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# Impact of Semi-Symmetric Metric Connection on Homology of Warped Product Pointwise Semi-Slant Submanifolds of an Odd-Dimensional Sphere

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**Abstract:** Our paper explores warped product pointwise semi-slant submanifolds with a semi-symmetric metric connection in an odd-dimensional sphere and uncovers fundamental results. We also demonstrate how our findings can be applied to the homology of these submanifolds. Notably, we prove that under a specific condition, there are no stable currents for these submanifolds. This work adds valuable insights into the stability and behavior of warped product pointwise semi-slant submanifolds and sets the foundation for further research in this field.

**Keywords:** semi-invariant; warped product manifolds; semi-symmetric; Sasakian manifolds; stable-currents

**MSC:** 53B50; 53C20; 53C40

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## 1. Introduction

Bishop and O'Neill [1] developed the concept of warped products to construct illustrations of Riemannian manifolds with negative curvature. In fact, the warped product $B \times_b F$ of two pseudo-Riemannian manifolds $(B, g_B)$ and $(F, g_F)$ with a positive valued smooth function $b$ on $B$ provided the metric tensor $g = g_B \oplus b^2 g_F$. Here, $(B, g_B)$ is known as the base manifold; however, $(F, g_F)$ is the fiber and $b$ is the warping function. Warped product manifolds with a conformal Killing vector have been studied in the context of Einstein–Weyl geometry; in this setting, the warping function plays the role of a conformal factor, and the geometry is determined by a conformal class of metrics. For more details, see the works of Leistner and Nurowski [2,3]. Warped product manifolds have been used to construct various examples of Ricci solitons, which are self-similar solutions to the Ricci flow. In particular, the so-called "cigar solitons" on the Euclidean space have been studied extensively [4,5].

B. Y. Chen [6] was the first to examine the notion of warped products in the submanifold theory. In fact, Chen developed a CR-warped product submanifold in the setting of almost Hermitian manifolds and provided an approximation for the norm of the second fundamental form in the expressions of the warping function. Inspired by Chen, Hesigawa and Mihai [7] explored the contact form of these submanifolds and obtained a comparable approximation for the second fundamental form of a contact CR-warped product submanifold of a Sasakian space form. In addition, in [8], the authors concluded that the homology groups were trivial and that there were no stable currents in a contact CR-warped product submanifold immersed in an odd-dimensional sphere due to the non-existence of stable integral currents and the vanishing of homology. As a step forward, F. Sahin [9,10] has shown that the CR-warped product submanifold in $R^n$ and $S^6$ yields identical results. However, several scholars obtained different findings on the topological and differentiable structures of submanifolds by imposing certain constraints on the second fundamental form [8,11–14].

An algebraic description of a manifold can be found in its homology groups, which are significant topological features. Besides other issues, these groups include extensive topological data on the related parts, holes, tunnels, and the structure of manifolds, and this theory has numerous applications. In fact, homology theory has implications for root construction, molecular mooring, segmentation of images, and genetic expression information. Homology theory has found applications in data analysis, particularly in topological data analysis. Persistent homology, a variant of homology theory, is used to analyze the topological features of complex data sets, such as point clouds or graphs. It provides a way to detect and quantify the presence of holes and voids in the data [15]. It is well known that submanifold theory and homological theory have a strong relationship. In this context, Federer and Fleming [16] demonstrated that any non-trivial integral homological group $H_p(M, Z)$ is connected by stable currents. Later, Lawson and Simon [17] extended the same study to the submanifold of a sphere and proved that there does not exist an integral current under a pinching condition of the second fundamental form. However, Leung [18] and Xin [12] expanded the results from a sphere to Euclidean space. Further, in a similar line of research, Zhang [19] studied the homology of the torus. In addition, Liu and Zhang [14] proved that stable integral currents do not exist for specific kinds of hypersurfaces in Euclidean spaces.

On the other hand, Friedmann and Schouten [20] first proposed the concept of a semi-symmetric linear connection on a Riemannian manifold. Afterwards, Hayden [21] defined a semi-symmetric connection as a linear connection $\nabla$ that exists on an $n-$dimensional Riemannian manifold $(M, g)$ and whose torsion tensor $T$ satisfies $T(\omega_1, \omega_2) = \eta(\omega_2)\omega_1 - \eta(\omega_1)\omega_2$, where $\eta$ is a 1-form and $\omega_1, \omega_2 \in TM$. K. Yano [22] investigated semi-symmetric metric connections and analyzed some of their features. He demonstrated that a conformally flat Riemannian manifold with a semi-symmetric connection has a vanishing curvature tensor. Further, Sular and Ozgur [23] investigated warped product manifolds with a semi-symmetric metric connection and took into account Einstein's warped product manifolds with a semi-symmetric metric connection. However, in [24], they also obtained some more results on warped product manifolds with a semi-symmetric metric connection. Furthermore, the studies mentioned in [25–38] are important contributions to soliton theory and submanifold theory, etc., related to the relevant topics. Motivated by these studies, we are interested in determining the impact of a semi-symmetric metric connection on the warped product pointwise semi-slant submanifolds and their homology in an odd-dimensional sphere.

