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Saturated Varieties of Semigroups

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Abstract: The complete characterization of saturated varieties of semigroups remains an unsolved problem. The primary objective of this paper is to make significant progress in this direction. We initially demonstrate that the variety of semigroups defined by the identity $axy = ayxa$ is saturated. The next main result establishes that the variety of semigroups determined by the identity $axy = ayax$ is saturated. Finally, we show that medial semigroups satisfying the identity $xy = xy^n$, where $n \geq 2$, are also saturated. These results collectively lead to the conclusion that epis from these saturated varieties are onto. This paper thus offers substantial progress towards the comprehensive characterization of saturated varieties of semigroups.

Keywords: dominions; epimorphisms; zigzag equations; saturated; identity; variety

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1. Introduction and Preliminaries

The study of semigroups and their varieties has been the subject of extensive research in algebraic structures. In particular, the concept of a saturated variety of semigroups is interesting, and its complete characterization remains an open problem. This paper aims to make a significant contribution towards the understanding of saturated varieties by focusing on some specific varieties of semigroups determined by certain identities, such as $axy = ayxa$ and $axy = ayax$, and proving that they are saturated. Furthermore, it is known that in the category of semigroups, epis are not onto in general. Therefore, by finding saturated varieties of semigroups, we can determine subvarieties within the variety of all semigroups for which epis are onto.

Consider semigroups A and B with A as a subsemigroup of a semigroup B. Following Howie and Isbell [1], the *dominion* of A in B is denoted as $Dom_B(A)$ and is defined as

$$Dom_B(A) = \{b \in B : \forall\, T, \forall\, \delta, \eta : B \to T, \text{ if } \delta|_A = \eta|_A \text{ then } b\delta = b\eta\}$$

i.e., A dominates an element b of B if, for any semigroup T and for any two semigroup morphisms δ, η of B that coincide on elements of A, they also coincide on b. It can be easily verified that $Dom_B(A)$ is a closure operator, i.e., $A \subseteq Dom_B(A) \subseteq B$. A subsemigroup A of a semigroup B is considered *closed* in B if $Dom_B(A) = A$, and it is considered *absolutely closed* if it is closed in every containing semigroup B. On the other hand, A is said to be saturated if $Dom_B(A) \neq B$ for every properly containing semigroup B.

A variety $\mathcal{V}$ of semigroups is considered saturated if each member of $\mathcal{V}$ is saturated. Additionally, $\mathcal{V}$ is epimorphically closed if, for any semigroup $B \in \mathcal{V}$ and an epimorphism $\alpha : B \to T$, it implies that T also belongs to $\mathcal{V}$. Equivalently, for any semigroup A of a semigroup B where $A \in \mathcal{V}$ and $Dom_B(A) = B$, it implies $B \in \mathcal{V}$.

It is evident that every absolutely closed variety is saturated, and likewise, every saturated variety is epimorphically closed. But the converse is not true, as Higgins ([2]

Theorem 4) has shown that generalized inverse semigroups are saturated. This result in particular shows that the variety of rectangular bands is saturated but not absolutely closed as follows from Howie ([1] Theorem 2.9).

Khan [3] has proved that the variety of permutative semigroups is epimorphically closed. However, it is important to highlight that not all epimorphically closed varieties are saturated. For example, the variety of commutative semigroups is not saturated, as demonstrated by the fact that infinite monogenic semigroup is epimorphically embeddable in an infinite cyclic group ([4], Chap VIII, Ex. 6(a)).

Throughout this paper, we denote mappings to the right of their arguments. Let $\delta : B \to T$ be a semigroup morphism. We say δ is an *epimorphism* (epis for short) if, for every pair of morphisms $\eta, \theta : T \to S$, $\delta\eta = \delta\theta$ implies $\eta = \theta$. It is straightforward to verify that a morphism $\delta : B \to T$ is an epimorphism if and only if $Dom_T(B\delta) = T$. While every surjective morphism is an epimorphism, the converse is not true in general. It depends on the category under consideration. For example, in the category of groups, epimorphisms are indeed surjective. However, this does not hold true for all categories. In the category of semigroups, there exist non-surjective epimorphisms. For instance, consider the mapping $i : (0, 1] \to (0, \infty)$ regarding both the intervals as multiplicative semigroups.

