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Applications of Euler Sums and Series Involving the Zeta Functions

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Abstract: A very recent article delved into and expanded the four parametric linear Euler sums, revealing that two well-established subjects—Euler sums and series involving the zeta functions—display particular correlations. In this study, we present several closed forms of series involving zeta functions by using formulas for series associated with the zeta functions detailed in the aforementioned paper. Another closed form of series involving Riemann zeta functions is provided by utilizing a known identity for a series of rational functions in the series index, expressed in terms of Gamma functions. Furthermore, we demonstrate a myriad of applications and relationships of series involving the zeta functions and the extended parametric linear Euler sums. These include connections with Wallis's infinite product formula for π, Mathieu series, Mellin transforms, determinants of Laplacians, certain integrals expressed in terms of Euler sums, representations and evaluations of some integrals, and certain parametric Euler sum identities. The use of Mathematica for various approximation values and certain integral formulas is elaborated upon. Symmetry naturally occurs in Euler sums.

Keywords: gamma function; harmonic numbers; generalized harmonic numbers; alternating harmonic numbers; generalized alternating harmonic numbers; Riemann zeta function; Hurwitz (generalized) zeta function; eta function; Dirichlet beta function; Catalan's constant; generalized eta function; psi function; polygamma functions; dilogarithm; polylogarithm; linear Euler sums; nonlinear Euler sums; parametric linear Euler sums; series involving the zeta functions; Wallis's infinite product formula for π; Mathieu series; Mellin transform; determinants of Laplacians

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1. Introduction and Preliminaries

In 1775, Euler found the following series of harmonic numbers:

$$\sum_{\xi=1}^{\infty} \frac{\mathsf{H}_\xi}{(\xi+1)^2} = \frac{1}{2} \sum_{\xi=1}^{\infty} \frac{\mathsf{H}_\xi}{\xi^2} = \zeta(3), \tag{1}$$

which have a long pedigree (consult, for instance, ([1] p. 252 and ensuing) ; see also [2]). Here $\zeta(z)$ is the Riemann zeta function defined by

$$\zeta(z) := \lim_{\xi\to\infty} \mathsf{H}_\xi^{(z)} = \sum_{\eta=1}^{\infty} \frac{1}{\eta^z} \quad (\Re(z) > 1), \tag{2}$$

where $\mathsf{H}_\xi^{(z)}$ denote harmonic numbers of order z given by

$$\mathsf{H}_\xi^{(z)} := \sum_{\eta=1}^{\xi} \frac{1}{\eta^z} \quad (z \in \mathbb{C},\ \xi \in \mathbb{Z}_{\geqslant 1}), \tag{3}$$

and $\mathsf{H}_\xi := \mathsf{H}_\xi^{(1)}$ $(\xi \in \mathbb{Z}_{\geqslant 1})$ are the harmonic numbers. The well-known link Euler found between the Riemann zeta function and the Bernoulli numbers is as follows: (consult, for instance, [3], p. 166)

$$\zeta(2\eta) = (-1)^{\eta+1} \frac{(2\pi)^{2\eta}}{2\,(2\eta)!} B_{2\eta} \quad (\eta \in \mathbb{Z}_{\geqslant 0}). \tag{4}$$

In this work, as in others, an empty sum is considered to be zero; thus, $\mathsf{H}_0^{(z)} = 0$. We denote by $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{Z}$, respectively, the sets of complex numbers, real numbers, and integers. Also, let

$$\mathbb{E}_{<\eta}, \quad \mathbb{E}_{\leqslant\eta}, \quad \mathbb{E}_{>\eta}, \quad \text{and} \quad \mathbb{E}_{\geqslant\eta}$$

be the subsets of $\mathbb{E}$ that are less than η, less than or equal to η, greater than η, and greater than or equal to η, respectively, for some $\eta \in \mathbb{E}$, where $\mathbb{E}$ is either $\mathbb{R}$ or $\mathbb{Z}$.

The extended harmonic numbers $\mathsf{H}_\xi^{(z)}(u)$ are defined by

$$\mathsf{H}_\xi^{(z)}(u) := \sum_{\eta=1}^{\xi} \frac{1}{(\eta+u)^z} \quad (z \in \mathbb{C},\ u \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1},\ \xi \in \mathbb{Z}_{\geqslant 1}), \tag{5}$$

and $\mathsf{H}_\xi^{(z)}(0) = \mathsf{H}_\xi^{(z)}$. The generalized (or Hurwitz) zeta function $\zeta(z,u)$ is defined by

$$\zeta(z,u) := \lim_{\xi\to\infty} \mathsf{H}_\xi^{(z)}(u-1) = \sum_{\nu=0}^{\infty} \frac{1}{(\nu+u)^z} \quad (u \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0},\ \Re(z) > 1). \tag{6}$$

By means of (2) and (6), we find

$$\zeta(z) = \zeta(z,1) = (2^z-1)^{-1}\,\zeta(z,1/2) = 1 + \zeta(z,2). \tag{7}$$

The following captivating identity comes to mind (see [4], also consult ([5] Equation (2.16)), ([6] p. 280), ([7] Equation (9))):

$$\sum_{\xi=1}^{\infty} \frac{\mathsf{H}_\xi^2}{(\xi+1)^2} = \frac{11}{17} \sum_{\xi=1}^{\infty} \frac{\mathsf{H}_\xi^2}{\xi^2} = \frac{11}{4}\,\zeta(4). \tag{8}$$

Euler, during his correspondence with Goldbach in 1742, initiated this line of investigation, and he pioneered the study of linear harmonic sums (see, for instance, [8,9]):

$$\mathbf{S}_{a,b} := \sum_{\xi=1}^{\infty} \frac{\mathsf{H}_\xi^{(a)}}{\xi^b}. \tag{9}$$

Euler's research, which was completed by Nielsen in 1906 (consult [10]), demonstrated that the linear harmonic sums in (9) are established in the following cases: $a = 1$; $a = b$; $a+b$ odd; and $a+b$ even, yet the couple (a,b) belongs to the set $\{(2,4),\ (4.2)\}$. Along with each of these situations, if $\mathbf{S}_{a,b}$ is known in the ones with $a \neq b$, then $\mathbf{S}_{b,a}$ is found using the symmetric connection:

$$\mathbf{S}_{a,b} + \mathbf{S}_{b,a} = \zeta(a)\,\zeta(b) + \zeta(a+b) \tag{10}$$

and conversely (see, for instance, [11]). The numerical analysis of the linear correlations between polynomials with zeta values and linear Euler sums (see [9,12]) unambiguously states that Euler identified all viable evaluations of linear harmonic sums, for example:

$$2\,\mathbf{S}_{1,\xi} = (\xi+2)\,\zeta(\xi+1) - \sum_{\eta=1}^{\xi-2} \zeta(\xi-\eta)\,\zeta(\eta+1) \quad (\xi \in \mathbb{Z}_{\geqslant 2}). \tag{11}$$

Nonlinear harmonic sums are generated by multiplying at least two (extended) harmonic numbers together. Let $A = (a_1, \ldots, a_\ell)$ be a partition of an integer a into ℓ summands, so that $a = a_1 + \cdots + a_\ell$ and $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_\ell$. The nonlinear Euler sum of index A, b is defined by

$$\mathbf{S}_{A;b} := \sum_{\xi=1}^{\infty} \frac{\mathsf{H}_\xi^{(a_1)} \mathsf{H}_\xi^{(a_2)} \cdots \mathsf{H}_\xi^{(a_\ell)}}{\xi^b}. \tag{12}$$

Here, the amount $b + a_1 + \cdots + a_\ell$ is named the weight, and the number ℓ is the degree. To make things simple, powers are used to denote repeating summands in partitions, such as

$$\mathbf{S}_{1^3,3^2,6;b} = \mathbf{S}_{1,1,1,3,3,6;b} = \sum_{\xi=1}^{\infty} \frac{(\mathsf{H}_\xi)^3 \left\{\mathsf{H}_\xi^{(3)}\right\}^2 \mathsf{H}_\xi^{(6)}}{\xi^b}.$$

An expository cum survey [13] provided a comprehensive assessment of publications on Euler sums of varying degrees and multiple zeta values. Since then, there has been much interest in Euler sums and multiple zeta values (see, for instance, [14–36]).

Flajolet and Salvy [9] proposed the following notations for a total of four distinct types of linear Euler sums:

$$\begin{aligned} \mathbf{S}_{a,b}^{++} &= \sum_{\xi=1}^{\infty} \frac{\mathsf{H}_\xi^{(a)}}{\xi^b}, \qquad & \mathbf{S}_{a,b}^{+-} &= \sum_{\xi=1}^{\infty} (-1)^{\xi+1} \frac{\mathsf{H}_\xi^{(a)}}{\xi^b}, \\ \mathbf{S}_{a,b}^{-+} &= \sum_{\xi=1}^{\infty} \frac{\mathsf{A}_\xi^{(a)}}{\xi^b}, \qquad & \mathbf{S}_{a,b}^{--} &= \sum_{\xi=1}^{\infty} (-1)^{\xi+1} \frac{\mathsf{A}_\xi^{(a)}}{\xi^b}. \end{aligned} \tag{13}$$

Obviously, $\mathbf{S}_{a,b}^{++} = \mathbf{S}_{a,b}$. Here $\mathcal{A}_\xi^{(z)}$ are alternating harmonic numbers of order z, given by

$$\mathsf{A}_\xi^{(z)} := \sum_{\ell=1}^{\xi} \frac{(-1)^{\ell+1}}{\ell^z} \quad (z \in \mathbb{C},\ \xi \in \mathbb{Z}_{\geqslant 1}), \tag{14}$$

and $\mathsf{A}_\xi := \mathsf{A}_\xi^{(1)}$. There exists the following connection between numbers $\mathsf{A}_\xi^{(z)}$ and $\mathsf{H}_\xi^{(z)}$:

$$\mathsf{A}_\xi^{(z)} = \mathsf{H}_\xi^{(z)} - 2^{1-z} \mathsf{H}_{[\xi/2]}^{(z)}. \tag{15}$$

In this and other instances, $[\xi]$ is the integral component of $\xi \in \mathbb{R}$.

Like (5), the generalized types $\mathsf{A}_\xi^{(z)}(u)$ of the numbers $\mathsf{A}_\xi^{(z)}$ are denoted by

$$\mathsf{A}_\xi^{(z)}(u) := \sum_{\ell=1}^{\xi} \frac{(-1)^{\ell+1}}{(\ell+u)^z} \quad (z \in \mathbb{C},\, u \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1},\ \xi \in \mathbb{Z}_{\geqslant 1}). \tag{16}$$

The Dirichlet eta function $\eta(z)$ is defined as follows:

$$\eta(z) := \lim_{\xi \to \infty} \mathsf{A}_\xi^{(z)} = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{\ell^z} \quad (\Re(z) > 0). \tag{17}$$

In particular,

$$\eta(1) = \ln 2 \quad \text{and} \quad \eta(0) = \frac{1}{2}. \tag{18}$$

The extended eta function $\eta(z,u)$ is defined by

$$\eta(z,u) := \lim_{\xi \to \infty} \mathcal{A}_\xi^{(z)}(u-1) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(\nu+u)^z} \quad (u \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0},\ \Re(z) > 0). \tag{19}$$

One finds from (19) that

$$\eta(z,u+1)=\frac{1}{u^z}-\eta(z,u) \quad \text{and} \quad \eta\left(z,\frac{1}{2}\right)=\frac{1}{2^z}\left\{\zeta\left(z,\frac{1}{4}\right)-\zeta\left(z,\frac{3}{4}\right)\right\}. \tag{20}$$

By using (17) and (19), one has

$$\eta(z,1)=\eta(z) \quad \text{and} \quad \eta(z,2)=1-\eta(z). \tag{21}$$

The Dirichlet beta function $\beta(z)$ is defined by

$$\beta(z)=\sum_{\tau=0}^{\infty}\frac{(-1)^\tau}{(2\tau+1)^z} \quad (\Re(z)>0). \tag{22}$$

Among several different expressions for $\beta(z)$, the following is written in terms of the generalized zeta function (6) and the extended eta function (19):

$$\begin{aligned}\beta(z)&=4^{-z}\left(\zeta\left(z,\frac{1}{4}\right)-\zeta\left(z,\frac{3}{4}\right)\right)\\&=2^{-z}\,\eta\left(z,\frac{1}{2}\right) \quad (z\in\mathbb{C}).\end{aligned} \tag{23}$$

It is highlighted here that

$$\eta\left(1,\frac{1}{2}\right)=\frac{\pi}{2} \qquad \text{and} \qquad \beta(1)=\frac{\pi}{4}. \tag{24}$$

Catalan's constant G is given by

$$G=\beta(2)=\sum_{\tau=0}^{\infty}\frac{(-1)^\tau}{(2\tau+1)^2}\simeq 0.9159655941772190\ldots. \tag{25}$$

Polylogarithm $\mathrm{Li}_k(\alpha)$ is given by (consult, for example, [3] (p. 185)):

$$\begin{aligned}\mathrm{Li}_k(\alpha)&:=\sum_{\xi=1}^{\infty}\frac{\alpha^\xi}{\xi^k} \quad (k\in\mathbb{Z}_{\geqslant 2},\ |\alpha|\leqslant 1)\\&=\int_0^\alpha\frac{\mathrm{Li}_{k-1}(\tau)}{\tau}\,d\tau \quad (k\in\mathbb{Z}_{\geqslant 3}).\end{aligned} \tag{26}$$

