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Novel Approach to Multi-Criteria Decision-Making Based on the n,mPR-Fuzzy Weighted Power Average Operator

Tareq Hamadneh [1], Hariwan Z. Ibrahim [2,*], Mayada Abualhomos [3], Maha Mohammed Saeed [4], Gharib Gharib [5], Maha Al Soudi [6] and Abdallah Al-Husban [7]

1 Department of Mathematics, Faculty of Science, Al Zaytoonah University of Jordan, Amman 11733, Jordan; t.hamadneh@zuj.edu.jo
2 Department of Mathematics, Faculty of Education, University of Zakho, Zakho 42002, Iraq
3 Department Basic Science, Applied Science Private University Applied Science Research Center (ASRC), Amman 11733, Jordan; abuhomos@asu.edu.jo
4 Department of Mathematics, Faculty of Sciences, King Abdulaziz University, Jeddah 21589, Saudi Arabia; mmmohammed@kau.edu.sa
5 Department of Mathematics, Faculty of Science, Zarqa University, Zarqa 13110, Jordan; ggharib@zu.edu.jo
6 Department of Basic Scientific Sciences, Applied Science Private University, Amman 11931, Jordan; m_alsoudi@asu.edu.jo
7 Department of Mathematics, Faculty of Science and Technology, Irbid National University, Irbid 21110, Jordan; dralhosban@inu.edu.jo
* Correspondence: hariwan_math@yahoo.com

Abstract: A significant addition to fuzzy set theory for expressing uncertain data is an n,m-th power root fuzzy set. Compared to the n^{th} power root, Fermatean, Pythagorean, and intuitionistic fuzzy sets, n,m-th power root fuzzy sets can cover more uncertain situations due to their greater range of displayed membership grades. When discussing the symmetry between two or more objects, the innovative concept of an n,m-th power root fuzzy set over dual universes is more flexible than the current notion of an intuitionistic fuzzy set, a Pythagorean fuzzy set, and a n^{th} power root fuzzy set. In this study, we demonstrate a number of additional operations on n,m-th power root fuzzy sets along with a number of their special aspects. Additionally, to deal with choice information, we create a novel weighted aggregated operator called the n,m-th power root fuzzy weighted power average ($FWPA_m^n$) across n,m-th power root fuzzy sets and demonstrate some of its fundamental features. To rank n,m-th power root fuzzy sets, we also define the score and accuracy functions. Moreover, we use this operator to identify the countries with the best standards of living and show how we can select the best option by contrasting aggregate results using score values. Finally, we contrast the results of the $FWPA_m^n$ operator with the square-root fuzzy weighted power average (SR-FWPA), the n^{th} power root fuzzy weighted power average (nPR-FWPA), the Fermatean fuzzy weighted power average (FFWPA), and the n,m-rung orthopair fuzzy weighted power average (n,m-ROFWPA) operators.

Keywords: n,mPR-fuzzy sets; operations; score function; aggregation operator

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1. Introduction

Decision-making is a procedure of resolving real issues by selecting the best option from a set of appropriate alternatives. Throughout the span of a typical day, people make a lot of decisions. When there is only one option, no decision must be made; nevertheless, if there are two or more possibilities, choosing is advantageous. In the branch of operations research known as multi-criteria decision-making (MCDM), the optimal solution is determined after considering all potential options in light of various criteria. There are lots of applications of MCDM issues in various fields. Numerous issues with ambiguity and uncertainty exist in real life. To deal with this ambiguous and uncertain information, Zadeh [1] developed fuzzy set (FS) theory. This set garnered a lot of attention for representing data with uncertainty shortly after it was introduced, and it is still in the spotlight. It has been utilized in numerous multi-criteria decision-making situations. Al-Husban et al. [2] used the multi-fuzzy space to create a new algebraic system. Chen and Phuong [3] suggested a new fuzzy time series forecasting approach based on optimal partitions of intervals in the universe of discourse and optimal weighting vectors of two-factor second-order fuzzy-trend logical relationship groups. Er and Jebril [4] advanced research based on the fuzzy controller. Vovan [5] used the fuzzy clustering technique to develop a predictive model for interval time series and used several benchmark data series to demonstrate practical applications. Atanassov [6] expanded fuzzy sets to intuitionistic fuzzy sets (IFSs) by assigning a degree of membership and non-membership to the items, meeting the requirement $\widehat{\Omega}_\Gamma(t) + \Omega_\Gamma(t) \leq 1$. In light of the inclusion of membership and non-membership grades, IFSs became wider, more significant, and more usable. In order to manage complicated ambiguity and uncertainty with the condition $(\widehat{\Omega}_\Gamma(t))^2 + (\Omega_\Gamma(t))^2 \leq 1$, Yager [7] created a new extension of IFSs called Pythagorean fuzzy sets (PFSs). Thereafter, the concept of Pythagorean fuzzy numbers was proposed by Zhang and Xu [8]. There are numerous uses for IFSs in various sectors, such as image fusion and reservoir flood control [9], optimization problems, medical analysis [10], and decision-making [11,12]. PFSs are more strong and practicable than IFSs, since they can accommodate greater unpredictability than IFSs. Senapati et al. [13] established Fermatean fuzzy sets (FFSs) and fundamental operations on them, along with a Fermatean fuzzy TOPSIS method for resolving multi-criteria decision-making issues. Yager [14] proposed the notion of q-rung orthopair fuzzy sets (q-ROFSs) in order to widen the range of member and non-membership degrees. Lately, many techniques for handling the input data have been proposed, motivated by the observation that the weights of membership and non-membership degrees may not always be equal in general situations. These methods can be applied to characterize some real-world problems and broaden the data sets being studied. In this light, Al-shami et al. [15] created a novel class of fuzzy sets called SR-fuzzy sets and thoroughly investigated their properties. The n,m-rung orthopair fuzzy sets were described by Ibrahim and Alshammari [16] as a different variety of the generalized q-rung orthopair fuzzy set. Al-shami et al. [17] presented the idea of an nth power root fuzzy set (nPR-FS) and offered its core set of operations.

In medical science, engineering, economics, the environment, artificial intelligence, and other fields, the majority of situations are unknown in some way. Action-oriented research has so far created a wide range of competing models for the representation of vaguely defined scenarios, which include ambiguous situations like those connected to a lack of comprehensive and accurate knowledge. Probability theory, fuzzy sets, rough sets, and soft sets are a few examples of these models. The application of rough sets and soft rough sets has proven to be a significant tool for managing uncertainty and vagueness in data that has widespread use in the medical and economic domains [18–23].

The major issue with decision-making challenges is the combination of numerous elements from different sources to produce results or conclusions. Researchers have used a variety of strategies to achieve the best aggregation by taking rules into account and employing diverse procedures. Therefore, aggregation operators were created for this purpose. These aggregation operations are very significant, since they combine the input data into a unified value. These operators for data aggregation are crucial for the development of data analysis findings. Averaging operators such as intuitionistic fuzzy weighted, ordered weighted, and hybrid ones were suggested by Xu [24] to handle intuitionistic fuzzy information. Additionally, weighted, ordered weighted, and hybrid geometric operators based on IFSs were described by Xu and Yager [25]. Zeng and Sua [26] combined aggregation operators and distance measures to create the intuitionistic fuzzy ordered weighted distance operator. In the context of the Pythagorean fuzzy weighted, weighted power, and ordered weighted operators, Yager [27] introduced various geometric aggregation and averaging operations. In a further study, Peng and Yuan [28] looked at some fundamental aspects of Pythagorean fuzzy aggregation operators. The correlation coefficients between Pythagorean fuzzy sets, linguistic Pythagorean fuzzy sets, and generalized Pythagorean fuzzy geometric interactive aggregation operators employing Einstein operations were all taken into consideration by Garg in [29–31]. Several decision-making techniques were described by [32,33]. Regarding Fermatean fuzzy sets, Senapati et al. [34] created the Fermatean fuzzy weighted power average operator. Al-shami et al. [15] suggested and applied the SR-fuzzy weighted power average operator to select the best university. Ibrahim and Alshammari [16] proposed the n,m-rung orthopair fuzzy weighted power average. Ibrahim et al. [35] developed a new weighted aggregated operator via n^{th} power root fuzzy sets.

The examination of livable urban environment modeling is of the highest significance for successfully implementing the livable city idea at various spatial scales. The choice of an appropriate MCDM model for assessing cities' livable environments in China was investigated by Chen [36]. The TOPSIS technique and fuzzy-AHP approaches were suggested by Rashmi et al. [37] to address the problem of choosing the best travel destination in India. Genç and Filipe [38] created a methodology in order to have a multi-criteria approach for choosing a tourist destination region or place in Portugal. Wu et al. [39] proposed a decision-making model based on the combination of two fuzzy AHP and fuzzy TOPSIS methods, which are capable of determining the optimal agritourism location for investors in Vietnam.

The concept of the n,m-th power root fuzzy set (n,mPR-FS) was demonstrated by Saeed and Ibrahim [40], who also provided its fundamental set of operations. It is more likely to be used in uncertain situations than other types of fuzzy sets due to its wider range of displayed membership grades. They also investigated the idea of topology for n,m-th power root fuzzy sets. In this paper, we continue to look into several other concepts motivated by this kind of fuzzy set extension and demonstrate how this class of fuzzy set extension allows us to analyze information data of various significance for grades of membership and non-membership, which is suitable for some real problems.

The motives of the current study are summarized as follows: When dealing with two-dimensional uncertainty, n,mPR-FSs have a wider range of applications than IFSs, PFSs, FFSs, and nPR-FSs. To better understand this argument, consider a pair containing membership and non-membership degrees $(0.99, 0.31)$; then, it is apparent that $0.99^n + 0.31^n > 1$ for $n < 4$ and $0.99^n + \sqrt[n]{0.31} > 1$, for example, $n = 2, 3, 80, 112$. But, $0.99^n + \sqrt[m]{0.31} \leq 1$, for example, $n > 80$ and $m = 2$, or $n > 112$ and $m = 3$.

Motivated by the above analysis, in this research paper, the notion of nPR-FS is extended to n,mPR-FS, thus allowing more uncertainties to be handled easily, as the order of uncertainty is increased from 1 to the n^{th} power of the membership degree and 1 to the m^{th} power of the non-membership degree.

The aims of this study are (1) to offer a novel fundamental operational; (2) to provide alternative score and accuracy functions for comparing n,mPR-fuzzy numbers; (3) to introduce an n,mPR-fuzzy weighted power average aggregation operator and to discuss some of its features; and (4) to present a multi-criteria decision-making technique based on this aggregation operator, which depends on an n,mPR-fuzzy environment.

This research contributes the following:

1. A number of a fresh operations on n,mPR-FSs are provided and supported with examples;
2. A real-life multi-criteria decision-making problem, including the choice of an adequate best country for life, is solved using one more effective algorithm that operates in an n,mPR-fuzzy environment;
3. A comparison of the developed group decision-making method under n,mPR-fuzzy sets with few existing approaches is also given.

This manuscript is formatted as follows. In Section 2, we present some definitions and results related to this article. In Section 3, we outline several operations for the n,m-th power root fuzzy set and look into some of their key traits. In Section 4, we illustrate the idea of a weighted power average operator that is defined across the category of n,m-th power root fuzzy sets. Then, using an empirical example, we discuss the MADM problems that can occur when utilizing this operator. It is clear that one of the n,m-th power root fuzzy set's main advantages is that it can be used in a wide range of decision-making situations. In Section 5, we provide a comparative analysis of the proposed FWPA^n_m operator with other popular operators such as the SR-FWPA operator [15], nPR-FWPA operator [35], FFWPA operator [34], and n,m-ROFWPA operator [16]. In the final section, we summarize the paper's main achievements and make some recommendations for future research.

The objectives of this study are to provide a novel weighted aggregating operator and analyze its salient features as well as to investigate the MCDM techniques that employ this operator.

2. Preliminaries

In this section, we review a few definitions and results that are pertinent to this study.

Definition 1. *Let θ be the universal set and $\mathbb{N}$ be a set of all natural numbers. Then, $\Gamma = \{\left\langle t, \widehat{\Omega}_\Gamma(t), \Omega_\Gamma(t) \right\rangle : t \in \theta\}$ is called*

1. *A q-rung orthopair fuzzy set (q-ROFS) [14] if $0 \le (\widehat{\Omega}_\Gamma(t))^q + (\Omega_\Gamma(t))^q \le 1$ for $q \ge 1$;*
2. *An n,m-rung orthopair fuzzy set (n,m-ROFS) [16] if $0 \le (\widehat{\Omega}_\Gamma(t))^n + (\Omega_\Gamma(t))^m \le 1$ for $n, m \in \mathbb{N}$;*
3. *An n^{th} power root fuzzy set (nPR-FS) [17] if $0 \le (\widehat{\Omega}_\Gamma(t))^n + \sqrt[n]{\Omega_\Gamma(t)} \le 1$ for $n \in \mathbb{N} \setminus \{1\}$; and*
4. *An n,m-th power root fuzzy set (n,mPR-FS) [40] if $0 \le (\widehat{\Omega}_\Gamma(t))^n + \sqrt[m]{\Omega_\Gamma(t)} \le 1$ for $n, m \in \mathbb{N} \setminus \{1\}$,*

where $\widehat{\Omega}_\Gamma(t)$ (resp. $\Omega_\Gamma(t)$) $: \theta \to [0,1]$ is the degree of membership (resp. non-membership) of $t \in \theta$ to Γ

To keep things simple, we denote the n,mPR-FS $\Gamma = \{\left\langle t, \widehat{\Omega}_\Gamma(t), \Omega_\Gamma(t) \right\rangle : t \in \theta\}$ sign as $\Gamma = (\widehat{\Omega}_\Gamma, \Omega_\Gamma)$.