Isbell [5] presented one of the most insightful characterizations of semigroup dominions, known as Isbell's Zigzag Theorem, and is stated as follows:

Theorem 1 ([5] (Theorem 2.3))**.** *Let A be subsemigroup of a semigroup B and $d \in B$. Then $d \in Dom_B(A)$ if and only if $d \in A$ or there exists a system of equalities for d as follows:*

$$\begin{aligned} d &= a_0 y_1 & a_0 &= t_1 a_1 \\ a_1 y_1 &= a_2 y_2 & t_1 a_2 &= t_2 a_3 \\ &\vdots & &\vdots \\ a_{2i-1} y_i &= a_{2i} y_{i+1} & t_i a_{2i} &= t_{i+1} a_{2i+1} \quad (i = 1, 2, \cdots, m-1) \\ a_{2m-1} y_m &= a_{2m} & t_m a_{2m} &= d \end{aligned} \tag{1}$$

where $a_i \in A, (0 \leq i \leq 2m)$ and $t_i, y_i \in B, (1 \leq i \leq m)$.

The system of equalities given in (1) above is referred to as the *zigzag of length m* in B over A with value d. In whatever follows, by zigzag equations, we shall mean equations of type (1). Furthermore, the bracketed statements shall mean the statements dual to each other.

The following theorems are from Khan [6].

Theorem 2 ([6] (Result 3))**.** *Let A be a subsemigroup of a semigroup B and let $d \in Dom_B(A) \setminus A$. If (1) is a zigzag of minimal length m in B over A with value d, then $x_i, y_i \in B \setminus A$ for all $i = 1, 2, \ldots m$.*

Theorem 3 ([6] (Result 4))**.** *Let A be a subsemigroup of a semigroup B such that $Dom_B(A) = B$. Then, for any $d \in B \setminus A$ and any positive integer k, if (1) is a zigzag of minimal length m in B over A with value d, then there exist $b_1, b_2, \ldots, b_k \in A$ and $d_k \in B \setminus A$ such that $d = b_1 b_2 \ldots b_k d_k$ $[d = d_k b_k b_{k-1} \cdots b_1]$.*

2. Saturated Varieties of Semigroups

Definition 1. *Let u be any word. The* content *of u is the (necessarily finite) set of all variables appearing in u, and will be denoted by $C(u)$.*

Definition 2. *A* semigroup identity *$u = v$ is the formal equality of two words u and v formed by letters over an alphabet set X.*

Definition 3. *An identity* $u(x_1, x_2, \ldots, x_n) = v(x_1, x_2, \ldots, x_n)$ *in the variables* $x_1, x_2, \ldots, x_n$ *is called* homotypical *if* $C(u) = C(v)$ *and* heterotypical *if* $C(u) \neq C(v)$.

Definition 4. *A semigroup B is said to satisfy an identity if for every substitution of elements from B for the letters forming the words of the identity, the resulting words are equal in B.*

Definition 5. *We call semigroup B a* medial semigroup *if it satisfies the identity* $pqrs = prsq$ *for every* $p, q, r, s \in B$.

Definition 6. *An identity* $u = v$ *is said to be* preserved under epis *if, for any semigroups A and B,* $Dom_B(A) = B$ *and A satisfies* $u = v$*, which implies B also satisfies* $u = v$.

Remark 1. *Let* $\mathcal{A}$ *be a class of semigroups, and let* $\delta : A \to T$ *be any epimorphism, where* $A \in \mathcal{V}$ *and T is any semigroup. Then,* δ *is onto if* $A\delta \in \mathcal{V}$ *(i.e., morphically closed), and* $Dom_B(A) \neq B$ *for any semigroup B properly containing A.*

In [7], the sufficient condition for a homotypical variety of semigroups to be saturated is as follows.

Theorem 4 ([7] (Theorem 16))**.** *A sufficient condition for a homotypical variety* $\mathcal{V}$ *of semigroups to be saturated is that* $\mathcal{V}$ *admits an identity*

$$\phi : x_1 x_2 \cdots x_n = f(x_1, x_2, \ldots, x_n)$$

for which $|x_i|_f > 1$*, for some* $1 \leq i \leq n$*, and such that f neither begins with* x_1 *nor ends at* x_n.

Furthermore, Khan in [6] has provided necessary and sufficient information for a permutative variety of semigroups to be saturated. Thus, we have the following.

Theorem 5 ([6] (Theorem 5.4))**.** *A permutative variety* $\mathcal{V}$ *is saturated if and only if it admits an identity I such that*

(i) I is not a permutation identity, and
(ii) At least one side of I has no repeated variable.

Inverse semigroups, generalized inverse semigroups, and locally inverse semigroups are some well known examples of saturated classes of semigroups. Alam [8] has demonstrated that certain classes of permutative semigroups, which satisfy specific homotypical identities, are also saturated. Moreover, Shah et al. [9] have shown that classes of structurally (n, m)-generalized inverse semigroups are saturated as well. Furthermore, Alam et al. [10] have extended Howie's and Isbell's result to show that any H-commutative semigroup satisfying the minimum condition on principal ideals is saturated. In a related direction, Ahanger et al. [11] have established the saturation of generalized left [right] regular semigroups. Additionally, Alam et al. [12] have identified some saturated classes of H-commutative, left [right] regular semigroups, medial semigroups, and paramedial semigroups Recently, in [13] the authors have identified several saturated classes of structurally regular semigroups. However, the complete identification of all saturated varieties of semigroups is an open problem. Solving this question holds significant importance in the realm of identifying saturated semigroup varieties. Therefore, it is interesting to characterize saturated homotypical varieties of semigroups that do not belong to the class of varieties described in Theorem 4.

