\Psi(y)}{2} < \Psi(x) < \Psi(y)$, we also easily obtain

$$\mathcal{M}\{\eta_1 \le x|\eta_2 \le y\} = \frac{\Psi(x) + \Psi(y) - 1}{\Psi(y)}.$$

Then,

$$\alpha = \frac{(1+exp(\frac{\pi(e-x)}{\sqrt{3}\sigma}))^{(-1)} + (1+exp(\frac{\pi(e-y)}{\sqrt{3}\sigma}))^{(-1)} - 1}{(1+exp(\frac{\pi(e-y)}{\sqrt{3}\sigma}))^{(-1)}}.$$

Thus

$$\Psi^{-1}_{x|y}(\alpha) = x = 2e - y - \frac{\sqrt{3}\sigma}{\pi} ln(1-\alpha + \alpha exp(\frac{\pi(e-y)}{\sqrt{3}\sigma})).$$

Remark 3. *Assume Ψ is the normal uncertainty distribution for uncertain variables η_1, η_2, that $\eta_1 \sim \mathcal{N}(e,\sigma)$,$\eta_2 \sim \mathcal{N}(e,\sigma)$, and $\Psi(y) < 1$ for a real number y. Then, the conditional inverse uncertainty distribution of $\mathcal{M}\{\eta_1 \le x|\eta_2 > y\}$ defined by Definition 11 is*

$$\Psi^{-1}_{x|y}(\alpha) = \begin{cases} 2e - y - \frac{\sqrt{3}\sigma}{\pi}(ln(1/\alpha + exp(\frac{\pi(e-y)}{\sqrt{3}\sigma})(1/\alpha - 1))), & if \quad \Psi(y) < \Psi(x) < \frac{1-\Psi(y)}{2} \\ y - \frac{\sqrt{3}\sigma}{\pi}(ln(1-\alpha) - ln(\alpha exp(\frac{\pi(e-y)}{\sqrt{3}\sigma}) + 1)), & if \quad \frac{1+\Psi(y)}{2} \le \Psi(x). \end{cases}$$

Example 3. *Assume $\eta_1 \sim \mathcal{N}(2,9)$, $\eta_2 \sim \mathcal{N}(2,9)$ and Ψ is the normal uncertainty distribution of η_1,η_2, and $\Psi(y) < 1$ for a real number y. Then, the conditional inverse uncertainty distribution of $\mathcal{M}\{\eta_1 \le x|\eta_2 > y\}$ defined by Definition 11 is*

$$\Psi^{-1}_{x|y}(\alpha) = \begin{cases} 4 - y - \frac{3\sqrt{3}}{\pi}(ln(1/\alpha + exp(\frac{\pi(2-y)}{3\sqrt{3}})(1/\alpha - 1))), & if \quad y < x < 4 - y - \frac{3\sqrt{3}}{\pi}ln[3 + (exp(\frac{\pi(2-y)}{3\sqrt{3}})] \\ y - \frac{3\sqrt{3}}{\pi}(ln(1-\alpha) - ln(\alpha exp(\frac{\pi(2-y)}{3\sqrt{3}}) + 1)), & if \quad y + \frac{3\sqrt{3}}{\pi}ln(2 + exp(\frac{\pi(2-y)}{3\sqrt{3}}) \leq x. \end{cases}$$

for $0 \leq \alpha \leq 1$.

Remark 4. *Assume* Ψ *is the normal uncertainty distribution for uncertain variables* η_1, η_2, *i.e.,* $\eta_1 \sim \mathcal{N}(e,\sigma)$,$\eta_2 \sim \mathcal{N}(e,\sigma)$, *and* $\Psi(y) > 0$ *for a real number y. Then, the conditional inverse uncertainty distribution of* $\mathcal{M}\{\eta_1 \leq x|\eta_2 \leq y\}$*defined by Definition 11 is*

$$\Psi^{-1}_{x|y}(\alpha) = \begin{cases} e - \frac{\sqrt{3}\sigma}{\pi}ln(1 - \alpha + exp(\frac{\pi(e-y)}{\sqrt{3}\sigma})), & if \quad \Psi(x) < \frac{\Psi(y)}{2} \\ 2e - y - \frac{\sqrt{3}\sigma}{\pi}ln(1 - \alpha + \alpha exp(\frac{\pi(e-y)}{\sqrt{3}\sigma})), & if \quad 1 - \frac{\Psi(y)}{2} < \Psi(x) < \Psi(y). \end{cases}$$

Example 4. *Assume* $\eta_1 \sim \mathcal{N}(3{,}16)$, $\eta_2 \sim \mathcal{N}(3{,}16)$, *and* Ψ *is the normal uncertainty distribution of* η_1,η_2, *and* $\Psi(y) > 0$ *for a real number y. Then, the conditional inverse uncertainty distribution of* $\mathcal{M}\{\eta_1 \leq x|\eta_2 \leq y\}$ *defined by Definition 11 is*

$$\Psi^{-1}_{x|y}(\alpha) = \begin{cases} 3 - \frac{4\sqrt{3}}{\pi}ln(1 - \alpha + exp(\frac{\pi(3-y)}{4\sqrt{3}})), & if \quad x < 3 - \frac{4\sqrt{3}}{\pi}ln[1 + 2exp(\frac{\pi(3-y)}{4\sqrt{3}})] \\ 6 - y - \frac{4\sqrt{3}}{\pi}ln(1 - \alpha + \alpha exp(\frac{\pi(3-y)}{4\sqrt{3}})), & if \quad 3 + \frac{4\sqrt{3}}{\pi}ln[\frac{1+2exp(\frac{\pi(3-y)}{4\sqrt{3}})}{1-4exp(\frac{\pi(3-y)}{4\sqrt{3}})}] < x < y. \end{cases}$$

for $0 \leq \alpha \leq 1$.

5. Conclusions

The employment of conditional uncertain measures as well as the conditional uncertainty distribution is crucial when dealing with uncertain problems and nonindependent uncertain variable problems. Therefore, this study has successfully established theorems concerning the conditional uncertainty distribution involving two uncertain variables. Similarly, the utility of the conditional inverse uncertainty distributions holds immense importance in various analytical and estimative contexts. So, a clear definition of the conditional inverse uncertainty distribution has been presented, alongside the derivation of specific instances of such distributions in conjunction with particular conditional uncertainty scenarios.

Looking ahead, our research trajectory will involve an exploration of uncertain variables characterized by interdependence rather than independence. Additionally, we will delve into the practical applications of conditional inverse uncertainty distributions. Our focus will extend to constructing multidimensional functions involving dependent variables. This framework will facilitate predictive modeling through the utilization of conditional inverse uncertainty distributions, allowing for subsequent adjustments of distribution functions in the presence of actual data.

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