Definition 2 ([40]). *Let $\Gamma = (\widehat{\Omega}_\Gamma, \Omega_\Gamma)$, $\Gamma_1 = (\widehat{\Omega}_{\Gamma_1}, \Omega_{\Gamma_1})$ and $\Gamma_2 = (\widehat{\Omega}_{\Gamma_2}, \Omega_{\Gamma_2})$ be n,mPR-FSs. Then,*

1. *$\Gamma_1 = \Gamma_2$ if and only if $\widehat{\Omega}_{\Gamma_1} = \widehat{\Omega}_{\Gamma_2}$ and $\Omega_{\Gamma_1} = \Omega_{\Gamma_2}$.*
2. *$\Gamma_1 \ge \Gamma_2$ if and only if $\widehat{\Omega}_{\Gamma_1} \ge \widehat{\Omega}_{\Gamma_2}$ and $\Omega_{\Gamma_1} \le \Omega_{\Gamma_2}$.*
3. *$\Gamma^c = (\sqrt[n.m]{\Omega_\Gamma}, (\widehat{\Omega}_\Gamma)^{n.m})$.*
4. *$\Gamma_1 \cap \Gamma_2 = (min\{\widehat{\Omega}_{\Gamma_1}, \widehat{\Omega}_{\Gamma_2}\}, max\{\Omega_{\Gamma_1}, \Omega_{\Gamma_2}\})$.*
5. *$\Gamma_1 \cup \Gamma_2 = (max\{\widehat{\Omega}_{\Gamma_1}, \widehat{\Omega}_{\Gamma_2}\}, min\{\Omega_{\Gamma_1}, \Omega_{\Gamma_2}\})$.*

Figure 1 shows different types of n,mPR-fuzzy membership grade spaces.

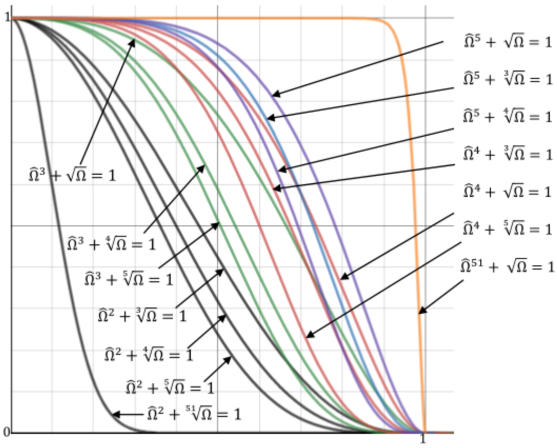

Figure 1. Several n,mPR-FS-type grade spaces.

Remark 1 ([40]). *For every n,mPR-FS $\Gamma = (\widehat{\Omega}_\Gamma, \Omega_\Gamma)$, we have*

1. *Γ is both an n,m-ROFS and an n-ROFS, where $m, n > 1$.*
2. *Γ is an nPR-FS, where $m > n > 1$.*

Remark 2 ([40]). *If $\Gamma = (\widehat{\Omega}_\Gamma, \Omega_\Gamma)$ is an nPR-FS and $1 < m < n$, then Γ is an n,mPR-FS.*

3. Several New Operations on n,mPR-FSs

In this section, we suggest a number of fresh operations on n,mPR-FSs and focus on a few of their attributes. Exactly four decimal digits are used for the calculations throughout the full document.

Definition 3. *Let $\Gamma = (\widehat{\Omega}_\Gamma, \Omega_\Gamma)$, $\Gamma_1 = (\widehat{\Omega}_{\Gamma_1}, \Omega_{\Gamma_1})$ and $\Gamma_2 = (\widehat{\Omega}_{\Gamma_2}, \Omega_{\Gamma_2})$ be n,mPR-FSs. Then,*

1. $\Gamma_1 \oplus \Gamma_2 = \left(\sqrt[n]{\widehat{\Omega}^n_{\Gamma_1} + \widehat{\Omega}^n_{\Gamma_2} - \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_2}}, \Omega_{\Gamma_1}\Omega_{\Gamma_2}\right)$,
2. $\Gamma_1 \otimes \Gamma_2 = \left(\widehat{\Omega}_{\Gamma_1}\widehat{\Omega}_{\Gamma_2}, (\sqrt[m]{\Omega_{\Gamma_1}} + \sqrt[m]{\Omega_{\Gamma_2}} - \sqrt[m]{\Omega_{\Gamma_1}}\sqrt[m]{\Omega_{\Gamma_2}})^m\right)$,
3. $\alpha\Gamma = \left(\sqrt[n]{1 - (1 - \widehat{\Omega}^n_\Gamma)^\alpha}, \Omega^\alpha_\Gamma\right)$, *and*
4. $\Gamma^\alpha = \left(\widehat{\Omega}^\alpha_\Gamma, (1 - (1 - \sqrt[m]{\Omega_\Gamma})^\alpha)^m\right)$,

where α is a positive real number.

Example 1. *Consider the 3,2PR-FSs $\Gamma_1 = (0.53, 0.48)$ and $\Gamma_2 = (0.35, 0.71)$ for $\theta = \{t\}$. Then,*

1. $$\begin{aligned}\Gamma_1 \oplus \Gamma_2 &= \left(\sqrt[n]{\widehat{\Omega}^n_{\Gamma_1} + \widehat{\Omega}^n_{\Gamma_2} - \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_2}}, \Omega_{\Gamma_1}\Omega_{\Gamma_2}\right)\\ &= \left(\sqrt[3]{0.53^3 + 0.35^3 - (0.53)^3(0.35)^3}, (0.48)(0.71)\right) \approx (0.5702, 0.3408).\end{aligned}$$
2. $$\begin{aligned}\Gamma_1 \otimes \Gamma_2 &= \left(\widehat{\Omega}_{\Gamma_1}\widehat{\Omega}_{\Gamma_2}, (\sqrt[m]{\Omega_{\Gamma_1}} + \sqrt[m]{\Omega_{\Gamma_2}} - \sqrt[m]{\Omega_{\Gamma_1}}\sqrt[m]{\Omega_{\Gamma_2}})^m\right)\\ &= \left((0.53)(0.35), (\sqrt{0.48} + \sqrt{0.71} - \sqrt{0.48}\sqrt{0.71})^2\right) \approx (0.1855, 0.9056).\end{aligned}$$
3. $\alpha\Gamma_1 = \left(\sqrt[n]{1 - (1 - \widehat{\Omega}^n_{\Gamma_1})^\alpha}, \Omega^\alpha_{\Gamma_1}\right) = \left(\sqrt[3]{1 - (1 - 0.53^3)^4}, 0.48^4\right) \approx (0.7804, 0.0531)$, *for* $\alpha = 4$.
4. $\Gamma_1^\alpha = \left(\widehat{\Omega}^\alpha_{\Gamma_1}, (1 - (1 - \sqrt[m]{\Omega_{\Gamma_1}})^\alpha)^n\right) = \left(0.53^4, (1 - (1 - \sqrt{0.48})^4)^2\right) \approx (0.0789, 0.9823)$, *for* $\alpha = 4$.

Theorem 1. *If $\Gamma_1 = (\widehat{\Omega}_{\Gamma_1}, \Omega_{\Gamma_1})$ and $\Gamma_2 = (\widehat{\Omega}_{\Gamma_2}, \Omega_{\Gamma_2})$ are n,mPR-FSs, then $\Gamma_1 \oplus \Gamma_2$ and $\Gamma_1 \otimes \Gamma_2$ are also n,mPR-FSs.*

Proof. For n,mPR-FSs $\Gamma_1 = (\widehat{\Omega}_{\Gamma_1}, \Omega_{\Gamma_1})$ and $\Gamma_2 = (\widehat{\Omega}_{\Gamma_2}, \Omega_{\Gamma_2})$, the relationships shown below are clear:

$$0 \le \widehat{\Omega}^n_{\Gamma_1} \le 1, 0 \le \sqrt[m]{\Omega_{\Gamma_1}} \le 1, 0 \le (\widehat{\Omega}_{\Gamma_1})^n + \sqrt[m]{\Omega_{\Gamma_1}} \le 1;$$

and

$$0 \le \widehat{\Omega}^n_{\Gamma_2} \le 1, 0 \le \sqrt[m]{\Omega_{\Gamma_2}} \le 1, 0 \le (\widehat{\Omega}_{\Gamma_2})^n + \sqrt[m]{\Omega_{\Gamma_2}} \le 1.$$

Thus, we obtain

$$\widehat{\Omega}^n_{\Gamma_1} \ge \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_2}, \widehat{\Omega}^n_{\Gamma_2} \ge \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_2}, 1 \ge \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_2} \ge 0$$

and

$$\sqrt[m]{\Omega_{\Gamma_1}} \ge \sqrt[m]{\Omega_{\Gamma_1}}\sqrt[m]{\Omega_{\Gamma_2}}, \sqrt[m]{\Omega_{\Gamma_2}} \ge \sqrt[m]{\Omega_{\Gamma_1}}\sqrt[m]{\Omega_{\Gamma_2}}, 1 \ge \sqrt[m]{\Omega_{\Gamma_1}}\sqrt[m]{\Omega_{\Gamma_2}} \ge 0$$

which shows that

$\widehat{\Omega}^n_{\Gamma_1} + \widehat{\Omega}^n_{\Gamma_2} - \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_2} \geq 0$ implies $\sqrt[n]{\widehat{\Omega}^n_{\Gamma_1} + \widehat{\Omega}^n_{\Gamma_2} - \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_2}} \geq 0,$

and

$\sqrt[m]{\Omega_{\Gamma_1}} + \sqrt[m]{\Omega_{\Gamma_2}} - \sqrt[m]{\Omega_{\Gamma_1}}\sqrt[m]{\Omega_{\Gamma_2}} \geq 0$ implies $(\sqrt[m]{\Omega_{\Gamma_1}} + \sqrt[m]{\Omega_{\Gamma_2}} - \sqrt[m]{\Omega_{\Gamma_1}}\sqrt[m]{\Omega_{\Gamma_2}})^m \geq 0.$

Since $\widehat{\Omega}^n_{\Gamma_2} \leq 1$ and $0 \leq 1 - \widehat{\Omega}^n_{\Gamma_1}$, then $\widehat{\Omega}^n_{\Gamma_2}(1 - \widehat{\Omega}^n_{\Gamma_1}) \leq (1 - \widehat{\Omega}^n_{\Gamma_1})$, we obtain $\widehat{\Omega}^n_{\Gamma_1} + \widehat{\Omega}^n_{\Gamma_2} - \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_2} \leq 1$, and hence, $\sqrt[n]{\widehat{\Omega}^n_{\Gamma_1} + \widehat{\Omega}^n_{\Gamma_2} - \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_2}} \leq 1.$

Similarly, we can acquire

$$(\sqrt[m]{\Omega_{\Gamma_1}} + \sqrt[m]{\Omega_{\Gamma_2}} - \sqrt[m]{\Omega_{\Gamma_1}}\sqrt[m]{\Omega_{\Gamma_2}})^m \leq 1.$$

It is clear that

$$0 \leq \sqrt[m]{\Omega_{\Gamma_1}} \leq 1 - \widehat{\Omega}^n_{\Gamma_1} \text{ and } 0 \leq \sqrt[m]{\Omega_{\Gamma_2}} \leq 1 - \widehat{\Omega}^n_{\Gamma_2}.$$

Thus, we can acquire

$$(\sqrt[n]{\widehat{\Omega}^n_{\Gamma_1} + \widehat{\Omega}^n_{\Gamma_2} - \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_2}})^n + \sqrt[m]{\Omega_{\Gamma_1}\Omega_{\Gamma_2}}$$
$$\leq \widehat{\Omega}^n_{\Gamma_1} + \widehat{\Omega}^n_{\Gamma_2} - \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_2} + (1 - \widehat{\Omega}^n_{\Gamma_1})(1 - \widehat{\Omega}^n_{\Gamma_2}) = 1$$

Hence,

$$0 \leq \sqrt[n]{\widehat{\Omega}^n_{\Gamma_1} + \widehat{\Omega}^n_{\Gamma_2} - \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_2}} \leq 1, 0 \leq \Omega_{\Gamma_1}\Omega_{\Gamma_2} \leq 1 \text{ and}$$
$$0 \leq (\sqrt[n]{\widehat{\Omega}^n_{\Gamma_1} + \widehat{\Omega}^n_{\Gamma_2} - \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_2}})^n + \sqrt[m]{\Omega_{\Gamma_1}\Omega_{\Gamma_2}} \leq 1.$$

Similarly, we have

$$0 \leq \widehat{\Omega}_{\Gamma_1}\widehat{\Omega}_{\Gamma_2} \leq 1, 0 \leq (\sqrt[m]{\Omega_{\Gamma_1}} + \sqrt[m]{\Omega_{\Gamma_2}} - \sqrt[m]{\Omega_{\Gamma_1}}\sqrt[m]{\Omega_{\Gamma_2}})^m \leq 1 \text{ and}$$
$$0 \leq (\widehat{\Omega}_{\Gamma_1}\widehat{\Omega}_{\Gamma_2})^n + \sqrt[m]{(\sqrt[m]{\Omega_{\Gamma_1}} + \sqrt[m]{\Omega_{\Gamma_2}} - \sqrt[m]{\Omega_{\Gamma_1}}\sqrt[m]{\Omega_{\Gamma_2}})^m} \leq 1.$$

Thus, $\Gamma_1 \oplus \Gamma_2$ and $\Gamma_1 \otimes \Gamma_2$ are n,mPR-FSs. □

Theorem 2. *Let $\Gamma = (\widehat{\Omega}_\Gamma, \Omega_\Gamma)$ be a n,mPR-FS and $\alpha > 0$. Then, $\alpha\Gamma$ and Γ^α are n,mPR-FSs.*

Proof. Since $0 \leq \widehat{\Omega}^n_\Gamma \leq 1$, $0 \leq \sqrt[m]{\Omega_\Gamma} \leq 1$ and $0 \leq (\widehat{\Omega}_\Gamma)^n + \sqrt[m]{\Omega_\Gamma} \leq 1$, then

$$0 \leq \sqrt[m]{\Omega_\Gamma} \leq 1 - \widehat{\Omega}^n_\Gamma$$
$$\Rightarrow 0 \leq (1 - \widehat{\Omega}^n_\Gamma)^\alpha$$
$$\Rightarrow 1 - (1 - \widehat{\Omega}^n_\Gamma)^\alpha \leq 1$$
$$\Rightarrow 0 \leq \sqrt[n]{1 - (1 - \widehat{\Omega}^n_\Gamma)^\alpha} \leq \sqrt[n]{1} = 1.$$

It is clear that $0 \leq \Omega^\alpha_\Gamma \leq 1$. Then, we have

$$0 \leq (\sqrt[n]{1 - (1 - \widehat{\Omega}^n_\Gamma)^\alpha})^n + \sqrt[m]{\Omega^\alpha_\Gamma} \leq 1 - (1 - \widehat{\Omega}^n_\Gamma)^\alpha + (1 - \widehat{\Omega}^n_\Gamma)^\alpha = 1.$$

Similarly, we have

$$0 \leq (\widehat{\Omega}^\alpha_\Gamma)^n + \sqrt[m]{(1 - (1 - \sqrt[m]{\Omega_\Gamma})^\alpha)^m} \leq 1.$$

Hence, $\alpha\Gamma$ and Γ^α are n,mPR-FSs. □

Theorem 3. *Let* $\Gamma_1 = (\widehat{\Omega}_{\Gamma_1}, \Omega_{\Gamma_1})$ *and* $\Gamma_2 = (\widehat{\Omega}_{\Gamma_2}, \Omega_{\Gamma_2})$ *be n,mPR-FSs. Then,*

1. $\Gamma_1 \oplus \Gamma_2 = \Gamma_2 \oplus \Gamma_1$.
2. $\Gamma_1 \otimes \Gamma_2 = \Gamma_2 \otimes \Gamma_1$.