## 2. Preliminaries

Let $(\bar{M}, g)$ be an odd-dimensional Riemannian manifold. Then, $\bar{M}$ is said to be an almost contact metric manifold if there exists on $\bar{M}$ a tensor field $\phi$ of type $(1, 1)$ and a global vector field $\xi$ such that

$$\phi^2\omega_1 = -\omega_1 + \eta(\omega_1)\xi, \quad g(\omega_1, \xi) = \eta(\omega_1)$$

$$g(\phi\omega_1, \phi\omega_2) = g(\omega_1, \omega_2) - \eta(\omega_1)\eta(\omega_2)$$

where $\eta$ is the dual 1-form of $\xi$. It is well known that an almost contact metric manifold is a Sasakian manifold if and only if

$$(\bar{\bar{\nabla}}_{\omega_1}\phi)\omega_2 = g(\omega_1, \omega_2)\xi - \eta(\omega_2)\omega_1. \tag{1}$$

On a Sasakian manifold $\bar{M}$, it is easy to see that

$$\bar{\bar{\nabla}}_{\omega_1}\xi = -\phi\omega_1, \tag{2}$$

where $\omega_1, \omega_2 \in T\bar{M}$, and $\bar{\bar{\nabla}}$ is the Riemannian connection with respect to $g$.

Now, defining a connection $\bar{\nabla}$ as

$$\bar{\nabla}_{\omega_1}\omega_2 = \bar{\bar{\nabla}}_{\omega_1}\omega_2 + \eta(\omega_2)\omega_1 - g(\omega_1, \omega_2)\xi \tag{3}$$

such that $\bar{\nabla}g = 0$ for any $\omega_1, \omega_2 \in TM$, where $\bar{\bar{\nabla}}$ is the Riemannian connection with respect to $g$. The connection $\bar{\nabla}$ is semi-symmetric because $T(\omega_1, \omega_2) = \eta(\omega_2)\omega_1 - \eta(\omega_1)\omega_2$. Using (3) in (1), we have

$$(\bar{\nabla}_{\omega_1}\phi)\omega_2 = g(\omega_1, \omega_2)\xi - g(\omega_1, \phi\omega_2)\xi - \eta(\omega_2)\omega_1 - \eta(\omega_2)\phi\omega_1 \tag{4}$$

and

$$\bar{\nabla}_{\omega_1}\xi = \omega_1 - \eta(\omega_1)\xi - \phi\omega_1. \tag{5}$$

A Sasakian manifold $\bar{M}$ is said to be a Sasakian space form if it has a constant $\phi-$holomorphic sectional curvature $c$ and is denoted by $\bar{M}(c)$. The curvature tensor $\bar{R}$ with respect to the semi-symmetric metric connection $\bar{\nabla}$ is

$$\bar{R}(\omega_1, \omega_2)\omega_3 = \bar{\nabla}_{\omega_1}\bar{\nabla}_{\omega_2}\omega_3 - \bar{\nabla}_{\omega_2}\bar{\nabla}_{\omega_1}\omega_3 - \bar{\nabla}_{[\omega_1,\omega_2]}\omega_3. \tag{6}$$

Similarly, we can also define the curvature tensor $\bar{\bar{R}}$ for the Riemannian connection $\bar{\bar{\nabla}}$. Let

$$\beta(\omega_1, \omega_2) = (\bar{\nabla}_{\omega_1}\eta)\omega_2 - \eta(\omega_1)\eta(\omega_2) + \frac{1}{2}g(\omega_1, \omega_2)\eta(P). \tag{7}$$

Now, by the application of (3), (6) and (7), we get

$$\begin{aligned} \bar{R}(\omega_1, \omega_2, \omega_3, \omega_4) = {} & \bar{\bar{R}}(\omega_1, \omega_2, \omega_3, \omega_4) + \beta(\omega_1, \omega_3)g(\omega_2, \omega_4) \\ & - \beta(\omega_2, \omega_3)g(\omega_1, \omega_4) + \beta(\omega_2, \omega_4)g(\omega_1, \omega_3) - \beta(\omega_1, \omega_4)g(\omega_2, \omega_3). \end{aligned} \tag{8}$$

When utilizing the value of $\bar{\bar{R}}(\omega_1, \omega_2, \omega_3, \omega_4)$, which is further elaborated in [39], we obtain the subsequent expression for the curvature tensor $\bar{R}$ of a Sasakian space form $\bar{M}(c)$ endowed with a semi-symmetric metric connection, as mentioned in [40].

$$\begin{aligned} \bar{R}(\omega_1, \omega_2, \omega_3, \omega_4) = {} & \frac{c+3}{4}\{g(\omega_2, \omega_3)g(\omega_1, \omega_4) - g(\omega_1, \omega_3)g(\omega_2, \omega_4)\} \\ & + \frac{c-1}{4}\{\eta(\omega_1)\eta(\omega_3)g(\omega_2, \omega_4) - \eta(\omega_2)\eta(\omega_3)g(\omega_1, \omega_4) \\ & + g(\omega_1, \omega_3)\eta(\omega_2)\eta(\omega_4) - g(\omega_2, \omega_3)\eta(\omega_1)\eta(\omega_4) \\ & + g(\phi\omega_2, \omega_3)g(\phi\omega_1, \omega_4) + g(\phi\omega_3, \omega_1)g(\phi\omega_2, \omega_4) \\ & - 2g(\phi\omega_1, \omega_2)g(\phi\omega_3, \omega_4)\} + \beta(\omega_1, \omega_3)g(\omega_2, \omega_4) \\ & - \beta(\omega_2, \omega_3)g(\omega_1, \omega_4) + \beta(\omega_2, \omega_4)g(\omega_1, \omega_3) \\ & - \beta(\omega_1, \omega_4)g(\omega_2, \omega_3), \end{aligned} \tag{9}$$

for all $\omega_1, \omega_2, \omega_3, \omega_4 \in T\bar{M}$.