Clearly,

$$\mathrm{Li}_k(1)=\zeta(k) \quad (k\in\mathbb{Z}_{\geqslant 2}), \tag{27}$$

and

$$\mathrm{Li}_k(-1)=-\eta(k) \quad (k\in\mathbb{Z}_{\geqslant 1}). \tag{28}$$

Dilogarithm $\mathrm{Li}_2(\alpha)$ is defined by

$$\begin{aligned}\mathrm{Li}_2(\alpha)&:=\sum_{\xi=1}^{\infty}\frac{\alpha^\xi}{\xi^2} \quad (|\alpha|\leqslant 1)\\&=-\int_0^\alpha\frac{\log(1-\tau)}{\tau}\,d\tau.\end{aligned} \tag{29}$$

Polylogarithm $\mathrm{Li}_k(\alpha)$ can be extended as follows (consult, for example, [3] (p. 198, Equation (28))):

$$\mathrm{Li}_z(\alpha):=\sum_{\xi=1}^{\infty}\frac{\alpha^\xi}{\xi^z} \quad (|\alpha|<1,\ z\in\mathbb{C};\ |\alpha|=1,\ \Re(z)>1). \tag{30}$$

Recall mingling interactions that are similar to (10) (refer to [9] (p. 33)):

$$\mathbf{S}^{-+}_{a,b} + \mathbf{S}^{+-}_{b,a} = \eta(a)\,\zeta(b) + \eta(a+b); \tag{31}$$

$$\mathbf{S}^{--}_{a,b} + \mathbf{S}^{--}_{b,a} = \eta(a)\,\eta(b) + \zeta(a+b). \tag{32}$$

In terms of extensions of Flajolet and Salvy's linear Euler sums (13), Alzer and Choi [37] developed four different types of parameterized linear Euler sums:

$$\begin{aligned}
\mathbf{S}^{++}_{\mu,z}(\alpha,\beta) &:= \sum_{\xi=1}^{\infty} \frac{\mathsf{H}^{(\mu)}_{\xi}(\alpha)}{(\xi+\beta)^z}, &\quad \mathbf{S}^{+-}_{\mu,z}(\alpha,\beta) &:= \sum_{\xi=1}^{\infty} (-1)^{\xi+1} \frac{\mathsf{H}^{(\mu)}_{\xi}(\alpha)}{(\xi+\beta)^z}, \\
\mathbf{S}^{-+}_{\mu,z}(\alpha,\beta) &:= \sum_{\xi=1}^{\infty} \frac{\mathsf{A}^{(\mu)}_{\xi}(\alpha)}{(\xi+\beta)^z}, &\quad \mathbf{S}^{--}_{\mu,z}(\alpha,\beta) &:= \sum_{\xi=1}^{\infty} (-1)^{\xi+1} \frac{\mathsf{A}^{(\mu)}_{\xi}(\alpha)}{(\xi+\beta)^z}.
\end{aligned} \tag{33}$$

Obviously,

$$\mathbf{S}^{++}_{\mu,z}(0,0) = \mathbf{S}^{++}_{\mu,z},\ \ \mathbf{S}^{+-}_{\mu,z}(0,0) = \mathbf{S}^{+-}_{\mu,z},\ \ \mathbf{S}^{-+}_{\mu,z}(0,0) = \mathbf{S}^{-+}_{\mu,z},\ \ \mathbf{S}^{--}_{\mu,z}(0,0) = \mathbf{S}^{--}_{\mu,z}.$$

The authors of [37] investigated a variety of intriguing properties and identities of the four parameterized linear Euler sums in (33), including their analytic continuations and mingling relations. Different Euler sums with parameters have been studied (consult, for instance, [18,21,22,28,38,39]).

Very recently, Sofo and Choi [40] extended the four parameterized linear Euler sums in (33) as follows:

$$\begin{aligned}
\mathbf{S}^{++}_{\mu,z}(\alpha,\beta;q) &:= \sum_{\xi=1}^{\infty} \frac{\mathsf{H}^{(\mu)}_{q\xi}(\alpha)}{(\xi+\beta)^z}, &\quad \mathbf{S}^{+-}_{\mu,z}(\alpha,\beta;q) &:= \sum_{\xi=1}^{\infty} (-1)^{\xi+1} \frac{\mathsf{H}^{(\mu)}_{q\xi}(\alpha)}{(\xi+\beta)^z}, \\
\mathbf{S}^{-+}_{\mu,z}(\alpha,\beta;q) &:= \sum_{\xi=1}^{\infty} \frac{\mathsf{A}^{(\mu)}_{q\xi}(\alpha)}{(\xi+\beta)^z}, &\quad \mathsf{S}^{--}_{\mu,z}(\alpha,\beta;q) &:= \sum_{\xi=1}^{\infty} (-1)^{\xi+1} \frac{\mathsf{A}^{(\mu)}_{q\xi}(\alpha)}{(\xi+\beta)^z}.
\end{aligned} \tag{34}$$

Here, $q \in \mathbb{Z}_{\geqslant 1}$, $\alpha,\ \beta \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$ and $\mu,\ z \in \mathbb{C}$ are modified, such that the involved defining series can converge. Clearly,

$$\begin{aligned}
\mathbf{S}^{++}_{\mu,z}(\alpha,\beta;1) &= \mathbf{S}^{++}_{\mu,z}(\alpha,\beta), &\qquad \mathbf{S}^{+-}_{\mu,z}(\alpha,\beta;1) &= \mathbf{S}^{+-}_{\mu,z}(\alpha,\beta), \\
\mathbf{S}^{-+}_{\mu,z}(\alpha,\beta;1) &= \mathbf{S}^{-+}_{\mu,z}(\alpha,\beta), &\qquad \mathbf{S}^{--}_{\mu,z}(\alpha,\beta;1) &= \mathbf{S}^{--}_{\mu,z}(\alpha,\beta).
\end{aligned}$$

Using these expanded sums, the authors of [40] analyzed some of their characteristics and identities. Specifically, the authors of [40] observed that two well-known and popular subjects, namely Euler sums and series involving zeta functions, had some surprising relationships.

In this study, we propose establishing several closed forms of series involving zeta functions by using the formulae for series linked with zeta functions from [40], see Theorem 4.2. Another closed form of series involving Riemann zeta functions is also provided by utilizing a known formula for a series of rational functions, where the series index is expressed in terms of Gamma functions. Numerous applications and relationships of series involving zeta functions and extended parametric linear Euler sums, such as their connections with the Mathieu series, Mellin transforms, determinants of Laplacians, certain integrals expressed in terms of Euler sums, representations and evaluations of some integrals, and certain parametric Euler sum identities, are also demonstrated. The use of Mathematica 13.0 (Home Edition) for various approximation values and integral formulae is addressed.

2. Series Involving the Zeta Functions

This section establishes closed-form expressions of several new families of series associated with zeta functions by making use of Theorem 1 ([40], Theorem 4.2) and a known infinite product formula expressed in terms of Gamma functions.

Zeta function series have piqued the curiosity of numerous academics. The interested reader can refer to, for instance, the monograph [3] for information on the subject's history and an astoundingly large number of identities. The series involving zeta functions in ([40] Theorem 4.1)) are clearly of different types from those previously presented (such as [3], Chapter 3) and from those recollected in Theorem 1 (see [40], Theorem 4.2).

Let us recall two parametric and two variable summations (see [40], Theorem 4.2):

$$\mathbf{S}^{+}_{\mu,\nu}(x,y) := \sum_{\xi=1}^{\infty} \frac{1}{(\xi+x)^{\mu}\,(\xi+y)^{\nu}}, \tag{35}$$

and

$$\mathbf{S}^{-}_{\mu,\nu}(x,y) := \sum_{\xi=1}^{\infty} \frac{(-1)^{\xi+1}}{(\xi+x)^{\mu}\,(\xi+y)^{\nu}}. \tag{36}$$

The psi (or digamma) function $\psi(u)$ is given by

$$\psi(u) := \frac{d}{du}\{\log\Gamma(u)\} = \frac{\Gamma'(u)}{\Gamma(u)} \quad \text{or} \quad \log\Gamma(u) = \int_1^u \psi(\xi)\,d\xi. \tag{37}$$

Here, Γ represents the renowned Gamma function (consult, for instance, [3], Section 1.1). The psi function $\psi(u)$ gratifies

$$\psi(u) = -\gamma - \frac{1}{u} + \sum_{\xi=1}^{\infty} \frac{u}{\xi(u+\xi)} \quad (u \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}). \tag{38}$$

Here, and in other places, γ signifies the Euler–Mascheroni constant (see, for instance, [3], Section 1.2; see also [41,42]).

The polygamma functions $\psi^{(k)}(u)$ are provided by

$$\psi^{(\eta)}(u) := \frac{d^{\eta+1}}{du^{\eta+1}}\log\Gamma(u) = \frac{d^{\eta}}{du^{\eta}}\psi(u) \quad (u \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0},\ \eta \in \mathbb{Z}_{\geqslant 1}). \tag{39}$$

The following relationship between the zeta function of Hurwitz $\zeta(z,u)$ and the polygamma functions $\psi^{(\eta)}(u)$ is worth noting:

$$\psi^{(\eta)}(u) = (-1)^{\eta+1}\,\eta!\sum_{\xi=0}^{\infty}\frac{1}{(\xi+u)^{\eta+1}} = (-1)^{\eta+1}\,\eta!\,\zeta(\eta+1,u) \tag{40}$$

$$(u \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0},\ \eta \in \mathbb{Z}_{\geqslant 1}).$$

One finds the subsequent identity:

$$\begin{aligned}\psi^{(r)}(u+\eta) - \psi^{(r)}(u) &= (-1)^{r}\,r!\sum_{\xi=1}^{\eta}\frac{1}{(u+\xi-1)^{r+1}} \\ &= (-1)^{r}\,r!\,H_{\eta}^{(r+1)}(u-1) \quad (r,\ \eta \in \mathbb{Z}_{\geqslant 0}).\end{aligned} \tag{41}$$

The Pochhammer symbol $(\alpha)_{\beta}$ is given (for $\alpha,\ \beta \in \mathbb{C}$) by

$$(\alpha)_{\beta} := \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} = \begin{cases} 1 & (\beta = 0;\ \alpha \in \mathbb{C}\setminus\{0\}) \\ \alpha(\alpha+1)\cdots(\alpha+\ell-1) & (\beta = \ell \in \mathbb{Z}_{\geqslant 0};\ \alpha \in \mathbb{C}), \end{cases} \tag{42}$$

accepting that $(0)_0 := 1$. Also, the falling factorial $\langle \alpha \rangle_\ell$ is defined (for $\alpha \in \mathbb{C}$) by

$$\langle \alpha \rangle_\ell := \alpha(\alpha-1)\cdots(\alpha-\ell+1) \quad (\ell \in \mathbb{Z}_{\geqslant 1}) \quad \text{and} \quad \langle \alpha \rangle_0 := 1. \tag{43}$$

Theorem 1 ([40], Theorem 4.2)**.** *Assume that $x, y \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$ satisfy $x \neq y$ and $|x-y| < |1+x|$. Also, put $\mu, \nu \in \mathbb{Z}_{\geqslant 1}$. Then,*

$$\begin{aligned}
\mathbf{S}^{+}_{\mu,\nu}(x,y) &= \sum_{\xi=0}^{\infty} \frac{(\nu)_\xi}{\xi!}\,(x-y)^{\xi}\,\zeta(\mu+\nu+\xi, 1+x) \\
&= (-1)^{\nu}\binom{\mu+\nu-2}{\mu-1}\frac{\psi(y+1)-\psi(x+1)}{(x-y)^{\mu+\nu-1}} \\
&\quad + (-1)^{\nu}\sum_{j=1}^{\nu-1}\frac{1}{j!}\binom{\mu+\nu-j-2}{\mu-1}\frac{\psi^{(j)}(y+1)}{(x-y)^{\mu+\nu-1-j}} \\
&\quad + (-1)^{\nu}\sum_{j=1}^{\mu-1}\frac{(-1)^{1+j}}{j!}\binom{\mu+\nu-j-2}{\nu-1}\frac{\psi^{(j)}(x+1)}{(x-y)^{\mu+\nu-1-j}},
\end{aligned} \tag{44}$$

and

$$\begin{aligned}
\mathbf{S}^{-}_{\mu,\nu}(x,y) &= \sum_{\xi=0}^{\infty} \frac{(\nu)_\xi}{\xi!}\,(x-y)^{\xi}\,\eta(\mu+\nu+\xi, 1+x) \\
&= (y-x)^{-\nu}\sum_{j=0}^{\mu-1}\binom{\nu+j-1}{\nu-1}(x-y)^{-j}\,\eta(\mu-j, x+1) \\
&\quad + (x-y)^{-\mu}\sum_{j=0}^{\nu-1}\binom{\mu+j-1}{\mu-1}(y-x)^{-j}\,\eta(\nu-j, y+1).
\end{aligned} \tag{45}$$

Lemma 1. *Let $n, \ell, m \in \mathbb{Z}_{\geqslant 0}$ with $\ell \geqslant m$ and $x, y \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$. Also, the principal values of the involved logarithms are assumed. Then*

$$\int_x^{x+n} \log\Gamma(t+1)\,dt = \sum_{k=1}^{n}(x+k)\,\log(x+k) - nx - \frac{1}{2}n(n+1) + \frac{1}{2}n\,\log(2\pi); \tag{46}$$

$$\begin{aligned}
I(\ell, m; x, y) &:= \int_x^y (x-t)^m\,\psi^{(\ell)}(t+1)\,dt \\
&= \sum_{k=0}^{m-1}(-1)^{m+k}\langle m \rangle_k\,(y-x)^{m-k}\,\psi^{(m-k-1)}(y+1) \\
&\quad + m!\left[\psi^{(\ell-m-1)}(y+1) - \psi^{(\ell-m-1)}(x+1)\right].
\end{aligned} \tag{47}$$

where

$$\psi^{(-1)}(t+1) := \log\Gamma(t+1) \quad \textit{and} \quad \psi^{(0)}(t+1) = \psi(t+1). \tag{48}$$

Proof. Formula (46) follows from the known one (see, for instance, ([43], p. 24, Equation (20)), ([3], p. 29, Equation (41))).