Proof. 1. $\Gamma_1 \oplus \Gamma_2 = \left(\sqrt[n]{\widehat{\Omega}^n_{\Gamma_1} + \widehat{\Omega}^n_{\Gamma_2} - \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_2}}, \Omega_{\Gamma_1}\Omega_{\Gamma_2} \right)$
$\left(\sqrt[n]{\widehat{\Omega}^n_{\Gamma_2} + \widehat{\Omega}^n_{\Gamma_1} - \widehat{\Omega}^n_{\Gamma_2}\widehat{\Omega}^n_{\Gamma_1}}, \Omega_{\Gamma_2}\Omega_{\Gamma_1} \right) = \Gamma_2 \oplus \Gamma_1$.

2. $\Gamma_1 \otimes \Gamma_2 = \left(\widehat{\Omega}_{\Gamma_1}\widehat{\Omega}_{\Gamma_2}, (\sqrt[m]{\Omega_{\Gamma_1}} + \sqrt[m]{\Omega_{\Gamma_2}} - \sqrt[m]{\Omega_{\Gamma_1}}\sqrt[m]{\Omega_{\Gamma_2}})^m \right)$
$= \left(\widehat{\Omega}_{\Gamma_2}\widehat{\Omega}_{\Gamma_1}, (\sqrt[m]{\Omega_{\Gamma_2}} + \sqrt[m]{\Omega_{\Gamma_1}} - \sqrt[m]{\Omega_{\Gamma_2}}\sqrt[m]{\Omega_{\Gamma_1}})^m \right) = \Gamma_2 \otimes \Gamma_1$.
□

Theorem 4. *Let* $\Gamma_1 = (\widehat{\Omega}_{\Gamma_1}, \Omega_{\Gamma_1})$, $\Gamma_2 = (\widehat{\Omega}_{\Gamma_2}, \Omega_{\Gamma_2})$ *and* $\Gamma_3 = (\widehat{\Omega}_{\Gamma_3}, \Omega_{\Gamma_3})$ *be n,mPR-FSs. Then,*

1. $\Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3 = \Gamma_1 \oplus \Gamma_3 \oplus \Gamma_2$.
2. $\Gamma_1 \otimes \Gamma_2 \otimes \Gamma_3 = \Gamma_1 \otimes \Gamma_3 \otimes \Gamma_2$.

Proof. 1. $\Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3$
$= (\widehat{\Omega}_{\Gamma_1}, \Omega_{\Gamma_1}) \oplus (\widehat{\Omega}_{\Gamma_2}, \Omega_{\Gamma_2}) \oplus (\widehat{\Omega}_{\Gamma_3}, \Omega_{\Gamma_3})$
$= \left(\sqrt[n]{\widehat{\Omega}^n_{\Gamma_1} + \widehat{\Omega}^n_{\Gamma_2} - \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_2}}, \Omega_{\Gamma_1}\Omega_{\Gamma_2} \right) \oplus (\widehat{\Omega}_{\Gamma_3}, \Omega_{\Gamma_3})$
$= \left(\sqrt[n]{\widehat{\Omega}^n_{\Gamma_1} + \widehat{\Omega}^n_{\Gamma_2} - \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_2} + \widehat{\Omega}^n_{\Gamma_3} - \widehat{\Omega}^n_{\Gamma_3}(\widehat{\Omega}^n_{\Gamma_1} + \widehat{\Omega}^n_{\Gamma_2} - \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_2})}, \Omega_{\Gamma_1}\Omega_{\Gamma_2}\Omega_{\Gamma_3} \right)$
$= \left(\sqrt[n]{\widehat{\Omega}^n_{\Gamma_1} + \widehat{\Omega}^n_{\Gamma_2} + \widehat{\Omega}^n_{\Gamma_3} - \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_2} - \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_3} - \widehat{\Omega}^n_{\Gamma_2}\widehat{\Omega}^n_{\Gamma_3} + \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_2}\widehat{\Omega}^n_{\Gamma_3}}, \Omega_{\Gamma_1}\Omega_{\Gamma_2}\Omega_{\Gamma_3} \right)$
$= \left(\sqrt[n]{\widehat{\Omega}^n_{\Gamma_1} + \widehat{\Omega}^n_{\Gamma_3} - \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_3} + \widehat{\Omega}^n_{\Gamma_2} - \widehat{\Omega}^n_{\Gamma_2}(\widehat{\Omega}^n_{\Gamma_1} + \widehat{\Omega}^n_{\Gamma_3} - \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_3})}, \Omega_{\Gamma_1}\Omega_{\Gamma_2}\Omega_{\Gamma_3} \right)$
$= \left(\sqrt[n]{\widehat{\Omega}^n_{\Gamma_1} + \widehat{\Omega}^n_{\Gamma_3} - \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_3}}, \Omega_{\Gamma_1}\Omega_{\Gamma_3} \right) \oplus (\widehat{\Omega}_{\Gamma_2}, \Omega_{\Gamma_2})$
$= \Gamma_1 \oplus \Gamma_3 \oplus \Gamma_2$.

2. We apply the same strategy as described in (1).
□

Theorem 5. *Let* $\Gamma = (\widehat{\Omega}_{\Gamma}, \Omega_{\Gamma})$, $\Gamma_1 = (\widehat{\Omega}_{\Gamma_1}, \Omega_{\Gamma_1})$ *and* $\Gamma_2 = (\widehat{\Omega}_{\Gamma_2}, \Omega_{\Gamma_2})$ *be n,mPR-FSs. Then,*

1. $\alpha(\Gamma_1 \oplus \Gamma_2) = \alpha\Gamma_1 \oplus \alpha\Gamma_2$, *for* $\alpha > 0$.
2. $(\alpha_1 + \alpha_2)\Gamma = \alpha_1\Gamma \oplus \alpha_2\Gamma$, *for* $\alpha_1, \alpha_2 > 0$.
3. $(\Gamma_1 \otimes \Gamma_2)^\alpha = \Gamma_1^\alpha \otimes \Gamma_2^\alpha$, *for* $\alpha > 0$.
4. $\Gamma^{\alpha_1} \otimes \Gamma^{\alpha_2} = \Gamma^{(\alpha_1+\alpha_2)}$, *for* $\alpha_1, \alpha_2 > 0$.

Proof. 1. $\alpha(\Gamma_1 \oplus \Gamma_2) = \alpha\left(\sqrt[n]{\widehat{\Omega}^n_{\Gamma_1} + \widehat{\Omega}^n_{\Gamma_2} - \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_2}}, \Omega_{\Gamma_1}\Omega_{\Gamma_2} \right)$
$= \left(\sqrt[n]{1 - (1 - \widehat{\Omega}^n_{\Gamma_1} - \widehat{\Omega}^n_{\Gamma_2} + \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_2})^\alpha}, (\Omega_{\Gamma_1}\Omega_{\Gamma_2})^\alpha \right)$
$= \left(\sqrt[n]{1 - (1 - \widehat{\Omega}^n_{\Gamma_1})^\alpha(1 - \widehat{\Omega}^n_{\Gamma_2})^\alpha}, \Omega^\alpha_{\Gamma_1}\Omega^\alpha_{\Gamma_2} \right)$.
And $\alpha\Gamma_1 \oplus \alpha\Gamma_2 = \left(\sqrt[n]{1 - (1 - \widehat{\Omega}^n_{\Gamma_1})^\alpha}, \Omega^\alpha_{\Gamma_1} \right) \oplus \left(\sqrt[n]{1 - (1 - \widehat{\Omega}^n_{\Gamma_2})^\alpha}, \Omega^\alpha_{\Gamma_2} \right)$
$= \left(\sqrt[n]{1 - (1 - \widehat{\Omega}^n_{\Gamma_1})^\alpha + 1 - (1 - \widehat{\Omega}^n_{\Gamma_2})^\alpha - (1 - (1 - \widehat{\Omega}^n_{\Gamma_1})^\alpha)(1 - (1 - \widehat{\Omega}^n_{\Gamma_2})^\alpha)}, \Omega^\alpha_{\Gamma_1}\Omega^\alpha_{\Gamma_2} \right)$
$= \left(\sqrt[n]{1 - (1 - \widehat{\Omega}^n_{\Gamma_1})^\alpha(1 - \widehat{\Omega}^n_{\Gamma_2})^\alpha}, \Omega^\alpha_{\Gamma_1}\Omega^\alpha_{\Gamma_2} \right) = \alpha(\Gamma_1 \oplus \Gamma_2)$.

2. $(\alpha_1 + \alpha_2)\Gamma = (\alpha_1 + \alpha_2)(\widehat{\Omega}_\Gamma, \Omega_\Gamma) = \left(\sqrt[n]{1 - (1 - \widehat{\Omega}^n_\Gamma)^{\alpha_1+\alpha_2}}, \Omega_\Gamma^{\alpha_1+\alpha_2} \right)$
$= \left(\sqrt[n]{1 - (1 - \widehat{\Omega}^n_\Gamma)^{\alpha_1}(1 - \widehat{\Omega}^n_\Gamma)^{\alpha_2}}, \Omega_\Gamma^{\alpha_1+\alpha_2} \right)$
$= \left(\sqrt[n]{1 - (1 - \widehat{\Omega}^n_\Gamma)^{\alpha_1} + 1 - (1 - \widehat{\Omega}^n_\Gamma)^{\alpha_2} - (1 - (1 - \widehat{\Omega}^n_\Gamma)^{\alpha_1})(1 - (1 - \widehat{\Omega}^n_\Gamma)^{\alpha_2})}, \Omega_\Gamma^{\alpha_1}\Omega_\Gamma^{\alpha_2} \right)$
$= \left(\sqrt[n]{1 - (1 - \widehat{\Omega}^n_\Gamma)^{\alpha_1}}, \Omega_\Gamma^{\alpha_1} \right) \oplus \left(\sqrt[n]{1 - (1 - \widehat{\Omega}^n_\Gamma)^{\alpha_2}}, \Omega_\Gamma^{\alpha_2} \right)$
$= \alpha_1\Gamma \oplus \alpha_2\Gamma$.

3. $(\Gamma_1 \otimes \Gamma_2)^\alpha = \left(\widehat{\Omega}_{\Gamma_1}\widehat{\Omega}_{\Gamma_2}, (\sqrt[m]{\Omega_{\Gamma_1}} + \sqrt[m]{\Omega_{\Gamma_2}} - \sqrt[m]{\Omega_{\Gamma_1}}\sqrt[m]{\Omega_{\Gamma_2}})^m\right)^\alpha$
$= \left((\widehat{\Omega}_{\Gamma_1}\widehat{\Omega}_{\Gamma_2})^\alpha, (1-(1-\sqrt[m]{\Omega_{\Gamma_1}} - \sqrt[m]{\Omega_{\Gamma_2}} + \sqrt[m]{\Omega_{\Gamma_1}}\sqrt[m]{\Omega_{\Gamma_2}})^\alpha)^m\right)$
$= \left(\widehat{\Omega}^\alpha_{\Gamma_1}\widehat{\Omega}^\alpha_{\Gamma_2}, (1-(1-\sqrt[m]{\Omega_{\Gamma_1}})^\alpha(1-\sqrt[m]{\Omega_{\Gamma_2}})^\alpha)^m\right)$
$= \left(\widehat{\Omega}^\alpha_{\Gamma_1}, (1-(1-\sqrt[m]{\Omega_{\Gamma_1}})^\alpha)^m\right) \otimes \left(\widehat{\Omega}^\alpha_{\Gamma_2}, (1-(1-\sqrt[m]{\Omega_{\Gamma_2}})^\alpha)^m\right)$
$= \Gamma_1^\alpha \otimes \Gamma_2^\alpha.$
4. $\Gamma^{\alpha_1} \otimes \Gamma^{\alpha_2} = \left(\widehat{\Omega}^{\alpha_1}_\Gamma, (1-(1-\sqrt[m]{\Omega_\Gamma})^{\alpha_1})^m\right) \otimes \left(\widehat{\Omega}^{\alpha_2}_\Gamma, (1-(1-\sqrt[m]{\Omega_\Gamma})^{\alpha_2})^m\right)$
$= (\widehat{\Omega}^{\alpha_1+\alpha_2}_\Gamma, [1-(1-\sqrt[m]{\Omega_\Gamma})^{\alpha_1} + 1 - (1-\sqrt[m]{\Omega_\Gamma})^{\alpha_2} - (1-(1-\sqrt[m]{\Omega_\Gamma})^{\alpha_1})$
$(1-(1-\sqrt[m]{\Omega_\Gamma})^{\alpha_2})]^m)$
$= \left(\widehat{\Omega}^{\alpha_1+\alpha_2}_\Gamma, (1-(1-\sqrt[m]{\Omega_\Gamma})^{\alpha_1+\alpha_2})^m\right)$
$= \Gamma^{(\alpha_1+\alpha_2)}.$
□

Theorem 6. *Let* $\Gamma_1 = (\widehat{\Omega}_{\Gamma_1}, \Omega_{\Gamma_1})$, $\Gamma_2 = (\widehat{\Omega}_{\Gamma_2}, \Omega_{\Gamma_2})$ *and* $\Gamma_3 = (\widehat{\Omega}_{\Gamma_3}, \Omega_{\Gamma_3})$ *be n,mPR-FSs. Then,*

1. $(\Gamma_1 \cap \Gamma_2) \oplus \Gamma_3 = (\Gamma_1 \oplus \Gamma_3) \cap (\Gamma_2 \oplus \Gamma_3)$.
2. $(\Gamma_1 \cup \Gamma_2) \oplus \Gamma_3 = (\Gamma_1 \oplus \Gamma_3) \cup (\Gamma_2 \oplus \Gamma_3)$.
3. $(\Gamma_1 \cap \Gamma_2) \otimes \Gamma_3 = (\Gamma_1 \otimes \Gamma_3) \cap (\Gamma_2 \otimes \Gamma_3)$.
4. $(\Gamma_1 \cup \Gamma_2) \otimes \Gamma_3 = (\Gamma_1 \otimes \Gamma_3) \cup (\Gamma_2 \otimes \Gamma_3)$.