For a submanifold $M$ isometrically immersed in a differentiable manifold $\bar{M}$, by a routine calculation, the Gauss and Weingarten formulae for a semi-symmetric metric connection are $\bar{\nabla}_{\omega_1}\omega_2 = \nabla_{\omega_1}\omega_2 + h(\omega_1, \omega_2)$ and $\bar{\nabla}_{\omega_1}N = -A_N\omega_1 + \nabla^{\perp}_{\omega_1}N + \eta(N)\omega_1$, where $\nabla$ is the induced semi-symmetric metric connection on $M$, $N \in T^{\perp}M$, $h$ is the second fundamental form of $M$, $\nabla^{\perp}$ is the normal connection on the normal bundle $T^{\perp}M$, and $A_N$ is the shape operator. The second fundamental form $h$ and the shape operator are associated by the following formula:

$$g(h(\omega_1, \omega_2), N) = g(A_N\omega_1, \omega_2).$$

For the vector fields $\omega_1 \in TM$ and $\omega_3 \in T^{\perp}M$, we have the following decomposition:

$$\phi\omega_1 = T\omega_1 + F\omega_1 \tag{10}$$

and

$$\phi\omega_3 = t\omega_3 + f\omega_3 \tag{11}$$

where $T\omega_1(t\omega_3)$ and $F\omega_1(f\omega_3)$ are the tangential and normal parts of $\phi\omega_1(\phi\omega_3)$, respectively.

Let $R$ be the Riemannian curvature tensor of $M$. Then, the equation of Gauss for a semi-symmetric connection is given by

$$\bar{R}(\omega_1,\omega_2,\omega_3,\omega_4) = R(\omega_1,\omega_2,\omega_3,\omega_4) - g(h(\omega_1,\omega_4),h(\omega_2,\omega_3)) + g(h(\omega_2,\omega_4),h(\omega_1,\omega_3)) \tag{12}$$

for $\omega_1,\omega_2,\omega_3,\omega_4 \in TM$.

In [23], Sular and Ožgur considered the warped products of the type $M_1 \times_f M_2$, admitting a semi-symmetric metric connection with associated vector field $P$ on $M_1 \times_f M_2$, where $M_1, M_2$ are the Riemannian manifolds and $f$ is a positive differentiable function on $M_1$, called the warping function. Now, we compile some results of [23] in the form of the following lemma, which is important for the subsequent study.

**Lemma 1.** *Let $M_1 \times_f M_2$ be a warped product manifold with a semi-symmetric metric connection $\bar{\nabla}$.*

$(i)$ *If the associated vector field $P \in TM_1$, then*

$$\bar{\nabla}_{\omega_1}\omega_3 = \frac{\omega_1 f}{f}\omega_3 \quad \text{and} \quad \bar{\nabla}_{\omega_3}\omega_1 = \frac{\omega_1 f}{f}\omega_3 + \eta(\omega_1)\omega_3$$

$(ii)$ *If $P \in TM_2$, then*

$$\bar{\nabla}_{\omega_1}\omega_3 = \frac{\omega_1 f}{f}\omega_3 \quad \text{and} \quad \bar{\nabla}_{\omega_3}\omega_1 = \frac{\omega_1 f}{f}\omega_3,$$

*where $\omega_1 \in TM_1$, $\omega_3 \in TM_2$ and $\eta$ is the 1-form associated with the vector field $P$.*

Let us consider the warped product submanifold $M = M_1 \times_f M_2$ of a Sasakian manifold $\bar{M}$. In this case, we have the curvature tensors $R$ and $\tilde{R}$ associated with the submanifold $M$ and its induced semi-symmetric metric connection $\nabla$ and induced Riemannian connection $\tilde{\nabla}$, respectively. Then,

$$\begin{aligned} R(\omega_1,\omega_2)\omega_3 =& \tilde{R}(\omega_1,\omega_2)\omega_3 + g(\omega_3,\nabla_{\omega_1}P)\omega_2 - g(\omega_3,\nabla_{\omega_2}P)\omega_1 \\ &+ g(\omega_1,\omega_3)\nabla_{\omega_2}P - g(\omega_2,\omega_3)\nabla_{\omega_1}P \\ &+ \eta(P)[g(\omega_1,\omega_3)\omega_2 - g(\omega_2,\omega_3)\omega_1] \\ &+ [g(\omega_2,\omega_3)\eta(\omega_1) - g(\omega_1,\omega_3)\eta(\omega_2)]P \\ &+ \eta(\omega_3)[\eta(\omega_2)\omega_1 - \eta(\omega_1)\omega_2], \end{aligned} \tag{13}$$

for any vector field $\omega_1,\omega_2,\omega_3$ on $M$ [23].

For the warped product submanifold $M = M_1 \times_f M_2$, from part (ii) of Lemma 3.2 of [23], we have

$$\tilde{R}(\omega_1,\omega_2)\omega_3 = \frac{H^f(\omega_1,\omega_2)}{f}\omega_3, \tag{14}$$

where $\omega_1,\omega_2 \in TM_1$, $\omega_3 \in TM_2$, respectively, and $H^f$ is the Hessian of the warping function.

By considering Equations (13) and (14), we can deduce the following:

$$R(\omega_1,\omega_3)\omega_2 = \frac{H^f(\omega_1,\omega_2)}{f} + \frac{Pf}{f}g(\omega_1,\omega_2)\omega_3 + \eta(P)g(\omega_1,\omega_2)\omega_3 + g(\omega_2,\nabla_{\omega_1}P)\omega_3 - \eta(\omega_1)\eta(\omega_2)\omega_3, \tag{15}$$

for the vector fields $\omega_1,\omega_2 \in TM_1$, $\omega_3 \in TM_2$, and $P \in TM_1$.