In Lemmas 1–3, let A and B be any semigroups with A as a subsemigroup of B satisfying $Dom_B(A) = B$. Consider $d \in B \setminus A$ with a zigzag of type (1) in B over A, with value d of minimal length m. From Theorem 3 together with Theorem 2, it follows that for every $i = 1, 2, \ldots, m$, there exists $x'_i, y'_i \in B \setminus A$ and $u_1, u_2, v_{2i-1} \in A$ such that

$$y_1 = u_1 u_2 y'_1, \quad x_i = x'_i v_{2i-1} \tag{2}$$

In order to prove Theorem 6, we begin by proving the following lemmas in which A satisfies the given identity

$$axy = ayxa \tag{3}$$

Lemma 1. *For all* $k = 1, \ldots, m-1$,

$$d = x_{k+1}a_{2k+1}u_2\left(\prod_{i=1}^{k} v_{2i-1}a_{2i-1}\right)a_{2k+1}y_{k+1}.$$

Proof. We prove the lemma by using induction on k. For $k = 1$, we have

$$\begin{aligned}
d &= x_1a_1y_1 \text{ (by zigzag equations);}\\
&= x_1a_1u_1u_2y_1' \text{ (by Equation (2));}\\
&= x_1a_1u_2a_1u_1u_2y_1' \text{ (since } A \text{ satisfies identity (3));}\\
&= x_1a_1u_2a_1y_1 \text{ (by Equation (2));}\\
&= x_1a_1u_2a_2y_2 \text{ (by zigzag equations);}\\
&= x_1'v_1a_1u_2a_2y_2 \text{ (by Equation (2));}\\
&= x_1'v_1a_2a_1u_2v_1y_2 \text{ (since } A \text{ satisfies identity (3));}\\
&= x_1a_2a_1u_2v_1y_2 \text{ (by Equation (2));}\\
&= x_2a_3a_1u_2v_1y_2 \text{ (by zigzag equations);}\\
&= x_2a_3u_2v_1a_1a_3y_2 \text{ (since } A \text{ satisfies identity (3)).}
\end{aligned}$$

Thus, the result is true for $k = 1$. Assume for the sake of induction that the result is true for $k = j$, where $j < m-1$. We show that the result is also true for $k = j+1$. Now

$$\begin{aligned}
d &= x_{j+1}a_{2j+1}u_2\left(\prod_{i=1}^{j} v_{2i-1}a_{2i-1}\right)a_{2j+1}y_{j+1} \text{ (by inductive hypothesis);}\\
&= x_{j+1}a_{2j+1}u_2\left(\prod_{i=1}^{j} v_{2i-1}a_{2i-1}\right)a_{2j+2}y_{j+2} \text{ (by zigzag equations);}\\
&= x_{j+1}'v_{2j+1}a_{2j+1}u_2\left(\prod_{i=1}^{j} v_{2i-1}a_{2i-1}\right)a_{2j+2}y_{j+2} \text{ (by Equation (2));}\\
&= x_{j+1}'v_{2j+1}a_{2j+2}a_{2j+1}u_2\left(\prod_{i=1}^{j} v_{2i-1}a_{2i-1}\right)v_{2j+1}y_{j+2} \text{ (since } A \text{ satisfies identity (3));}\\
&= x_{j+1}a_{2j+2}a_{2j+1}u_2\left(\prod_{i=1}^{j} v_{2i-1}a_{2i-1}\right)v_{2j+1}y_{j+2} \text{ (by Equation (2));}\\
&= x_{j+2}a_{2j+3}a_{2j+1}u_2\left(\prod_{i=1}^{j} v_{2i-1}a_{2i-1}\right)v_{2j+1}y_{j+2} \text{ (by zigzag equations);}\\
&= x_{j+2}a_{2j+3}u_2\left(\prod_{i=1}^{j} v_{2i-1}a_{2i-1}\right)v_{2j+1}a_{2j+1}a_{2j+3}y_{j+2} \text{ (since } A \text{ satisfies identity (3));}\\
&= x_{j+2}a_{2j+3}u_2\left(\prod_{i=1}^{j+1} v_{2i-1}a_{2i-1}\right)a_{2j+3}y_{j+2}.
\end{aligned}$$