Proof. Definitions 3 and 2 give us

1. $(\Gamma_1 \cap \Gamma_2) \oplus \Gamma_3 = (min\{\widehat{\Omega}_{\Gamma_1}, \widehat{\Omega}_{\Gamma_2}\}, max\{\Omega_{\Gamma_1}, \Omega_{\Gamma_2}\}) \oplus (\widehat{\Omega}_{\Gamma_3}, \Omega_{\Gamma_3})$
$= \left(\sqrt[n]{min\{\widehat{\Omega}^n_{\Gamma_1}, \widehat{\Omega}^n_{\Gamma_2}\} + \widehat{\Omega}^n_{\Gamma_3} - \widehat{\Omega}^n_{\Gamma_3} min\{\widehat{\Omega}^n_{\Gamma_1}, \widehat{\Omega}^n_{\Gamma_2}\}}, max\{\Omega_{\Gamma_1}, \Omega_{\Gamma_2}\}\Omega_{\Gamma_3}\right)$
$= \left(\sqrt[n]{(1-\widehat{\Omega}^n_{\Gamma_3}) min\{\widehat{\Omega}^n_{\Gamma_1}, \widehat{\Omega}^n_{\Gamma_2}\} + \widehat{\Omega}^n_{\Gamma_3}}, max\{\Omega_{\Gamma_1}\Omega_{\Gamma_3}, \Omega_{\Gamma_2}\Omega_{\Gamma_3}\}\right)$.
And $(\Gamma_1 \oplus \Gamma_3) \cap (\Gamma_2 \oplus \Gamma_3)$
$= \left(\sqrt[n]{\widehat{\Omega}^n_{\Gamma_1} + \widehat{\Omega}^n_{\Gamma_3} - \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_3}}, \Omega_{\Gamma_1}\Omega_{\Gamma_3}\right) \cap \left(\sqrt[n]{\widehat{\Omega}^n_{\Gamma_2} + \widehat{\Omega}^n_{\Gamma_3} - \widehat{\Omega}^n_{\Gamma_2}\widehat{\Omega}^n_{\Gamma_3}}, \Omega_{\Gamma_2}\Omega_{\Gamma_3}\right)$
$= \left(min\{\sqrt[n]{\widehat{\Omega}^n_{\Gamma_1} + \widehat{\Omega}^n_{\Gamma_3} - \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_3}}, \sqrt[n]{\widehat{\Omega}^n_{\Gamma_2} + \widehat{\Omega}^n_{\Gamma_3} - \widehat{\Omega}^n_{\Gamma_2}\widehat{\Omega}^n_{\Gamma_3}}\}, max\{\Omega_{\Gamma_1}\Omega_{\Gamma_3}, \Omega_{\Gamma_2}\Omega_{\Gamma_3}\}\right)$
$= \left(min\{\sqrt[n]{(1-\widehat{\Omega}^n_{\Gamma_3})\widehat{\Omega}^n_{\Gamma_1} + \widehat{\Omega}^n_{\Gamma_3}}, \sqrt[n]{(1-\widehat{\Omega}^n_{\Gamma_3})\widehat{\Omega}^n_{\Gamma_2} + \widehat{\Omega}^n_{\Gamma_3}}\}, max\{\Omega_{\Gamma_1}\Omega_{\Gamma_3}, \Omega_{\Gamma_2}\Omega_{\Gamma_3}\}\right)$
$= \left(\sqrt[n]{(1-\widehat{\Omega}^n_{\Gamma_3}) min\{\widehat{\Omega}^n_{\Gamma_1}, \widehat{\Omega}^n_{\Gamma_2}\} + \widehat{\Omega}^n_{\Gamma_3}}, max\{\Omega_{\Gamma_1}\Omega_{\Gamma_3}, \Omega_{\Gamma_2}\Omega_{\Gamma_3}\}\right)$.
Thus, $(\Gamma_1 \cap \Gamma_2) \oplus \Gamma_3 = (\Gamma_1 \oplus \Gamma_3) \cap (\Gamma_2 \oplus \Gamma_3)$.
2. We apply the same strategy as described in (1).
3. $(\Gamma_1 \cap \Gamma_2) \otimes \Gamma_3 = (min\{\widehat{\Omega}_{\Gamma_1}, \widehat{\Omega}_{\Gamma_2}\}, max\{\Omega_{\Gamma_1}, \Omega_{\Gamma_2}\}) \otimes \Gamma_3$
$= \left(min\{\widehat{\Omega}_{\Gamma_1}, \widehat{\Omega}_{\Gamma_2}\}\widehat{\Omega}_{\Gamma_3}, (max\{\sqrt[m]{\Omega_{\Gamma_1}}, \sqrt[m]{\Omega_{\Gamma_2}}\} + \sqrt[m]{\Omega_{\Gamma_3}} - \sqrt[m]{\Omega_{\Gamma_3}} max\{\sqrt[m]{\Omega_{\Gamma_1}}, \sqrt[m]{\Omega_{\Gamma_2}}\})^m\right)$
$= \left(min\{\widehat{\Omega}_{\Gamma_1}\widehat{\Omega}_{\Gamma_3}, \widehat{\Omega}_{\Gamma_2}\widehat{\Omega}_{\Gamma_3}\}, ((1-\sqrt[m]{\Omega_{\Gamma_3}}) max\{\sqrt[m]{\Omega_{\Gamma_1}}, \sqrt[m]{\Omega_{\Gamma_2}}\} + \sqrt[m]{\Omega_{\Gamma_3}})^m\right)$.
And $(\Gamma_1 \otimes \Gamma_3) \cap (\Gamma_2 \otimes \Gamma_3) = \left(\widehat{\Omega}_{\Gamma_1}\widehat{\Omega}_{\Gamma_3}, (\sqrt[m]{\Omega_{\Gamma_1}} + \sqrt[m]{\Omega_{\Gamma_3}} - \sqrt[m]{\Omega_{\Gamma_1}}\sqrt[m]{\Omega_{\Gamma_3}})^m\right)$
$\cap \left(\widehat{\Omega}_{\Gamma_2}\widehat{\Omega}_{\Gamma_3}, (\sqrt[m]{\Omega_{\Gamma_2}} + \sqrt[m]{\Omega_{\Gamma_3}} - \sqrt[m]{\Omega_{\Gamma_2}}\sqrt[m]{\Omega_{\Gamma_3}})^m\right)$
$= \left(\widehat{\Omega}_{\Gamma_1}\widehat{\Omega}_{\Gamma_3}, ((1-\sqrt[m]{\Omega_{\Gamma_3}})\sqrt[m]{\Omega_{\Gamma_1}} + \sqrt[m]{\Omega_{\Gamma_3}})^m\right) \cap \left(\widehat{\Omega}_{\Gamma_2}\widehat{\Omega}_{\Gamma_3}, ((1-\sqrt[m]{\Omega_{\Gamma_3}})\sqrt[m]{\Omega_{\Gamma_2}} + \sqrt[m]{\Omega_{\Gamma_3}})^m\right)$
$= (min\{\widehat{\Omega}_{\Gamma_1}\widehat{\Omega}_{\Gamma_3}, \widehat{\Omega}_{\Gamma_2}\widehat{\Omega}_{\Gamma_3}\}, max\{((1-\sqrt[m]{\Omega_{\Gamma_3}})\sqrt[m]{\Omega_{\Gamma_1}} + \sqrt[m]{\Omega_{\Gamma_3}})^m,$
$((1-\sqrt[m]{\Omega_{\Gamma_3}})\sqrt[m]{\Omega_{\Gamma_2}} + \sqrt[m]{\Omega_{\Gamma_3}})^m\})$
$= \left(min\{\widehat{\Omega}_{\Gamma_1}\widehat{\Omega}_{\Gamma_3}, \widehat{\Omega}_{\Gamma_2}\widehat{\Omega}_{\Gamma_3}\}, ((1-\sqrt[m]{\Omega_{\Gamma_3}}) max\{\sqrt[m]{\Omega_{\Gamma_1}}, \sqrt[m]{\Omega_{\Gamma_2}}\} + \sqrt[m]{\Omega_{\Gamma_3}})^m\right)$.
Thus, $(\Gamma_1 \cap \Gamma_2) \otimes \Gamma_3 = (\Gamma_1 \otimes \Gamma_3) \cap (\Gamma_2 \otimes \Gamma_3)$.
4. We apply the same strategy as described in (3).

□

Theorem 7. *Let* $\Gamma_1 = (\widehat{\Omega}_{\Gamma_1}, \Omega_{\Gamma_1})$ *and* $\Gamma_2 = (\widehat{\Omega}_{\Gamma_2}, \Omega_{\Gamma_2})$ *be n,mPR-FSs, and* $\alpha > 0$. *Then,*

1. $\alpha(\Gamma_1 \cup \Gamma_2) = \alpha\Gamma_1 \cup \alpha\Gamma_2$.
2. $(\Gamma_1 \cup \Gamma_2)^\alpha = \Gamma_1^\alpha \cup \Gamma_2^\alpha$.

Proof. Definitions 2 and 3 give us

1. $\alpha(\Gamma_1 \cup \Gamma_2) = \alpha(max\{\widehat{\Omega}_{\Gamma_1}, \widehat{\Omega}_{\Gamma_2}\}, min\{\Omega_{\Gamma_1}, \Omega_{\Gamma_2}\})$
$= \left(\sqrt[n]{1-(1-max\{\widehat{\Omega}^n_{\Gamma_1}, \widehat{\Omega}^n_{\Gamma_2}\})^\alpha}, min\{\Omega^\alpha_{\Gamma_1}, \Omega^\alpha_{\Gamma_2}\}\right)$.
And $\alpha\Gamma_1 \cup \alpha\Gamma_2 = \left(\sqrt[n]{1-(1-\widehat{\Omega}^n_{\Gamma_1})^\alpha}, \Omega^\alpha_{\Gamma_1}\right) \cup \left(\sqrt[n]{1-(1-\widehat{\Omega}^n_{\Gamma_2})^\alpha}, \Omega^\alpha_{\Gamma_2}\right)$
$= \left(max\{\sqrt[n]{1-(1-\widehat{\Omega}^n_{\Gamma_1})^\alpha}, \sqrt[n]{1-(1-\widehat{\Omega}^n_{\Gamma_2})^\alpha}\}, min\{\Omega^\alpha_{\Gamma_1}, \Omega^\alpha_{\Gamma_2}\}\right)$
$= \left(\sqrt[n]{1-(1-max\{\widehat{\Omega}^n_{\Gamma_1}, \widehat{\Omega}^n_{\Gamma_2}\})^\alpha}, min\{\Omega^\alpha_{\Gamma_1}, \Omega^\alpha_{\Gamma_2}\}\right) = \alpha(\Gamma_1 \cup \Gamma_2)$.
2. This can be demonstrated similarly to (1).

□

Theorem 8. *Let* $\Gamma = (\widehat{\Omega}_\Gamma, \Omega_\Gamma)$, $\Gamma_1 = (\widehat{\Omega}_{\Gamma_1}, \Omega_{\Gamma_1})$ *and* $\Gamma_2 = (\widehat{\Omega}_{\Gamma_2}, \Omega_{\Gamma_2})$ *be n,mPR-FSs, and* $\alpha > 0$. *Then,*

1. $(\Gamma_1 \oplus \Gamma_2)^c = \Gamma_1^c \otimes \Gamma_2^c$.
2. $(\Gamma_1 \otimes \Gamma_2)^c = \Gamma_1^c \oplus \Gamma_2^c$.
3. $(\Gamma^c)^\alpha = (\alpha\Gamma)^c$.
4. $\alpha(\Gamma)^c = (\Gamma^\alpha)^c$.

Proof. Definitions 2 and 3 (3) give us

1. $(\Gamma_1 \oplus \Gamma_2)^c = \left(\sqrt[n]{\widehat{\Omega}^n_{\Gamma_1} + \widehat{\Omega}^n_{\Gamma_2} - \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_2}}, \Omega_{\Gamma_1}\Omega_{\Gamma_2}\right)^c$
$= \left(\sqrt[n.m]{\Omega_{\Gamma_1}\Omega_{\Gamma_2}}, (\sqrt[n]{\widehat{\Omega}^n_{\Gamma_1} + \widehat{\Omega}^n_{\Gamma_2} - \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_2}})^{n.m}\right)$
$= \left(\sqrt[n.m]{\Omega_{\Gamma_1}}\sqrt[n.m]{\Omega_{\Gamma_2}}, (\widehat{\Omega}^n_{\Gamma_1} + \widehat{\Omega}^n_{\Gamma_2} - \widehat{\Omega}^n_{\Gamma_1}\widehat{\Omega}^n_{\Gamma_2})^m\right)$
$= (\sqrt[n.m]{\Omega_{\Gamma_1}}, (\widehat{\Omega}_{\Gamma_1})^{n.m}) \otimes (\sqrt[n.m]{\Omega_{\Gamma_2}}, (\widehat{\Omega}_{\Gamma_2})^{n.m})$
$= \Gamma_1^c \otimes \Gamma_2^c$.
2. $(\Gamma_1 \otimes \Gamma_2)^c = \left(\widehat{\Omega}_{\Gamma_1}\widehat{\Omega}_{\Gamma_2}, (\sqrt[m]{\Omega_{\Gamma_1}} + \sqrt[m]{\Omega_{\Gamma_2}} - \sqrt[m]{\Omega_{\Gamma_1}}\sqrt[m]{\Omega_{\Gamma_2}})^m\right)^c$
$= \left(\sqrt[n.m]{(\sqrt[m]{\Omega_{\Gamma_1}} + \sqrt[m]{\Omega_{\Gamma_2}} - \sqrt[m]{\Omega_{\Gamma_1}}\sqrt[m]{\Omega_{\Gamma_2}})^m}, (\widehat{\Omega}_{\Gamma_1}\widehat{\Omega}_{\Gamma_2})^{n.m}\right)$
$= \left(\sqrt[n]{(\sqrt[m]{\Omega_{\Gamma_1}} + \sqrt[m]{\Omega_{\Gamma_2}} - \sqrt[m]{\Omega_{\Gamma_1}}\sqrt[m]{\Omega_{\Gamma_2}})}, (\widehat{\Omega}_{\Gamma_1})^{n.m}(\widehat{\Omega}_{\Gamma_2})^{n.m}\right)$
$= (\sqrt[n.m]{\Omega_{\Gamma_1}}, (\widehat{\Omega}_{\Gamma_1})^{n.m}) \oplus (\sqrt[n.m]{\Omega_{\Gamma_2}}, (\widehat{\Omega}_{\Gamma_2})^{n.m})$
$= \Gamma_1^c \oplus \Gamma_2^c$.
3. $(\Gamma^c)^\alpha = (\sqrt[n.m]{\Omega_\Gamma}, (\widehat{\Omega}_\Gamma)^{n.m})^\alpha$
$= \left((\sqrt[n.m]{\Omega_\Gamma})^\alpha, (1-(1-\widehat{\Omega}^n_\Gamma)^\alpha)^m\right)$
$= \left(\sqrt[n]{1-(1-\widehat{\Omega}^n_\Gamma)^\alpha}, \Omega^\alpha_\Gamma\right)^c$
$= (\alpha\Gamma)^c$.
4. $\alpha(\Gamma)^c = \alpha(\sqrt[n.m]{\Omega_\Gamma}, (\widehat{\Omega}_\Gamma)^{n.m}) = \left(\sqrt[n]{1-(1-\sqrt[m]{\Omega_\Gamma})^\alpha}, ((\widehat{\Omega}_\Gamma)^{n.m})^\alpha\right)$
$= \left(\widehat{\Omega}^\alpha_\Gamma, (1-(1-\sqrt[m]{\Omega_\Gamma})^\alpha)^m\right)^c = (\Gamma^\alpha)^c$.