Since we defined the semi-symmetric connection in Equation (3) by taking $P = \xi$, therefore, for a warped product submanifold $M = M_1 \times_f M_2$ of a Riemannian manifold $\bar{M}$, we deduce the following relation by part $(i)$ of Lemma 1:

$$\nabla_{\omega_1}\omega_3 = \omega_1 lnf \omega_3 \tag{16}$$

and

$$\nabla_{\omega_3}\omega_1 = \omega_1 lnf \omega_3 + \eta(\omega_1)\omega_3. \tag{17}$$

Furthermore, by utilizing the Gauss formula in Equation (15) along with Equation (5), we obtain

$$R(\omega_1,\omega_3)\omega_2 = \frac{H^f(\omega_1,\omega_2)}{f}\omega_3 + \frac{\xi f}{f}g(\omega_1,\omega_2)\omega_3 + 2g(\omega_1,\omega_2)\omega_3 - 2\eta(\omega_1)\eta(\omega_2)\omega_3 - g(\omega_2,\phi\omega_1)\omega_3, \tag{18}$$

for $\xi,\omega_1,\omega_2 \in TM_1$, and $\omega_3 \in TM_2$.

For the Laplacian $\Delta f$ of the warping function, it is easy to see the following expression:

$$\frac{\Delta f}{f} = \Delta lnf - \|\nabla lnf\|^2. \tag{19}$$

**3. Warped Product Pointwise Semi-Slant Submanifolds and Their Homology**

The concept of semi-invariant submanifolds in an almost contact metric manifold was introduced by A. Bejancu in 1981 [41]. According to this concept, an $m-$dimensional Riemannian submanifold $M$ of a Sasakian manifold $\bar{M}$ is referred to as a semi-invariant submanifold if the characteristic vector field $\xi$ is tangent to $M$, and if there exists a differentiable distribution $D$ on $M$ such that $D_x$, the distribution at $x \in M$ is invariant under $\phi$. The distribution $D_x^{\perp}$, which is the orthogonal complementary distribution of $D_x$ on $M$, is anti-invariant, meaning that $\phi D_x^{\perp} \subseteq T_x^{\perp}M$, where $T_xM$ and $T_x^{\perp}M$ are the tangent space and normal space at point $x \in M$, respectively. In [7], Hesigawa and Mihai studied the warped product submanifold of the form $M_T \times_f M_{\perp}$ of a Sasakian manifold $\bar{M}$, where $M_T$ is an invariant submanifold, $M_{\perp}$ is an anti-invariant submanifold, and $\xi$ is an element of $TM_T$. They referred to this type of submanifold as a contact CR-submanifold and provided some fundamental results on it.

In recent times, the concept of pointwise slant submanifolds was introduced by F. Etayo in a paper in which these submanifolds were referred to as quasi-slant submanifolds [42]. Subsequently, B. -Y Chen and O. J. Garay [6] studied the properties of pointwise slant submanifolds in the context of almost Hermitian manifolds. Afterward, Park [43] defined the concept of the pointwise slant submanifold in the setting of an almost contact metric manifold. In fact, he defined pointwise slant submanifolds as follows: a submanifold $M$ of an almost contact metric manifold $(\bar{M},\phi,\xi,\eta,g)$ is considered to be pointwise slant if, for every point $x \in M$, the angle $\theta = \theta(X)$ between $\phi X$ and $T_xM$ remains constant regardless of the choice of non zero vector field $X \in T_xM$, where $g(X,\xi(x)) = 0$. This angle is denoted by $\theta(X)$ and is called the slant function as a function on $M$. The necessary and sufficient condition for the submanifold $M$ to be a pointwise slant is whether the endomorphism $T$ satisfies the following relation:

$$T^2\omega = -\lambda(\omega - \eta(\omega)\xi) \tag{20}$$

for any $\omega \in TM$ and $\lambda \in [0,1]$ such that $\lambda = \cos^2\theta$. The following formulae can be deduced by using (10) and (20):

$$g(T\omega_1, T\omega_2) = \cos^2\theta[g(\omega_1,\omega_2) - \eta(\omega_1)\eta(\omega_2)], \tag{21}$$

$$g(F\omega_1, F\omega_2) = \sin^2\theta[g(\omega_1,\omega_2) - \eta(\omega_1)\eta(\omega_2)]. \tag{22}$$

Further, in the same paper, he introduced the notion of the pointwise semi-slant submanifold in the frame of an almost contact metric manifold. Further, he studied different types of warped product pointwise semi-slant submanifolds of almost contact metric manifolds. Basically, he proved the non-existence of the warped product of the types $N_T \times_f N_\theta$, with $\xi \in T^\perp M$, where $N_T$ is an invariant submanifold and $N_\theta$ is the pointwise slant submanifold. After that, he considered the vector field $\xi$ tangential to $N_T$ and proved the existence of the warped products $N_T \times_f N_\theta$.

We begin our analysis by examining a particular type of submanifold, namely the warped product pointwise semi-slant submanifolds of the form $N_\theta \times_f N_T$ in a Sasakian manifold equipped with a semi-symmetric metric connection. Here, $N_\theta$ is a pointwise slant submanifold, and $N_T$ is an invariant submanifold that satisfies $\xi \in TN_T$.