Integrating by parts repeatedly, with the aid of (39), we derive

$$\begin{aligned}
(-1)^m\,I(\ell, m; x, y) &= \sum_{k=0}^{m-1}(-1)^k\langle m \rangle_k\,(y-x)^{m-k}\,\psi^{(m-k-1)}(y+1) \\
&\quad + (-1)^m\,m!\int_x^y (t-x)^{m-m}\,\psi^{(\ell-m)}(t+1)\,dt \\
&= \sum_{k=0}^{m-1}(-1)^k\langle m \rangle_k\,(y-x)^{m-k}\,\psi^{(m-k-1)}(y+1) \\
&\quad + (-1)^m\,m!\left[\psi^{(\ell-m-1)}(y+1) - \psi^{(\ell-m-1)}(x+1)\right].
\end{aligned}$$

□

Theorem 2. *Let $x, y \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$ be such that $|x-y| < |1+x|$. Also, let $\mu, \nu \in \mathbb{Z}_{\geqslant 1}$. Furthermore, the principal values of the involved logarithms are assumed. Then,*

$$\begin{aligned}
&\sum_{\tau=0}^{\infty} \frac{(\nu)_\tau}{\tau!\,(\tau+\mu+\nu)}\,\zeta(\mu+\nu+\tau,1+x)\,(x-y)^{\tau+\mu+\nu} \\
&= (-1)^{\nu+1}\binom{\mu+\nu-2}{\mu-1}\left\{\log\frac{\Gamma(y+1)}{\Gamma(x+1)} + (x-y)\,\psi(x+1)\right\} \\
&\quad + (-1)^{\nu+1}\sum_{j=1}^{\nu-1}\frac{1}{j!}\binom{\mu+\nu-j-2}{\mu-1}\,I(j,j;x,y) \\
&\quad + (-1)^{\nu+1}\sum_{j=1}^{\mu-1}\frac{(-1)^j}{(j+1)!}\binom{\mu+\nu-j-2}{\nu-1}\,(x-y)^{j+1}\psi^{(j)}(x+1),
\end{aligned} \tag{49}$$

where

$$\begin{aligned}
I(j,j;x,y) &= \int_x^y \psi^{(j)}(t+1)\,(x-t)^j\,dt \\
&= \sum_{k=0}^{j-1}(-1)^{j+k}\,\langle j\rangle_k\,(y-x)^{j-k}\,\psi^{(j-k-1)}(y+1) \\
&\quad + j!\,\log\frac{\Gamma(y+1)}{\Gamma(x+1)}.
\end{aligned} \tag{50}$$

Proof. Multiplying each side of (44) by $(x-y)^{\xi}$ ($\xi \in \mathbb{Z}_{\geqslant 0}$), replacing y by t in the resulting identity, and integrating the second resulting identity with respect to t from x to y, we obtain

$$\begin{aligned}
&\sum_{\tau=0}^{\infty} \frac{(\nu)_\tau}{\tau!\,(\tau+\xi+1)}\,\zeta(\mu+\nu+\tau,1+x)\,(x-y)^{\tau+\xi+1} \\
&= (-1)^{\nu+1}\binom{\mu+\nu-2}{\mu-1}\left\{\int_x^y \psi(t+1)\,(x-t)^{\xi-\mu-\nu+1}\,dt + \psi(x+1)\,\frac{(x-y)^{\xi-\mu-\nu+2}}{\xi-\mu-\nu+2}\right\} \\
&\quad + (-1)^{\nu+1}\sum_{j=1}^{\nu-1}\frac{1}{j!}\binom{\mu+\nu-j-2}{\mu-1}\int_x^y \psi^{(j)}(t+1)\,(x-t)^{\xi-\mu-\nu+1+j}\,dt \\
&\quad + (-1)^{\nu+1}\sum_{j=1}^{\mu-1}\frac{(-1)^j}{j!}\binom{\mu+\nu-j-2}{\nu-1}\,\psi^{(j)}(x+1)\,\frac{(x-y)^{\xi-\mu-\nu+2+j}}{\xi-\mu-\nu+2+j}.
\end{aligned} \tag{51}$$

In order to apply (47) to evaluate the integral

$$\int_x^y \psi(t+1)\,(x-t)^{\xi-\mu-\nu+1}\,dt,$$

the following restriction is required:

$$0 \leqslant \xi-\mu-\nu+1 \leqslant 0 \quad\Longleftrightarrow\quad \xi+1 = \mu+\nu. \tag{52}$$

Using (52) in (51), we obtain

$$\begin{aligned}
&\sum_{\tau=0}^{\infty} \frac{(\nu)_\tau}{\tau!\,(\tau+\mu+\nu)}\,\zeta(\mu+\nu+\tau,1+x)\,(x-y)^{\tau+\mu+\nu} \\
&= (-1)^{\nu+1}\binom{\mu+\nu-2}{\mu-1}\left\{\log\frac{\Gamma(y+1)}{\Gamma(x+1)} + (x-y)\,\psi(x+1)\right\} \\
&\quad + (-1)^{\nu+1}\sum_{j=1}^{\nu-1}\frac{1}{j!}\binom{\mu+\nu-j-2}{\mu-1}\int_x^y \psi^{(j)}(t+1)\,(x-t)^j\,dt \\
&\quad + (-1)^{\nu+1}\sum_{j=1}^{\mu-1}\frac{(-1)^j}{(j+1)!}\binom{\mu+\nu-j-2}{\nu-1}\,(x-y)^{j+1}\psi^{(j)}(x+1).
\end{aligned} \tag{53}$$

Employing (47), we can evaluate

$$I(j,j;x,y) = \int_x^y \psi^{(j)}(t+1)\,(x-t)^j\,dt$$

as in (50). □

Corollary 1. *Let $y \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$ be such that $|y| < 1$. Also, let $\mu,\, \nu \in \mathbb{Z}_{\geqslant 1}$. Furthermore, the principal values of the involved logarithms are assumed. Then,*

$$\begin{aligned}
&\sum_{\tau=0}^{\infty} \frac{(\nu)_\tau}{\tau!\,(\tau+\mu+\nu)}\,\zeta(\mu+\nu+\tau)\,(-y)^{\tau+\mu+\nu} \\
&= (-1)^{\nu+1}\binom{\mu+\nu-2}{\mu-1}\{\log\Gamma(y+1)+\gamma\,y\} \\
&\quad + (-1)^{\nu+1}\sum_{j=1}^{\nu-1}\frac{1}{j!}\binom{\mu+\nu-j-2}{\mu-1} I(j,j;0,y) \\
&\quad + (-1)^{\nu+1}\sum_{j=1}^{\mu-1}\frac{(-1)^j}{j+1}\binom{\mu+\nu-j-2}{\nu-1}\zeta(j+1)\,y^{j+1},
\end{aligned} \tag{54}$$

where

$$\begin{aligned}
I(j,j;0,y) = &\sum_{k=0}^{j-1}(-1)^{j+k}\,\langle j\rangle_k\,y^{j-k}\,\psi^{(j-k-1)}(y+1) \\
&+ j!\,\log\Gamma(y+1),
\end{aligned} \tag{55}$$

and γ is the Euler–Mascheroni constant;

$$\sum_{\tau=2}^{\infty}\frac{(-1)^\tau}{\tau}\,\zeta(\tau)\,y^\tau = \log\Gamma(1+y)+\gamma\,y \quad (|y|<1); \tag{56}$$

$$\sum_{\tau=2}^{\infty}\frac{(-1)^\tau}{\tau(\tau+1)}\,\zeta(\tau) = -1+\frac{\gamma}{2}+\frac{1}{2}\log(2\pi). \tag{57}$$

Proof. Setting $x = 0$ in (54), with the aid of (7), (38), and (40), yields (54).

Taking $\mu = \nu = 1$ in (54) gives (56), which is a known identity (consult, for instance, ([3], p. 269)).

Furthermore, integrating both sides of (56) with respect to variable y from 0 to 1, and using (46), we obtain (57). The closed-form evaluation of the series involving zeta functions in (57) is also a known formula (see, for instance, [3], p. 324, Equation (568)). □

Theorem 3. *Assume that $x,\, y \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$ gratify $x \neq y$ and $|x-y| < |1+x|$. Also, set $\mu,\, \nu \in \mathbb{Z}_{\geqslant 1}$ and $s \in \mathbb{Z}_{\geqslant 0}$. Then,*

$$\begin{aligned}
&\sum_{\xi=0}^{\infty}\frac{(\xi+1)_s\,(\nu+s)_\xi}{(s+\xi)!}\,(x-y)^\xi\,\zeta(\mu+\nu+s+\xi,1+x) \\
&= (-1)^{\nu+s}\binom{\mu+\nu+s-2}{\mu-1}\frac{\psi(y+1)-\psi(x+1)}{(x-y)^{\mu+\nu+s-1}} \\
&\quad + (-1)^{\nu+s}\sum_{j=1}^{\nu+s-1}\frac{1}{j!}\binom{\mu+\nu+s-j-2}{\mu-1}\frac{\psi^{(j)}(y+1)}{(x-y)^{\mu+\nu+s-1-j}} \\
&\quad + (-1)^{\nu+s}\sum_{j=1}^{\mu-1}\frac{(-1)^{1+j}}{j!}\binom{\mu+\nu+s-j-2}{\nu+s-1}\frac{\psi^{(j)}(x+1)}{(x-y)^{\mu+\nu+s-1-j}};
\end{aligned} \tag{58}$$

$$\begin{aligned}
&\sum_{\xi=0}^{\infty} \frac{(\xi+1)_s\,(\nu+s)_\xi}{(s+\xi)!}\,(x-y)^\xi\,\eta(\mu+\nu+s+\xi,1+x) \\
&\quad=(y-x)^{-\nu-s}\sum_{j=0}^{\mu-1}\binom{\nu+s+j-1}{\nu+s-1}(x-y)^{-j}\,\eta(\mu-j,x+1) \\
&\qquad+(x-y)^{-\mu}\sum_{j=0}^{\nu+s-1}\binom{\mu+j-1}{\mu-1}(y-x)^{-j}\,\eta(\nu+s-j,y+1).
\end{aligned} \tag{59}$$

Proof. First consider the following formula:

$$\frac{d^\ell}{dt^\ell}(\xi+t)^{-\kappa}=(-1)^\ell\,(\kappa)_\ell\,(\xi+t)^{-\kappa-\ell}\quad(\ell\in\mathbb{Z}_{\geqslant 0}). \tag{60}$$

Differentiating both sides of (35) s times, with respect to y, with the aid of (60), we obtain

$$\begin{aligned}
\frac{\partial^s}{\partial y^s}\mathbf{S}^+_{\mu,\nu}(x,y)&=(-1)^s\,(\nu)_s\sum_{\xi=1}^{\infty}\frac{1}{(\xi+x)^\mu\,(\xi+y)^{\nu+s}} \\
&=(-1)^s\,(\nu)_s\,\mathbf{S}^+_{\mu,\nu+s}(x,y).
\end{aligned} \tag{61}$$

Using the first equality of (44), we have

$$\begin{aligned}
\frac{\partial^s}{\partial y^s}\mathbf{S}^+_{\mu,\nu}(x,y)&=\frac{\partial^s}{\partial y^s}\sum_{\xi=0}^{\infty}\frac{(\nu)_\xi}{\xi!}\,(x-y)^\xi\,\zeta(\mu+\nu+\xi,1+x) \\
&=\sum_{\xi=s}^{\infty}\frac{(-1)^s\,\langle\xi\rangle_s\,(\nu)_\xi}{\xi!}\,(x-y)^{\xi-s}\,\zeta(\mu+\nu+\xi,1+x).
\end{aligned} \tag{62}$$

Setting $\xi-s=\xi'$ in the last summation in (62), and dropping the prime on ξ, we obtain

$$\frac{\partial^s}{\partial y^s}\mathbf{S}^+_{\mu,\nu}(x,y)=\sum_{\xi=0}^{\infty}\frac{(-1)^s\,\langle s+\xi\rangle_s\,(\nu)_{s+\xi}}{(s+\xi)!}\,(x-y)^\xi\,\zeta(\mu+\nu+s+\xi,1+x). \tag{63}$$

Using the following two identities

$$\langle s+\xi\rangle_s=(\xi+1)_s\quad\text{and}\quad(\nu)_{s+\xi}=(\nu)_s(\nu+s)_\xi \tag{64}$$

in (63), and matching the resulting identity and the right member of (61), we derive

$$\sum_{\xi=0}^{\infty}\frac{(\xi+1)_s\,(\nu+s)_\xi}{(s+\xi)!}\,(x-y)^\xi\,\zeta(\mu+\nu+s+\xi,1+x)=\mathbf{S}^+_{\mu,\nu+s}(x,y), \tag{65}$$

which, upon substituting $\nu+s$ for ν in the right member of (44), yields the desired identity (58).