□

4. n,mPR-Fuzzy Weighted Power Average Operator

Definition 4. *Let* $\Gamma_i = (\widehat{\Omega}_{\Gamma_i}, \Omega_{\Gamma_i})$ $(i = 1, 2, ..., r)$ *be a value of n,mPR-FSs and* $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)^T$ *be a weight vector of* Γ_i *with* $\lambda_i > 0$, $\sum_{i=1}^r \lambda_i = 1$, *and* $n, m > 1$. *Then, an n,mPR-fuzzy weighted power average (*$FWPA^n_m$*) operator is a function* $FWPA^n_m$: $\Gamma^r \to \Gamma$, *where*

$$FWPA^n_m(\Gamma_1, \Gamma_2, ..., \Gamma_r) = ((\textstyle\sum_{i=1}^r \lambda_i \widehat{\Omega}^n_{\Gamma_i})^{\frac{1}{n}}, (\sum_{i=1}^r \lambda_i \Omega^{\frac{1}{m}}_{\Gamma_i})^m).$$

Example 2. *Let* $\Gamma_1 = (0.4, 0.1), \Gamma_2 = (0.3, 0.2)$ *and* $\Gamma_3 = (0.1, 0.8)$ *be n,mPR-FSs. If* $\lambda = (0.3, 0.5, 0.2)^T$ *is a weight vector of* Γ_i *(i= 1, 2, 3), then*
$FWPA^n_m(\Gamma_1, \Gamma_2, \Gamma_3) = ((0.3 \times 0.4^n + 0.5 \times 0.3^n + 0.2 \times 0.1^n)^{\frac{1}{n}}, (0.3 \times 0.1^{\frac{1}{m}} + 0.5 \times 0.2^{\frac{1}{m}} + 0.2 \times 0.8^{\frac{1}{m}})^m)$

$$\approx \begin{cases} (0.3082, 0.2188) & \text{for } n = 2 \text{ and } m = 13, \\ (0.3508, 0.2192) & \text{for } n = 8 \text{ and } m = 12, \\ (0.3805, 0.2196) & \text{for } n = 24 \text{ and } m = 11, \\ (0.3865, 0.2202) & \text{for } n = 35 \text{ and } m = 10, \\ (0.3936, 0.2208) & \text{for } n = 75 \text{ and } m = 9, \\ (0.3992, 0.2217) & \text{for } n = 567 \text{ and } m = 8, \\ (0.3994, 0.2228) & \text{for } n = 800 \text{ and } m = 7, \\ (0.3995, 0.2242) & \text{for } n = 885 \text{ and } m = 6, \\ (0.3996, 0.2264) & \text{for } n = 1100 \text{ and } m = 5, \\ (0.3998, 0.2296) & \text{for } n = 2000 \text{ and } m = 4, \\ (0.3292, 0.2352) & \text{for } n = 4 \text{ and } m = 3, \\ (0.3204, 0.2474) & \text{for } n = 3 \text{ and } m = 2. \end{cases}$$

Theorem 9. *Let* $\Gamma_i = (\widehat{\Omega}_{\Gamma_i}, \Omega_{\Gamma_i})$ $(i = 1, 2, ..., r)$ *be a value of n,mPR-FSs and* $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)^T$ *be a weight vector of* Γ_i *with* $\lambda_i > 0$ *and* $\sum_{i=1}^r \lambda_i = 1$. *Then,* $FWPA^n_m(\Gamma_1, \Gamma_2, ..., \Gamma_r)$ *is an n,mPR-FS.*

Proof. For any n,mPR-FS $\Gamma_i = (\widehat{\Omega}_{\Gamma_i}, \Omega_{\Gamma_i})$, we have

$$0 \le \widehat{\Omega}^n_{\Gamma_i} \le 1$$

$$0 \le \Omega^{\frac{1}{m}}_{\Gamma_i} \le 1$$

and

$$0 \le \widehat{\Omega}^n_{\Gamma_i} + \Omega^{\frac{1}{m}}_{\Gamma_i} \le 1;$$

Hence,

$$0 \le \lambda_1 \widehat{\Omega}^n_{\Gamma_1} + \lambda_1 \Omega^{\frac{1}{m}}_{\Gamma_1} \le \lambda_1$$

$$0 \le \lambda_2 \widehat{\Omega}^n_{\Gamma_2} + \lambda_2 \Omega^{\frac{1}{m}}_{\Gamma_2} \le \lambda_2$$

$$\vdots$$

$$0 \le \lambda_r \widehat{\Omega}^n_{\Gamma_r} + \lambda_r \Omega^{\frac{1}{m}}_{\Gamma_r} \le \lambda_r$$

and then

$$0 \le (\lambda_1 \widehat{\Omega}^n_{\Gamma_1} + \lambda_1 \Omega^{\frac{1}{m}}_{\Gamma_1}) + (\lambda_2 \widehat{\Omega}^n_{\Gamma_2} + \lambda_2 \Omega^{\frac{1}{m}}_{\Gamma_2}) + ... + (\lambda_r \widehat{\Omega}^n_{\Gamma_r} + \lambda_r \Omega^{\frac{1}{m}}_{\Gamma_r}) \le \lambda_1 + \lambda_2 + ... + \lambda_r,$$

which implies that

$$0 \le \sum_{i=1}^r \lambda_i \widehat{\Omega}^n_{\Gamma_i} + \sum_{i=1}^r \lambda_i \Omega^{\frac{1}{m}}_{\Gamma_i} \le \sum_{i=1}^r \lambda_i = 1.$$

Thus,

$$0 \le ((\sum_{i=1}^r \lambda_i \widehat{\Omega}^n_{\Gamma_i})^{\frac{1}{n}})^n + ((\sum_{i=1}^r \lambda_i \Omega^{\frac{1}{m}}_{\Gamma_i})^m)^{\frac{1}{m}}$$

$$= \sum_{i=1}^r \lambda_i \widehat{\Omega}^n_{\Gamma_i} + \sum_{i=1}^r \lambda_i \Omega^{\frac{1}{m}}_{\Gamma_i} \le 1.$$

It is clear that

$$0 \leq (\sum_{i=1}^{r} \lambda_i \widehat{\Omega}_{\Gamma_i}^{n})^{\frac{1}{n}} \leq 1$$

and

$$0 \leq (\sum_{i=1}^{r} \lambda_i \Omega_{\Gamma_i}^{\frac{1}{m}})^{m} \leq 1$$

Then, $\mathrm{FWPA}_m^n(\Gamma_1, \Gamma_2, ..., \Gamma_r)$ is an n,mPR-FS. □

Theorem 10. ***(Boundedness)*** *Let $\Gamma_i = (\widehat{\Omega}_{\Gamma_i}, \Omega_{\Gamma_i})$ $(i = 1, 2, ..., r)$ be a number of n,mPR-FSs and $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)^T$ be a weight vector of Γ_i with $\sum_{i=1}^{r} \lambda_i = 1$. Suppose that $\widehat{\Omega}_{\Gamma}^{\circ} = min_{1 \leq i \leq r}\{\widehat{\Omega}_{\Gamma_i}\}$, $\widehat{\Omega}_{\Gamma}^{\bullet} = max_{1 \leq i \leq r}\{\widehat{\Omega}_{\Gamma_i}\}$, $\Omega_{\Gamma}^{\circ} = min_{1 \leq i \leq r}\{\Omega_{\Gamma_i}\}$ and $\Omega_{\Gamma}^{\bullet} = max_{1 \leq i \leq r}\{\Omega_{\Gamma_i}\}$. Then,*

$$(\widehat{\Omega}_{\Gamma}^{\circ}, \Omega_{\Gamma}^{\bullet}) \leq FWPA_m^n(\Gamma_1, \Gamma_2, ..., \Gamma_r) \leq (\widehat{\Omega}_{\Gamma}^{\bullet}, \Omega_{\Gamma}^{\circ}).$$

Proof. For any $\Gamma_i = (\widehat{\Omega}_{\Gamma_i}, \Omega_{\Gamma_i})$ $(i = 1, 2, ..., r)$, we can obtain $\widehat{\Omega}_{\Gamma}^{\circ} \leq \widehat{\Omega}_{\Gamma_i} \leq \widehat{\Omega}_{\Gamma}^{\bullet}$ and $\Omega_{\Gamma}^{\circ} \leq \Omega_{\Gamma_i} \leq \Omega_{\Gamma}^{\bullet}$. Then, the inequalities for the membership value are

$$\widehat{\Omega}_{\Gamma}^{\circ} = (\sum_{i=1}^{r} \lambda_i \widehat{\Omega}_{\Gamma}^{\circ n})^{\frac{1}{n}} \leq (\sum_{i=1}^{r} \lambda_i \widehat{\Omega}_{\Gamma_i}^{n})^{\frac{1}{n}} \leq (\sum_{i=1}^{r} \lambda_i \widehat{\Omega}_{\Gamma}^{\bullet n})^{\frac{1}{n}} = \widehat{\Omega}_{\Gamma}^{\bullet}.$$

Similarly, for the non-membership value

$$\Omega_{\Gamma}^{\circ} = (\sum_{i=1}^{r} \lambda_i \Omega_{\Gamma}^{\circ \frac{1}{m}})^{m} \leq (\sum_{i=1}^{r} \lambda_i \Omega_{\Gamma_i}^{\frac{1}{m}})^{m} \leq (\sum_{i=1}^{r} \lambda_i \Omega_{\Gamma}^{\bullet \frac{1}{m}})^{m} = \Omega_{\Gamma}^{\bullet}.$$

Therefore, $(\widehat{\Omega}_{\Gamma}^{\circ}, \Omega_{\Gamma}^{\bullet}) \leq \mathrm{FWPA}_m^n(\Gamma_1, \Gamma_2, ..., \Gamma_r) \leq (\widehat{\Omega}_{\Gamma}^{\bullet}, \Omega_{\Gamma}^{\circ})$. □

Theorem 11. ***(Monotonicity)*** *Let $\Gamma_i = (\widehat{\Omega}_{\Gamma_i}, \Omega_{\Gamma_i})$ and $L_i = (\widehat{\Omega}_{L_i}, \Omega_{L_i})$ $(i = 1, 2, ..., r)$ be numbers of n,mPR-FSs. If $\widehat{\Omega}_{\Gamma_i} \leq \widehat{\Omega}_{L_i}$ and $\Omega_{\Gamma_i} \geq \Omega_{L_i}$ $\forall i$, then*

$$FWPA_m^n(\Gamma_1, \Gamma_2, ..., \Gamma_r) \leq FWPA_m^n(L_1, L_2, ..., L_r).$$

Proof. Since for all i we have $\widehat{\Omega}_{\Gamma_i} \leq \widehat{\Omega}_{L_i}$ and $\Omega_{\Gamma_i} \geq \Omega_{L_i}$, then $(\sum_{i=1}^{r} \lambda_i \widehat{\Omega}_{\Gamma_i}^{n})^{\frac{1}{n}} \leq (\sum_{i=1}^{r} \lambda_i \widehat{\Omega}_{L_i}^{n})^{\frac{1}{n}}$ and $(\sum_{i=1}^{r} \lambda_i \Omega_{\Gamma_i}^{\frac{1}{m}})^{m} \geq (\sum_{i=1}^{r} \lambda_i \Omega_{L_i}^{\frac{1}{m}})^{m}$. Therefore,
$\mathrm{FWPA}_m^n(\Gamma_1, \Gamma_2, ..., \Gamma_r) = ((\sum_{i=1}^{r} \lambda_i \widehat{\Omega}_{\Gamma_i}^{n})^{\frac{1}{n}}, (\sum_{i=1}^{r} \lambda_i \Omega_{\Gamma_i}^{\frac{1}{m}})^{m})$
$\leq ((\sum_{i=1}^{r} \lambda_i \widehat{\Omega}_{L_i}^{n})^{\frac{1}{n}}, (\sum_{i=1}^{r} \lambda_i \Omega_{L_i}^{\frac{1}{m}})^{m}) = \mathrm{FWPA}_m^n(L_1, L_2, ..., L_r)$. □

Theorem 12. ***(Idempotency)*** *Let $\Gamma_i = (\widehat{\Omega}_{\Gamma_i}, \Omega_{\Gamma_i})$ $(i = 1, 2, ..., r)$ be a number of n,mPR-FSs, such that $\Gamma_i = \Gamma = (\widehat{\Omega}_{\Gamma}, \Omega_{\Gamma})$, and let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)^T$ be a weight vector of Γ_i with $\sum_{i=1}^{r} \lambda_i = 1$. Then, $FWPA_m^n(\Gamma_1, \Gamma_2, ..., \Gamma_r) = \Gamma$.*

Proof. Since $\Gamma_i = \Gamma = (\widehat{\Omega}_{\Gamma}, \Omega_{\Gamma})$ $(i = 1, 2, ..., r)$, then $\mathrm{FWPA}_m^n(\Gamma_1, \Gamma_2, ..., \Gamma_r) =$
$((\sum_{i=1}^{r} \lambda_i \widehat{\Omega}_{\Gamma_i}^{n})^{\frac{1}{n}}, (\sum_{i=1}^{r} \lambda_i \Omega_{\Gamma_i}^{\frac{1}{m}})^{m}) = ((\sum_{i=1}^{r} \lambda_i \widehat{\Omega}_{\Gamma}^{n})^{\frac{1}{n}}, (\sum_{i=1}^{r} \lambda_i \Omega_{\Gamma}^{\frac{1}{m}})^{m}) = (\widehat{\Omega}_{\Gamma}, \Omega_{\Gamma}) = \Gamma$. □

