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Editorial

Foundations of Continuum Mechanics and Mathematical Physics—Editorial 2021–2023

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1. Introduction

It is well known that "Physics and Symmetry/Asymmetry" is a topical Section of *Symmetry*. Accordingly, we sought to give a brief overview of the activities of the Special Issue "Foundations of Continuum Mechanics and Mathematical Physics" carried out from 2021 to 2022, and there is no better method of doing so than providing a detailed description of the published papers. These papers, to varying degrees, represent the scope and subject matter of the articles we included in the description of the Special Issue.

Foundations of Continuum Mechanics: As is well known, the basic equations of continuum mechanics are obtained by imposing suitable invariance properties on Lagrangian functionals under suitable symmetry groups. Invariance in the Galilean symmetry group is involved in the equations of classical continuum mechanics, while invariance in the Lorentz group is involved in relativity.

Differential Equations of Mathematical Physics: In the section on Differential Equations of Mathematical Physics, questions related to the solvability, regularity, stability, and asymptotic behavior of solutions to the equations of mathematical physics and PDE, including the hydrodynamic (Stokes equations) and Helmgotz equations, were proposed for consideration.

In addition, this Special Issue addressed other qualitative properties of linear and nonlinear equations and systems of mathematical physics, such as scattering theory, inverse problems, variational methods, and variational calculus.

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Citation: dell'Isola, F.; Matevossian, H.A. Foundations of Continuum Mechanics and Mathematical Physics—Editorial 2021–2023. *Symmetry* **2023**, *15*, 1643. https://doi.org/10.3390/sym15091643

Received: 11 August 2023
Accepted: 23 August 2023
Published: 25 August 2023

2. Description of Articles [1,2]

First, let us briefly discuss the papers that demonstrate the importance of analyzing the general properties of symmetry in the mechanics of deformable media. As examples, we can conditionally isolate three areas where the consideration of symmetry properties is not only sometimes decisive in the construction of deformation models but also an important factor from the perspective of new, effective models of extended thermodynamics and complicated models of media accounting for the effects of the connectivity of physical fields (for example, heat and mass transfer) and mechanical deformation fields.

In [3–7], it is shown that for the extensive class of elasticity models, including those regarding gradient elasticity, additional symmetry conditions from classical elasticity models must be met. Otherwise, models intended to simulate a deformable, defect-free medium become erroneous because the condition of the continuity of distortions, i.e., the condition of the absence of a field of defects, is not satisfied.

In [5], it was demonstrated that this symmetry condition is essential to ensuring the validity of free variational formulations commonly employed for deriving the field equations of strain gradient elasticity. Using this symmetry condition, a symmetry-related unified theory of isotropic strain gradient elasticity (dubbed GL (Gusev–Lurie) theory in the literature) with two independent strain gradient material coefficients was explicitly derived. The presented theory has simple stability criteria, and its factorized displacement form equations of equilibrium allow for the expedient identification of the fundamental solutions operative in specific theoretical and practical studies.

In [3], a recent paper, an analysis of the symmetry conditions for the tensor components of the generalized elastic moduli of gradient elasticity is provided, and the symmetry conditions are established, which are characteristic only for the gradient models. It is shown that when constructing a boundary value problem based on the variational approach using the generalized Lagrange functional, the symmetry conditions can be lost in the boundary conditions. The authors obtained a nontrivial result revealing that the energetically insignificant components of the tensor of gradient elastic moduli can lead to an erroneous form of static boundary conditions for the vector gradient models.

This result is purely geometric and is, in fact, related to strain compatibility equations. Therefore, it also extends to non-gradient models of anisotropic media, for which the general structure of elastic modulus tensors is very complex. These issues were partly considered in [4,6]. The same papers discuss the issues of symmetry in the structure of elastic modulus tensors related to the potentiality of the strain energy density and the issues of the inheritance of the symmetrical anisotropy properties from the classical theory in a gradient anisotropic medium.

The authors of [1,2] present a solution to a three-dimensional non-stationary problem regarding the action of a moving source of heat flux induced by laser radiation on the surface of a half-space using the superposition principle and non-stationary functions.

In [1], the corresponding solution is based on a Green's function method, according to which the influence function of a surface-concentrated heat source is derived at the first stage. The influence function has axial symmetry, and the problem of determining the influence function is axisymmetric. To determine the Green's function, Laplace and Fourier integral transforms were used.