Our investigation yields the following result:

**Theorem 1.** *Let $(\bar{M}, \phi, \xi, \eta, g)$ be a Sasakian manifold with a semi-symmetric metric connection. Then, there does not exist a warped product pointwise semi-slant submanifold of the type $N_\theta \times_f N_T$, such that $\xi \in TN_T$.*

**Proof.** For any $\omega_1, \omega_2 \in TN_T$ and $\omega_3 \in TN_\theta$, by part $(ii)$ of Lemma 1, the Gauss formula, (10) and (4), we have

$$\begin{aligned}\omega_3 lnf g(\omega_1,\omega_2) = g(\bar{\nabla}_{\omega_1}\omega_3, \omega_2) &= g(\phi\bar{\nabla}_{\omega_1}\omega_3, \phi\omega_2) + \eta(\bar{\nabla}_{\omega_1}, \omega_3)\eta(\omega_2)\\ &= g(\bar{\nabla}_{\omega_1}(T\omega_3 + F\omega_3), \phi\omega_2) - g((\bar{\nabla}_{\omega_1}\phi)\omega_3, \phi\omega_2)\\ &+ \omega_3 lnf\eta(\omega_1)\eta(\omega_2).\end{aligned} \tag{23}$$

On further simplification, we get

$$\omega_3 lnf g(\omega_1,\omega_2) = -(\phi\bar{\nabla}_{\omega_1}T\omega_3, \omega_2) - g(h(\omega_1, \phi\omega_2), F\omega_3) + \omega_3 lnf\eta(\omega_1)\eta(\omega_2). \tag{24}$$

Again, by Equations (4) and (21), the Weingarten formula, and part $(ii)$ of Lemma 1, the preceding equation yields

$$\sin^2\theta\omega_3 lnf g(\omega_1,\omega_2) = g(h(\omega_1,\omega_2), FT\omega_3) - g(h(\omega_1,\phi\omega_2), F\omega_3) + \omega_3 lnf\eta(\omega_1)\eta(\omega_2). \tag{25}$$

Replacing $\omega_1$ and $\omega_2$ by $\xi$ in the above equation and using Equation (5), we obtain $\cos^2\theta\omega_3 lnf = 0$; this means that the warping function $f$ is constant and proves the result. □

Throughout this study, we focus on warped product pointwise semi-slant submanifolds $N_T \times_f N_\theta$ that admit semi-symmetric metric connections, such that $\xi \in TN_T$. With that in mind, we begin by presenting the following initial results.

**Lemma 2.** *Let $M = N_T \times_f N_\theta$ be a non-trivial warped product proper pointwise semi-slant submanifold of a Sasakian manifold admitting a semi-symmetric metric connection; then,*

*(i)* $g(A_{F\omega_3}\omega_4, \omega_1) = g(A_{F\omega_4}\omega_3, \omega_1)$,
*(ii)* $\xi lnf = 0$,

*for $\xi, \omega_1 \in TN_T$ and $\omega_3, \omega_4 \in TN_\theta$.*

**Proof.** Making use of the Weingarten formula along with (10), we have

$$g(A_{F\omega_3}\omega_4, \omega_1) = g(\bar{\nabla}_{\omega_1}\phi\omega_3, \omega_4) + g(\bar{\nabla}_{\omega_1}T\omega_3, \omega_4). \tag{26}$$

Now, using (4) and (16), we get the required result. To prove part $(ii)$, by Equations (5) and (10), we have $\nabla_{\omega_3}\xi = \omega_3 - T\omega_3$; applying Equation (17), we get the required result. □

**Lemma 3.** *Let $M = N_T \times_f N_\theta$ be a non-trivial warped product proper pointwise semi-slant submanifold of a Sasakian manifold admitting semi-symmetric metric connection; then,*

$$g(h(\omega_1, \omega_4), FT\omega_3) = -\phi\omega_1 lnf g(T\omega_3, \omega_4) - \eta(\omega_1)g(T\omega_3, \omega_4) - \omega_1 lnf \cos^2\theta g(\omega_3, \omega_4) \tag{27}$$

*for $\omega_1 \in TN_T$ and $\omega_3, \omega_4 \in TN_\theta$.*

**Proof.** From part $(i)$ of Lemma 2 and the Weingarten equation, we find

$$g(A_{FT\omega_3}\omega_4, \omega_1) = g(A_{F\omega_4}T\omega_3, \omega_1) = -g(\bar{\nabla}_{T\omega_3}F\omega_4, \omega_1).$$

Using (4), (10), and (17), we obtain

$$\begin{aligned} g(A_{FT\omega_1}\omega_4, \omega_1) = -g(g(T\omega_3, \omega_4)\xi - g(T\omega_3, \phi\omega_4)\xi, \omega_1) + g(\nabla_{T\omega_3}\omega_4, \phi\omega_1) \\ - g(T\omega_4, \omega_1 lnf T\omega_3 + \eta(\omega_1)T\omega_3). \end{aligned} \tag{28}$$

Solving further, the above equation can be reduced to

$$\begin{aligned} g(A_{FT\omega_3}\omega_4, \omega_1) = -g(T\omega_3,\omega_4)\eta(\omega_1) + \cos^2\theta\eta(\omega_1)g(\omega_3, \omega_4) - g(\omega_4, \phi\omega_1 lnf T\omega_3) \\ - \cos^2\theta\omega_1 lnf\eta(\omega_1)g(\omega_3, \omega_4) - \eta(\omega_1)\cos^2\theta g(\omega_3, \omega_4). \end{aligned} \tag{29}$$