Therefore, the result is true for $k = j+1$, and hence the lemma follows. □

Lemma 2. *For all $k = 1, 2, \ldots, m-1$,*

$$d = x_{m-k} a_{2m-2k-1} u_2 \left(\prod_{i=1}^{m-k-1} v_{2i-1} a_{2i-1} \right) (a_{2m-2k} v_{2m-2k-1}) (a_{2m-2k+2} v_{2m-2k+1}) \cdots (a_{2m-2} v_{2m-3}) a_{2m}. \quad (4)$$

Proof. We will prove this by induction on k. For $k = 1$, we have

$$\begin{aligned}
d &= x_m a_{2m-1} u_2 \left(\prod_{i=1}^{m-1} v_{2i-1} a_{2i-1} \right) a_{2m} \text{ (by Lemma 1 for } k = m-1\text{)}; \\
&= x_{m-1} a_{2m-2} u_2 (\prod_{i=1}^{m-1} v_{2i-1} a_{2i-1}) a_{2m} \text{ (by zigzag equations)}; \\
&= x'_{m-1} v_{2m-3} a_{2m-2} u_2 \left(\prod_{i=1}^{m-2} v_{2i-1} a_{2i-1} \right) v_{2m-3} a_{2m-3} a_{2m} \text{ (by Equation (2))}; \\
&= x'_{m-1} v_{2m-3} a_{2m-3} u_2 \left(\prod_{i=1}^{m-2} v_{2i-1} a_{2i-1} \right) a_{2m-2} v_{2m-3} a_{2m} \text{ (since } A \text{ satisfies identity (3))}; \\
&= x_{m-1} a_{2m-3} u_2 \left(\prod_{i=1}^{m-2} v_{2i-1} a_{2i-1} \right) a_{2m-2} v_{2m-3} a_{2m} \text{ (by Equation (2))}.
\end{aligned}$$

Assume that (4) holds for $k = j < m-1$. We show that it also holds for $k = j+1$. Now

$$\begin{aligned}
d &= x_{m-j} a_{2m-2j-1} u_2 \left(\prod_{i=1}^{m-j-1} v_{2i-1} a_{2i-1} \right) (a_{2m-2j} v_{2m-2j-1}) \\
&\quad (a_{2m-2j+2} v_{2m-2j+1}) \cdots (a_{2m-2} v_{2m-3}) a_{2m} \text{ (by inductive hypothesis)}; \\
&= x_{m-j-1} a_{2m-2j-2} u_2 \left(\prod_{i=1}^{m-j-1} v_{2i-1} a_{2i-1} \right) (a_{2m-2j} v_{2m-2j-1}) \\
&\quad (a_{2m-2j+2} v_{2m-2j+1}) \cdots (a_{2m-2} v_{2m-3}) a_{2m} \text{ (by zigzag equations)}; \\
&= x'_{m-j-1} v_{2m-2j-3} a_{2m-2j-2} u_2 \left(\prod_{i=1}^{m-j-2} v_{2i-1} a_{2i-1} \right) v_{2m-2j-3} a_{2m-2j-3} z a_{2m} \\
&\quad \text{(by Equation (2) and } z = (a_{2m-2j} v_{2m-2j-1})(a_{2m-2j+2} v_{2m-2j+1}) \cdots (a_{2m-2} v_{2m-3})\text{)}; \\
&= x'_{m-j-1} v_{2m-2j-3} a_{2m-2j-3} u_2 \left(\prod_{i=1}^{m-j-2} v_{2i-1} a_{2i-1} \right) a_{2m-2j-2} v_{2m-2j-3} z a_{2m} \\
&\quad \text{(since } A \text{ satisfies identity (3))}; \\
&= x_{m-j-1} a_{2m-2j-3} u_2 \left(\prod_{i=1}^{m-j-2} v_{2i-1} a_{2i-1} \right) a_{2m-2j-2} v_{2m-2j-3} z a_{2m} \text{ (by Equation (2))}; \\
&= x_{m-j-1} a_{2m-2j-3} u_2 \left(\prod_{i=1}^{m-j-2} v_{2i-1} a_{2i-1} \right) a_{2m-2j-2} v_{2m-2j-3} \\
&\quad (a_{2m-2j} v_{2m-2j-1})(a_{2m-2j+2} v_{2m-2j+1}) \cdots (a_{2m-2} v_{2m-3}) a_{2m}.
\end{aligned}$$

Hence, the lemma follows. □

Theorem 6. *Any semigroup A satisfying the identity $axy = ayxa$ is saturated.*

Proof. Suppose, to the contrary, that A is not saturated, then there exists a semigroup B that properly contains A and satisfies $Dom_B(A) = B$. Now

$$\begin{aligned} d &= x_1 a_1 u_2 (a_2 v_1)(a_4 v_3) \cdots (a_{2m-2} v_{2m-3}) a_{2m} \text{ (by Lemma 2 for } k = m-1); \\ &= a_0 u_2 (a_2 v_1)(a_4 v_3) \cdots (a_{2m-2} v_{2m-3}) a_{2m} \text{ (by zigzag equations).} \end{aligned}$$