The novelty of the obtained analytical solution is that the heat transfer at the free surface of the half-space is taken into account. The Green's function that was obtained is used to construct an analytical solution to the moving heat-source problem in the integral form. The kernel of the advising integral operator is the constructed Green's function. The Gaussian distribution is used to analytically calculate the integrals of spatial variables.

Gaussian law models the distribution of heat flux in a laser beam. As a result, the corresponding integrals of the spatial variables can be calculated analytically. A convenient formula that allows one to study the non-stationary temperature distribution when the heat source moves along arbitrary trajectories can be obtained.

In [2], the hyperbolic equation of transient thermal conductivity, accounting for relaxation time, was used to model the laser-heating process. It is assumed that the heat flux is distributed symmetrically with respect to the center of the heating spot. A combined numerical and analytical algorithm was developed and implemented, allowing one to determine the temperature distribution on both the surface of and at deep regions in the half-space. In this case, the principle of superposition was used along with a special symmetric Gaussian distribution to describe the model of a source of high-intensity heat flux. The use of such a symmetric distribution allowed the authors to analytically calculate the integrals of the spatial variables.

The results of this paper could be used to estimate the contribution of the conductive component in the overall heat transfer of materials exposed to intense heat flows (e.g., as in laser surface treatment, laser additive technologies, the streamlining and heating of materials by high-enthalpy gases, etc.).

3. Description of Article [8]

The concept of combining the theories of gravitation and electromagnetism has interested many researchers. Approaches to the unification of these theories have been proposed by Einstein, Eddington, Weyl, Cartan, and others. The corresponding papers use the four-dimensional signature manifold $(1, 3)$, which is the space we observe in everyday life, with one temporal and three spatial coordinates. However, these unifying theories have fundamental shortcomings that have forced researchers to introduce additional dimensions. Thus, the five-dimensional Kaluza–Klein model emerged, combining gravitation and electromagnetism, and so did its various generalizations, such as super-symmetric models, super-gravity theory, and others.

In these models, some compact manifold is added to the basic four-dimensional space-time manifold in the form of writing the direct product. The resulting manifold represents an extended space, and unified theories of gauge fields are constructed on this basis. The gauge fields are induced by symmetry groups.

The number of dimensions introduced by the additional manifold can be quite large. For example, the minimum number of additional dimensions required to construct a gauge theory of super-unification is seven. Furthermore, the geometrical nature of these dimensions of super-space is ambiguous. The commonly accepted way of arranging the basic and additional manifolds is in terms of the bundle theory. The typical consideration in this regard is that the additional manifold is not directly related to the geometric structure of the basic manifold, so the additional dimensions do not have an interpretation associated with the nature of the basic manifolds. This greatly complicates the task of creating a unified and purely geometric theory of interactions. To the best of the authors' knowledge, no sufficiently convincing geometrical constructions have been proposed so far.

The authors attempt to develop the foundations of a unified theory of gravitation and electromagnetism based on single, uniform space-time. The most symmetric case is chosen, that is, the manifold with a symmetric signature (i.e., an equal number of spatial and temporal dimensions), which extends the usual space of a signature $(1, 3)$. The smallest extension of this kind is obtained by adding two temporal dimensions, resulting in a signature of $(3, 3)$. It turns out that such a construction is sufficient for developing a unified theory.

The structure of the cited work is as follows. A manifold combining elements of the structures of Riemann, Weyl, and Finsler spaces (called RWF-space) is introduced for a uniform description of gravitational and electromagnetic interactions. The RWF-space is supplied with a metric tensor of a special kind depending on the coordinates and local velocities. Setting the tensor induces a corresponding field in this space. A definition of the geodesic in RWF-space is given. Geodesic lines are defined by second-order differential equations, whose coefficients can be divided into those depending on the metric tensor (relating to the gravitational interaction) and those depending on the vector field (relating to the electromagnetic interaction). It is shown that when moving along the geodesic, the space remains homogeneous and isotropic.