Finally, we get the following equation:

$$g(h(\omega_1, \omega_4), FT\omega_3) = -\eta(\omega_3)g(T\omega_3, \omega_4) - \phi\omega_1 lnf g(T\omega_3, \omega_4) - \omega_1 lnf \cos^2\theta g(\omega_3, \omega_4), \tag{30}$$

which is the required result. □

**Lemma 4.** *Let $M = N_T \times_f N_\theta$ be a non-trivial warped product proper pointwise semi-slant submanifold of a Sasakian manifold admitting a semi-symmetric metric connection. Then,*

*(i)* $g(h(\omega_1, \omega_3), FT\omega_3) = -\omega_1 lnf \cos^2\theta\|\omega_3\|^2$,
*(ii)* $g(h(\phi\omega_1, \omega_3), F\omega_3) = \omega_1 lnf\|\omega_3\|^2$,

*for $\omega_1 \in TN_T$ and $\omega_3 \in TN_\theta$.*

**Proof.** Replacing $\omega_4$ by $\omega_3$ in Equation (27), we get part $(i)$, and using the Gauss formula along with Equation (10), we obtain

$$g(h(\phi\omega_1, \omega_3), F\omega_3) = g(\bar{\nabla}_{\omega_3}\phi\omega_1, \phi\omega_3) - g(\bar{\nabla}_{\omega_3}\phi\omega_1, T\omega_3). \tag{31}$$

Finally, on applying the Gauss formula and Equation (17), we obtain $g(h(\phi\omega_1, \omega_3), F\omega_3) = \omega_1 lnf\|\omega_3\|^2$, which is the required result. □

Now, we study the stable currents on warped product pointwise semi-slant submanifolds. In fact, we prove that under some specific conditions, there do not exist stable currents. Now, we exhibit the well-known results of Simons, Xin, and Lang

**Lemma 5** ([14,17]). *For a compact submanifold $M^n$ with dimension $n$ of a space form $\bar{M}(c)$ with a positive curvature $c$, if the second fundamental form satisfies the following inequality,*

$$\sum_{i=1}^{p}\sum_{s=p+1}^{n}(2|h(u_i,u_j)|^2 - g(h(u_i,u_i),h(u_i,u_s))) < pqc, \tag{32}$$

*then there do not exist stable currents in $M^n$, where $p,q \in \mathbb{Z}^+$, such that $p+q=n$, $\{u_1,\dots,u_n\}$ is a set of orthonormal bases at $T_xM$, $x \in M$. In addition, $\tilde{H}_p(M^n,Z)=0$, $\tilde{H}_q(M^n,Z)=0$, such that $H_j(M,Z)$ denotes the $j-th$ homology of $M$ with integer coefficients.*

It is well known that the odd-dimensional sphere $S^{2n+1}(1)$ is the Sasakian manifold with constant sectional curvature one [39]. Now, we have the following theorem:

**Theorem 2.** *Let $M^{p+q+1} = N_T^{p+1} \times_f N_\theta^q$ be a compact warped product pointwise semi-slant submanifold of $S^{2(\frac{p}{2}+q)+1}(1)$ with a semi-symmetric metric connection. If the following inequality holds,*

$$\Delta f - \sum_{i=1}^{p+1}\beta(u_i,u_i) + \frac{p+1}{q}\sum_{j=1}^{q}\beta(u_j,u_j) > (\csc^2\theta + \cot^2\theta + 1 - q)\|\nabla lnf\|^2 - \frac{q}{f}\eta(\nabla f) - 1, \tag{33}$$

*then the $(p+1)-$stable currents are absent in $M^{p+1+q}$. In addition, $H_{p+1}(M^n,Z)=0$, $H_q(M^n,Z)=0$, where $H_j(M,Z)$ is the $j-th$ homology group of $M$, and $p+1,q$ are the dimensions of the invariant submanifold $M_T^{p+1}$ and the pointwise slant submanifold $M_\theta^q$, respectively.*

**Proof.** Suppose $\dim N_T^{p+1} = p+1 = 2\alpha+1$ and $\dim N_\theta^q = q = 2\beta$, where $N_T$ and $N_\theta$ are the integral manifolds of invariant distribution $D_T$ and the pointwise slant distribution $D_\theta$. Let $\{u_0=\xi, u_1, u_2, \dots, u_\alpha, u_{\alpha+1}=\phi u_1, \dots, u_{2\alpha}=\phi u_\alpha\}$ and $\{u_{2\alpha+1}=u_1^*, \dots, u_{2\alpha+\beta}=u_\beta^*, u_{2\alpha+\beta+1}=u_{\beta+1}^*=\sec\theta T u_1^*, \dots, u_{p+q}=u_q^*=\sec\theta T u_\beta^*\}$ be orthonormal bases of $TN_T^{p+1}$ and $TN_\theta^q$, respectively. Therefore, the orthonormal basis for the normal subbundle $FD_\theta$ is $\{u_{n+1}=\bar{u}_1=\csc\theta F u_1^*, \dots, u_{n+\beta}=\bar{u}_\beta=\csc\theta F u_1^*, u_{n+\beta+1}=\bar{u}_{\beta+1}=\csc\theta\sec\theta F T u_1^*, \dots, u_{n+2\beta}=\bar{u}_{2\beta}=\csc\theta\sec\theta F T u_\beta^*\}$.