Likewise, using (36) and (45), we obtain

$$\sum_{\xi=0}^{\infty}\frac{(\xi+1)_s\,(\nu+s)_\xi}{(s+\xi)!}\,(x-y)^\xi\,\eta(\mu+\nu+s+\xi,1+x)=\mathbf{S}^-_{\mu,\nu+s}(x,y), \tag{66}$$

which, upon replacing ν with $\nu+s$ in the right member of (45), leads to the desired identity (59). □

Setting $x=0$ (58) and (59), with the aid of (7), (21), (38) and (40), we obtain series involving Riemann zeta and eta functions, without proofs, asserted in the ensuing corollary.

Corollary 2. *Let $y \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$ be such that $|y| < 1$. Also, let μ, $\nu \in \mathbb{Z}_{\geqslant 1}$ and $s \in \mathbb{Z}_{\geqslant 0}$. Then*

$$\begin{aligned}
&\sum_{\xi=0}^{\infty} \frac{(\xi+1)_s\,(\nu+s)_\xi}{(s+\xi)!}\,(-y)^\xi\,\zeta(\mu+\nu+s+\xi) \\
&\quad = (-1)^{1-\mu}\binom{\mu+\nu+s-2}{\mu-1}\frac{\psi(y+1)+\gamma}{y^{\mu+\nu+s-1}} \\
&\qquad + (-1)^{1-\mu}\sum_{j=1}^{\nu+s-1}\frac{(-1)^j}{j!}\binom{\mu+\nu+s-j-2}{\mu-1}\frac{\psi^{(j)}(y+1)}{y^{\mu+\nu+s-1-j}} \\
&\qquad + (-1)^{1-\mu}\sum_{j=1}^{\mu-1}(-1)^j\binom{\mu+\nu+s-j-2}{\nu+s-1}\frac{\zeta(j+1)}{y^{\mu+\nu+s-1-j}};
\end{aligned} \tag{67}$$

$$\begin{aligned}
&\sum_{\xi=0}^{\infty} \frac{(\xi+1)_s\,(\nu+s)_\xi}{(s+\xi)!}\,(-y)^\xi\,\eta(\mu+\nu+s+\xi) \\
&\quad = (y)^{-\nu-s}\sum_{j=0}^{\mu-1}\binom{\nu+s+j-1}{\nu+s-1}(-y)^{-j}\,\eta(\mu-j) \\
&\qquad + (-y)^{-\mu}\sum_{j=0}^{\nu+s-1}\binom{\mu+j-1}{\mu-1}y^{-j}\,\eta(\nu+s-j,y+1).
\end{aligned} \tag{68}$$

The subsequent lemma provides formulas for derivatives of the generalized (or Hurwitz) zeta function $\zeta(z,u)$ in (6) and the extended eta function $\eta(z,u)$ (19), which are easily derivable (consult, for example, [3] , p. 159, Equation (18)).

Lemma 2. *The following differential formulas hold:*

$$\frac{\partial^k}{\partial u^k}\zeta(z,u) = (-1)^k\,(z)_k\,\zeta(z+k,u) \quad (k \in \mathbb{Z}_{\geqslant 0}) \tag{69}$$

and

$$\frac{\partial^k}{\partial u^k}\eta(z,u) = (-1)^k\,(z)_k\,\eta(z+k,u) \quad (k \in \mathbb{Z}_{\geqslant 0}). \tag{70}$$

Theorem 4. *Suppose that x, $y \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$ satisfy $x \neq y$ and $|x-y| < |1+x|$. Also, put μ, $\nu \in \mathbb{Z}_{\geqslant 1}$ and $s \in \mathbb{Z}_{\geqslant 0}$. Then,*

$$\begin{aligned}
&\sum_{\xi=0}^{\infty} \frac{(\nu)_\xi}{(\mu)_s\,\xi!}\sum_{\substack{j=0\\ j\leqslant\xi}}^{s}(-1)^j\binom{s}{j}\langle\xi\rangle_j\,(\mu+\nu+\xi)_{s-j} \\
&\qquad \times (x-y)^{\xi-j}\,\zeta(\mu+\nu+\xi+s-j,1+x) \\
&\quad = (-1)^\nu\binom{\mu+\nu+s-2}{\mu+s-1}\frac{\psi(y+1)-\psi(x+1)}{(x-y)^{\mu+\nu+s-1}} \\
&\qquad + (-1)^\nu\sum_{j=1}^{\nu-1}\frac{1}{j!}\binom{\mu+\nu+s-j-2}{\mu+s-1}\frac{\psi^{(j)}(y+1)}{(x-y)^{\mu+\nu+s-1-j}} \\
&\qquad + (-1)^\nu\sum_{j=1}^{\mu+s-1}\frac{(-1)^{1+j}}{j!}\binom{\mu+\nu+s-j-2}{\nu-1}\frac{\psi^{(j)}(x+1)}{(x-y)^{\mu+\nu+s-1-j}}
\end{aligned} \tag{71}$$

and

$$\begin{aligned}
&\sum_{\xi=0}^{\infty} \frac{(\nu)_\xi}{(\mu)_s\, \xi!} \sum_{\substack{j=0\\ j\leqslant \xi}}^{s} (-1)^j \binom{s}{j} \langle \xi \rangle_j\, (\mu+\nu+\xi)_{s-j} \\
&\qquad \times (x-y)^{\xi-j}\, \eta(\mu+\nu+\xi+s-j, 1+x) \\
&= (y-x)^{-\nu} \sum_{j=0}^{\mu+s-1} \binom{\nu+j-1}{\nu-1} (x-y)^{-j}\, \eta(\mu+s-j, x+1) \\
&\quad + (x-y)^{-\mu-s} \sum_{j=0}^{\nu-1} \binom{\mu+s+j-1}{\mu+s-1} (y-x)^{-j}\, \eta(\nu-j, y+1).
\end{aligned} \tag{72}$$

Proof. Using the similar process of proof Theorem 3, with the aid of the identities (69) and (70), and (44) and (45), we prove (71) and (72). The involved details are omitted. □

Like Corollary 2, putting $x = 0$ in (71) and (72), we obtain series involving Riemann zeta and eta functions, without proof, asserted in the subsequent corollary.

Corollary 3. *Let $y \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$ be such that $|y| < 1$. Also, set μ, $\nu \in \mathbb{Z}_{\geqslant 1}$ and $s \in \mathbb{Z}_{\geqslant 0}$. Then,*

$$\begin{aligned}
&\sum_{\xi=0}^{\infty} \frac{(-1)^\xi\, (\nu)_\xi}{(\mu)_s\, \xi!} \sum_{\substack{j=0\\ j\leqslant \xi}}^{s} \binom{s}{j} \langle \xi \rangle_j\, (\mu+\nu+\xi)_{s-j} \\
&\qquad \times y^{\xi-j}\, \zeta(\mu+\nu+\xi+s-j) \\
&= (-1)^\nu \binom{\mu+\nu+s-2}{\mu+s-1} \frac{\psi(y+1)+\gamma}{(-y)^{\mu+\nu+s-1}} \\
&\quad + (-1)^\nu \sum_{j=1}^{\nu-1} \frac{1}{j!} \binom{\mu+\nu+s-j-2}{\mu+s-1} \frac{\psi^{(j)}(y+1)}{(-y)^{\mu+\nu+s-1-j}} \\
&\quad + (-1)^\nu \sum_{j=1}^{\mu+s-1} \binom{\mu+\nu+s-j-2}{\nu-1} \frac{\zeta(j+1)}{(-y)^{\mu+\nu+s-1-j}}
\end{aligned} \tag{73}$$

and

$$\begin{aligned}
&\sum_{\xi=0}^{\infty} \frac{(-1)^\xi\, (\nu)_\xi}{(\mu)_s\, \xi!} \sum_{\substack{j=0\\ j\leqslant \xi}}^{s} \binom{s}{j} \langle \xi \rangle_j\, (\mu+\nu+\xi)_{s-j} \\
&\qquad \times y^{\xi-j}\, \eta(\mu+\nu+\xi+s-j) \\
&= y^{-\nu} \sum_{j=0}^{\mu+s-1} \binom{\nu+j-1}{\nu-1} (-y)^{-j}\, \eta(\mu+s-j) \\
&\quad + (-y)^{-\mu-s} \sum_{j=0}^{\nu-1} \binom{\mu+s+j-1}{\mu+s-1} y^{-j}\, \eta(\nu-j, y+1).
\end{aligned} \tag{74}$$

The following theorem offers an interesting closed-form evaluation of series involving Riemann zeta functions with parameters.

Theorem 5. *Let $\ell \in \mathbb{Z}_{>0}$, $\alpha_k \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$ and $\beta_k \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$ $(k = 1, 2, \ldots, \ell)$ be such that*

$$\max_{1\leqslant k\leqslant \ell} \{|\alpha_k|, |\beta_k|\} < 1. \tag{75}$$

and

$$\sum_{k=1}^{\ell} \alpha_k = \sum_{k=1}^{\ell} \beta_k. \tag{76}$$

Here and elsewhere, the principal value of $\log(\cdot)$ *is assumed to be taken. Then,*

$$\sum_{j=2}^{\infty} \frac{(-1)^j}{j} \sum_{k=1}^{\ell} \left\{ (\beta_k)^j - (\alpha_k)^j \right\} \zeta(j) = \log\left\{ \prod_{k=1}^{\ell} \frac{\Gamma(1+\beta_k)}{\Gamma(1+\alpha_k)} \right\}. \tag{77}$$

We also have

$$\sum_{j=2}^{\infty} \frac{1}{j} \sum_{k=1}^{\ell} \left\{ (\beta_k)^j - (\alpha_k)^j \right\} \zeta(j) = \log\left\{ \prod_{k=1}^{\ell} \frac{\Gamma(1-\beta_k)}{\Gamma(1-\alpha_k)} \right\}; \tag{78}$$

$$\sum_{j=1}^{\infty} \frac{1}{j} \sum_{k=1}^{\ell} \left\{ (\beta_k)^{2j} - (\alpha_k)^{2j} \right\} \zeta(2j) = \log\left\{ \prod_{k=1}^{\ell} \frac{\beta_k \sin(\pi\,\alpha_k)}{\alpha_k \sin(\pi\,\beta_k)} \right\}; \tag{79}$$

$$\begin{aligned} &\sum_{j=1}^{\infty} \frac{1}{2j+1} \sum_{k=1}^{\ell} \left\{ (\beta_k)^{2j+1} - (\alpha_k)^{2j+1} \right\} \zeta(2j+1) \\ &\qquad = \frac{1}{2} \log\left\{ \prod_{k=1}^{\ell} \frac{\Gamma(1+\alpha_k)\Gamma(1-\beta_k)}{\Gamma(1-\alpha_k)\Gamma(1+\beta_k)} \right\}. \end{aligned} \tag{80}$$

Proof. Under constraint (76), we find (see, for instance, [43], pp. 6–7)

$$\prod_{n=1}^{\infty} \frac{(n+\alpha_1)(n+\alpha_2)\cdots(n+\alpha_\ell)}{(n+\beta_1)(n+\beta_2)\cdots(n+\beta_\ell)} = \prod_{k=1}^{\ell} \frac{\Gamma(1+\beta_k)}{\Gamma(1+\alpha_k)}. \tag{81}$$

Let $\mathcal{L}_1$ be the left member of (81). Then,

$$\mathcal{L}_1 = \prod_{n=1}^{\infty} \frac{\left(1+\frac{\alpha_1}{n}\right)\left(1+\frac{\alpha_2}{n}\right)\cdots\left(1+\frac{\alpha_\ell}{n}\right)}{\left(1+\frac{\beta_1}{n}\right)\left(1+\frac{\beta_2}{n}\right)\cdots\left(1+\frac{\beta_\ell}{n}\right)}. \tag{82}$$

Taking logarithms on both sides of (82) gives

$$\log \mathcal{L}_1 = \sum_{n=1}^{\infty} \sum_{k=1}^{\ell} \left\{ \log\left(1+\frac{\alpha_k}{n}\right) - \log\left(1+\frac{\beta_k}{n}\right) \right\}. \tag{83}$$

Applying the Maclaurin series

$$\log(1+z) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} z^j \quad (|z|<1)$$

to each log-term in (83), we obtain

$$\begin{aligned} \log \mathcal{L}_1 &= \sum_{n=1}^{\infty} \sum_{k=1}^{\ell} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \left\{ (\alpha_k)^j - (\beta_k)^j \right\} \frac{1}{n^j} \\ &= \sum_{n=1}^{\infty} \sum_{j=2}^{\infty} \frac{(-1)^{j+1}}{j} \sum_{k=1}^{\ell} \left\{ (\alpha_k)^j - (\beta_k)^j \right\} \frac{1}{n^j} \\ &= \sum_{j=2}^{\infty} \frac{(-1)^{j+1}}{j} \sum_{k=1}^{\ell} \left\{ (\alpha_k)^j - (\beta_k)^j \right\} \zeta(j), \end{aligned}$$

where restrictions (75) and (76) are used for the first and second equalities, respectively, and (2) is employed for the third equality. Also, since one can observe that the second multiple series converges absolutely, the order of summations is interchangeable. Hence, upon matching the logarithm on the right member of (81) and the last expression of $\log \mathcal{L}_1$, the result (77) easily follows.