Thus, $d \in U$, which is a contradiction as required. □

Corollary 1. *The variety $\mathcal{V}_1 = [axy = ayxa]$ of semigroups is saturated.*

Corollary 2. *In the category of all semigroups, any epi from a semigroup $A \in \mathcal{V}_1$ is onto.*

Example 1. *Let $B = \{a_1, a_2, a_3, a_4, a_5\}$ be a five-element semigroup and $A = \{a_1, a_2, a_3, a_4\}$ be a subsemigroup of B. The Cayley's table for B is given below:*

.	a_1	a_2	a_3	a_4	a_5
a_1	a_1	a_2	a_2	a_2	a_2
a_2	a_2	a_2	a_2	a_2	a_2
a_3	a_4	a_2	a_3	a_4	a_5
a_4	a_4	a_2	a_2	a_2	a_4
a_5	a_5	a_2	a_2	a_2	a_5.

For any $a, x, y \in B$, one can easily check that $axy = ayxa$. Also, $Dom_B(A) \subsetneq B$, as $a_5 \notin Dom_B(A)$.

To prove Theorem 7, we first prove the following lemma in which A satisfies the given identity

$$axy = ayax \tag{5}$$

Lemma 3. *For all $k = 1, \ldots, m-1$,*

$$d = x_{k+1} a_{2k+1} \left(\prod_{i=1}^{k} a_{2k-(2i-1)} \right) v_1 a_{2k-1} a_{2k+1} w y_{k+1}.$$

Proof. We will prove this lemma by using induction on k. For $k = 1$, we have

$$\begin{aligned} d &= x_1 a_1 y_1 \text{ (by zigzag equations);} \\ &= x_1 a_1 u_1 u_2 y_1' \text{ (by Equation (2));} \\ &= x_1 a_1 u_2 u_1 a_1 u_1 u_2 y_1' \text{ (since } A \text{ satisfies identity (5));} \\ &= x_1 a_1 u_2 u_1 a_2 y_2 \text{ (by Equation (2) and zigzag equations);} \\ &= x_1' v_1 a_1 w a_2 y_2 \text{ (by Equation (2), where } w = u_2 u_1); \\ &= x_1' v_1 a_2 v_1 a_1 w y_2 \text{ (since } A \text{ satisfies identity (5));} \\ &= x_1 a_2 v_1 a_1 w y_2 \text{ (by Equation (2));} \\ &= x_2 (a_3 v_1 a_1) w y_2 \text{ (by zigzag equations);} \\ &= x_2 a_3 a_1 v_1 a_1 a_3 w y_2 \text{ (since } A \text{ satisfies identity (5)).} \end{aligned}$$

Thus the lemma holds for $k = 1$. Assume that it holds for $k = l < m-1$. We will prove that it also holds for $k = l+1$. Now

$$\begin{aligned} d &= x_{l+1} a_{2l+1} \left(\prod_{i=1}^{l} a_{2l-(2i-1)} \right) v_1 a_{2l-1} a_{2l+1} w y_{l+1}; \\ &= x_{l+1}' v_{l+1} a_{2l+1} \left(\prod_{i=1}^{l} a_{2l-(2i-1)} \right) v_1 a_{2l-1} a_{2l+1} w y_{l+1} \text{ (by Equation (2));} \end{aligned}$$

$$= x'_{l+1}v_{l+1}\left(\prod_{i=1}^{l} a_{2l-(2i-1)}\right)v_1a_{2l-1}a_{2l+1}wv_{l+1}a_{2l+1}y_{l+1} \text{ (since } A \text{ satisfies identity (5));}$$

$$= x'_{l+1}v_{l+1}\left(\prod_{i=1}^{l} a_{2l-(2i-1)}\right)v_1a_{2l-1}a_{2l+1}wv_{l+1}a_{2l+2}y_{l+2} \text{ (by zigzag equations);}$$

$$= x'_{l+1}v_{l+1}a_{2l+2}\left(\prod_{i=1}^{l} a_{2l-(2i-1)}\right)v_1a_{2l-1}a_{2l+1}wy_{l+2} \text{ (since } A \text{ satisfies identity (5));}$$

$$= x_{l+2}a_{2l+3}a_{2l-1}\left(\prod_{i=2}^{l} a_{2l-(2i-1)}\right)v_1a_{2l-1}a_{2l+1}wy_{l+2} \text{ (by zigzag equations);}$$