If there is no gravity, RWF-space transforms into a pseudo-Euclidean space with a signature of $(3, 3)$, and the geodesic equations take the form of the Lorentz equation describing the motion of a unit charge in an electromagnetic field. The connection between the six-dimensional electrodynamics and and the traditional four-dimensional system of Maxwell's equations is outlined. Mapping from six to four dimensions allows for the introduction of the notions of charge and current densities, which have a purely geometric nature. The appearance of the point electric charge is associated with the circulation of the vector potential around a dedicated time axis in the three-dimensional time subspace. Thus, electric charge formation occurs in the unobservable three-dimensional temporal region of six-dimensional space-time, and its existence is manifested in the effects observed in the real three-dimensional physical subspace. These properties of six-dimensional electrodynamics enable the abandonment of the concept of an electric charge in favor of operating exclusively with the components of the electromagnetic tensor in six-dimensional space-time. Traditionally, Maxwell's equations in the four-dimensional theory of electromagnetism are interpreted as relationships between the spatial distribution of the charge density and the electromagnetic field density, i.e., relationships between phenomenological objects without a clear mathematical definition. In six-dimensional electrodynamics, this interpretation changes to a more rigorous and consistent understanding of the equations of electromagnetism as solely relations between components of the electromagnetic tensor in six-dimensional space.

Apparently, the proposed six-dimensional model of classical electrodynamics may aid the adoption of a new perspective of the renormalization problem in quantum electrodynamics. It is well known that permutation functions and Green's functions have singularities on the light cone of four-dimensional space-time. In six-dimensional electrodynamics, due to accounting for the mechanism of electric charge formation, the light cone is replaced by a one-band hyperboloid, which should lead to a revision of the calculation technique. It is probable that the occurrence of meaningless expressions in calculations within traditional four-dimensional quantum electrodynamics is intimately connected with an inappropriate choice of the dimensionality and structure of real physical space-time.

It is also shown that the Maxwell equations are invariant with respect to the group of local eigenmovements of the Minkowski metric, which is wider than the Lorentz group. A mutually unambiguous relation has been established between the admissible currents included in the Maxwell equations and the local eigenmovements of the Minkowski metric. An attempt to extend these results to arbitrary currents led to a change in the form of Maxwell's equations. Thus, Maxwell's equations turn out to be valid not for arbitrary currents, as is currently accepted, but only for a certain class of currents defined by the maximum local group of eigenmovements of the Minkowski metric.

4. Description of Article [9]

As is known, for some areas of theoretical physics, such as wave mechanics, the theory of oscillations, etc., the solutions of problems are reduced to the problem of eigenvalues. In addition, the question of the unambiguous definition of a mechanical system, i.e., the Hamilton function, through the spectrum of the eigenvalues of the linear differential equation with which it is associated it is critical.

Considering the case where a string is vibrating and the boundary conditions are natural, it was shown in [10] that the spectrum of eigenvalues uniquely determines the differential equation, which, in Schrödinger's theory, is called the "amplitude equation".

One paper, [11], deals with the problem of determining the Hill equation (or the one-dimensional Schrödinger equation) based on its spectrum and deriving the Hill equation from the specific properties of its discriminant. A great deal is known about the analytic structure of the discriminant (see, for example, [12,13]).

Numerous fundamental and applied scientific papers and books are devoted to the asymptotic behavior and spectral properties of the Schrödinger operator, and we will note some of them (see, [14–22]).

In particular, in [14,15], the spectral properties of the Schrödinger operator in domains with an infinite boundary, as well as the behavior of the solution in non-stationary problems as $t \to \infty$, are studied.

In [22], regarding a one-dimensional Schrödinger equation with a quasi-periodic analytic potential on its shell, it was shown that the equation presents an Floquet representation for almost any energy value, E, in the upper part of the spectrum. It was also proved that the upper part of the spectrum is purely continuous, and, for the general potential, this is the Cantor set. In addition, the authors also show that for a small potential, these results can be extended to the entire spectrum.

In [9], the asymptotic behavior (as $t \to \infty$) of solutions to an initial-boundary value problem for a second-order hyperbolic equation with periodic coefficients on the half-axis $x > 0$ was considered. The main approach to studying the problem under consideration was based on the spectral theory of differential operators as well as on the properties of the spectrum $\sigma(H_0)$ of the one-dimensional Schrödinger operator H_0 with periodic coefficients $p(x)$ and $q(x)$.

In [23,24], similar questions were considered for the Cauchy problem with initial conditions, as in the case of a positive Hill operator $H_0 > 0$ and the case when the left end of the spectrum $\sigma(H_0)$ of the Hill operator H_0 is non-positive.

We hope that this Special Issue will inspire young talents with natural ambition to make important discoveries in the field of symmetry in the foundations of continuum mechanics and mathematical physics.

Author Contributions: Both authors contributed equally to the review. All authors have read and agreed to the published version of the manuscript.

Acknowledgments: We want to express our gratitude to all the Authors, Assistants, Associate Editors, and Editors for their great work on this Special Issue.

Conflicts of Interest: The authors declare no conflict of interest.

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