Therefore, we are able to express the following relationship:

$$\begin{aligned}\sum_{i=0}^{p}\sum_{j=p+1}^{n}\{2\|h(u_i,u_j)\|^2 - g(h(u_i,u_i),h(u_j,u_j))\} &= \sum_{r=n+1}^{p+2q+1}\sum_{i=0}^{p}\sum_{j=p+1}^{n}(h_{ij}^r)^2 \\ &+ \sum_{i=0}^{p}\sum_{j=p+1}^{n}\{\|h(u_i,u_j)\|^2 - g(h(u_i,u_i),h(u_j,u_j))\}.\end{aligned} \tag{34}$$

Applying Gauss Equation (12) for a sphere of odd dimension,

$$\begin{aligned}\sum_{i=0}^{p}\sum_{j=p+1}^{n}\{2\|h(u_i,u_j)\|^2 - g(h(u_i,u_i),h(u_j,u_j))\} &= \sum_{r=n+1}^{p+2q+1}\sum_{i=0}^{p}\sum_{j=p+1}^{n}(h_{ij}^r)^2 \\ &+ \sum_{i=0}^{p}\sum_{j=1}^{q}g(R(u_i,u_j)u_i,u_j) - \sum_{i=0}^{p}\sum_{j=1}^{q}g(\bar{R}(u_i,u_j)u_i,u_j).\end{aligned} \tag{35}$$

On making use of formula (9) for an odd-dimensional sphere,

$$\sum_{i=0}^{p}\sum_{j=p+1}^{n}\{2\|h(u_i,u_j)\|^2 - g(h(u_i,u_i),h(u_j,u_j))\} = \sum_{r=n+1}^{p+2q+1}\sum_{i=0}^{p}\sum_{j=p+1}^{n}(h_{ij}^r)^2 - (p+1)q - (p+1)\sum_{j=1}^{q}\beta(u_j,u_j) - q\sum_{i=0}^{p}\beta(u_i,u_i) + \sum_{i=0}^{p+1}\sum_{j=1}^{q} g(R(u_i,u_j)u_i,u_j). \tag{36}$$

From Equation (18), for the submanifold $N_T^p \times_f N_\theta^q$ of $S^{2(\frac{p}{2}+q)+1}$,

$$\begin{aligned}\sum_{i=0}^{p}\sum_{j=1}^{q} g(R(u_i,u_j)u_i,u_j) &= \sum_{i=0}^{p}\sum_{j=1}^{q}\frac{H^f(u_i,u_i)}{f}g(u_j,u_j)\\ &+\sum_{i=0}^{p}\sum_{j=1}^{q}\{2g(u_i,u_i)g(u_j,u_j)\\ &-2\eta(u_i)\eta(u_i)g(u_j,u_j) - g(u_j,\phi u_i)g(u_j,u_j)\}\\ &=\frac{q}{f}\sum_{i=0}^{p} g(\nabla_{u_i}\nabla f,u_i) + 2(p+1)q - 2q.\end{aligned} \tag{37}$$

Ultimately, the subsequent equation is obtained.

$$\sum_{i=0}^{p}\sum_{j=1}^{q} g(R(u_i,u_j)u_i,u_j) = \frac{q}{f}\sum_{i=0}^{p} g(\nabla_{u_i}\nabla f,u_i) + 2pq. \tag{38}$$

Initially, the computation of the term $\Delta f$ is performed, which is the Laplacian of $f$, resulting in the following derivation.

$$\Delta f = -\sum_{k=1}^{n} g(\nabla_{u_k}\nabla f,u_k) = -\sum_{i=0}^{p} g(\nabla_{u_i}\nabla f,u_i) - \sum_{j=1}^{q} g(\nabla_{u_j^*}\nabla f,u_j^*) \tag{39}$$

By utilizing the adapted orthonormal frame, the components of $N_\theta^q$ can be expressed as follows:

$$\Delta f = -\sum_{i=0}^{p} g(\nabla_{u_i}\nabla f,u_i) - \sum_{j=1}^{\beta} g(\nabla_{u_j^*}\nabla f,u_j^*) - \sec^2\theta\sum_{j=1}^{\beta} g(\nabla_{Tu_j^*}\nabla f,Tu_j^*). \tag{40}$$

Since $N_T^p$ is totally geodesic in $M^n$ and $\nabla f \in TN_T$, we obtain

$$\Delta f = -\frac{1}{f}\sum_{j=1}^{\beta}(g(u_j^*,u_j^*) + \sec^2\theta g(Tu_j^*,Tu_j^*))\|\nabla f\|^2 - \sum_{i=0}^{p} g(\nabla_{u_i}\nabla f,u_i) - q\eta(\nabla f), \tag{41}$$

or

$$\frac{\Delta f}{f} = -q\|\nabla f\|^2 - \frac{1}{f}\sum_{i=0}^{p} g(\nabla_{u_i}\nabla f,u_i) - \frac{q}{f}\eta(\nabla f). \tag{42}$$

Making use of (19), we find that

$$\frac{1}{f}\sum_{i=0}^{p} g(\nabla_{u_i}\nabla f,u_i) = -\Delta(lnf) + (1-q)\|\nabla lnf\|^2 - \frac{q}{f}\eta(\nabla f) \tag{43}$$

or

$$\frac{q}{f}\sum_{i=0}^{p} g(\nabla_{u_i}\nabla f, u_i) = -q\Delta(lnf) + q(1-q)\|\nabla lnf\|^2 - \frac{q^2}{f}\eta(\nabla f). \tag{44}$$

Substituting the aforementioned value into Equation (38), we obtain

$$\sum_{i=0}^{p}\sum_{j=1}^{q} R((u_i,u_j)u_i,u_j) = -q\Delta lnf + q(1-q)\|\nabla lnf\|^2 + 2pq. \tag{45}$$