Theorem 13. *Let $\Gamma_i = (\widehat{\Omega}_{\Gamma_i}, \Omega_{\Gamma_i})$ $(i = 1, 2, ..., r)$ be a value of n,mPR-FSs, $\Gamma = (\widehat{\Omega}_{\Gamma}, \Omega_{\Gamma})$ be n,mPR-FS, and $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)^T$ be a weight vector of Γ_i with $\sum_{i=1}^{r} \lambda_i = 1$. Then,*

$$FWPA_m^n(\Gamma_1 \oplus \Gamma, \Gamma_2 \oplus \Gamma, ..., \Gamma_r \oplus \Gamma) \geq FWPA_m^n(\Gamma_1 \otimes \Gamma, \Gamma_2 \otimes \Gamma, ..., \Gamma_r \otimes \Gamma).$$

Proof. For any $\Gamma_i = (\widehat{\Omega}_{\Gamma_i}, \Omega_{\Gamma_i})$ $(i = 1, 2, ..., r)$ and $\Gamma = (\widehat{\Omega}_{\Gamma}, \Omega_{\Gamma})$, we have

$$\widehat{\Omega}_{\Gamma_i}^{n} + \widehat{\Omega}_{\Gamma}^{n} - \widehat{\Omega}_{\Gamma_i}^{n}\widehat{\Omega}_{\Gamma}^{n} \geq 2\widehat{\Omega}_{\Gamma_i}^{n}\widehat{\Omega}_{\Gamma}^{n} - \widehat{\Omega}_{\Gamma_i}^{n}\widehat{\Omega}_{\Gamma}^{n} = \widehat{\Omega}_{\Gamma_i}^{n}\widehat{\Omega}_{\Gamma}^{n}$$

and

$\Omega_{\Gamma_i}^{\frac{1}{m}} + \Omega_{\Gamma}^{\frac{1}{m}} - \Omega_{\Gamma_i}^{\frac{1}{m}}\Omega_{\Gamma}^{\frac{1}{m}} \geq 2\Omega_{\Gamma_i}^{\frac{1}{m}}\Omega_{\Gamma}^{\frac{1}{n}} - \Omega_{\Gamma_i}^{\frac{1}{m}}\Omega_{\Gamma}^{\frac{1}{m}} = \Omega_{\Gamma_i}^{\frac{1}{m}}\Omega_{\Gamma}^{\frac{1}{m}}$.

That is,

$\sum_{i=1}^{r} \lambda_i(\widehat{\Omega}_{\Gamma_i}^{n} + \widehat{\Omega}_{\Gamma}^{n} - \widehat{\Omega}_{\Gamma_i}^{n}\widehat{\Omega}_{\Gamma}^{n}) \geq \sum_{i=1}^{r} \lambda_i\widehat{\Omega}_{\Gamma_i}^{n}\widehat{\Omega}_{\Gamma}^{n}$

$\Rightarrow (\sum_{i=1}^{r} \lambda_i(\widehat{\Omega}_{\Gamma_i}^{n} + \widehat{\Omega}_{\Gamma}^{n} - \widehat{\Omega}_{\Gamma_i}^{n}\widehat{\Omega}_{\Gamma}^{n}))^{\frac{1}{n}} \geq (\sum_{i=1}^{r} \lambda_i\widehat{\Omega}_{\Gamma_i}^{n}\widehat{\Omega}_{\Gamma}^{n})^{\frac{1}{n}}$ ——(1)

and

$\sum_{i=1}^{r} \lambda_i(\Omega_{\Gamma_i}^{\frac{1}{m}} + \Omega_{\Gamma}^{\frac{1}{m}} - \Omega_{\Gamma_i}^{\frac{1}{m}}\Omega_{\Gamma}^{\frac{1}{m}}) \geq \sum_{i=1}^{r} \lambda_i\Omega_{\Gamma_i}^{\frac{1}{m}}\Omega_{\Gamma}^{\frac{1}{m}}$

$\Rightarrow (\sum_{i=1}^{r} \lambda_i(\Omega_{\Gamma_i}^{\frac{1}{m}} + \Omega_{\Gamma}^{\frac{1}{m}} - \Omega_{\Gamma_i}^{\frac{1}{m}}\Omega_{\Gamma}^{\frac{1}{m}}))^{m} \geq (\sum_{i=1}^{r} \lambda_i\Omega_{\Gamma_i}^{\frac{1}{m}}\Omega_{\Gamma}^{\frac{1}{m}})^{m}$. ——(2)

Thus, we have

$\mathrm{FWPA}_m^n(\Gamma_1 \oplus \Gamma, \Gamma_2 \oplus \Gamma, ..., \Gamma_r \oplus \Gamma) = ((\sum_{i=1}^{r} \lambda_i(\widehat{\Omega}_{\Gamma_i}^{n} + \widehat{\Omega}_{\Gamma}^{n} - \widehat{\Omega}_{\Gamma_i}^{n}\widehat{\Omega}_{\Gamma}^{n}))^{\frac{1}{n}}, (\sum_{i=1}^{r} \lambda_i\Omega_{\Gamma_i}^{\frac{1}{m}}\Omega_{\Gamma}^{\frac{1}{m}})^{m})$

and

$\mathrm{FWPA}_m^n(\Gamma_1 \otimes \Gamma, \Gamma_2 \otimes \Gamma, ..., \Gamma_r \otimes \Gamma) = ((\sum_{i=1}^{r} \lambda_i\widehat{\Omega}_{\Gamma_i}^{n}\widehat{\Omega}_{\Gamma}^{n})^{\frac{1}{n}}, (\sum_{i=1}^{r} \lambda_i(\Omega_{\Gamma_i}^{\frac{1}{m}} + \Omega_{\Gamma}^{\frac{1}{m}} - \Omega_{\Gamma_i}^{\frac{1}{m}}\Omega_{\Gamma}^{\frac{1}{m}}))^{m})$.

Then, from (1) and (2), we obtain
$\mathrm{FWPA}_m^n(\Gamma_1 \oplus \Gamma, \Gamma_2 \oplus \Gamma, ..., \Gamma_r \oplus \Gamma) \geq \mathrm{FWPA}_m^n(\Gamma_1 \otimes \Gamma, \Gamma_2 \otimes \Gamma, ..., \Gamma_r \otimes \Gamma)$.
□

Theorem 14. *Let $\Gamma_i = (\widehat{\Omega}_{\Gamma_i}, \Omega_{\Gamma_i})$ $(i = 1, 2, ..., r)$ be a value of n,mPR-FSs, $\Gamma = (\widehat{\Omega}_{\Gamma}, \Omega_{\Gamma})$ be n,mPR-FS, and $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)^T$ be a weight vector of Γ_i with $\sum_{i=1}^{r} \lambda_i = 1$. Then,*

1. *$FWPA_m^n(\Gamma_1 \oplus \Gamma, \Gamma_2 \oplus \Gamma, ..., \Gamma_r \oplus \Gamma) \geq FWPA_m^n(\Gamma_1, \Gamma_2, ..., \Gamma_r) \otimes \Gamma$.*
2. *$FWPA_m^n(\Gamma_1, \Gamma_2, ..., \Gamma_r) \oplus \Gamma \geq FWPA_m^n(\Gamma_1, \Gamma_2, ..., \Gamma_r) \otimes \Gamma$.*

Proof. We provide proof of (1). The other assumption is verified in a similar way. Since for any $\Gamma_i = (\widehat{\Omega}_{\Gamma_i}, \Omega_{\Gamma_i})$ $(i = 1, 2, ..., r)$ and $\Gamma = (\widehat{\Omega}_{\Gamma}, \Omega_{\Gamma})$, we have

$(\sum_{i=1}^{r} \lambda_i(\widehat{\Omega}_{\Gamma_i}^{n} + \widehat{\Omega}_{\Gamma}^{n} - \widehat{\Omega}_{\Gamma_i}^{n}\widehat{\Omega}_{\Gamma}^{n}))^{\frac{1}{n}} \geq (\sum_{i=1}^{r} \lambda_i\widehat{\Omega}_{\Gamma_i}^{n}\widehat{\Omega}_{\Gamma}^{n})^{\frac{1}{n}} = (\sum_{i=1}^{r} \lambda_i\widehat{\Omega}_{\Gamma_i}^{n})^{\frac{1}{n}}\widehat{\Omega}_{\Gamma}$. ——(1)

Similarly,

$(\sum_{i=1}^{r} \lambda_i(\Omega_{\Gamma_i}^{\frac{1}{m}} + \Omega_{\Gamma}^{\frac{1}{m}} - \Omega_{\Gamma_i}^{\frac{1}{m}}\Omega_{\Gamma}^{\frac{1}{m}}))^{m} \geq (\sum_{i=1}^{r} \lambda_i\Omega_{\Gamma_i}^{\frac{1}{m}}\Omega_{\Gamma}^{\frac{1}{m}})^{m} = (\sum_{i=1}^{r} \lambda_i\Omega_{\Gamma_i}^{\frac{1}{m}})^{m}\Omega_{\Gamma}$. ——(2)

Thus, we have

$$\mathrm{FWPA}_m^n(\Gamma_1 \oplus \Gamma, \Gamma_2 \oplus \Gamma, ..., \Gamma_r \oplus \Gamma)$$
$$= ((\sum_{i=1}^{r} \lambda_i(\widehat{\Omega}_{\Gamma_i}^{n} + \widehat{\Omega}_{\Gamma}^{n} - \widehat{\Omega}_{\Gamma_i}^{n}\widehat{\Omega}_{\Gamma}^{n}))^{\frac{1}{n}}, (\sum_{i=1}^{r} \lambda_i\Omega_{\Gamma_i}^{\frac{1}{m}}\Omega_{\Gamma}^{\frac{1}{m}})^{m})$$

and

$$\mathrm{FWPA}_m^n(\Gamma_1, \Gamma_2, ..., \Gamma_r) \otimes \Gamma$$
$$= ((\sum_{i=1}^{r} \lambda_i\widehat{\Omega}_{\Gamma_i}^{n})^{\frac{1}{n}}, (\sum_{i=1}^{r} \lambda_i\Omega_{\Gamma_i}^{\frac{1}{m}})^{m}) \otimes (\widehat{\Omega}_{\Gamma}, \Omega_{\Gamma})$$
$$= ((\sum_{i=1}^{r} \lambda_i\widehat{\Omega}_{\Gamma_i}^{n})^{\frac{1}{n}}\widehat{\Omega}_{\Gamma}, (\sum_{i=1}^{r} \lambda_i\Omega_{\Gamma_i}^{\frac{1}{m}} + \Omega_{\Gamma}^{\frac{1}{m}} - \sum_{i=1}^{r} \lambda_i\Omega_{\Gamma_i}^{\frac{1}{m}}\Omega_{\Gamma}^{\frac{1}{m}})^{m}).$$

Therefore, from (1) and (2), we have

$$\mathrm{FWPA}_m^n(\Gamma_1 \oplus \Gamma, \Gamma_2 \oplus \Gamma, ..., \Gamma_r \oplus \Gamma) \geq \mathrm{FWPA}_m^n(\Gamma_1, \Gamma_2, ..., \Gamma_r) \otimes \Gamma.$$
□

Theorem 15. *Let $\Gamma_i = (\widehat{\Omega}_{\Gamma_i}, \Omega_{\Gamma_i})$ and $L_i = (\widehat{\Omega}_{L_i}, \Omega_{L_i})$ $(i = 1, 2, ..., r)$ be values of n,mPR-FSs and $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)^T$ be a weight vector of them with $\sum_{i=1}^r \lambda_i = 1$. Then,*

1. $FWPA_m^n(\Gamma_1 \oplus L_1, \Gamma_2 \oplus L_2, ..., \Gamma_r \oplus L_r) \geq FWPA_m^n(\Gamma_1 \otimes L_1, \Gamma_2 \otimes L_2, ..., \Gamma_r \otimes L_r)$.
2. $FWPA_m^n(\Gamma_1, \Gamma_2, ..., \Gamma_r) \oplus FWPA_m^n(L_1, L_2, ..., L_r) \geq$ $FWPA_m^n(\Gamma_1, \Gamma_2, ..., \Gamma_r) \otimes FWPA_m^n(L_1, L_2, ..., L_r)$.

Proof. For any $\Gamma_i = (\widehat{\Omega}_{\Gamma_i}, \Omega_{\Gamma_i})$ and $L_i = (\widehat{\Omega}_{L_i}, \Omega_{L_i})$ $(i = 1, 2, ..., r)$, we have