Replacing β_k and α_k with $-\beta_k$ and $-\alpha_k$ $(k = 1, \ldots, \ell)$ in (77) yields (78). Recall the following well-known formula:

$$\Gamma(s)\,\Gamma(1-s) = \pi \csc(\pi s) \quad (s \in \mathbb{C} \setminus \mathbb{Z}). \tag{84}$$

Adding (78) and (77), and subtracting (77) from (78), side by side, with the aid of (84), respectively, shows (79) and (80). □

3. Applications

Euler sums and series involving zeta functions have been associated with and used in various research subjects (consult, for example, [3]). Several applications of Euler sums and series involving zeta functions are demonstrated in this section.

To prevent any confusion, throughout this section, the pure imaginary unit $i = \sqrt{-1}$ is indicated as $\mathbf{i} := \sqrt{-1}$.

3.1. Wallis's Infinite Product Formula for π

In 1655, Wallis [44] presented his renowned infinite product for π:

$$\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\frac{8}{7}\frac{8}{9}\frac{10}{9}\frac{10}{11}\frac{12}{11}\frac{12}{13}\frac{14}{13}\frac{14}{15}\frac{16}{15}\frac{16}{17}\cdots =: \Omega. \tag{85}$$

Since then, various basic and advanced proofs of (85), as well as related products and fascinating anecdotes have been presented (see, e.g., [45–54]). In 1873, Catalan [50] proved the Wallis-type formulas

$$\frac{\pi}{2\sqrt{2}} = \frac{4}{3}\frac{4}{5}\frac{8}{7}\frac{8}{9}\frac{12}{11}\frac{12}{13}\frac{16}{15}\frac{16}{17}\cdots =: \Lambda_1 \tag{86}$$

and

$$\sqrt{2} = \frac{2}{1}\frac{2}{3}\frac{6}{5}\frac{6}{7}\frac{10}{9}\frac{10}{11}\frac{14}{13}\frac{14}{15}\cdots =: \Lambda_2. \tag{87}$$

Together, they provide an elegant factorization of Wallis's formula, which is written as $\Omega = \Lambda_1 \times \Lambda_2$ (see also [52]).

Here, (85), (86) and (87) are shown using closed-form evaluations of specific series involving zeta functions. We find that

$$\Omega = \prod_{k=1}^{\infty} \frac{4k^2}{4k^2-1}, \quad \Lambda_1 = \prod_{k=1}^{\infty} \frac{(4k)^2}{(4k)^2-1}, \quad \Lambda_2 = \prod_{k=1}^{\infty} \frac{(4k-2)^2}{(4k-3)\,(4k-1)}. \tag{88}$$

Taking the principal logarithms on each member in (88), we obtain

$$\log \Omega = -\sum_{k=1}^{\infty} \log\left(1 - \frac{1}{4k^2}\right) = \sum_{k=1}^{\infty} \sum_{\tau=1}^{\infty} \frac{1}{\tau\,(4k^2)^{\tau}} = \sum_{\tau=1}^{\infty} \frac{\zeta(2\tau)}{\tau\, 2^{2\tau}}; \tag{89}$$

$$\log \Lambda_1 = -\sum_{k=1}^{\infty} \log\left(1 - \frac{1}{(4k)^2}\right) = \sum_{\tau=1}^{\infty} \frac{\zeta(2\tau)}{\tau\, 4^{2\tau}}; \tag{90}$$

$$\begin{aligned}
\log \Lambda_2 &= \sum_{k=1}^{\infty} \left\{ 2\log\left(1 - \frac{1}{2k}\right) - \log\left(1 - \frac{3}{4k}\right) - \log\left(1 - \frac{1}{4k}\right) \right\} \\
&= \sum_{k=1}^{\infty} \left\{ -2\sum_{\tau=1}^{\infty} \frac{1}{\tau\,(2k)^{\tau}} + \sum_{\tau=1}^{\infty} \frac{1}{\tau}\left(\frac{3}{4k}\right)^{\tau} + \sum_{\tau=1}^{\infty} \frac{1}{\tau}\left(\frac{1}{4k}\right)^{\tau} \right\} \\
&= \sum_{k=1}^{\infty} \left\{ -2\sum_{\tau=2}^{\infty} \frac{1}{\tau\,(2k)^{\tau}} + \sum_{\tau=2}^{\infty} \frac{1}{\tau}\left(\frac{3}{4k}\right)^{\tau} + \sum_{\tau=2}^{\infty} \frac{1}{\tau}\left(\frac{1}{4k}\right)^{\tau} \right\} \\
&= -2\sum_{\tau=2}^{\infty} \frac{\zeta(\tau)}{\tau\, 2^{\tau}} + \sum_{\tau=2}^{\infty} \frac{\zeta(\tau)}{\tau}\left(\frac{3}{4}\right)^{\tau} + \sum_{\tau=2}^{\infty} \frac{\zeta(\tau)}{\tau}\left(\frac{1}{4}\right)^{\tau} =: \Theta.
\end{aligned} \tag{91}$$

For expression (89), also see ([51], Equation (6)). Replacing y with $-y$ in (56) gives

$$\sum_{\tau=2}^{\infty} \frac{\zeta(\tau)}{\tau}\, y^{\tau} = \log\Gamma(1-y) - \gamma\, y \quad (|y|<1). \tag{92}$$

Adding (56) and (92) side by side, with the help of (84), gives

$$\sum_{\tau=1}^{\infty} \frac{\zeta(2\tau)}{\tau}\, y^{2\tau} = \log\left(\frac{\pi y}{\sin \pi y}\right) \quad (|y|<1), \tag{93}$$

which is a known formula (see, e.g., [3], p. 271, Equation (17)). Setting $y = \frac{1}{2}$ and $y = \frac{1}{4}$ in (93), respectively, gives

$$\log\Omega = \log\left(\frac{\pi}{2}\right) \quad \text{and} \quad \log\Lambda_1 = \log\left(\frac{\pi}{2\sqrt{2}}\right),$$

which are found to be equivalent to the product values in (85) and (86). Putting $y = \frac{1}{2}$, $y = \frac{3}{4}$, and $y = \frac{1}{4}$ in (92), with the aid of (84), provides the value Θ in (91):

$$\Theta = -2\,\log\Gamma\left(\frac{1}{4}\right) + \log\left\{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)\right\} = \log\sqrt{2},$$

which proves (87).

3.2. Mathieu Series

Émile Leonard Mathieu (1835–1890) [55] explored the infinite series

$$\mathbf{S}(\tau) = \sum_{\xi=1}^{\infty} \frac{2\xi}{(\xi^2+\tau^2)^2} \quad (\tau \in \mathbb{R}_{>0}) \tag{94}$$

in research on the elasticity of solid bodies (also consult [56]). Pogány et al. [57] proposed an alternate representation of the Mathieu series (94):

$$\tilde{\mathbf{S}}(\tau) = \sum_{\xi=1}^{\infty} (-1)^{\xi-1} \frac{2\xi}{(\xi^2+\tau^2)^2} \quad (\tau \in \mathbb{R}_{>0}). \tag{95}$$

Since Mathieu's era, numerous researchers have explored various facets of the Mathieu series, such as (94), as well as (95), in a range of techniques (see, for instance, [55–74]).

Pogány et al. [57] offered the integral representations of the Mathieu series (94) and the alternating Mathieu series (95), as follows:

$$\mathbf{S}(\tau) = \frac{1}{\tau}\int_0^{\infty} \frac{u\,\sin(\tau u)}{e^u-1}\,du, \tag{96}$$

and

$$\tilde{\mathbf{S}}(\tau) = \frac{1}{\tau}\int_0^{\infty} \frac{u\,\sin(\tau u)}{e^u+1}\,du. \tag{97}$$

Choi and Srivastava [75] formulated (94) and (95) as series associated with the Riemann zeta function, which is evaluated by the Trigamma function and, hence, by the generalized zeta function:

$$\mathbf{S}(\tau) = 2\sum_{n=1}^{\infty} (-1)^{n-1}\, n\,\zeta(2n+1)\,\tau^{2(n-1)} \quad (|\tau|<1), \tag{98}$$

and

$$\tilde{\mathbf{S}}(\tau) = 2\sum_{n=1}^{\infty} (-1)^{n-1}\, n\,\eta(2n+1)\,\tau^{2(n-1)} \quad (|\tau|<1): \tag{99}$$

$$\mathbf{S}(\tau) = \frac{\mathbf{i}}{2\tau}\left\{\psi'(1+\mathbf{i}\tau) - \psi'(1-\mathbf{i}\tau)\right\} \quad (0 < |\tau| < 1); \tag{100}$$

$$\mathbf{S}(\tau) = \frac{\mathbf{i}}{2\tau}\left\{\zeta(2, 1+\mathbf{i}\tau) - \zeta(2, 1-\mathbf{i}\tau)\right\} \quad (0 < |\tau| < 1); \tag{101}$$

$$\tilde{\mathbf{S}}(\tau) = \mathbf{S}(\tau) - \frac{1}{4}\,\mathbf{S}\left(\frac{\tau}{2}\right) \quad (|\tau| < 1). \tag{102}$$

Here, it is noted that identities (100) and (101) are very useful because of numerous properties and formulas of the psi function $\psi(z)$ (and so the Trigamma function $\psi'(z)$) and the Hurwitz zeta function $\zeta(s,a)$ have been presented. In this connection, among other things, Choi and Srivastava [75] offered several integral representations for $\mathbf{S}(\tau)$.

Recall the following power series representations of variable a for $\zeta(s,a)$ (see [76], p. 25):

$$\zeta(s,a) = \sum_{n=0}^{\infty} (-1)^n \frac{(a-1)^n}{n!}\,\Gamma(n+s)\,\zeta(s+n) \quad (|a-1| < 1). \tag{103}$$

Applying (103) to (101) yields (98).

Using the following formula (see, for instance, [76], p. 14):

$$\psi^{(n)}(z) = (-1)^{n+1} \int_0^\infty \frac{u^n}{1-e^{-u}}\,e^{-zu}\,du \quad (\Re(z) > 0;\, n \in \mathbb{Z}_{>0}) \tag{104}$$

in (100) gives (96).

Employing an integral representation (see, for instance, [76], p. 16):

$$\psi(z) = -\gamma + 2\int_0^\infty e^{-zt}\,\frac{\sinh[(z-1)t]}{\sinh t}\,dt \quad (\Re(z) > 0) \tag{105}$$

in (100) affords

$$\mathbf{S}(\tau) = \frac{2}{\tau}\int_0^\infty t\,e^{-t}\,\frac{\sin(2\tau t)}{\sinh t}\,dt, \tag{106}$$

which is found to be the same formula in (96).

3.3. Mellin Transforms

Taking the Mellin transform in (96), with the aid of ([77], p. 37, Entry 2.4.1-1 and [78], p. 307, Equation (3)), we obtain

$$\mathfrak{M}\{\mathbf{S}(\tau);s\} = -2^{2-s}\cos\left(\frac{\pi}{2}s\right)\Gamma(s-1)\int_0^\infty \frac{e^{-x}\,x^{2-s}}{\sinh x}\,dx \quad (-1 < \Re(s) < 1). \tag{107}$$

Applying a known integral formula (see, e.g., [79], p. 381, Entry 3.552-1):

$$\int_0^\infty \frac{x^{\mu-1}\,e^{-\beta x}}{\sin hx}\,dx = 2^{1-\mu}\,\Gamma(\mu)\,\zeta\left[\mu, \frac{1}{2}(\beta+1)\right] \quad (\Re(\mu) > 1,\ \Re(\beta) > -1) \tag{108}$$

to (108) yields

$$\begin{aligned}\mathfrak{M}\{\mathbf{S}(\tau);s\} &= -\cos\left(\frac{\pi}{2}s\right)\Gamma(s-1)\,\Gamma(3-s)\,\zeta(3-s)\\ &= -\frac{\pi}{2}\,\csc\left(\frac{\pi}{2}s\right)\zeta(3-s) \quad (-1 < \Re(s) < 1),\end{aligned} \tag{109}$$

where Formula (84) is employed.

Similarly,

$$\mathfrak{M}\{\tilde{\mathbf{S}}(\tau);s\} = -2^{2-s}\cos\left(\frac{\pi}{2}s\right)\Gamma(s-1)\int_0^\infty \frac{e^{-x}\,x^{2-s}}{\cosh x}\,dx \quad (-1 < \Re(s) < 1). \tag{110}$$

Applying a known integral formula (consult, for example, [79], p. 381, Entry 3.552-3):

$$\int_0^\infty \frac{x^{\mu-1}\, e^{-x}}{\cos hx}\, dx = 2^{1-\mu} \left(1 - 2^{1-\mu}\right) \Gamma(\mu)\, \zeta(\mu) \quad (\Re(\mu) > 0,\ \mu \neq 1) \tag{111}$$

to (110) produces

$$\mathfrak{M}\{\tilde{\mathbf{S}}(\tau); s\} \quad = -\frac{\pi}{2} \csc\left(\frac{\pi}{2} s\right) \left(1 - 2^{s-2}\right) \zeta(3-s) \quad (-1 < \Re(s) < 1). \tag{112}$$

Recall a Mellin transform, which was input into Mathematica 13.0 (Home Edition):

$$\mathfrak{M}\left\{\tau^{2(n-1)}; s\right\} = 2\pi\, \delta[\mathbf{i}\{2(n-1) + s\}] \quad (n \in \mathbb{Z}_{>0}), \tag{113}$$

where $\delta(x)$ is the Dirac delta function (see, for instance, [80], pp. 28, 30, 80).