Therefore, by Equation (36),

$$\begin{aligned}\sum_{i=0}^{p}\sum_{j=p+1}^{n}\{2\|h(u_i,u_j)\|^2 - g(h(u_i,u_i),h(u_j,u_j))\} &= \sum_{r=n+1}^{p+2q+1}\sum_{i=0}^{p}\sum_{j=p+1}^{n}(h_{ij}^r)^2 \\ &- (p+1)q - (p+1)\sum_{j=1}^{q}\beta(u_j,u_j) - q\sum_{i=0}^{p}\beta(u_i,u_i) \\ &- q\Delta lnf + q(1-q)\|\nabla lnf\|^2 + 2pq - \frac{q^2}{f}\eta(\nabla f),\end{aligned} \tag{46}$$

or equivalently,

$$\begin{aligned}\sum_{i=0}^{p}\sum_{j=p+1}^{n}\{2\|h(u_i,u_j)\|^2 - g(h(u_i,u_i),h(u_j,u_j))\} &= \sum_{r=n+1}^{p+2q+1}\sum_{i=0}^{p}\sum_{j=p+1}^{n}(h_{ij}^r)^2 \\ &- (1-p)q - (p+1)\sum_{j=1}^{q}\beta(u_j,u_j) - q\sum_{i=0}^{p}\beta(u_i,u_i) \\ &- q\Delta lnf + q(1-q)\|\nabla lnf\|^2 - \frac{q^2}{f}\eta(\nabla f),\end{aligned} \tag{47}$$

Now, let $\omega_1 = u_\alpha (1 \leq \alpha \leq p+1)$ and $\omega_3 = u_\beta^* (1 \leq \beta \leq q)$

$$\begin{aligned}\sum_{r=n+1}^{p+2q+1}\sum_{i=0}^{p}\sum_{j=p+1}^{n}(h_{ij}^r)^2 &= \sum_{r=n+1}^{p+2q+1}\sum_{i=0}^{p}\sum_{j=p+1}^{n} g(h(u_i,u_j^*),\bar{u}_r^*)^2 \\ &= \sum_{i=0}^{p}\sum_{j,r=1}^{\beta}\{g(h(u_i,u_j^*),\csc\theta F u_r^*)^2 \\ &+ g(h(u_i,u_j^*),\csc\theta\sec\theta FTu_r^*)^2\} \\ &= \sum_{i=0}^{p}\sum_{j,r=1}^{\beta}\{g(h(u_i,u_j^*),\csc\theta F u_r^*)^2 \\ &+ g(h(u_i,u_j^*),\csc\theta\sec\theta FTu_r^*)^2\} \\ &+ \sum_{i=0}^{\alpha}\sum_{j,r=1}^{\beta}\{g(\phi u_i,u_j^*),\csc\theta F u_r^*)^2 \\ &+ g(h)\phi u_i,u_j^*),\csc\theta\sec\theta FTu_r^*)^2\}.\end{aligned} \tag{48}$$

Applying Lemma 4 to the aforementioned equation yields

$$\sum_{r=n+1}^{p+2q+1}\sum_{i=0}^{p}\sum_{j=p+1}^{n}(h_{ij}^{r})^2 = (\csc^2\theta+\cot^2\theta)\sum_{i=1}^{\alpha+1}\sum_{j=1}^{\beta}(u_i lnf)^2 g(u_j^*,u_j^*)^2 + (\csc^2\theta+\cot^2\theta)\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}(\phi u_i lnf)^2 g(u_j^*,u_j^*)^2 \tag{49}$$

or equivalently,

$$\sum_{r=n+1}^{p+2q+1}\sum_{i=0}^{p}\sum_{j=p+1}^{n}(h_{ij}^{r})^2 = q(\csc^2\theta+\cot^2\theta)\|\nabla lnf\|^2. \tag{50}$$

Using Equations (47) and (50), we can see that

$$\sum_{i=0}^{p}\sum_{j=p+1}^{n}\{2\|h(u_i,u_j)\|^2 - g(h(u_i,u_i),h(u_j,u_j))\} - pq = q(\csc^2\theta+\cot^2\theta + 1 - q)\|\nabla lnf\|^2 - q\Delta lnf - q - (p+1)\sum_{j=1}^{q}\beta(u_j,u_j) - q\sum_{i=1}^{p}\beta(u_i,u_i) - \frac{q^2}{f}\eta(\nabla f). \tag{51}$$

Assuming that condition (33) is satisfied, we can derive the following inequality:

$$\sum_{i=0}^{p}\sum_{j=p+1}^{n}\{2\|h(u_i,u_j)\|^2 - g(h(u_i,u_i),h(u_j,u_j))\} < pq. \tag{52}$$

Using Lemma 5 on the odd-dimensional sphere with $c = 1$ leads us to the final conclusion of our theorem. □

## 4. Conclusions

In the context of Riemannian manifolds, there are two well-known types of differentiable connections: Levi–Civita connections and semi-symmetric metric connections. These connections exhibit fundamental differences, and substantial effort has been devoted to comparing and contrasting the geometry of submanifolds in relation to the Levi–Civita connection and the semi-symmetric metric connection. While the Levi–Civita connection has been extensively studied in the literature for the homology of warped product submanifolds, the homology of these submanifolds with semi-symmetric connections remains unknown. In light of this, we investigate the homology and stable currents of semi-invariant warped product submanifolds of Sasakian manifolds with a semi-symmetric connection in this paper. It is our hope that our study will spur further research into generalized warped product submanifolds and their topological properties.

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