1. $\widehat{\Omega}_{\Gamma_i}^n + \widehat{\Omega}_{L_i}^n - \widehat{\Omega}_{\Gamma_i}^n \widehat{\Omega}_{L_i}^n \geq 2\widehat{\Omega}_{\Gamma_i}^n \widehat{\Omega}_{L_i}^n - \widehat{\Omega}_{\Gamma_i}^n \widehat{\Omega}_{L_i}^n = \widehat{\Omega}_{\Gamma_i}^n \widehat{\Omega}_{L_i}^n$
   and
   $$\Omega_{\Gamma_i}^{\frac{1}{m}} + \Omega_{L_i}^{\frac{1}{m}} - \Omega_{\Gamma_i}^{\frac{1}{m}} \Omega_{L_i}^{\frac{1}{m}} \geq 2\Omega_{\Gamma_i}^{\frac{1}{m}} \Omega_{L_i}^{\frac{1}{m}} - \Omega_{\Gamma_i}^{\frac{1}{m}} \Omega_{L_i}^{\frac{1}{m}} = \Omega_{\Gamma_i}^{\frac{1}{m}} \Omega_{L_i}^{\frac{1}{m}}.$$
   That is,
   $$\sum_{i=1}^r \lambda_i (\widehat{\Omega}_{\Gamma_i}^n + \widehat{\Omega}_{L_i}^n - \widehat{\Omega}_{\Gamma_i}^n \widehat{\Omega}_{L_i}^n) \geq \sum_{i=1}^r \lambda_i \widehat{\Omega}_{\Gamma_i}^n \widehat{\Omega}_{L_i}^n$$
   $$\Rightarrow (\sum_{i=1}^r \lambda_i (\widehat{\Omega}_{\Gamma_i}^n + \widehat{\Omega}_{L_i}^n - \widehat{\Omega}_{\Gamma_i}^n \widehat{\Omega}_{L_i}^n))^{\frac{1}{n}} \geq (\sum_{i=1}^r \lambda_i \widehat{\Omega}_{\Gamma_i}^n \widehat{\Omega}_{L_i}^n)^{\frac{1}{n}} \text{ ——(*)}$$
   and
   $$\sum_{i=1}^r \lambda_i (\Omega_{\Gamma_i}^{\frac{1}{m}} + \Omega_{L_i}^{\frac{1}{m}} - \Omega_{\Gamma_i}^{\frac{1}{m}} \Omega_{L_i}^{\frac{1}{m}}) \geq \sum_{i=1}^r \lambda_i \Omega_{\Gamma_i}^{\frac{1}{m}} \Omega_{L_i}^{\frac{1}{m}}$$
   $$\Rightarrow (\sum_{i=1}^r \lambda_i (\Omega_{\Gamma_i}^{\frac{1}{m}} + \Omega_{L_i}^{\frac{1}{m}} - \Omega_{\Gamma_i}^{\frac{1}{m}} \Omega_{L_i}^{\frac{1}{m}}))^m \geq (\sum_{i=1}^r \lambda_i \Omega_{\Gamma_i}^{\frac{1}{m}} \Omega_{\Gamma}^{\frac{1}{m}})^m. \text{ ——(**)}$$
   Hence, we have
   $$\text{FWPA}_m^n(\Gamma_1 \oplus L_1, \Gamma_2 \oplus L_2, ..., \Gamma_r \oplus L_r)$$
   $$= ((\sum_{i=1}^r \lambda_i (\widehat{\Omega}_{\Gamma_i}^n + \widehat{\Omega}_{L_i}^n - \widehat{\Omega}_{\Gamma_i}^n \widehat{\Omega}_{L_i}^n))^{\frac{1}{n}}, (\sum_{i=1}^r \lambda_i \Omega_{\Gamma_i}^{\frac{1}{m}} \Omega_{L_i}^{\frac{1}{m}})^m)$$
   and
   $$\text{FWPA}_m^n(\Gamma_1 \otimes L_1, \Gamma_2 \otimes L_2, ..., \Gamma_r \otimes L_r)$$
   $$= ((\sum_{i=1}^r \lambda_i \widehat{\Omega}_{\Gamma_i}^n \widehat{\Omega}_{L_i}^n)^{\frac{1}{n}}, (\sum_{i=1}^r \lambda_i (\Omega_{\Gamma_i}^{\frac{1}{m}} + \Omega_{L_i}^{\frac{1}{m}} - \Omega_{\Gamma_i}^{\frac{1}{m}} \Omega_{L_i}^{\frac{1}{m}}))^m)$$
   Thus, from (*) and (**) we obtain
   $$\text{FWPA}_m^n(\Gamma_1 \oplus L_1, \Gamma_2 \oplus L_2, ..., \Gamma_r \oplus L_r) \geq \text{FWPA}_m^n(\Gamma_1 \otimes L_1, \Gamma_2 \otimes L_2, ..., \Gamma_r \otimes L_r).$$
2. Since
   $$\sum_{i=1}^r \lambda_i \widehat{\Omega}_{\Gamma_i}^n \geq \sum_{i=1}^r \lambda_i \widehat{\Omega}_{\Gamma_i}^n \sum_{i=1}^r \lambda_i \widehat{\Omega}_{L_i}^n$$
   and
   $$\sum_{i=1}^r \lambda_i \widehat{\Omega}_{L_i}^n \geq \sum_{i=1}^r \lambda_i \widehat{\Omega}_{\Gamma_i}^n \sum_{i=1}^r \lambda_i \widehat{\Omega}_{L_i}^n,$$
   $$\sum_{i=1}^r \lambda_i \widehat{\Omega}_{\Gamma_i}^n + \sum_{i=1}^r \lambda_i \widehat{\Omega}_{L_i}^n \geq \sum_{i=1}^r \lambda_i \widehat{\Omega}_{\Gamma_i}^n \sum_{i=1}^r \lambda_i \widehat{\Omega}_{L_i}^n + \sum_{i=1}^r \lambda_i \widehat{\Omega}_{\Gamma_i}^n \sum_{i=1}^r \lambda_i \widehat{\Omega}_{L_i}^n,$$
   which implies that
   $$\sum_{i=1}^r \lambda_i \widehat{\Omega}_{\Gamma_i}^n + \sum_{i=1}^r \lambda_i \widehat{\Omega}_{L_i}^n - \sum_{i=1}^r \lambda_i \widehat{\Omega}_{\Gamma_i}^n \sum_{i=1}^r \lambda_i \widehat{\Omega}_{L_i}^n \geq \sum_{i=1}^r \lambda_i \widehat{\Omega}_{\Gamma_i}^n \sum_{i=1}^r \lambda_i \widehat{\Omega}_{L_i}^n,$$
   and hence

$$\sqrt[n]{\sum_{i=1}^r \lambda_i \widehat{\Omega}_{\Gamma_i}^n + \sum_{i=1}^r \lambda_i \widehat{\Omega}_{L_i}^n - \sum_{i=1}^r \lambda_i \widehat{\Omega}_{\Gamma_i}^n \sum_{i=1}^r \lambda_i \widehat{\Omega}_{L_i}^n} \geq \sqrt[n]{\sum_{i=1}^r \lambda_i \widehat{\Omega}_{\Gamma_i}^n \sum_{i=1}^r \lambda_i \widehat{\Omega}_{L_i}^n} \text{ ——(***)}$$

Similarly,

$$(\sum_{i=1}^{r}\lambda_i\Omega_{\Gamma_i}^{\frac{1}{m}}+\sum_{i=1}^{r}\lambda_i\Omega_{L_i}^{\frac{1}{m}}-\sum_{i=1}^{r}\lambda_i\Omega_{\Gamma_i}^{\frac{1}{m}}\sum_{i=1}^{r}\lambda_i\Omega_{L_i}^{\frac{1}{m}})^m \geq (\sum_{i=1}^{r}\lambda_i\Omega_{\Gamma_i}^{\frac{1}{m}}\sum_{i=1}^{r}\lambda_i\Omega_{L_i}^{\frac{1}{m}})^m \text{ ——(****)}$$

Hence, we have

$$\begin{aligned}
&\mathrm{FWPA}_m^n(\Gamma_1,\Gamma_2,\ldots,\Gamma_r)\oplus nPR\text{-}FWPA(L_1,L_2,\ldots,L_r)\\
&=((\sum_{i=1}^{r}\lambda_i\widehat{\Omega}_{\Gamma_i}^n)^{\frac{1}{n}},(\sum_{i=1}^{r}\lambda_i\Omega_{\Gamma_i}^{\frac{1}{m}})^m)\oplus((\sum_{i=1}^{r}\lambda_i\widehat{\Omega}_{L_i}^n)^{\frac{1}{n}},(\sum_{i=1}^{r}\lambda_i\Omega_{L_i}^{\frac{1}{m}})^m)\\
&=(\sqrt[n]{\sum_{i=1}^{r}\lambda_i\widehat{\Omega}_{\Gamma_i}^n+\sum_{i=1}^{r}\lambda_i\widehat{\Omega}_{L_i}^n-\sum_{i=1}^{r}\lambda_i\widehat{\Omega}_{\Gamma_i}^n\sum_{i=1}^{r}\lambda_i\widehat{\Omega}_{L_i}^n},(\sum_{i=1}^{r}\lambda_i\Omega_{\Gamma_i}^{\frac{1}{m}})^m(\sum_{i=1}^{r}\lambda_i\Omega_{L_i}^{\frac{1}{m}})^m)
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{FWPA}_m^n(\Gamma_1,\Gamma_2,\ldots,\Gamma_r)\otimes nPR\text{-}FWPA(L_1,L_2,\ldots,L_r)\\
&=((\sum_{i=1}^{r}\lambda_i\widehat{\Omega}_{\Gamma_i}^n)^{\frac{1}{n}},(\sum_{i=1}^{r}\lambda_i\Omega_{\Gamma_i}^{\frac{1}{m}})^m)\otimes((\sum_{i=1}^{r}\lambda_i\widehat{\Omega}_{L_i}^n)^{\frac{1}{n}},(\sum_{i=1}^{r}\lambda_i\Omega_{L_i}^{\frac{1}{m}})^m)\\
&=((\sum_{i=1}^{r}\lambda_i\widehat{\Omega}_{\Gamma_i}^n)^{\frac{1}{n}}(\sum_{i=1}^{r}\lambda_i\widehat{\Omega}_{L_i}^n)^{\frac{1}{n}},(\sum_{i=1}^{r}\lambda_i\Omega_{\Gamma_i}^{\frac{1}{m}}+\sum_{i=1}^{r}\lambda_i\Omega_{L_i}^{\frac{1}{m}}-\sum_{i=1}^{r}\lambda_i\Omega_{\Gamma_i}^{\frac{1}{m}}\sum_{i=1}^{r}\lambda_i\Omega_{L_i}^{\frac{1}{m}})^m).
\end{aligned}$$

Thus, from (***) and (****), we obtain

$$\begin{aligned}
&\mathrm{FWPA}_m^n(\Gamma_1,\Gamma_2,\ldots,\Gamma_r)\oplus nPR\text{-}FWPA(L_1,L_2,\ldots,L_r)\geq\\
&\mathrm{FWPA}_m^n(\Gamma_1,\Gamma_2,\ldots,\Gamma_r)\otimes nPR\text{-}FWPA(L_1,L_2,\ldots,L_r).
\end{aligned}$$

□

For the purpose of ranking the n,mPR-FSs, we present the score and accuracy functions of the n,mPR-FS.

Definition 5. *Let $\Gamma=(\widehat{\Omega}_\Gamma,\Omega_\Gamma)$ be an n,mPR-FS. Then,*

1. *the score of Γ is given as $\triangle(\Gamma)=\widehat{\Omega}_\Gamma^n-\sqrt[m]{\Omega_\Gamma}$, and*
2. *the accuracy of Γ is given as $\bigtriangledown(\Gamma)=\widehat{\Omega}_\Gamma^n+\sqrt[m]{\Omega_\Gamma}$.*

Example 3. *Consider that $\Gamma=(0.7,0.4)$ is an n,mPR-FS. Then,*

$$\triangle(\Gamma)\approx\begin{cases}-0.2895 & \text{for } n=3 \text{ and } m=2,\\ -0.4967 & \text{for } n=4 \text{ and } m=3,\\ -0.7149 & \text{for } n=6 \text{ and } m=5,\end{cases}$$

and

$$\bigtriangledown(\Gamma)\approx\begin{cases}0.9755 & \text{for } n=3 \text{ and } m=2,\\ 0.9769 & \text{for } n=4 \text{ and } m=3,\\ 0.9502 & \text{for } n=6 \text{ and } m=5.\end{cases}$$

Theorem 16. *Let $\Gamma=(\widehat{\Omega}_\Gamma,\Omega_\Gamma)$ be any n,mPR-FS. Then,*

1. $\triangle(\Gamma)\in[-1,1]$.
2. $\bigtriangledown(\Gamma)\in[0,1]$.

Proof. 1. For any n,mPR-FS Γ, we have $\widehat{\Omega}_\Gamma^n+\Omega_\Gamma^{\frac{1}{m}}\leq 1$. Hence, $\widehat{\Omega}_\Gamma^n-\Omega_\Gamma^{\frac{1}{m}}\leq\widehat{\Omega}_\Gamma^n\leq 1$ and $\widehat{\Omega}_\Gamma^n-\Omega_\Gamma^{\frac{1}{m}}\geq-\Omega_\Gamma^{\frac{1}{m}}\geq-1$. Thus, $-1\leq\widehat{\Omega}_\Gamma^n-\Omega_\Gamma^{\frac{1}{m}}\leq 1$, namely $\triangle(\Gamma)\in[-1,1]$. If $\Gamma=(0,1)$, then $\triangle(\Gamma)=-1$, and if $\Gamma=(1,0)$, then $\triangle(\Gamma)=1$.

2. The proof is clear.

□

Definition 6. *For any n,mPR-FSs $\Gamma_i = (\widehat{\Omega}_{\Gamma_i}, \Omega_{\Gamma_i})$, the comparative approach is designed as the following:*

1. *if $\triangle(\Gamma_1) < \triangle(\Gamma_2)$, then $\Gamma_1 \prec \Gamma_2$,*
2. *if $\triangle(\Gamma_1) > \triangle(\Gamma_2)$, then $\Gamma_1 \succ \Gamma_2$,*
3. *if $\triangle(\Gamma_1) = \triangle(\Gamma_2)$, then*
   (a) if $\bigtriangledown(\Gamma_1) < \bigtriangledown(\Gamma_2)$, then $\Gamma_1 \prec \Gamma_2$,
   (b) if $\bigtriangledown(\Gamma_1) > \bigtriangledown(\Gamma_2)$, then $\Gamma_1 \succ \Gamma_2$,
   (c) if $\bigtriangledown(\Gamma_1) = \bigtriangledown(\Gamma_2)$, then $\Gamma_1 \approx \Gamma_2$.

Application of n,mPR-fuzzy sets:

The following involves the application of an FWPA^n_m operator to MCDM problems in order to evaluate alternatives using n,mPR-fuzzy information. The following steps are generally incorporated within the suggested approach:

Step 1: For an MCDM problem, we build the n,mPR-fuzzy decision matrix $R = (a_{ij})_{t_2 \times t_1}$ containing the values of n,mPR-FSs, where the elements a_{ij} $(j = 1, 2, ..., t_1, i = 1, 2, ..., t_2)$ are the appraisals of the alternative $C_i \in C$ regarding the criterion $G_j \in G$.

Step 2: Create a normalized n,mPR-fuzzy decision matrix from the n,mPR-fuzzy decision matrix $R = (a_{ij})_{t_2 \times t_1}$.

Step 3: The proposed FWPA^n_m operator is used to calculate alternative choice values with related weights.

Step 4: Determine the scores for the n,mPR-FSs values that were acquired in Step 3.

Step 5: The best option can be found by utilizing Definition 6 to establish the optimal ranking order of the alternatives.