Taking Mellin transforms on both sides of (96) and (97), with the aid of (98), (99), (109), (112) and (113), we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty} (-1)^{n-1}\, n\, \zeta(2n+1)\, \delta[\mathbf{i}\{2(n-1) + s\}] \\ &\quad = -\frac{1}{8} \csc\left(\frac{\pi}{2} s\right) \zeta(3-s) \quad (-1 < \Re(s) < 1), \end{aligned} \tag{114}$$

and

$$\begin{aligned} &\sum_{n=1}^{\infty} (-1)^{n-1}\, n\, \eta(2n+1)\, \tau^{2(n-1)}\, \delta[\mathbf{i}\{2(n-1) + s\}] \\ &\quad = -\frac{1}{8} \csc\left(\frac{\pi}{2} s\right) \left(1 - 2^{s-2}\right) \zeta(3-s) \quad (-1 < \Re(s) < 1). \end{aligned} \tag{115}$$

Differentiating both sides of (56) gives a known formula (see, for instance, [3], p. 271, Equation (14)):

$$\sum_{\tau=1}^{\infty} (-1)^{\tau+1}\, \zeta(\tau+1)\, y^{\tau} = \psi(y+1) + \gamma \quad (|y| < 1). \tag{116}$$

Taking the Mellin transform on both sides of (116), with the aid of known identities (see (113) and [77], p. 98, Entry 3.1.2-1), we obtain

$$\sum_{\tau=1}^{\infty} (-1)^{\tau+1}\, \zeta(\tau+1)\, \delta[\mathbf{i}(\tau + s)] = \frac{\zeta(1-s)}{2\, \sin(\pi s)} \quad (-1 < \Re(s) < 0). \tag{117}$$

3.4. Determinants of Laplacians

Numerous authors, including [81,82], Sarnak [83], and Voros [84], have paid considerable attention to the problem of evaluating the determinants of the Laplacians on Riemann manifolds over the last four decades. They computed the determinants of the Laplacians on compact Riemann surfaces of constant curvatures in terms of special values of the Selberg zeta function. Although interest in the Laplacian determinants began with Riemann surfaces, it is equally fascinating and possibly beneficial to calculate these determinants for higher-dimensional classical Riemannian manifolds, such as spheres. The assessment of the functional determinant for the n-dimensional unit sphere $\mathbf{S}^n$ $(n \in \mathbb{Z}_{>0})$ with the standard metric has received special attention (see, for instance, [85–90]).

Assume $\{\alpha_k\}_{k=0}^{\infty}$ is a nonnegative, increasing, and unbounded real sequence; that is,

$$0 = \alpha_0 < \alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_k \leqq \cdots; \quad \alpha_k \uparrow \infty \quad (k \to \infty); \tag{118}$$

for the remainder of this section, we will discuss only such nonnegative growing sequences. Then we can show that

$$Z(s) := \sum_{k=1}^{\infty} \frac{1}{\alpha_k^s}, \tag{119}$$

which is recognized to converge absolutely in a half-plane $\Re(s) > \sigma$ for some $\sigma \in \mathbb{R}$.

The determinant of the Laplacian Δ on the compact manifold M is defined to be (cf. Osgood et al. [87])

$$\det{}'\Delta := \prod_{\alpha_k \neq 0} \alpha_k, \tag{120}$$

where $\{\alpha_k\}$ is the sequence of eigenvalues of the Laplacian Δ on M. The sequence $\{\alpha_k\}$ is recognized to gratify the restriction, as in (118), but the product in (120) is always divergent. Thus, some kind of regularization must be applied in order for the phrase (120) to make sense (consult, for example, [89]). It is straightforward to deduce that $e^{-Z'(0)}$ is the product of Δ's nonzero eigenvalues. Although this product does not converge, $Z(s)$ can be continued analytically to a neighborhood of $s = 0$. As a result, we can provide a proper definition:

$$\det{}'\Delta := e^{-Z'(0)}, \tag{121}$$

which is referred to as the Laplacian Δ's functional determinant on M.

The order μ of the sequence $\{\alpha_k\}$ is defined by

$$\mu := \inf\left\{\rho > 0 \,\middle|\, \sum_{k=1}^{\infty} \frac{1}{\alpha_k^{\rho}} < \infty\right\}. \tag{122}$$

The analogous and shifted analogous Weierstrass canonical products $E(\alpha)$ and $E(\alpha, b)$ of the sequence $\{\alpha_k\}$ are defined, respectively, by

$$E(\alpha) := \prod_{k=1}^{\infty}\left\{\left(1 - \frac{\alpha}{\alpha_k}\right)\exp\left(\frac{\alpha}{\alpha_k} + \frac{\alpha^2}{2\alpha_k^2} + \cdots + \frac{\alpha^{[\mu]}}{[\mu]\alpha_k^{[\mu]}}\right)\right\} \tag{123}$$

and

$$E(\alpha, b) := \prod_{k=1}^{\infty}\left\{\left(1 - \frac{\alpha}{\alpha_k + b}\right)\exp\left(\frac{\alpha}{\alpha_k + b} + \cdots + \frac{\alpha^{[\mu]}}{[\mu](\alpha_k + b)^{[\mu]}}\right)\right\}, \tag{124}$$

where $[\mu]$ signifies the biggest integer that is less than or equal to the order μ of the sequence $\{\alpha_k\}$.

$E(\alpha)$ and $E(\alpha, b)$ have the following relationship (see Voros [84]):

$$E(\alpha, b) = \exp\left(\sum_{m=1}^{[\mu]} \mathcal{R}_{m-1}(-b)\frac{\alpha^m}{m!}\right)\frac{E(\alpha - b)}{E(-b)}, \tag{125}$$

where, for the sake of convenience,

$$\mathcal{R}_{[\mu]}(\alpha - b) := \frac{d^{[\mu]+1}}{d\alpha^{[\mu]+1}}\{-\log E(\alpha, b)\}. \tag{126}$$

The shifted series $Z(s, b)$ of $Z(s)$ in (119) by b is given by

$$Z(s, b) := \sum_{k=1}^{\infty} \frac{1}{(\alpha_k + b)^s}. \tag{127}$$

Actually, we obtain

$$Z'(0, -\alpha) = -\sum_{k=1}^{\infty} \log(\alpha_k - \alpha),$$

on a formal level, which entails that

$$D(\alpha) = \prod_{k=1}^{\infty}(\alpha_k - \alpha)$$

immediately upon defining

$$D(\alpha) := \exp\big[-Z'(0, -\alpha)\big]. \tag{128}$$

Indeed, Voros [84] established the following link between $D(\alpha)$ and $E(\alpha)$:

$$\begin{aligned} D(\alpha) = &\exp[-Z'(0)]\ \exp\left[-\sum_{k=1}^{[\mu]} \mathrm{FP}Z(k)\frac{\alpha^k}{k}\right] \\ &\times \exp\left(-\sum_{k=2}^{[\mu]} \Omega_{-k}\,\mathsf{H}_{k-1}\,\frac{\alpha^k}{k!}\right) E(\alpha). \end{aligned} \tag{129}$$

The finite part prescription of a function h is implemented as follows (see Voros [84], p. 446):

$$\mathrm{FP}h(s) := \begin{cases} h(s) & \text{if } s \text{ is not a pole,} \\ \lim\limits_{\epsilon\to 0}\left(h(s+\epsilon) - \dfrac{\text{Residue}}{\epsilon}\right) & \text{if } s \text{ is a simple pole,} \end{cases} \tag{130}$$

and

$$Z(-k) = (-1)^k\, k!\, \Omega_{-k}. \tag{131}$$

Take now the sequence of eigenvalues on the standard Laplacian Δ_n on $\mathbf{S}^n$. Vardi's work [90] (see also Terras [91]) established that the standard Laplacian Δ_n $(n \in \mathbb{Z}_{>0})$ possesses eigenvalues

$$\eta_k := k(k+n-1) \tag{132}$$

with multiplicity

$$\begin{aligned} q_n(k) := \binom{k+n}{n} - \binom{k+n-2}{n} &= \frac{(2k+n-1)\,(k+n-2)!}{k!\,(n-1)!} \\ &= \frac{2k+n-1}{(n-1)!}\prod_{j=1}^{n-2}(k+j) \quad (k \in \mathbb{Z}_{\geqslant 0}). \end{aligned} \tag{133}$$

From here on, we will refer to the sequence $\{\alpha_k\}$ of $\{\eta_k\}$ in (132), which is shifted by $\left(\frac{n-1}{2}\right)^2$, as a basic sequence. Then, the sequence $\{\alpha_k\}$ is expressed in the following concise form:

$$\alpha_k = \eta_k + \left(\frac{n-1}{2}\right)^2 = \left(k + \frac{n-1}{2}\right)^2 \tag{134}$$

with the same multiplicity as in (133).

Here, the order μ_n $(n \in \mathbb{Z}_{>0})$ of the sequence $\{\alpha_k\}$ in (134) is given by

$$\mu_n = \begin{cases} m & (n = 2m), \\ m + \frac{1}{2} & (n = 2m+1), \end{cases} \tag{135}$$

for $m \in \mathbb{Z}_{>0}$.

We shall exclude the zero mode; that is, we will begin the sequence at $k = 1$ for further analysis. Additionally, in order to emphasize n on $\mathbf{S}^n$, we use the notations $Z_n(s)$, $Z_n(s,b)$, $E_n(\alpha)$, $E_n(\alpha,b)$, and $D_n(\alpha)$ instead of $Z(s)$, $Z(s,b)$, $E(\alpha)$, $E(\alpha,b)$, and $D(\alpha)$, respectively.

We readily observe from (128) that

$$D_n\left(\left(\frac{n-1}{2}\right)^2\right) = \det{}'\Delta_n, \tag{136}$$

where $\det' \Delta_n$ denotes the *determinants of the Laplacians* on $\mathbf{S}^n$ $(n \in \mathbb{Z}_{>0})$.

Several authors (see Choi [85], Kumagai [86], Vardi [90], and Voros [84]) employed the theory of multiple gamma functions (see Barnes [92–95]) to evaluate the determinants of the Laplacians on the n-dimensional unit sphere $\mathbf{S}^n$ $(n \in \mathbb{Z}_{>0})$. Quine and Choi [88] utilized zeta regularized products to evaluate $\det' \Delta_n$ and the determinant of the conformal Laplacian, $\det(\Delta_{\mathbf{S}^n} + n(n-2)/4)$. Choi and Srivastava [96,97], Choi et al. [98], and Choi [99] used certain closed-form evaluations of the series associated with zeta functions (see [3], Chapter 3) for the computation of the determinants of the Laplacians on $\mathbf{S}^n$ $(n = 2, 3, 4, 5, 6, 7, 8, 9)$. Choi [100] presented a general explicit formula for the determinants of the Laplacians on $\mathbf{S}^{2n+1}$ $(n \in \mathbb{Z}_{>0})$ by mainly using a closed-form expression of certain series involving zeta functions.

Question: As in [100], can one establish a general explicit formula for the determinants of the Laplacians on $\mathbf{S}^{2n}$ $(n \in \mathbb{Z}_{>0})$ by mainly using closed-form evaluations of certain series involving zeta functions (for instance, (54), (57))?

Here, we attempt to only evaluate

$$D_{10}(\alpha) = \det{}'\Delta_{10},$$

where $\alpha := (9/2)^2$.