$$= x_{l+2}a_{2l+3}a_{2l+1}a_{2l-1}\left(\prod_{i=2}^{l} a_{2l-(2i-1)}\right)v_1a_{2l+1}a_{2l+3}wy_{l+2} \text{ (since } A \text{ satisfies identity (5));}$$

$$= x_{l+2}a_{2l+3}\left(\prod_{i=1}^{l+1} a_{2(l+1)-(2i-1)}\right)v_1a_{2l+1}a_{2l+3}wy_{l+2},$$

as required. □

Theorem 7. *Any semigroup A satisfying the identity $axy = ayax$ is saturated.*

Proof. Suppose on the contrary that A satisfying the identity $axy = ayax$ is not saturated. Therefore, there exists a semigroup B containing A properly such that $Dom_B(A) = B$. Now, we have

$$d = x_ma_{2m-1}\left(\prod_{i=1}^{m-1} a_{2(m-1)-(2i-1)}\right)v_1a_{2m-3}a_{2m-1}wy_m \text{ (by Lemma 3 for } k = m-1\text{);}$$

$$= x_ma_{2m-1}\left(\prod_{i=1}^{m-1} a_{2(m-1)-(2i-1)}\right)v_1a_{2m-3}wa_{2m-3}a_{2m-1}y_m \text{ (since } A \text{ satisfies identity (5);}$$

$$= x_ma_{2m-1}\left(\prod_{i=1}^{m-1} a_{2(m-1)-(2i-1)}\right)u \text{ (where } u = v_1a_{2m-3}wa_{2m-3}a_{2m-1}y_m\text{);}$$

$$= x_{m-1}a_{2m-2}a_{2m-3}\left(\prod_{i=2}^{m-1} a_{2(m-1)-(2i-1)}\right)u \text{ (by zigzag equations);}$$

$$= x'_{m-1}v_{2m-3}a_{2m-2}a_{2m-3}\left(\prod_{i=2}^{m-1} a_{2(m-1)-(2i-1)}\right)u \text{ (by Equation (2));}$$

$$= x'_{m-1}v_{2m-3}a_{2m-3}v_{2m-3}a_{2m-2}\left(\prod_{i=2}^{m-1} a_{2(m-1)-(2i-1)}\right)u \text{ (since } A \text{ satisfies identity (5));}$$

$$= x_{m-1}a_{2m-3}v_{2m-3}a_{2m-2}\left(\prod_{i=2}^{m-1} a_{2(m-1)-(2i-1)}\right)u \text{ (by Equation (2));}$$

$$= x_{m-2}a_{2m-4}v_{2m-3}a_{2m-2}\left(\prod_{i=2}^{m-1} a_{2(m-1)-(2i-1)}\right)u \text{ (by zigzag equations);}$$

$$= x'_{m-2}(v_{2m-5})(a_{2m-4}v_{2m-3}a_{2m-2})(a_{2m-5})\left(\prod_{i=3}^{m-1} a_{2(m-1)-(2i-1)}\right)u \text{ (by Equations (2));}$$

$$= x'_{m-2}v_{2m-5}a_{2m-5}v_{2m-5}a_{2m-4}v_{2m-3}a_{2m-2}\left(\prod_{i=3}^{m-1} a_{2(m-1)-(2i-1)}\right)u$$

(since A satisfies identity (5));

$$= x_{m-2}a_{2m-5}v_{2m-5}a_{2m-4}v_{2m-3}a_{2m-2}\left(\prod_{i=3}^{m-1} a_{2(m-1)-(2i-1)}\right)u \text{ (by Equation (2));}$$

$$
\begin{aligned}
&\vdots\\
&= x_2a_3v_3a_4v_5a_6\cdots v_{2m-3}a_{2m-2}a_1u;\\
&= x_1a_2v_3a_4v_5a_6\cdots v_{2m-3}a_{2m-2}a_1u \ \text{(by zigzag equations);}\\
&= x_1'(v_1)(a_2v_3a_4v_5a_6\cdots v_{2m-3}a_{2m-2})(a_1)u \ \text{(by Equation (2));}\\
&= x_1'v_1a_1v_1a_2v_3a_4v_5a_6\cdots v_{2m-3}a_{2m-2}u \ \text{(since } A \text{ satisfies identity (5));}\\
&= x_1a_1v_1a_2v_3a_4v_5a_6\cdots v_{2m-3}a_{2m-2}u \ \text{(by Equation (2));}\\
&= a_0v_1a_2v_3a_4v_5a_6\cdots v_{2m-3}a_{2m-2}u,
\end{aligned}
$$

which is in A. Thus $d \in A$, a contradiction as required. □

Corollary 3. *The variety $\mathcal{V}_2 = [axy = ayax]$ of semigroups is saturated.*