We use a real-world example of choosing a particular country utilizing n,mPR-fuzzy data to illustrate the proposed approach. Based on statistical comparisons of each country's performance on a number of important business, economic, and quality of life variables, the best countries for living can be determined. We can determine the best country for life based on the ten criteria stated below:

1. **Agility:** is adaptable, dynamic, responsive, and so on.
2. **Cultural Influence:** is culturally significant in terms of entertainment, fashion, influential culture, prestige, and so on.
3. **Open for Business:** has cheap manufacturing costs, a favorable tax environment, transparent government practices, and so on.
4. **Social Purpose:** cares about human rights, cares about the environment, gender equality, religious freedom, respects property rights, and so on.
5. **Power:** a leader, economically influential, has strong exports, has strong international alliances, has strong military services, and so on.
6. **Movers:** different, distinctive, dynamic, unique, and so on.
7. **Adventure:** friendly, good for tourism, pleasant climate, scenic, and so on.
8. **Heritage:** has a rich history, many cultural attractions, many geographical attractions, and so on.
9. **Entrepreneurship:** has a skilled labor force, technological expertise, well-developed infrastructure, and so on.
10. **Quality of Life:** has a good job market and income equality, is politically stable and safe, and has a well-developed public education system, well-developed public health system, and so on.

The aforementioned criteria were created by grouping country characteristics from the study's findings that showed comparable global tendencies.

Let $C = \{C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8\}$ be a set of alternatives (countries), where
C_1 = Japan,
C_2 = Germany,
C_3 = Sweden,
C_4 = Switzerland,

C_5 = Australia,
C_6 = Canada,
C_7 = United States, and
C_8 = United Kingdom.

And, let $G = \{G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8, G_9, G_{10}\}$ be a set of criteria for the selection of countries (category), where
G_1 = Adventure,
G_2 = Agility,
G_3 = Cultural Influence,
G_4 = Entrepreneurship,
G_5 = Heritage,
G_6 = Movers,
G_7 = Open for Business,
G_8 = Power,
G_9 = Quality of Life, and
G_{10} = Social Purpose.

The construction of the n,mPR-fuzzy set decision-making matrix is shown in Table 1. It is shown that the degree to which country C_i meets those requirements G_i is $\widehat{\Omega}_{C_i}$ and the degree to which country C_i fails to meet those requirements G_i is Ω_{C_i}, such that $(\widehat{\Omega}_{C_i})^n + \Omega_{C_i}^{\frac{1}{m}} \leq 1$ for $\widehat{\Omega}_{C_i}, \Omega_{C_i} \in [0,1]$. The following was decided upon as the weight vector for the criteria: $\lambda = (0.0548, 0.1396, 0.1036, 0.1417, 0.0309, 0.1057, 0.0935, 0.0502, 0.1452, 0.1348)^T$. G_9 is given more importance, while G_5 is given a lower value.

Table 1. n,mPR-fuzzy values.

Countries/Category	G_1	G_2	G_3	G_4	G_5
Japan	(0.45, 0.28)	(0.83, 0.03)	(0.81, 0.04)	(0.96, 0)	(0.76, 0.07)
Germany	(0.26, 0.42)	(0.85, 0.02)	(0.60, 0.09)	(1, 0)	(0.49, 0.21)
Sweden	(0.65, 0.09)	(0.73, 0.06)	(0.56, 0.08)	(0.73, 0.09)	(0.52, 0.02)
Switzerland	(0.61, 0.14)	(0.70, 0.13)	(0.63, 0.08)	(0.81, 0.04)	(0.45, 0.24)
Australia	(0.75, 0.08)	(0.79, 0.05)	(0.58, 0.10)	(0.66, 0.14)	(0.52, 0.19)
Canada	(0.56, 0.17)	(0.82, 0.03)	(0.55, 0.15)	(0.76, 0.07)	(0.39, 0.28)
United States	(0.42, 0.32)	(1, 0)	(0.85, 0.02)	(0.99, 0)	(0.51, 0.20)
United Kingdom	(0.37, 0.37)	(0.73, 0.10)	(0.78, 0.06)	(0.81, 0.04)	(0.61, 0.14)
Countries/Category	G_6	G_7	G_8	G_9	G_{10}
Japan	(0.63, 0.08)	(0.55, 0.33)	(0.63, 0.08)	(0.71, 0.12)	(0.25, 0.23)
Germany	(0.18, 0.44)	(0.62, 0.23)	(0.81, 0.04)	(0.86, 0.01)	(0.70, 0.13)
Sweden	(0.66, 0.17)	(0.71, 0.07)	(0.62, 0.22)	(1, 0)	(0.42, 0.08)
Switzerland	(0.33, 0.26)	(1, 0)	(0.26, 0.20)	(0.88, 0.01)	(0.86, 0.01)
Australia	(0.30, 0.29)	(0.68, 0.15)	(0.33, 0.16)	(0.68, 0.06)	(0.79, 0.05)
Canada	(0.36, 0.63)	(0.74, 0.01)	(0.66, 0.05)	(0.85, 0.02)	(0.88, 0.01)
United States	(0.32, 0.27)	(0.43, 0.53)	(1, 0)	(0.53, 0.21)	(0.46, 0.30)
United Kingdom	(0.12, 0.65)	(0.56, 0.32)	(0.79, 0.05)	(0.71, 0.12)	(0.69, 0.13)

Now, using the weight vectors
$\lambda = (0.0548, 0.1396, 0.1036, 0.1417, 0.0309, 0.1057, 0.0935, 0.0502, 0.1452, 0.1348)^T$, $n = 2, 3, 4$ and $m = 2, 3$, we use the FWPA_m^n operator in Table 2, as follows:

Now, as stated in Table 3, we determine the score value for each option and the rankings are presented in Table 4.

Table 2. Aggregated n,mPR-fuzzy information matrix.

Countries/Operator	$\mathbf{FWPA}_3^2$	$\mathbf{FWPA}_2^3$	$\mathbf{FWPA}_2^4$	$\mathbf{FWPA}_3^4$
Japan	(0.7039, 0.0706)	(0.7290, 0.0869)	(0.7487, 0.0869)	(0.7487, 0.0706)
Germany	(0.7326, 0.0631)	(0.7621, 0.0833)	(0.7838, 0.0833)	(0.7838, 0.0631)
Sweden	(0.7038, 0.0554)	(0.7231, 0.0660)	(0.7418, 0.0660)	(0.7418, 0.0554)
Switzerland	(0.7414, 0.0501)	(0.7647, 0.0615)	(0.7828, 0.0615)	(0.7828, 0.0501)
Australia	(0.6575, 0.1024)	(0.6720, 0.1056)	(0.6830, 0.1056)	(0.6830, 0.1024)
Canada	(0.7246, 0.0665)	(0.7406, 0.0780)	(0.7533, 0.0780)	(0.7533, 0.0665)
United States	(0.7202, 0.0660)	(0.7633, 0.1042)	(0.7980, 0.1042)	(0.7980, 0.0660)
United Kingdom	(0.6700, 0.1432)	(0.6884, 0.1529)	(0.7003, 0.1529)	(0.7003, 0.1432)

Table 3. Score values.

Scores/n,m	2,3	3,2	4,2	4,3
$\triangle(C_1)$	0.0821	0.0926	0.0195	−0.0991
$\triangle(C_2)$	0.1386	0.1540	0.0887	−0.0208
$\triangle(C_3)$	0.1141	0.1212	0.0459	−0.0785
$\triangle(C_4)$	0.1811	0.1993	0.1275	0.0069
$\triangle(C_5)$	−0.0355	−0.0215	−0.1073	−0.2503
$\triangle(C_6)$	0.1199	0.1269	0.0427	−0.0832
$\triangle(C_7)$	0.1145	0.1219	0.0828	0.0014
$\triangle(C_8)$	−0.0742	−0.0648	−0.1505	−0.2826

Table 4. Ranking using score values.

Operators	Ranking	Best Country
FWPA_3^2	$C_4 \succ C_2 \succ C_6 \succ C_7 \succ C_3 \succ C_1 \succ C_5 \succ C_8$	C_4
FWPA_2^3	$C_4 \succ C_2 \succ C_6 \succ C_7 \succ C_3 \succ C_1 \succ C_5 \succ C_8$	C_4
FWPA_2^4	$C_4 \succ C_2 \succ C_7 \succ C_3 \succ C_6 \succ C_1 \succ C_5 \succ C_8$	C_4
FWPA_3^4	$C_4 \succ C_7 \succ C_2 \succ C_3 \succ C_6 \succ C_1 \succ C_5 \succ C_8$	C_4

To show how the parameters n and m affected the final results of the MADM, we utilized several values of n and m to rank the choices. The outcomes of the ranking of the alternatives based on the FWPA_m^n operator are displayed in Table 4, as follows:

1. When $n = 2, 3$ and $m = 2, 3$, as a result, we obtained a ranking of options as follows: $C_4 \succ C_2 \succ C_6 \succ C_7 \succ C_3 \succ C_1 \succ C_5 \succ C_8$.
2. When $n = 4$ and $m = 2$, as a result, we obtained a ranking of options as follows: $C_4 \succ C_2 \succ C_7 \succ C_3 \succ C_6 \succ C_1 \succ C_5 \succ C_8$.
3. When $n = 4$ and $m = 3$, as a result, we obtained a ranking of options as follows: $C_4 \succ C_7 \succ C_2 \succ C_3 \succ C_6 \succ C_1 \succ C_5 \succ C_8$.

Thus, the finest option worldwide is Switzerland.

5. Comparison Analysis and Discussion

In order to illustrate the advantages of the suggested models, we compare the suggested FWPA_m^n operator with various well-known operators using n,mPR-fuzzy numbers.

Here, we use our data in accordance with the hybrid model by which we must compare our proposed model in order to verify the accuracy and efficacy of our generated hybrid model. Table 5 provides a summary of the computed results using the currently used square-root fuzzy weighted power average (SR-FWPA) operator [15], the nth power root fuzzy weighted power average (nPR-FWPA) operator [35], the Fermatean fuzzy weighted power average (FFWPA) operator [34], and the n,m-rung orthopair fuzzy weighted power average (n,m-ROFWPA) operator [16]. Therefore, the SR-FWPA, 3PR-FWPA, FFWPA, and n,m-ROFWPA operators are applied to our application and yield identical optimal results, which are presented in Table 6. When we use the operator

1. SR-FWPA, the ultimate order is $C_4 \succ C_2 \succ C_6 \succ C_3 \succ C_1 \succ C_7 \succ C_5 \succ C_8$;
2. 3PR-FWPA, the ultimate order is $C_4 \succ C_2 \succ C_7 \succ C_6 \succ C_3 \succ C_1 \succ C_5 \succ C_8$;
3. FFWPA, the ultimate order is $C_4 \succ C_2 \succ C_7 \succ C_1 \succ C_6 \succ C_3 \succ C_5 \succ C_8$;
4. 3,4-ROFWPA, the ultimate order is $C_4 \succ C_2 \succ C_7 \succ C_6 \succ C_1 \succ C_3 \succ C_8 \succ C_5$.

In this regard, the optimum alternative is Switzerland, which is the same as the indicated operator. As a consequence, our suggested technique is more flexible than the existing methods.

Table 5. Comparison of final scores for our application.

	SR-FWPA	3PR-FWPA	FFWPA	3,4-ROFWPA
C_1	(0.7039, 0.0869)	(0.7290, 0.0706)	(0.7290, 0.1871)	(0.7290, 0.2077)
$\triangle(C_1)$	0.2006	−0.0260	0.3809	0.3856
C_2	(0.7326, 0.0833)	(0.7621, 0.0631)	(0.7621, 0.2459)	(0.7621, 0.2787)
$\triangle(C_2)$	0.2481	0.0444	0.4277	0.4365
C_3	(0.7038, 0.0660)	(0.7231, 0.0554)	(0.7231, 0.1114)	(0.7231, 0.1235)
$\triangle(C_3)$	0.2385	−0.0032	0.3766	0.3778
C_4	(0.7414, 0.0615)	(0.7647, 0.0501)	(0.7647, 0.1475)	(0.7647, 0.1645)
$\triangle(C_4)$	0.3017	0.0786	0.4440	0.4465
C_5	(0.6575, 0.1056)	(0.6720, 0.1024)	(0.6720, 0.1574)	(0.6720, 0.1750)
$\triangle(C_5)$	0.1075	−0.1645	0.2995	0.3025
C_6	(0.7246, 0.0780)	(0.7406, 0.0665)	(0.7406, 0.3029)	(0.7406, 0.3608)
$\triangle(C_6)$	0.2458	0.0011	0.3784	0.3892
C_7	(0.7202, 0.1042)	(0.7633, 0.0660)	(0.7633, 0.2845)	(0.7633, 0.3157)
$\triangle(C_7)$	0.1958	0.0405	0.4217	0.4348
C_8	(0.6700, 0.1529)	(0.6884, 0.1432)	(0.6884, 0.3292)	(0.6884, 0.3805)
$\triangle(C_8)$	0.0579	−0.1969	0.2905	0.3052

Table 6. Comparison rankings for our application.

Operators	Ranking	Best Country
SR-FWPA	$C_4 \succ C_2 \succ C_6 \succ C_3 \succ C_1 \succ C_7 \succ C_5 \succ C_8$	C_4
3PR-FWPA	$C_4 \succ C_2 \succ C_7 \succ C_6 \succ C_3 \succ C_1 \succ C_5 \succ C_8$	C_4
FFWPA	$C_4 \succ C_2 \succ C_7 \succ C_1 \succ C_6 \succ C_3 \succ C_5 \succ C_8$	C_4
3,4-ROFWPA	$C_4 \succ C_2 \succ C_7 \succ C_6 \succ C_1 \succ C_3 \succ C_8 \succ C_5$	C_4

6. Conclusions

Aggregation operators are computational models that have developed into essential tools for combining multiple inputs into one valuable output. Additionally, the n,mPR-FS is an effective tool for characterizing the uncertainty present in decision-making issues where there are multiple perspectives on the same data source. To express the fuzziness of information, n,mPR-FS is a good tool. It has the parameters n and m, so it holds a wider range of fuzzy information than IFS, PFS, FFS, and nPR-FS. In this paper, several operators on n,mPR-fuzzy sets were explored, and their relationships were discovered. Furthermore, we developed a new weighted aggregated operator over n,mPR-fuzzy sets and thoroughly described its features. Moreover, we illustrated this process with an one fully applicable example. Finally, the results of the FWPA^n_m operator were compared to the results of other well-known operators such as the SR-FWPA, 3PR-FWPA, FFWPA, and 3,4-ROFWPA operators.

In the future, we can modify the aggregation operator described here to include models for uncertain data, such as n,mPR-fuzzy soft sets. Additionally, the weighted average operator, weighted geometric operator, and weighted power geometric operator over n,mPR-fuzzy sets may be investigated and MCDM methods may be discussed depending on these operators. Moreover, we wish to extend our research with the following ideas:

(1) complex n,mPR-FSs, (2) interval-valued n,mPR-FSs, and (3) bipolar n,mPR-FSs. Finally, decision models can be amalgamated with deep learning concepts to handle complex and critical decision-making problems.

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