To do this, from (133)–(135), we find that the shifted basic sequence of eigenvalues on the standard Laplacian Δ_{10} on $\mathbf{S}^{10}$ is given as

$$\alpha_k = \left(k + \frac{9}{2}\right)^2 \tag{137}$$

with multiplicity

$$q_{10}(k) = \frac{2k+9}{9!} \prod_{j=1}^{8} (k+j) \quad (k \in \mathbb{Z}_{\geqslant 0}). \tag{138}$$

From (135), the order of the sequence (137) is 5. Hence, in view of (129), it suffices to compute the following:

$$\begin{aligned} D_{10}\left(\frac{81}{4}\right) &= \exp[-Z'_{10}(0)]\, \exp\left[-\sum_{k=1}^{5} \mathrm{FP}Z_{10}(k) \frac{\left(\frac{81}{4}\right)^k}{k}\right] \\ &\quad \times \exp\left(-\sum_{k=2}^{5} \Omega_{-k}\, \mathcal{H}_{k-1} \frac{\left(\frac{81}{4}\right)^k}{k!}\right) E_{10}\left(\frac{81}{4}\right). \end{aligned} \tag{139}$$

Here, in order to show how $\det'\Delta_{10}$ can be involved in closed-form evaluations of series associated with the zeta functions, we only have to compute $E_{10}\left(\frac{81}{4}\right)$. From (123), we obtain

$$\begin{aligned} \log E_{10}(\alpha) &= \sum_{k=1}^{\infty} q_{10}(k) \left[\log\left(1 - \frac{\alpha}{\alpha_k}\right) + \sum_{j=1}^{5} \frac{1}{j}\left(\frac{\alpha}{\alpha_k}\right)^j\right] \\ &= -\sum_{k=1}^{\infty} q_{10}(k) \left[\sum_{\ell=6}^{\infty} \frac{1}{\ell}\left(\frac{\alpha}{\alpha_k}\right)^\ell\right], \end{aligned} \tag{140}$$

where the Maclaurin expansion is used:

$$\log(1-x) = -\sum_{\ell=1}^{\infty} \frac{x^{\ell}}{\ell} \quad (|x| < 1).$$

Using (137) and (138) in (140) gives

$$-8!\,\log E_{10}(\alpha) = \sum_{k=1}^{\infty} \left(\prod_{j=1}^{8} (k+j) \right) \left[\sum_{\ell=6}^{\infty} \frac{1}{\ell} \left(\frac{9}{2}\right)^{2\ell-1} \left(\frac{1}{k+9/2} \right)^{2\ell-1} \right]. \tag{141}$$

Let $\tau := k + 9/2$. Then,

$$\prod_{j=1}^{8} (k+j) = \prod_{j=1}^{4} \left[\tau^2 - \left(\frac{2j-9}{2} \right)^2 \right],$$

which, upon shortening the computation, yields

$$\prod_{j=1}^{8} (k+j) = \tau^8 - 21\,\tau^6 + \frac{987}{8}\,\tau^4 - \frac{3229}{16}\,\tau^2 + \frac{11025}{256}. \tag{142}$$

Setting (142) in (141) gives

$$\begin{aligned}
-8!\,\log E_{10}(81/4) = \sum_{\ell=6}^{\infty} \frac{1}{\ell} \left(\frac{9}{2}\right)^{2\ell-1} & \left[\sum_{k=0}^{\infty} \left(\frac{1}{k+11/2} \right)^{2\ell-9} - 21 \sum_{k=0}^{\infty} \left(\frac{1}{k+11/2} \right)^{2\ell-7} \right. \\
& + \frac{987}{8} \sum_{k=0}^{\infty} \left(\frac{1}{k+11/2} \right)^{2\ell-5} - \frac{3229}{16} \sum_{k=0}^{\infty} \left(\frac{1}{k+11/2} \right)^{2\ell-3} \\
& \left. + \frac{11025}{256} \sum_{k=0}^{\infty} \left(\frac{1}{k+11/2} \right)^{2\ell-1} \right],
\end{aligned}$$

which, in view of (6), is expressed in terms of series involving the generalized zeta functions:

$$\begin{aligned}
-\,8!\,\log E_{10}(81/4) = \sum_{\ell=6}^{\infty} \frac{1}{\ell} \left(\frac{9}{2}\right)^{2\ell-1} & \Big[\zeta(2\ell-9, 11/2) - 21\,\zeta(2\ell-7, 11/2) \\
+ \frac{987}{8}\,\zeta(2\ell-5, 11/2) - \frac{3229}{16}\,\zeta(2\ell-3, 11/2) & + \frac{11025}{256}\,\zeta(2\ell-1, 11/2) \Big].
\end{aligned} \tag{143}$$

The identity (143) is also rewritten as follows:

$$-8!\,\log E_{10}(81/4) = \sum_{k=1}^{5} \mathcal{Z}_k, \tag{144}$$

where

$$\mathcal{Z}_1 = \sum_{\ell=1}^{\infty} \frac{1}{\ell+5} \left(\frac{9}{2}\right)^{2\ell+9} \zeta(2\ell+1, 11/2),$$

$$\mathcal{Z}_2 = -21 \sum_{\ell=1}^{\infty} \frac{1}{\ell+5} \left(\frac{9}{2}\right)^{2\ell+9} \zeta(2\ell+3, 11/2),$$

$$\mathcal{Z}_3 = \frac{987}{8} \sum_{\ell=1}^{\infty} \frac{1}{\ell+5} \left(\frac{9}{2}\right)^{2\ell+9} \zeta(2\ell+5, 11/2),$$

$$\mathcal{Z}_4 = -\frac{3229}{16} \sum_{\ell=1}^{\infty} \frac{1}{\ell+5} \left(\frac{9}{2}\right)^{2\ell+9} \zeta(2\ell+7, 11/2),$$

$$\mathcal{Z}_5 = \frac{11025}{256} \sum_{\ell=1}^{\infty} \frac{1}{\ell+5} \left(\frac{9}{2}\right)^{2\ell+9} \zeta(2\ell+9, 11/2).$$

Here, $\mathcal{Z}_k$ $(k = 1, \ldots, 5)$ can be evaluated by using a formula for series involving zeta functions (see [3], p. 258, Equations (66) and (67)). For example,

$$\begin{aligned} \mathcal{Z}_3 = \frac{329}{12} \Big[&-\frac{1}{4}\left(\frac{9}{2}\right)^8 \zeta(3, 11/2) - \frac{1}{5}\left(\frac{9}{2}\right)^{19} \zeta(5, 11/2) \\ &+ \sum_{\ell=1}^{\infty} \frac{1}{\ell+3}\left(\frac{9}{2}\right)^{2\ell+6} \zeta(2\ell+1, 11/2) \Big]. \end{aligned} \tag{145}$$

Here, by using a known formula (see [3], p. 258, Equation (67)), we obtain

$$\begin{aligned} &\sum_{\ell=1}^{\infty} \frac{1}{\ell+3}\left(\frac{9}{2}\right)^{2\ell+6} \zeta(2\ell+1, 11/2) \\ &= \sum_{k=0}^{5} \binom{5}{k} \Big[\zeta'(k) - (-1)^k \zeta'(-k, 10) \Big] \left(\frac{9}{2}\right)^{5-k} \\ &+ \frac{82695519}{8960} - \frac{177147}{64}\gamma - \frac{177147}{32} \log 2, \end{aligned} \tag{146}$$

where some other formulas (see [3], p. 31, Equations (50) and (51); p. 151, Equation (17)) are used.

Also, if (7) and an identity (see, for instance, [3], p. 150, Equation (5)) are employed, the $\zeta(3, 11/2)$ and $\zeta(5, 11/2)$ are reduced to yield

$$\zeta(3, 11/2) = 7\,\zeta(3) - \frac{262380376}{31255875}; \quad \zeta(5, 11/2) = 31\,\zeta(5) - \frac{2^8 \cdot 1749037771}{(5 \cdot 7 \cdot 9)^5}.$$

Note that the other components in (139) can be readily computed (consult, for example, [99,100]).

3.5. Integrals Expressed in Terms of Euler Sums

When certain log–log integrals on the real half-line $x \geqslant 0$ are evaluated, their representations are expressed in terms of Euler sums (see the four notations in (34)) corresponding to the first three equations in [40], Theorem 2.1. In general, integrals have the form:

$$\int_0^{\infty} \frac{\log^p(x)\, \log(1 \pm x^q)}{x(1 \pm x)}\, dx.$$

Consider the family of integrals

$$J(p, q) := \int_0^{\infty} \frac{\log^p(x)\, \log(1 + x^q)}{x(1 + x)}\, dx, \tag{147}$$

where $(p, q) \in \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{>0}$. The simple case $(p, q) = (0, 1)$ of (147) gives the famous Euler's formula:

$$\zeta(2) = \int_0^{\infty} \frac{\log(1 + x)}{x(1 + x)}\, dx.$$

Decompose the integral in (147) as follows:

$$J(p, q) = \int_0^1 f(x)\, dx + \int_1^{\infty} f(x)\, dx \quad [(p, q) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}], \tag{148}$$

where

$$f(x) := \frac{\log^p(x)\, \log(1 + x^q)}{x(1 + x)}. \tag{149}$$

It is noted that the $f(x)$ in (149) is continuous and bounded on the interval $[0,\infty)$, and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to\infty} f(x) = 0.$$

In the last integral of (148), substituting $x = 1/y$ and then being the variable y replaced by x, we obtain

$$J(p,q) = \int_0^1 (1+(-1)^p x) f(x)dx + (-1)^{p+1} q \int_0^1 \frac{\log(x)^{p+1}}{(1+x)}\,dx. \tag{150}$$

Analyzing the integrals of (150) by Taylor series expansions, and taking into account their appropriate convergence regions, after some appropriate simplifications, we arrive at

$$\begin{aligned} J(p,q) =& \frac{p!}{2^{p+1}}((-1)^p - 1)\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n}\mathsf{H}^{(p+1)}_{\frac{qn}{2}-\frac{1}{2}} \\ &- \frac{p!}{2^{p+1}}((-1)^p - 1)\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n}\mathsf{H}^{(p+1)}_{\frac{qn}{2}} \\ &+ p!\left(\frac{(-1)^p}{q^{p+1}} + q(p+1)\right)\eta(p+2). \end{aligned} \tag{151}$$

Using the notation (34) in (151), we obtain

$$\begin{aligned} J(p,q) =& \frac{p!}{2^{p+1}}((-1)^p - 1)\left(\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n}\mathsf{H}^{(p+1)}_{\frac{qn}{2}-\frac{1}{2}} - \mathbf{S}^{+-}_{p+1,1}\left(0,0,\frac{q}{2}\right)\right) \\ &+ p!\left(\frac{(-1)^p}{q^{p+1}} + q(p+1)\right)\eta(p+2). \end{aligned} \tag{152}$$

The formula (152) can be separated into the following two cases:

(a) p is even:

$$J(p,q) = p!\left(\frac{1}{q^{p+1}} + q(p+1)\right)\eta(p+2). \tag{153}$$

(b) p is odd:

$$\begin{aligned} J(p,q) =& \frac{p!}{2^p}\left(\mathbf{S}^{+-}_{p+1,1}\left(0,0,\frac{q}{2}\right) - \sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n}\mathsf{H}^{(p+1)}_{\frac{qn}{2}-\frac{1}{2}}\right) \\ &+ p!\left(q(p+1) - \frac{1}{q^{p+1}}\right)\eta(p+2). \end{aligned} \tag{154}$$

The $J(p,q)$ seems to be input into Mathematica, whose several particular cases are recorded:

$$J(1,2) = \pi\, G + \frac{1}{8}\,\pi^2\,\ln 2 - \frac{3}{16}\,\zeta(3); \tag{155}$$

$$J(3,2) = 6\pi\,\beta(4) + \frac{3}{4}\,\pi^3\,G + \frac{7}{64}\pi^4\,\ln 2 - \frac{3}{64}\,\pi^2\,\zeta(3) - \frac{45}{128}\zeta(5); \tag{156}$$

$$\begin{aligned} J(5,2) =& \frac{25}{16}\pi^5 G + \frac{29295}{128}\zeta(6)\ln 2 + 15\pi^3\beta(4) + 120\pi\beta(6) \\ &- \frac{315}{128}\zeta(4)\zeta(3) - \frac{225}{128}\zeta(2)\zeta(5) - \frac{945}{512}\zeta(7), \end{aligned} \tag{157}$$

where G is Catalan's constant in (25) and $\beta(\cdot)$ is the Dirichlet Beta function in (22).

Also, consider the following family of integrals containing polylogarithmic functions $\mathrm{Li}_t(\pm x^q)$ in the integrand:

$$J(p,q,t) := \int_0^\infty \frac{\log^p(x)\,\mathrm{Li}_t(\pm x^q)}{x(1\pm x)}\,dx,$$

in which numerous other Euler-sum identities similar to (152) can be identified; however, the specifics will not be discussed here. For other log integrals, one can refer to [101–103].

3.6. Representations and Evaluations of Integrals

$\log\Gamma(y+1)$ is related to series involving zeta functions, as follows (see, for instance, [43], p. 46, Equation (9)):

$$\log\Gamma(y+1) = -\frac{1}{2}\log\left(\frac{\sin \pi y}{\pi y}\right) - \sum_{\tau=1}^{\infty} \frac{\zeta(2\tau+1)}{2\tau+1}\, y^{2\tau+1} - \gamma\, y \quad (|y|<1). \tag{158}$$

Applying (158) to (56) gives

$$\sum_{\tau=2}^{\infty} \frac{(-1)^\tau}{\tau}\,\zeta(\tau)\,y^\tau = -\frac{1}{2}\log\left(\frac{\sin \pi y}{\pi y}\right) - \sum_{\tau=1}^{\infty} \frac{\zeta(2\tau+1)}{2\tau+1}\, y^{2\tau+1} \quad (|y|<1). \tag{159}$$

Integrating both sides of (159) from 0 to 1 offers

$$\sum_{\tau=2}^{\infty} \frac{(-1)^\tau}{\tau(\tau+1)}\,\zeta(\tau) = -\frac{1}{2}\int_0^1 \log\left(\frac{\sin \pi y}{\pi y}\right) dy - \sum_{\tau=1}^{\infty} \frac{\zeta(2\tau+1)}{(2\tau+1)(2\tau+2)}. \tag{160}$$

Decomposing the summation on the left side of (160) into even and odd summation indices, and simplifying the resulting identity, we obtain

$$\int_0^1 \log\left(\frac{\sin \pi y}{\pi y}\right) dy = -\sum_{\tau=1}^{\infty} \frac{\zeta(2\tau)}{\tau(2\tau+1)} \approx -0.837877. \tag{161}$$

Recall a known formula (see, e.g., [3], p. 326, Equation (580)):

$$\sum_{\tau=1}^{\infty} \frac{\zeta(2\tau)-1}{\tau(2\tau+1)} = -3 + \log(8\,\pi). \tag{162}$$

Using the following easily-derivable formula

$$\sum_{\tau=1}^{\infty} \frac{1}{\tau(2\tau+1)} = 2 - 2\,\log 2 \tag{163}$$

in (162) affords

$$\sum_{\tau=1}^{\infty} \frac{\zeta(2\tau)}{\tau(2\tau+1)} = -1 + \log(2\,\pi). \tag{164}$$

Finally, employing (164) in (161), we obtain an integral formula

$$\int_0^1 \log\left(\frac{\sin \pi y}{\pi y}\right) dy = \frac{1}{\pi}\int_0^\pi \log\left(\frac{\sin y}{y}\right) dy = 1 - \log(2\,\pi), \tag{165}$$

which can be obtained by combining two known integral formulas ([79], Entries 4.215-1 and 4.224-1).