Corollary 4. *In the category of all semigroups, any epi from a semigroup $A \in \mathcal{V}_2$ is onto.*

In Lemma 4, consider A as a medial semigroup and B as an arbitrary semigroup with A being a subsemigroup of B, such that $Dom_B(A) = B$. Take any $d \in B \setminus A$ and let (1) be a zigzag in B over A with value d of minimal length m. Then by Theorem 2, $x_i, y_i \in B \setminus A$ for all $i = 1, 2, \ldots, m$. Now, by Theorem 3, there exists $x_i', y_i' \in B \setminus A$ and $u_i, v_i \in A$ such that

$$x_i = x_i'u_i, \ \ y_1 = v_iy_i' \tag{6}$$

To prove Theorem 8, we begin by proving the following lemma, wherein A satisfies the given identity

$$ab = ab^n \text{ with } n \geq 2, \ (n \in \mathbb{N}) \tag{7}$$

Lemma 4. *For all $k = 1, \ldots, m$,*

$$d = x_k\left(\prod_{i=1}^{k} a_{2i-1}^{n-1}\right)a_{2k-1}y_k.$$

Proof. We prove the lemma by using induction on k. For $k = 1$, we have

$$
\begin{aligned}
d &= x_1a_1y_1 \ \text{(by zigzag equations);}\\
&= x_1'u_1a_1y_1 \ \text{(by Equation (6));}\\
&= x_1'u_1a_1^ny_1 \ \text{(since } A \text{ satisfies (7));}\\
&= x_1a_1^{n-1}a_1y_1 \ \text{(by Equation (6)).}
\end{aligned}
$$

Thus, the result holds for $k = 1$. Assume for the sake of induction that the result holds for $k = l$, where $l < m$. Now we show that the result holds for $k = l + 1$. We have

$$
\begin{aligned}
d &= x_l\left(\prod_{i=1}^{l} a_{2l-1}^{n-1}\right)a_{2l-1}y_l \ \text{(by inductive hypothesis);}\\
&= x_l\left(\prod_{i=1}^{l} a_{2l-1}^{n-1}\right)a_{2l}y_{l+1} \ \text{(by zigzag equations);}\\
&= x_lu_l'\left(\prod_{i=1}^{l} a_{2l-1}^{n-1}\right)a_{2l}v_{l+1}y_{l+1}' \ \text{(by Equation (6));}\\
&= x_lu_l'a_{2l}\left(\prod_{i=1}^{l} a_{2l-1}^{n-1}\right)vl + 1y_{l+1}' \ \text{(since } A \text{ is medial);}
\end{aligned}
$$

$$
\begin{aligned}
&= x_l a_{2l} \left(\prod_{i=1}^{l} a_{2l-1}^{n-1} \right) y_{l+1} \text{ (by Equation (6));} \\
&= x_{l+1} a_{2l+1} \left(\prod_{i=1}^{l} a_{2l-1}^{n-1} \right) y_{l+1} \text{ (by zigzag equations);} \\
&= x'_{l+1} u_{l+1} a_{2l+1} \left(\prod_{i=1}^{l} a_{2l-1}^{n-1} \right) v_{l+1} y'_{l+1} \text{ (by Equation (6));} \\
&= x'_{l+1} u_{l+1} a_{2l+1}^{n} \left(\prod_{i=1}^{l} a_{2l-1}^{n-1} \right) v_{l+1} y'_{l+1} \text{ (since } A \text{ satisfies (7));} \\
&= x'_{l+1} u_{l+1} \left(\prod_{i=1}^{l} a_{2l-1}^{n-1} \right) a_{2l+1}^{n} v_{l+1} y'_{l+1} \text{ (since } A \text{ is medial);} \\
&= x_{l+1} \left(\prod_{i=1}^{l} a_{2l-1}^{n-1} \right) a_{2l+1}^{n} y_{l+1} \text{ (by Equation (6));} \\
&= x_{l+1} \left(\prod_{i=1}^{l+1} a_{2l-1}^{n-1} \right) a_{2l+1} y_{l+1} \text{ (since } A \text{ satisfies (7)).}
\end{aligned}
$$

Therefore, the lemma holds for $k = l + 1$, and hence the lemma follows. □

Theorem 8. *Medial semigroups satisfying the identity $ab = ab^n$ for $n \geq 2$ $(n \in \mathbb{N})$ are saturated.*

Proof. Consider a medial subsemigroup A of a semigroup B that satisfies the identity $xy = xy^n$ with $n \geq 2$ $(n \in \mathbb{N})$. On the contrary, let us assume that A is not saturated. Therefore, there exists a semigroup B that properly contains A such that $Dom_B(A) = B$. Now, we have

$$
\begin{aligned}
d &= x_m \left(\prod_{i=1}^{m} a_{2i-1}^{n-1} \right) a_{2m} \text{ (by Lemma 4 for } k = m\text{);} \\
&= x_m \left(\prod_{i=1}^{m-1} a_{2i-1}^{n-1} \right) a_{2m-1}^{n-1} a_{2m}; \\
&= x'_m u_m \left(\prod_{i=1}^{m-1} a_{2i-1}^{n-1} \right) a_{2m-1}^{n-1} a_{2m} \text{ (by Equation (6));} \\
&= x'_m u_m a_{2m-1} \left(\prod_{i=1}^{m-1} a_{2i-1}^{n-1} \right) a_{2m-1}^{n-2} a_{2m} \text{ (since } A \text{ is medial);} \\
&= x_m a_{2m-1} \left(\prod_{i=1}^{m-1} a_{2i-1}^{n-1} \right) a_{2m-1}^{n-2} a_{2m} \text{ (by Equation (6));} \\
&= x_{m-1} a_{2m-2} \left(\prod_{i=1}^{m-1} a_{2i-1}^{n-1} \right) a_{2m-1}^{n-2} a_{2m} \text{ (by zigzag equations);} \\
&\vdots \\
&= x_1 (\prod_{i=2}^{m} a_{2i-2}) a_1 \left(\prod_{i=1}^{m-1} a_{2i-1}^{n-2} \right) a_{2m}; \\
&= x'_1 u_1 (\prod_{i=2}^{m} a_{2i-2}) a_1 \left(\prod_{i=1}^{m-1} a_{2i-1}^{n-2} \right) a_{2m} \text{ (by Equation (6));} \\
&= x'_1 u_1 a_1 \left(\prod_{i=2}^{m} a_{2i-2} \right) (\prod_{i=1}^{m-1} a_{2i-1}^{n-2}) a_{2m} \text{ (since } A \text{ is medial);}
\end{aligned}
$$

$$= x_1 a_1 \left(\prod_{i=2}^{m} a_{2i-2}\right)\left(\prod_{i=1}^{m-1} a_{2i-1}^{n-2}\right) a_{2m} \text{ (by Equation (6))};$$
$$= a_0 \left(\prod_{i=2}^{m} a_{2i-2}\right)\left(\prod_{i=1}^{m-1} a_{2i-1}^{n-2}\right) a_{2m} \text{ (by zigzag equations)};$$
$$= \left(\prod_{i=1}^{m} a_{2i-2}\right)\left(\prod_{i=1}^{m-1} a_{2i-1}^{n-2}\right) a_{2m};$$

this is, in A, a contradiction. Thus, $DomB(A) \neq B$ and, so, A is saturated. □

Corollary 5. *The variety $\mathcal{V}_3 = [ab = ab^n, n \geq 2(n \in \mathbb{N}), pqrs = prsq]$ is saturated.*

Corollary 6. *In the category of all semigroups, any epi from a semigroup $A \in \mathcal{V}_3$ is onto.*

Example 2. *Let $A = \{a, b, c, d\}$ be subsemigroup of semigroup $B = \{a, b, c, d, e\}$. The Cayley's table for S is given below:*

.	a	b	c	d	e
a	a	b	b	b	b
b	b	b	b	b	b
c	d	b	c	d	e
d	d	b	b	b	d
e	d	b	b	b	e

For any $x, y \in B$, one can easily check that $xy = xy^n$. Clearly, $Dom_B(A) \subsetneq B$ as $e \notin Dom_B(A)$.

3. Conclusions

Higgins [7] gave a sufficient condition for a homotypical variety of semigroups to be saturated. The present paper contributes significantly to the field of saturated varieties of semigroups by successfully determining several such saturated varieties. Specifically, this paper focuses on uncovering homotypical varieties of semigroups that fall outside of the categories covered by previous results. The revelation that epis within these saturated varieties are onto offers valuable insights into the behavior of mappings in these semigroups. This highlights the significance of saturated varieties in comprehending the surjective properties of semigroup morphisms. Notably, the study of homomorphisms between semigroups holds great importance in diverse areas like signal processing and image recognition, making the exploration of homotypical varieties and their saturated counterparts particularly relevant.

This paper marks a significant advancement in characterizing saturated varieties of semigroups. However, ample work still remains in the exploration of other subvarieties within the variety of all semigroups and their saturation properties. The study also anticipates delving into broader classes of semigroups for which epis are onto. To this end, the following open problems stand out:

(i) Are structurally (n, m)-locally inverse semigroups saturated or not?
(ii) Given the current lack of a complete classification of all saturated classes of semigroups, further investigations can be directed in this area.

These open questions present exciting opportunities for extending the understanding of saturated varieties and their implications in semigroup theory.

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