Dividing both sides of (56) by y and integrating the resulting identity from 0 to 1, we find

$$\begin{aligned}\int_0^1 \frac{1}{y} \log \Gamma(y+1)\, dy &= -\gamma + \sum_{\tau=2}^{\infty} \frac{(-1)^{\tau}}{\tau^2}\, \zeta(\tau) \\ &\approx -0.25687,\end{aligned} \tag{166}$$

which, upon integrating by parts, also yields

$$\int_0^1 (\log y)\, \psi(y+1)\, dy = \gamma - \sum_{\tau=2}^{\infty} \frac{(-1)^{\tau}}{\tau^2}\, \zeta(\tau). \tag{167}$$

From (158) and (166), we readily establish the following formula:

$$\int_0^1 \frac{1}{y} \log\left(\frac{\sin \pi y}{\pi y}\right) dy = -\frac{1}{2} \sum_{\tau=1}^{\infty} \frac{\zeta(2\tau)}{\tau^2} \approx -1.15634. \tag{168}$$

Rearranging the terms in (159), we derive

$$\log\left(\frac{\sin \pi y}{\pi y}\right) = -2 \sum_{\tau=2}^{\infty} \frac{(-1)^{\tau}}{\tau}\, \zeta(\tau)\, y^{\tau} - 2 \sum_{\tau=1}^{\infty} \frac{\zeta(2\tau+1)}{2\tau+1}\, y^{2\tau+1} \quad (|y| < 1), \tag{169}$$

which implies

$$\frac{1}{y^2} \log\left(\frac{\sin \pi y}{\pi y}\right) = O(1) \quad (y \to 0).$$

Dividing both sides of (169) by y^2 and integrating both sides of the resulting identity from 0 to 1, we obtain

$$\int_0^1 \frac{1}{y^2} \log\left(\frac{\sin \pi y}{\pi y}\right) dy = -\sum_{\tau=1}^{\infty} \frac{\zeta(2\tau)}{\tau(2\tau-1)} \approx -2.04628. \tag{170}$$

It is noted that (161), (168) and (170) can be derived from a known series representation ([104], Entry (50.6.5)).

Using (4), the series representation ([104], Entry (50.6.4)) is written as follows:

$$\sum_{\tau=1}^{\infty} \frac{(-1)^{\tau+1}}{\tau}\, \zeta(2\tau)\, y^{2\tau} = \log\left(\frac{\sinh \pi y}{\pi y}\right) \quad (|y| < 1). \tag{171}$$

Taking the limit as $y \to 1$ in (171) gives

$$\sum_{\tau=1}^{\infty} \frac{(-1)^{\tau+1}}{\tau}\, \zeta(2\tau) = \log\left(\frac{e^{\pi} - e^{-\pi}}{2\pi}\right). \tag{172}$$

It is noted that the particular cases $n = 2$ and $a = 1$ of the identity ([11], p. 136, Proposition 3) (or [3], p. 263, Proposition 3.6) yields

$$\sum_{\tau=1}^{\infty} \frac{(-1)^{\tau}}{\tau}\, \zeta(2\tau) = \log[\Gamma(1-\mathbf{i})\, \Gamma(1+\mathbf{i})], \tag{173}$$

which is equivalent to (172) by recalling the following well-known formula (see, e.g., [3], p. 3, Equation (12)):

$$\Gamma(1-z)\, \Gamma(1+z) = \pi z\, \csc(\pi z) \quad (z \in \{0\} \cup \mathbb{C} \setminus \mathbb{Z}). \tag{174}$$

Using (171), we can obtain the following relationships between integrals and series involving zeta functions:

$$\int_0^1 \log\left(\frac{\sinh \pi y}{\pi y}\right) dy = \sum_{\tau=1}^{\infty} \frac{(-1)^{\tau+1}}{\tau(2\tau+1)}\, \zeta(2\tau) \approx 0.471417; \tag{175}$$

$$\int_0^1 \frac{1}{y} \log\left(\frac{\sinh \pi y}{\pi y}\right) dy = \frac{1}{2} \sum_{\tau=1}^{\infty} \frac{(-1)^{\tau+1}}{\tau^2}\, \zeta(2\tau) \approx 0.724263; \tag{176}$$

$$\int_0^1 \frac{1}{y^2} \log\left(\frac{\sinh \pi y}{\pi y}\right) dy = \sum_{\tau=1}^{\infty} \frac{(-1)^{\tau+1}}{\tau(2\tau-1)}\, \zeta(2\tau) \approx 1.50989. \tag{177}$$

Here, the integral in (175) is evaluated as follows:

$$\int_0^1 \log\left(\frac{\sinh \pi y}{\pi y}\right) dy = 1 + \frac{5\pi}{12} - \log(2\pi) + \frac{1}{2\pi}, \tag{178}$$

which was input into Mathematica.

Employing (172) in (175) and (177), respectively, gives

$$\int_0^1 \log\left(\frac{\sinh \pi y}{\pi y}\right) dy = \log\left(\frac{e^{\pi} - e^{-\pi}}{2\pi}\right) - 2 \sum_{\tau=1}^{\infty} \frac{(-1)^{\tau+1}}{2\tau+1}\, \zeta(2\tau), \tag{179}$$

and

$$\int_0^1 \frac{1}{y^2} \log\left(\frac{\sinh \pi y}{\pi y}\right) dy = 2 \sum_{\tau=1}^{\infty} \frac{(-1)^{\tau+1}}{2\tau-1}\, \zeta(2\tau) - \log\left(\frac{e^{\pi} - e^{-\pi}}{2\pi}\right). \tag{180}$$

From (178) and (179), we have

$$\sum_{\tau=1}^{\infty} \frac{(-1)^{\tau+1}}{2\tau+1}\, \zeta(2\tau) = \frac{1}{2} \log\left(e^{\pi} - e^{-\pi}\right) - \frac{1}{2} - \frac{5\,\pi}{24} - \frac{1}{4\pi} \mathrm{Li}_2\left(e^{-2\pi}\right). \tag{181}$$

Adding both sides of (179) and (180) offers

$$\int_0^1 \left(1 + \frac{1}{y^2}\right) \log\left(\frac{\sinh \pi y}{\pi y}\right) dy = 4 \sum_{\tau=1}^{\infty} \frac{(-1)^{\tau+1}}{4\tau^2 - 1}\, \zeta(2\tau). \tag{182}$$

Adding (168) and (176), and subtracting (176) from (168), respectively, we obtain

$$\int_0^1 \frac{1}{y} \log\left(\frac{\sin \pi y \sinh \pi y}{\pi^2 y^2}\right) dy = -\frac{1}{4} \sum_{\tau=1}^{\infty} \frac{\zeta(4\tau)}{\tau^2} \approx -0.432076, \tag{183}$$

and

$$\int_0^1 \frac{1}{y} \log\left(\frac{\sin \pi y}{\sinh \pi y}\right) dy = -\sum_{\tau=1}^{\infty} \frac{\zeta(4\tau-2)}{(2\tau-1)^2} \approx -1.8806. \tag{184}$$

3.7. Parametric Euler Sum Identities

Borwein et al. [38] showed several very interesting parameterized classes of multiple sums whose many specific instances reduce to well-known Euler (and related) sums by extensive use of computer algebra systems (as they noted). Their fundamental formulae are summarized here (see also [75], Equations (5.24) and (5.25)):

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\ell=1}^{n-1} \frac{1}{\ell} \arctan\left(\frac{\ell}{n^2 - \ell\, n + 1}\right) &= \sum_{n=1}^{\infty} \frac{\arctan(1/n)}{n^2} \\ &= \sum_{\tau=1}^{\infty} \frac{(-1)^{\tau+1}}{2\tau-1}\, \zeta(2\tau+1), \end{aligned} \tag{185}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\ell=1}^{n-1} \frac{1}{\ell} \log \frac{n^2\{(n-\ell)^2+1\}}{(n-\ell)^2\,(n^2+1)} = \sum_{\tau=1}^{\infty} \frac{(-1)^{\tau+1}}{\tau}\,\zeta(2\tau+2). \tag{186}$$

We attempt to express the series involving zeta functions in (185) and (186) in terms of integrals. Recall a known identity (consult, for example, [3], p. 270, Equation (11)):

$$\sum_{\tau=1}^{\infty} \frac{\zeta(2\tau)}{\tau}\,u^{2\tau} = \log[\Gamma(1+u)\,\Gamma(1-u)] \quad (|u|<1). \tag{187}$$

Using (174) in (187) and differentiating both sides of the resulting identity, we have

$$\sum_{\tau=1}^{\infty} \zeta(2\tau)\,u^{2\tau-1} = \frac{1}{2\,u} - \frac{\pi}{2}\,\cot(\pi u),$$

the first term on the left member, which transposes to yield

$$\sum_{\tau=2}^{\infty} \zeta(2\tau)\,u^{2\tau-1} = -\frac{\pi^2}{6}u + \frac{1}{2\,u} - \frac{\pi}{2}\,\cot(\pi u). \tag{188}$$

Dividing both sides of (188) by u^2, we obtain

$$\sum_{\tau=2}^{\infty} \zeta(2\tau)\,u^{2\tau-3} = -\frac{\pi^2}{6\,u} + \frac{1}{2\,u^3} - \frac{\pi}{2u^2}\,\cot(\pi u). \tag{189}$$

Integrating both sides of (189) from 0 to y, we have

$$\sum_{\tau=2}^{\infty} \frac{\zeta(2\tau+2)}{\tau}\,y^{2\tau} = 2\int_0^y \left[-\frac{\pi^2}{6\,u} + \frac{1}{2\,u^3} - \frac{\pi}{2u^2}\,\cot(\pi u)\right] du. \tag{190}$$

Setting $y=\mathbf{i}$ in (190), we have

$$\sum_{\tau=2}^{\infty} \frac{(-1)^{\tau+1}}{\tau}\,\zeta(2\tau+2) \neq 2\int_0^{\mathbf{i}} \left[\frac{\pi^2}{6\,u} - \frac{1}{2\,u^3} + \frac{\pi}{2u^2}\,\cot(\pi u)\right] du, \tag{191}$$

whose integral does diverge.

Using a similar method, as in obtaining (191) in the known identity (consult, for example, [3], p. 271, Equation (16)):

$$\sum_{\tau=1}^{\infty} \zeta(2\tau+1)\,u^{2\tau} = -\frac{1}{2}\{\psi(1+u)+\psi(1-u)\} - \gamma \quad (|u|<1), \tag{192}$$

we can readily obtain

$$\begin{aligned}\sum_{\tau=1}^{\infty} \frac{(-1)^{\tau+1}}{2\tau-1}\,\zeta(2\tau+1) &= \mathbf{i}\int_0^{\mathbf{i}} \left[\frac{\gamma}{u^2} + \frac{1}{2\,u^2}\{\psi(1+u)+\psi(1-u)\}\right] du \\ &\approx 0.97657.\end{aligned} \tag{193}$$

Using (4) to modify the known identity ([104], Entry (50.5.10)) gives

$$\sum_{\tau=0}^{\infty} (-1)^{\tau+1}\,\zeta(2\tau)\,u^{2\tau} = \frac{1}{2}\,\pi u\,\coth(\pi u) \quad (|u|<1). \tag{194}$$

Employing a similar method, as in obtaining (191), using (194), we find

$$\sum_{\tau=2}^{\infty} \frac{(-1)^{\tau+1}}{\tau} \zeta(2\tau+2) \neq 2 \int_0^1 \left[\frac{\pi^2}{6\,u} + \frac{1}{u^3} - \frac{\pi}{2u^2} \coth(\pi u) \right] du, \tag{195}$$

whose integral does not converge.

4. Concluding Remarks

Four forms of linear Euler sums were suggested and investigated by Flajolet and Salvy [9]. Alzer and Choi [37] constructed and studied the four parametric linear Euler sums, which are parametric expansions of Flajolet and Salvy's four types of linear Euler sums [9]. Very recently, Sofo and Choi [40] broadened and investigated the four parametric linear Euler sums [37], revealing that two well-established and well-known topics, Euler sums and series involving the zeta functions, exhibit specific relationships (consult, for example, [40], Theorem 4.1). Both topics—Euler sums and series involving the zeta functions—have lengthy histories and have piqued the curiosities of many scholars. In this study, we presented several closed forms of series involving zeta functions (see Theorems 2–4) by using formulas for series associated with the zeta functions in [40], Theorem 4.2. Also, several applications and relationships of series involving the zeta functions and the extended parametric linear Euler sums have been explored, such as the Mathieu series, Mellin transforms, determinants of Laplacians, specific integrals represented in terms of Euler sums, as well as the representation and evaluation of certain integrals and specific parametric Euler sum identities. The use of Mathematica 13.0 (Home Edition) for various approximation values and certain integral formulas is addressed (see, for instance, Equations (113), (161), (166), (168) and (170)).

As a result of this line of research, it is anticipated that interested and concerned scholars, including the authors, will continue to study the four extended parametric linear Euler sums (see (34)) and series associated with the zeta functions.

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